Black hole attractors and $\mathrm{U}(1)$ Fayet-Iliopoulos gaugings: analysis and classification

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ABSTRACT: We classify the critical points of the effective black hole potential which governs the attractor mechanism taking place at the horizon of static dyonic extremal black holes in $\mathcal{N} = 2$, $D = 4$ Maxwell-Einstein supergravity with U(1) Fayet-Iliopoulos gaugings. We use a manifestly symplectic covariant formalism, and we consider both spherical and hyperbolic horizons, recognizing the relevant sub-classes to which some representative examples belong. We also exploit projective special Kähler geometry of vector multiplets scalar manifolds, the *U*-duality-invariant quartic structure (and 2-polarizations thereof) in order to retrieve and generalize various expressions of the entropy of asymptotically AdS$_4$ BPS black holes, in the cases in which the scalar manifolds are symmetric spaces. Finally, we present a novel static extremal black hole solution to the *STU* model, in which the dilaton interpolates between an hyperbolic near-horizon geometry and AdS$_4$ at infinity.

Contents

1 Introduction

The relevance of black holes (BHs) in AdS spaces is known for several reasons. One is of course the AdS/CFT correspondence and its several applications, for instance to condensed matter physics (see e.g. [1]), Fermi liquids physics [2], and superconductivity [3], to name a few. In such frameworks, the coupling to electromagnetic charges and scalar fields is of utmost importance, at least in order to deal with for realistic physical models; as a consequence, gauged supergravity models including Abelian gauge fields and coupled to non-linear sigma models quite naturally acquire a key role. On the other hand, BPS solutions provide examples in which supersymmetric conformal field theories are defined on curved backgrounds, the conformal boundaries. However, non-BPS as well as non-extremal BH solutions turn out to have intriguing applications within the holographic paradigm, such as, for example, to finite temperature condensed matter systems. Another important and more recently established framework is the Kerr/CFT correspondence, which offer valuable insights into the microscopic description and computation of BH entropy (cf. e.g. [4, 5]).

The structure of single-center extremal BPS black holes in $\mathcal{N} = 2$, $D = 4$ ungauged supergravity is well known: they are asymptotically flat solutions to Maxwell-Einstein equations, preserving eight supersymmetries at spatial infinity (at which, due to the absence of a gauge potential, scalar fields are unfixed moduli), then breaking all supersymmetry when radially flowing towards the event horizon, and finally restoring half of the supersymmetries when the scalar fields, regardless of their asymptotical values, are attracted to fixed values, purely dependent on the conserved electric and magnetic BH charges, at the spherically symmetric horizon. This is the celebrated attractor mechanism [6–10]. In gauged supergravity, the physical scenarios open up to a wide range of possibilities, one of which will be the object of the present investigation. Recent years witnessed unanticipated progress in finding BPS, as well as non-BPS and non-extremal, thermal BH solutions in generally matter coupled $\mathcal{N} = 2$ gauged supergravity in $D = 4$ space-time dimensions; see for instance [11–27]. In presence of gauging (of the isometries of the scalar manifolds), the supersymmetric BH solutions may be asymptotically AdS$_4$ and preserve all eight supersymmetries, with the scalars being fixed at critical points (at least local minima) of the gauge potential itself. In this framework, the near-horizon geometry of extremal BHs is no more the Bertotti-Robinson conformally-flat AdS$_2 \times S^2$ geometry, but rather generalizes to spherical, hyperbolic and also flat configurations (with generally non-vanishing Weyl tensor).

Whereas the aforementioned attractor mechanism and its exploitation in terms of critical dynamics of an effective black hole potential [10] is well studied in the ungauged theory,

a systematic study of the attractor mechanism and of the corresponding (generalized) effective BH potential is still missing in gauged supergravity, notwithstanding the wealth of possibilities of the gauged scenario. Investigation of attractor flows in presence of gauging started to be carried out in the BPS case in [28] and [29], as well as in [30] and [31] in the effective black hole potential formalism (in [31] the coupling to hypermultiplets was considered, too).

Refs. [30, 31] and [32] provided the construction of an effective BH potential V_{eff} which depends on the gauge potential V, and moreover generalizes the BH potential V_{BH} of the ungauged case (to which it reduces in the limit of zero gauging). At the (unique) event horizon of extremal BHs, the (at least local minima) critical points of V_{eff} govern the attractor mechanism; despite scalar fields are not generally all stabilized at the BH horizon (and thus a moduli space of "flat" directions is present), the non-negative value of V_{eff} at the BH horizon provides the Bekenstein-Hawking BH entropy[1] (in units of π):

$$\frac{S}{\pi} = V_{\text{eff}}|_{\partial V_{\text{eff}}=0}. \tag{1.1}$$

The present paper concerns the classification of the critical points of V_{eff} and the study of the corresponding properties of the extremal BH solutions, in $\mathcal{N} = 2$ $D = 4$ supergravity coupled to vector multiplets in presence of a (generally dyonic) U(1) Fayet-Iliopoulos (FI) gauging. By denoting the number of vector multiplets with n_v, and developing on the findings of [32], we will exploit a manifestly symplectic, $\text{Sp}(2n_v+2, \mathbb{R})$-covariant formalism. Furthermore, we will use structural identities of the special Kähler geometry of vector multiplets' scalar manifolds in order to completely classify all the possible extremal BH solutions with spherical or hyperbolic near-geometries. As it will be evident from the treatment below, our analysis encompasses both BPS and non-BPS configurations, and we will provide detailed analysis of BPS sub-sectors throughout.

Upon extremizing V_{eff}, two main classes of critical points arise out; namely:

Class I, corresponding to critical points of V_{eff} which are also critical points of both V_{BH} and V (all placed at the horizon):

$$\left.\begin{array}{r}\partial_i V_{BH} = 0; \\ \partial_i V = 0;\end{array}\right\} \Rightarrow \partial_i V_{\text{eff}} = 0, \ \forall i. \tag{1.2}$$

Class II, corresponding to critical points of V_{eff} which are *not* critical points of V_{BH} *nor* of V, with the gradients of V_{BH} and of V being proportional:

$$\partial_i V_{BH} = \frac{(2V_{BH}V + \kappa\sqrt{1 - 4V_{BH}V} - 1)}{2V^2}\partial_i V, \ \forall i. \tag{1.3}$$

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[1] In general, proper extremal BH attractors are defined by (*at least* local) minima of the effective BH potential, both in the ungauged and gauged theory. For what concerns the ungauged case, in the symmetric cosets of special geometry, all critical points of V_{BH} are characterized by an Hessian matrix with strictly non-negative eigenvalues (with vanishing eigenvalues corresponding to "flat" directions of V_{BH} itself) [33]. In the gauged framework under consideration, we are assuming the same to hold for the critical points of V_{eff}; indeed, the zero Hessian eigenvalues seems to be ubiquitous also in presence of gauging (see e.g. [13]). We leave a detailed analysis of the Hessian modes at the critical points of V_{eff} for further future work.

For both classes, at least for *symmetric* scalar manifolds' geometries, the BH entropy can be related to suitable (2-)polarizations of a quartic structure, invariant under U-duality,[2] and primitive in all cases but for minimal coupling of the vector multiplets [36, 37]. Thus, our analysis provides an extension of the analysis of [38] (subsequently developed in [39]), in which algebraic BPS equations supported by generally dyonic charge configuration, and with a cubic prepotential function, were solved. Thence, we will recognize some examples among the currently known solutions as belonging to a corresponding sub-class of the aforementioned two main classes of critical points of V_{eff}. It should also be remarked here that an interesting outcome is provided by the explicit construction of a novel static extremal BH solution in U(1) FI gauged supergravity, supported by both non-BPS and BPS charge configurations.

All in all, the general structure of this paper splits up into three main parts:

1. In the first part (sections 2–4), we will exploit special Kähler geometry and the 2-polarizations of the quartic invariant structure in symmetric special cosets, in order to retrieve, and further generalize in various ways, some known results on the entropy of extremal BH attractors.

2. In the second part (sections 6–11), we consider the effective BH potential V_{eff} introduced in [30, 31, 40] and provide a complete classification of its critical points, pointing out the existence of various (yet undiscovered) BPS sub-sectors.

3. In the third part (section 12), we will provide some examples of known solutions, and determine their placement in the classification given in the second part. Furthermore, we will also present a novel static extremal BH solution to the *STU* model, in which the dilaton interpolates between an hyperbolic near-horizon geometry and AdS$_4$ at infinity.

Some final remarks and four appendices conclude the paper.

2 Identities and fluxes in projective special geometry

We start by introducing the symplectic vectors $\mathcal{Q}$ and $\mathcal{G}$ of electric-magnetic black hole fluxes resp. U(1) Fayet-Iliopoulos (FI) gaugings of $\mathcal{N} = 2$, $D = 4$ Maxwell-Einstein supergravity , which in the so-called 4D/5D special coordinates' symplectic frame can be written as

$$\mathcal{Q} := \left(p^0, p^i, q_0, q_i \right)^T ; \tag{2.1}$$

$$\mathcal{G} := \left(g^0, g^i, g_0, g_i \right)^T , \tag{2.2}$$

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[2]Here, U-duality is referred to as the "continuous" symmetries of [34]; their discrete versions are the U-duality non-perturbative string theory symmetries introduced by Hull and Townsend [35], which occur when the Dirac-Schwinger-Zwanzinger quantization condition is enforced.

where the naught index pertains to the graviphoton, and $i = 1, \ldots, n$, with n denoting the number of vector multiplets.[3] Moreover, κ is related to the constant scalar curvature $R = 2\kappa$ of the (unique) event horizon of the static extremal black hole solution under consideration. In the following treatment we will consider $\kappa = 1$ (spherical) or $\kappa = -1$ (hyperbolic) near-horizon geometry.[4]

The following identities holds in the projective special Kähler geometry of the vector multiplets' scalar manifold[5] M_v (with $\dim_{\mathbb{C}} M_v = n$; cf. e.g. [40–42], and refs. therein):

$$\mathcal{Q} = \mathbf{i}\overline{\mathcal{Z}}\mathcal{V} - \mathbf{i}\mathcal{Z}\overline{\mathcal{V}} + \mathbf{i}\mathcal{Z}^{\bar{\imath}}\overline{\mathcal{V}}_{\bar{\imath}} - \mathbf{i}\overline{\mathcal{Z}}^i\mathcal{V}_i; \tag{2.3}$$

$$\mathcal{G} = \mathbf{i}\overline{\mathcal{L}}\mathcal{V} - \mathbf{i}\mathcal{L}\overline{\mathcal{V}} + \mathbf{i}\mathcal{L}^{\bar{\imath}}\overline{\mathcal{V}}_{\bar{\imath}} - \mathbf{i}\overline{\mathcal{L}}^i\mathcal{V}_i, \tag{2.4}$$

where

$$\mathcal{Z} := \langle \mathcal{Q}, \mathcal{V} \rangle, \ \mathcal{Z}_i \equiv D_i\mathcal{Z} := \langle \mathcal{Q}, \mathcal{V}_i \rangle, \tag{2.5}$$

$$\mathcal{L} := \langle \mathcal{G}, \mathcal{V} \rangle, \ \mathcal{L}_i \equiv D_i\mathcal{L} := \langle \mathcal{G}, \mathcal{V}_i \rangle, \tag{2.6}$$

with $\langle \cdot, \cdot \rangle$ denoting the symplectic product defined in the flat symplectic bundle constructed over the special Kähler-Hodge manifold M_v. We adopt the notation of [40–42]; see also appendix C.

By using the results of [43–45], one can prove the following "two-centered" special Kähler identities:

$$\frac{1}{2} \langle \mathcal{Q}, \mathcal{G} \rangle = -\text{Im} \left(\mathcal{Z}\overline{\mathcal{L}} \right) + \text{Im} \left(\mathcal{Z}_i\overline{\mathcal{L}}^i \right) = -\text{Im} \left(\mathcal{Z}\overline{\mathcal{L}} - \mathcal{Z}_i\overline{\mathcal{L}}^i \right); \tag{2.7}$$

$$-\frac{1}{2}\mathcal{Q}^T \mathcal{M}\left(\mathcal{N}\right)\mathcal{G} = \text{Re} \left(\mathcal{Z}\overline{\mathcal{L}} \right) + \text{Re} \left(\mathcal{Z}_i\overline{\mathcal{L}}^i \right) = \text{Re} \left(\mathcal{Z}\overline{\mathcal{L}} + \mathcal{Z}_i\overline{\mathcal{L}}^i \right); \tag{2.8}$$

$$\frac{1}{2}\mathcal{Q}^T \mathcal{M}\left(\mathcal{F}\right)\mathcal{G} = -\text{Re} \left(\mathcal{Z}\overline{\mathcal{L}} \right) + \text{Re} \left(\mathcal{Z}_i\overline{\mathcal{L}}^i \right) = -\text{Re} \left(\mathcal{Z}\overline{\mathcal{L}} - \mathcal{Z}_i\overline{\mathcal{L}}^i \right), \tag{2.9}$$

where $\overline{\mathcal{L}}^i = g^{i\bar{\jmath}}\overline{\mathcal{L}}_{\bar{\jmath}}$, and $\mathcal{N} = \mathcal{N}_{\Lambda\Sigma}$ and $\mathcal{F} = F_{\Lambda\Sigma}$ respectively are the (complexified) kinetic vector matrix and the Hessian matrix of the prepotential F. The symplectic, real, symmetric $(2n + 2) \times (2n + 2)$ matrix $\mathcal{M}\left(\mathcal{N}\right)$ is defined as

$$\mathcal{M}\left(\mathcal{N}\right) = \begin{pmatrix} \text{Im}\left(\mathcal{N}\right) + \text{Re}\left(\mathcal{N}\right)\text{Im}^{-1}\left(\mathcal{N}\right)\text{Re}\left(\mathcal{N}\right) & -\text{Re}\left(\mathcal{N}\right)\text{Im}^{-1}\left(\mathcal{N}\right) \\ -\text{Im}^{-1}\left(\mathcal{N}\right)\text{Re}\left(\mathcal{N}\right) & \text{Im}^{-1}\left(\mathcal{N}\right) \end{pmatrix}, \tag{2.10}$$

and $\mathcal{M}\left(\mathcal{F}\right)$ is defined the same way, with $\mathcal{N}_{\Lambda\Sigma} \to F_{\Lambda\Sigma}$. In terms of the covariantly holomorphic sections $\mathcal{V}$ and of its covariant derivatives $\mathcal{V}_i$, such two matrices have the following

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[3]In order to compare our results to Halmagy's treatment [52], we here only deal with U(1) FI gauging (namely, only vector multiplets). After [32], it is however possible to straightforwardly include also hypermultiplets' Abelian gaugings, by simply replacing $\mathcal{G}$ with $\mathcal{P} := \mathcal{P}^x Q^x$. In this case, no assumptions on the geometry of the hypermultiplets' scalar manifold are needed. We leave the detailed treatment of such a framework to future investigation.

[4]The case $\kappa = 0$, corresponding to extremal black holes with flat horizon, deserves a separate treatment, which we leave for future investigation.

[5]We will henceforth denote the imaginary unit as $\mathbf{i}$.

expressions (see e.g. [44] and refs. therein):

$$\mathcal{M}(\mathcal{N}) = \Omega \left(\mathcal{V}\bar{\mathcal{V}}^T + \bar{\mathcal{V}}\mathcal{V}^T + \mathcal{V}_i\, g^{i\bar{j}}\bar{\mathcal{V}}_{\bar{j}}^T + \bar{\mathcal{V}}_{\bar{j}}g^{\bar{j}i}\mathcal{V}_i^T \right) \Omega \tag{2.11}$$

$$\mathcal{M}(\mathcal{F}) = \Omega \left(\mathcal{V}\bar{\mathcal{V}}^T + \bar{\mathcal{V}}\mathcal{V}^T - \mathcal{V}_i\, g^{i\bar{j}}\bar{\mathcal{V}}_{\bar{j}}^T - \bar{\mathcal{V}}_{\bar{j}}g^{\bar{j}i}\mathcal{V}_i^T \right) \Omega \tag{2.12}$$

where

$$\Omega := \begin{pmatrix} 0 & -\mathbb{I}_{n+1} \\ \mathbb{I}_{n+1} & 0 \end{pmatrix} \tag{2.13}$$

is the symplectic metric. Thus:

$$\frac{1}{2}\mathcal{Q}^T\mathcal{M}\left(\mathcal{F}\right)\mathcal{G} + \frac{\mathbf{i}}{2}\left\langle \mathcal{Q},\mathcal{G} \right\rangle = -\mathcal{Z}\overline{\mathcal{L}} + \mathcal{Z}_i\overline{\mathcal{L}}^i; \tag{2.14}$$

$$-\frac{1}{2}\mathcal{Q}^T\mathcal{M}\left(\mathcal{N}\right)\mathcal{G} - \frac{\mathbf{i}}{2}\left\langle \mathcal{Q},\mathcal{G} \right\rangle = \mathcal{Z}\overline{\mathcal{L}} + \mathcal{L}_i\overline{\mathcal{Z}}^i. \tag{2.15}$$

By denoting with

$$\mathfrak{H} := e^{-K/2}\mathcal{V}; \tag{2.16}$$

$$\mathfrak{H}_i := e^{-K/2}\mathcal{V}_i, \tag{2.17}$$

the holomorphic symplectic sections (such that $\partial_{\bar{i}}\mathfrak{H} = 0$ and $\partial_{\bar{i}}\mathfrak{H}_j = 0$), using the properties (cf. e.g. [41, 42])

$$\left\langle \mathfrak{H}, \overline{\mathfrak{H}} \right\rangle = -\mathbf{i}e^{-K}; \tag{2.18}$$

$$\left\langle \mathfrak{H}_i, \overline{\mathfrak{H}}_{\bar{j}} \right\rangle = \mathbf{i}e^{-K}g_{i\bar{j}}, \tag{2.19}$$

and defining the superpotential $\mathcal{W}$ and "gauging-superpotential" $\mathcal{Y}$ respectively as

$$\mathcal{W} := e^{-K/2}\mathcal{Z}, \ \mathcal{W}_i := e^{-K/2}\mathcal{Z}_i, \tag{2.20}$$

$$\mathcal{Y} := e^{-K/2}\mathcal{L}, \ \mathcal{Y}_i := e^{-K/2}\mathcal{L}_i, \tag{2.21}$$

Eqs. (2.7)–(2.9) can be rewritten as follows:

$$\frac{1}{2}\left\langle \mathcal{Q},\mathcal{G} \right\rangle = -\mathbf{i}e^{2K}\mathrm{Im}\left(\mathcal{W}\overline{\mathcal{Y}}\right)\left\langle \mathfrak{H},\overline{\mathfrak{H}} \right\rangle - \mathbf{i}e^{2K}\mathrm{Im}\left(\mathcal{W}_i\overline{\mathcal{Y}}_{\bar{j}}\right)\left\langle \mathfrak{H}^{\bar{j}},\overline{\mathfrak{H}}^i \right\rangle; \tag{2.22}$$

$$-\frac{1}{2}\mathcal{Q}^T\mathcal{M}\left(\mathcal{N}\right)\mathcal{G} = \mathbf{i}e^{2K}\mathrm{Re}\left(\mathcal{W}\overline{\mathcal{Y}}\right)\left\langle \mathfrak{H},\overline{\mathfrak{H}} \right\rangle - \mathbf{i}e^{2K}\mathrm{Re}\left(\mathcal{W}_i\overline{\mathcal{Y}}_{\bar{j}}\right)\left\langle \mathfrak{H}^{\bar{j}},\overline{\mathfrak{H}}^i \right\rangle; \tag{2.23}$$

$$\frac{1}{2}\mathcal{Q}^T\mathcal{M}\left(\mathcal{F}\right)\mathcal{G} = -\mathbf{i}e^{2K}\mathrm{Re}\left(\mathcal{W}\overline{\mathcal{Y}}\right)\left\langle \mathfrak{H},\overline{\mathfrak{H}} \right\rangle - \mathbf{i}e^{2K}\mathrm{Re}\left(\mathcal{W}_i\overline{\mathcal{Y}}_{\bar{j}}\right)\left\langle \mathfrak{H}^{\bar{j}},\overline{\mathfrak{H}}^i \right\rangle. \tag{2.24}$$

3 Symmetric very special geometry and quartic 2-polarizations

The above identities hold in the projective special Kähler geometry of the vector multiplets' scalar manifold M_v, regardless of the data specifying such a manifold.

Instead, we will now specialize the treatment by assuming M_v to be a *symmetric* (homogeneous) coset space, whose (local) geometry is given by a cubic holomorphic pre-potential

$$F\left(X\right) := \frac{1}{3!}d_{ijk}\frac{X^iX^jX^k}{X^0}, \tag{3.1}$$

with the cubic symmetric constant tensor d_{ijk} satisfying the so-called *adjoint identity*,

$$d_{i(kl}d_{j|mn)}d^{ijp} = \frac{4}{3}\delta^p_{(k}d_{lmn)}, \tag{3.2}$$

or equivalently

$$C_{i(kl}C_{j|mn)}\overline{C}^{ijp} = \frac{4}{3}\delta^p_{(k}C_{lmn)}, \tag{3.3}$$

where C_{ijk} is the Kähler covariantly holomorphic rank-3 symmetric tensor occurring in the identities

$$D_i\mathcal{V}_j = \mathbf{i}C_{ijk}\overline{\mathcal{V}}^k, \tag{3.4}$$

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}}g_{k\bar{l}} - g_{i\bar{l}}g_{k\bar{j}} + C_{ikm}\overline{C}_{\bar{j}\bar{l}\bar{p}}g^{m\bar{p}}, \tag{3.5}$$

with $R_{i\bar{j}k\bar{l}}$ denoting the Riemann tensor of M_v. The d_{ijk}'s and duality structures of the corresponding M_v's have been classified in [47] and [48].

In this framework, the ring of invariant homogeneous polynomials under the non-transitive action of the electric-magnetic duality group on its representation space **R** in which both the aforementioned symplectic vectors $\mathcal{Q}$ (2.1) and $\mathcal{G}$ (2.2) sit, is granted to be one-dimensional, and finitely generated by a *primitive*[6] quartic homogeneous polynomial, denoted by I_4 and associated to the rank-4 completely symmetric invariant tensor K_{MNPQ} [49] (see also [50] and [51]); for instance, considering the symplectic vector $\mathcal{Q}$ (2.1)$\in$ **R**, one can define

$$I_4(\mathcal{Q},\mathcal{Q},\mathcal{Q},\mathcal{Q}) := \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q. \tag{3.6}$$

The explicit expression of the rank-4 invariant symmetric tensor $K_{MNPQ} = K_{(MNPQ)}$ is given by (D.1) of [55] in the so-called 4D/5D special coordinate symplectic frame [48, 53], as well as by eq. (5.36) of [56] and by (4.4)-(4.14) of section 4.3 of [57] in a way independent from the symplectic frame (and manifestly invariant under diffeomorphisms in M_v).

For the treatment given in the present paper, we will need to explicitly compute the *2-polarizations* of I_4 [57–60]:

$$I_4(\mathcal{Q}+\mathcal{G},\mathcal{Q}+\mathcal{G},\mathcal{Q}+\mathcal{G},\mathcal{Q}+\mathcal{G}) =: \mathbf{I}_2 + 4\mathbf{I}_1 + 6\mathbf{I}_0 + 4\mathbf{I}_{-1} + \mathbf{I}_{-2}, \tag{3.7}$$

where[7]

$$\mathbf{I}_2 := I_4(\mathcal{Q},\mathcal{Q},\mathcal{Q},\mathcal{Q}) = \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q \tag{3.8}$$

$$= -\left(p^0q_0 + p^iq_i\right)^2 + \frac{2}{3}q_0d_{ijk}p^ip^jp^k - \frac{2}{3}p^0d^{ijk}q_iq_jq_k + d_{ijk}d^{ilm}p^jp^kq_lq_m \tag{3.9}$$

$$= \left(|\mathcal{Z}|^2 - |\mathcal{Z}_i|^2\right)^2 - \frac{4}{3}\text{Im}\left(\mathcal{Z}\overline{C}_{\bar{i}\bar{j}\bar{k}}\mathcal{Z}^{\bar{i}}\mathcal{Z}^{\bar{j}}\mathcal{Z}^{\bar{k}}\right) - g^{i\bar{j}}C_{ikl}\overline{C}_{\bar{j}\bar{m}\bar{n}}\overline{\mathcal{Z}}^k\overline{\mathcal{Z}}^l\mathcal{Z}^{\bar{m}}\mathcal{Z}^{\bar{n}}; \tag{3.10}$$

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[6]Primitivity of I_4, i.e. the fact that the corresponding invariant tensor K_{MNPQ} cannot be reduced in terms of other lower-rank tensors, generally holds for all symmetric M_v's characterized by the cubic holomorphic prepotential (3.1). However, in a peculiar sub-class of BPS black holes, treated in section 5.3.1, I_4 becomes a *perfect square* (and so are all its non-vanishing 2-polarizations; cf. (5.34)–(5.38)).

[7]Note that throughout our treatment $|\mathcal{Z}_i|^2$ and $|\mathcal{L}_i|^2$ are shorthand for $\sum_{i,j=1}^n \mathcal{Z}_i\overline{\mathcal{Z}}_{\bar{j}}g^{i\bar{j}}$ and $\sum_{i,j=1}^n \mathcal{L}_i\overline{\mathcal{L}}_{\bar{j}}g^{i\bar{j}}$, respectively (unless otherwise specified).

$$\mathbf{I_1} \; := \; I_4\left(\mathcal{Q},\mathcal{Q},\mathcal{Q},\mathcal{G}\right) = \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{G}^Q \tag{3.11}$$

$$= -\frac{1}{2}\left[\left(p^0\right)^2 q_0 g_0 + p^0 g^0 q_0^2\right] - \frac{1}{2}\left(p^i p^j q_i g_j + p^i g^j q_i q_j\right)$$

$$\quad - \frac{1}{2}\left(p^0 q_0 p^i g_i + p^0 q_0 g^i q_i + p^0 g_0 p^i q_i + g^0 q_0 p^i q_i\right)$$

$$\quad + \frac{1}{6}\left(g_0 d_{ijk} p^i p^j p^k + 3 q_0 d_{ijk} p^i p^j g^k\right) - \frac{1}{6}\left(g^0 d^{ijk} q_i q_j q_k + 3 p^0 d^{ijk} q_i q_j g_k\right)$$

$$\quad + \frac{1}{2}d_{ijk}d^{ilm}\left(p^j p^k q_l g_m + p^j g^k q_l q_m\right) \tag{3.12}$$

$$= \frac{1}{2}\left(|\mathcal{Z}|^2 - |\mathcal{Z}_i|^2\right)\left(\mathcal{Z}\overline{\mathcal{L}} + \overline{\mathcal{Z}}\mathcal{L} - \mathcal{Z}^{\bar{\jmath}}\overline{\mathcal{L}}_{\bar{\jmath}} - \overline{\mathcal{Z}}^j\mathcal{L}_j\right)$$

$$\quad - \frac{1}{3}\mathrm{Im}\left[\left(3\mathcal{Z}\mathcal{L}^{\bar{k}} + \mathcal{L}\mathcal{Z}^{\bar{k}}\right)\overline{C}_{\bar{\imath}\bar{\jmath}\bar{k}}\mathcal{Z}^{\bar{\imath}}\mathcal{Z}^{\bar{\jmath}}\right] - \frac{1}{2}g^{i\bar{\jmath}}C_{ikl}\overline{C}_{\bar{\jmath}\bar{m}\bar{n}}\overline{\mathcal{Z}}^k\mathcal{Z}^{\bar{n}}\left(\overline{\mathcal{Z}}^l\mathcal{L}^{\bar{m}} + \overline{\mathcal{L}}^l\mathcal{Z}^{\bar{m}}\right); \tag{3.13}$$

$$\mathbf{I_0} \; := \; I_4\left(\mathcal{Q},\mathcal{Q},\mathcal{G},\mathcal{G}\right) = \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{G}^P\mathcal{G}^Q \tag{3.14}$$

$$= -\frac{1}{6}\left[\left(p^0\right)^2 g_0^2 + \left(g^0\right)^2 q_0^2 + 4 p^0 g^0 q_0 g_0\right]$$

$$\quad - \frac{1}{6}\left[\left(p^i g_i\right)^2 + \left(g^i q_i\right)^2 + 2 p^i g^j q_i g_j + 2 p^i g^j q_j g_i\right]$$

$$\quad - \frac{1}{3}\left(p^0 q_0 g^i g_i + p^0 g_0 p^i g_i + p^0 g_0 g^i q_i + g^0 q_0 p^i g_i + g^0 q_0 g^i q_i + g^0 g_0 p^i q_i\right)$$

$$\quad + \frac{1}{3}\left(g_0 d_{ijk} p^i p^j g^k + q_0 d_{ijk} p^i g^j g^k\right) - \frac{1}{3}\left(g^0 d^{ijk} q_i q_j g_k + p^0 d^{ijk} q_i g_j g_k\right)$$

$$\quad + \frac{1}{6}d_{ijk}d^{ilm}\left(p^j p^k g_l g_m + 4 p^j g^k q_l g_m + g^j g^k q_l q_m\right) \tag{3.15}$$

$$= \frac{1}{3}\left(|\mathcal{Z}|^2 - |\mathcal{Z}_i|^2\right)\left(|\mathcal{L}|^2 - |\mathcal{L}_i|^2\right) + \frac{1}{6}\left(\mathcal{Z}\overline{\mathcal{L}} + \overline{\mathcal{Z}}\mathcal{L} - \mathcal{Z}^{\bar{\imath}}\overline{\mathcal{L}}_{\bar{\imath}} - \overline{\mathcal{Z}}^i\mathcal{L}_i\right)^2$$

$$\quad - \frac{2}{3}\mathrm{Im}\left[\left(\mathcal{Z}\mathcal{L}^{\bar{k}} + \mathcal{L}\mathcal{Z}^{\bar{k}}\right)\overline{C}_{\bar{\imath}\bar{\jmath}\bar{k}}\mathcal{Z}^{\bar{\imath}}\mathcal{L}^{\bar{\jmath}}\right]$$

$$\quad - \frac{1}{6}g^{i\bar{\jmath}}C_{ikl}\overline{C}_{\bar{\jmath}\bar{m}\bar{n}}\left(4\overline{\mathcal{Z}}^k\mathcal{Z}^{\bar{m}}\overline{\mathcal{L}}^l\mathcal{L}^{\bar{n}} + \overline{\mathcal{Z}}^k\overline{\mathcal{Z}}^l\mathcal{L}^{\bar{m}}\mathcal{L}^{\bar{n}} + \mathcal{Z}^{\bar{m}}\mathcal{Z}^{\bar{n}}\overline{\mathcal{L}}^k\overline{\mathcal{L}}^l\right); \tag{3.16}$$

$$\mathbf{I_{-1}} \; := \; I_4\left(\mathcal{Q},\mathcal{G},\mathcal{G},\mathcal{G}\right) = \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{G}^N\mathcal{G}^P\mathcal{G}^Q \tag{3.17}$$

$$= -\frac{1}{2}\left[\left(g^0\right)^2 q_0 g_0 + p^0 g^0 g_0^2\right] - \frac{1}{2}\left(p^i g^j g_i g_j + g^i g^j q_i g_j\right)$$

$$\quad - \frac{1}{2}\left(p^0 g_0 g^i g_i + g^0 q_0 g^i g_i + g^0 g_0 p^i g_i + g^0 g_0 g^i q_i\right)$$

$$\quad + \frac{1}{6}\left(3 g_0 d_{ijk} p^i g^j g^k + q_0 d_{ijk} g^i g^j g^k\right) - \frac{1}{6}\left(3 g^0 d^{ijk} q_i g_j g_k + p^0 d^{ijk} g_i g_j g_k\right)$$

$$\quad + \frac{1}{2}d_{ijk}d^{ilm}\left(p^j g^k g_l g_m + g^j g^k q_l g_m\right) \tag{3.18}$$

$$= \frac{1}{2}\left(|\mathcal{L}|^2 - |\mathcal{L}_i|^2\right)\left(\mathcal{Z}\overline{\mathcal{L}} + \overline{\mathcal{Z}}\mathcal{L} - \mathcal{Z}^{\bar{\jmath}}\overline{\mathcal{L}}_{\bar{\jmath}} - \overline{\mathcal{Z}}^j\mathcal{L}_j\right)$$

$$\quad - \frac{1}{3}\mathrm{Im}\left[\left(\mathcal{Z}\mathcal{L}^{\bar{k}} + 3\mathcal{L}\mathcal{Z}^{\bar{k}}\right)\overline{C}_{\bar{\imath}\bar{\jmath}\bar{k}}\mathcal{L}^{\bar{\imath}}\mathcal{L}^{\bar{\jmath}}\right] - \frac{1}{2}g^{i\bar{\jmath}}C_{ikl}\overline{C}_{\bar{\jmath}\bar{m}\bar{n}}\overline{\mathcal{L}}^k\mathcal{L}^{\bar{n}}\left(\mathcal{Z}^{\bar{m}}\overline{\mathcal{L}}^l + \overline{\mathcal{Z}}^l\mathcal{L}^{\bar{m}}\right); \tag{3.19}$$

$$\mathbf{I}_{-2} := I_4\left(\mathcal{G},\mathcal{G},\mathcal{G},\mathcal{G}\right) = \frac{1}{2}K_{MNPQ}\mathcal{G}^M\mathcal{G}^N\mathcal{G}^P\mathcal{G}^Q \tag{3.20}$$

$$= -\left(g^0 g_0 + g^i g_i\right)^2 + \frac{2}{3}g_0 d_{ijk}g^i g^j g^k - \frac{2}{3}g^0 d^{ijk}g_i g_j g_k + d_{ijk}d^{ilm}g^j g^k g_l g_m \tag{3.21}$$

$$= \left(|\mathcal{L}|^2 - |\mathcal{L}_i|^2\right)^2 - \frac{4}{3}\mathrm{Im}\left(\mathcal{L}\overline{C}_{\overline{ijk}}\mathcal{L}^{\bar{i}}\mathcal{L}^{\bar{j}}\mathcal{L}^{\bar{k}}\right) - g^{i\bar{j}}C_{ikl}\overline{C}_{\overline{jmn}}\overline{\mathcal{L}}^k\overline{\mathcal{L}}^l\mathcal{L}^{\overline{m}}\mathcal{L}^{\overline{n}}. \tag{3.22}$$

Notice that

$$\mathbf{I}_{-1} = \mathbf{I}_1|_{\mathcal{Q}\leftrightarrow\mathcal{G}}; \tag{3.23}$$

$$\mathbf{I}_{-2} = \mathbf{I}_2|_{\mathcal{Q}\to\mathcal{G}}; \tag{3.24}$$

$$\mathbf{I}_0 = \mathbf{I}_0|_{\mathcal{Q}\leftrightarrow\mathcal{G}}. \tag{3.25}$$

Furthermore, $\mathbf{I}_2$, $\mathbf{I}_1$, $\mathbf{I}_0$, $\mathbf{I}_{-1}$ and $\mathbf{I}_{-2}$ form a spin-2 (quintet) representation of the would-be horizontal symmetry $SL_h(2,\mathbb{R})$ acting on the doublet $(\mathcal{Q},\mathcal{G})^T$ [58].

4 BPS black hole entropy...

In [28], within the assumption of mutual non-locality[8]

$$\langle\mathcal{G},\mathcal{Q}\rangle = -\kappa, \tag{4.1}$$

the general form of the ($\frac{1}{4}$-)BPS conditions were obtained to read

$$\mathcal{Z} = \mathbf{i}\kappa S\mathcal{L}; \tag{4.2}$$

$$\mathcal{Z}_i = \mathbf{i}\kappa S\mathcal{L}_i, \tag{4.3}$$

where S denotes the Bekenstein-Hawking entropy[9] of the extremal BPS black hole solution. Note that, by virtue of the identity (2.7), the mutual non-locality condition (4.1) can be rewritten as

$$2\mathrm{Im}\left(\mathcal{Z}\overline{\mathcal{L}} - \mathcal{Z}_i\overline{\mathcal{L}}^i\right) = -\kappa. \tag{4.4}$$

From (4.2)–(4.3), one obtains the following expressions of the Bekenstein-Hawking entropy S of BPS extremal black holes (no sum on repeated indices)

$$S = -\mathbf{i}\kappa\frac{\mathcal{Z}}{\mathcal{L}} = -\mathbf{i}\kappa\frac{\mathcal{Z}_j}{\mathcal{L}_j}, \ \forall j, \tag{4.5}$$

implying

$$S = -\mathbf{i}\kappa\frac{\mathcal{Z}}{\mathcal{L}}\frac{\overline{\mathcal{L}}}{\overline{\mathcal{L}}} = -\mathbf{i}\kappa\frac{\mathcal{Z}\overline{\mathcal{L}}}{|\mathcal{L}|^2} = -\frac{\mathbf{i}\kappa}{2|\mathcal{L}|^2}\left(\mathcal{Z}\overline{\mathcal{L}} - \overline{\mathcal{Z}}\mathcal{L}\right) = \frac{\kappa\mathrm{Im}\left(\mathcal{Z}\overline{\mathcal{L}}\right)}{|\mathcal{L}|^2}, \tag{4.6}$$

and

$$S = -\mathbf{i}\kappa\frac{\mathcal{Z}_j\overline{\mathcal{L}}^j}{\mathcal{L}_j\overline{\mathcal{L}}^j} = \frac{-\mathbf{i}\kappa}{2\mathcal{L}_j\overline{\mathcal{L}}^j}\left(\mathcal{Z}_j\overline{\mathcal{L}}^j - \overline{\mathcal{Z}}_{\bar{j}}\mathcal{L}^{\bar{j}}\right) = \frac{\kappa\mathrm{Im}\left(\mathcal{Z}_j\overline{\mathcal{L}}^j\right)}{\mathcal{L}_j\overline{\mathcal{L}}^j}, \tag{4.7}$$

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[8]Actually, [28] only dealt with spherical horizon ($\kappa = 1$). The derivation of BPS conditions for hyperbolic horizon ($\kappa = -1$) was done in [32].

[9]In units of π, as understood throughout all the treatment.

where no summation on repeated indices is performed. From (4.6) and (4.7), one obtains

$$|\mathcal{L}|^2 \, S - |\mathcal{L}_i|^2 \, S = \kappa \text{Im} \left(\mathcal{Z}\overline{\mathcal{L}} - \mathcal{Z}_i \overline{\mathcal{L}}^i \right) = \frac{\kappa}{2} \left\langle \mathcal{G}, \mathcal{Q} \right\rangle; \tag{4.8}$$

$$\Updownarrow$$

$$S \left(|\mathcal{L}|^2 - |\mathcal{L}_i|^2 \right) = \frac{\kappa}{2} \left\langle \mathcal{G}, \mathcal{Q} \right\rangle, \tag{4.9}$$

where we recall that n is the number of vector multiplets (or, equivalently, the complex dimension of M_v), and the identity (2.7) has been used in the last step.

We should here also recall another expression for the BPS entropy in the case $\kappa = 1$, obtained in [28] by studying the near-horizon dynamics:

$$S = 2 \left(|\mathcal{Z}_i|^2 - |\mathcal{Z}|^2 \right) = \frac{1}{2 \left(|\mathcal{L}_i|^2 - |\mathcal{L}|^2 \right)} = \mathcal{Q}^T \mathcal{M}(\mathcal{F})\mathcal{Q} = \frac{1}{\mathcal{G}^T \mathcal{M}(\mathcal{F})\mathcal{G}}, \tag{4.10}$$

implying

$$S = \sqrt{\frac{|\mathcal{Z}|^2 - |\mathcal{Z}_i|^2}{|\mathcal{L}|^2 - |\mathcal{L}_i|^2}} = \sqrt{\frac{\mathcal{Q}^T \mathcal{M}(\mathcal{F})\mathcal{Q}}{\mathcal{G}^T \mathcal{M}(\mathcal{F})\mathcal{G}}}, \tag{4.11}$$

where we have recalled the symplectic-invariant quadratic form of projective special Kähler geometry defined by the matrix $\mathcal{M}(\mathcal{F})$ (see (2.10)–(2.12)). We also observe that (4.9) (with $\kappa = 1$) and (4.10) consistently imply (4.1) with $\kappa = 1$, because

$$S \left(|\mathcal{L}|^2 - |\mathcal{L}_i|^2 \right) = \frac{1}{2} \left\langle \mathcal{G}, \mathcal{Q} \right\rangle \Leftrightarrow \left\langle \mathcal{G}, \mathcal{Q} \right\rangle = -1. \tag{4.12}$$

By plugging (4.2)–(4.3) into (2.4), one obtains

$$\mathbf{i}\mathcal{Q} = \kappa \, S\mathcal{G} + 2\mathcal{Z}\overline{\mathcal{V}} - 2\mathcal{Z}^{\bar{i}}\overline{\mathcal{V}}_{\bar{i}}, \tag{4.13}$$

a relation which holds at the event horizon of the BPS extremal black hole.

Within these conventions, we can write the following symplectic products, that will be useful later,

$$\left\langle \mathcal{V}, \overline{\mathcal{V}} \right\rangle = -\mathbf{i}, \qquad \left\langle \mathcal{V}_i, \overline{\mathcal{V}}_{\bar{j}} \right\rangle = \mathbf{i} g_{i\bar{j}}, \quad \left\langle \mathcal{V}_i, \overline{\mathcal{V}}^j \right\rangle = \mathbf{i}\delta_i^j, \qquad \left\langle \overline{\mathcal{V}}_{\bar{i}}, \mathcal{V}^{\bar{j}} \right\rangle = -\mathbf{i}\delta_{\bar{i}}^{\bar{j}}, \tag{4.14}$$

$$\left\langle \overline{\mathcal{V}}, \mathcal{V}_i \right\rangle = \left\langle \overline{\mathcal{V}}_{\bar{i}}, \mathcal{V} \right\rangle = 0, \qquad \left\langle \mathcal{V}_i, \mathcal{V}_j \right\rangle = \left\langle \overline{\mathcal{V}}_{\bar{i}}, \overline{\mathcal{V}}_{\bar{j}} \right\rangle = 0, \quad \left\langle \mathcal{V}, \mathcal{V}_i \right\rangle = \left\langle \overline{\mathcal{V}}, \overline{\mathcal{V}}_{\bar{i}} \right\rangle = 0. \tag{4.15}$$

We note that, by using the BPS relation (4.13), the contractions between $\mathbf{i}\mathcal{Q}$ and the symplectic sections allow to retrieve again the BPS relations (4.2)–(4.3):

$$\mathbf{i} \left\langle \mathcal{V}, \mathcal{Q} \right\rangle = -\mathbf{i}\mathcal{Z} = \kappa S\mathcal{L} \Leftrightarrow \mathcal{Z} = \mathbf{i}\kappa S\mathcal{L}; \tag{4.16}$$

$$\mathbf{i} \left\langle \mathcal{V}_i, \mathcal{Q} \right\rangle = -\mathbf{i}\mathcal{Z}_i = \kappa S\mathcal{L}_i \Leftrightarrow \mathcal{Z}_i = \mathbf{i}\kappa S\mathcal{L}_i. \tag{4.17}$$

Moreover, $(\frac{1}{4}$-)BPS conditions (4.2)–(4.3) yield

$$\left\langle \mathcal{Q}, \mathcal{G} \right\rangle = -2\kappa S \left(|\mathcal{L}|^2 - |\mathcal{L}_i|^2 \right); \tag{4.18}$$

$$\mathcal{Q}^T \mathcal{M}\left(\mathcal{N} \right) \mathcal{G} = \mathcal{Q}^T \mathcal{M}\left(\mathcal{F} \right) \mathcal{G} = 0. \tag{4.19}$$

Note that (4.1) and (4.18) yield the generalization of the last term of the r.h.s. of (4.10) to $\kappa = \pm 1$, namely

$$S = \frac{\kappa}{2\left(|\mathcal{L}_i|^2 - |\mathcal{L}|^2\right)} = \frac{\kappa}{\mathcal{G}^T \mathcal{M}\left(\mathcal{N}\right)\mathcal{G}}. \tag{4.20}$$

Remarkably, in section 8 we will obtain a generalization, given by (8.2), of the first term in the r.h.s. of (4.10) holding for both cases $\kappa = \pm 1$.

5 ... and its relations with quartic 2-polarizations

The BPS conditions and their properties discussed above hold in the projective special Kähler geometry of the vector multiplets' scalar manifold M_v, regardless of the data specifying such a manifold.

Once again, we will now specialize the treatment by assuming M_v to be a *symmetric* (homogeneous) coset space, associated to the cubic holomorphic prepotential F (3.1). In this framework, we are going to determine the relations among the BPS black hole entropy S and the various 2-polarizations of the quartic invariant introduced in section 3.

In order to do this, we start and consider the contraction of the duality invariant quartic structure $\frac{1}{2}K_{MNPQ}$ with the "algebraic BPS conditions" given by (4.13). To this aim, from (4.13) we get

$$\mathcal{Q}^M + \mathbf{i}\kappa\, S\mathcal{G}^M = 2\mathbf{i}\left(-\mathcal{Z}\overline{\mathcal{V}}^M + \mathcal{Z}^{\bar{\imath}}\overline{\mathcal{V}}_{\bar{\imath}}^M\right), \tag{5.1}$$

whose l.h.s. and r.h.s. can then be contracted as follows:

$$\langle \mathcal{Q} + \mathbf{i}\kappa\, S\mathcal{G}, \mathcal{Q} + \mathbf{i}\kappa\, S\mathcal{G}\rangle = 0; \tag{5.2}$$

$$\left(\mathcal{Q}^M + \mathbf{i}\kappa\, S\mathcal{G}^M\right)^T \mathcal{M}(\mathcal{N})\left(\mathcal{Q}^M + \mathbf{i}\kappa\, S\mathcal{G}^M\right) = 0; \tag{5.3}$$

$$\left(\mathcal{Q}^M + \mathbf{i}\kappa\, S\mathcal{G}^M\right)^T \mathcal{M}(\mathcal{F})\left(\mathcal{Q}^M + \mathbf{i}\kappa\, S\mathcal{G}^M\right) = 0, \tag{5.4}$$

and

$$\frac{1}{2}K_{MNPQ}\left(\mathcal{Q}^M + \mathbf{i}\kappa\, S\mathcal{G}^M\right)\left(\mathcal{Q}^N + \mathbf{i}\kappa\, S\mathcal{G}^N\right)\left(\mathcal{Q}^P + \mathbf{i}\kappa\, S\mathcal{G}^P\right)\left(\mathcal{Q}^Q + \mathbf{i}\kappa\, S\mathcal{G}^Q\right) \tag{5.5}$$

$$= 8 \cdot \frac{1}{2}K_{MNPQ}\left(-\mathcal{Z}\overline{\mathcal{V}}^M + \mathcal{Z}^{\bar{\imath}}\overline{\mathcal{V}}_{\bar{\imath}}^M\right)\left(-\mathcal{Z}\overline{\mathcal{V}}^N + \mathcal{Z}^{\bar{\imath}}\overline{\mathcal{V}}_{\bar{\imath}}^N\right)\left(-\mathcal{Z}\overline{\mathcal{V}}^P + \mathcal{Z}^{\bar{\imath}}\overline{\mathcal{V}}_{\bar{\imath}}^P\right)\left(-\mathcal{Z}\overline{\mathcal{V}}^Q + \mathcal{Z}^{\bar{\imath}}\overline{\mathcal{V}}_{\bar{\imath}}^Q\right).$$

Let us start from the l.h.s. of eq. (5.5), which, by recalling the definitions (3.8)–(3.20), reads

$$\frac{1}{2}K_{MNPQ}\left(\mathcal{Q}^M + \mathbf{i}\kappa\, S\mathcal{G}^M\right)\left(\mathcal{Q}^N + \mathbf{i}\kappa\, S\mathcal{G}^N\right)\left(\mathcal{Q}^P + \mathbf{i}\kappa\, S\mathcal{G}^P\right)\left(\mathcal{Q}^Q + \mathbf{i}\kappa\, S\mathcal{G}^Q\right)$$

$$= \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q + 4\mathbf{i}\kappa\, S \cdot \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{G}^Q$$

$$- 6S^2 \cdot \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{G}^P\mathcal{G}^Q$$

$$- 4\mathbf{i}\kappa S^3 \cdot \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{G}^N\mathcal{G}^P\mathcal{G}^Q + S^4 \cdot \frac{1}{2}K_{MNPQ}\mathcal{G}^M\mathcal{G}^N\mathcal{G}^P\mathcal{G}^Q$$

$$= \mathbf{I}_2 - 6S^2\mathbf{I}_0 + S^4\mathbf{I}_{-2} + 4\mathbf{i}\kappa\, S\left(\mathbf{I}_1 - S^2\mathbf{I}_{-1}\right). \tag{5.6}$$

On the other hand, the r.h.s. of eq. (5.5) reads

$$8 \cdot \frac{1}{2} K_{MNPQ} \left(-\mathcal{Z}\overline{\mathcal{V}}^M + \mathcal{Z}^{\bar{\imath}}\overline{\mathcal{V}}_{\bar{\imath}}^M \right) \left(-\mathcal{Z}\overline{\mathcal{V}}^N + \mathcal{Z}^{\bar{\imath}}\overline{\mathcal{V}}_{\bar{\imath}}^N \right) \left(-\mathcal{Z}\overline{\mathcal{V}}^P + \mathcal{Z}^{\bar{\imath}}\overline{\mathcal{V}}_{\bar{\imath}}^P \right) \left(-\mathcal{Z}\overline{\mathcal{V}}^Q + \mathcal{Z}^{\bar{\imath}}\overline{\mathcal{V}}_{\bar{\imath}}^Q \right)$$

$$= 8\mathcal{Z}^4 \cdot \frac{1}{2} K_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}^N\overline{\mathcal{V}}^P\overline{\mathcal{V}}^Q - 32\mathcal{Z}^3\mathcal{Z}^{\bar{\imath}} \cdot \frac{1}{2}K_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}^N\overline{\mathcal{V}}^P\overline{\mathcal{V}}_{\bar{\imath}}^Q$$

$$+ 48\mathcal{Z}^2\mathcal{Z}^{\bar{\imath}}\mathcal{Z}^{\bar{\jmath}} \cdot \frac{1}{2}K_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}^N\overline{\mathcal{V}}_{\bar{\imath}}^P\overline{\mathcal{V}}_{\bar{\jmath}}^Q - 32\mathcal{Z}\mathcal{Z}^{\bar{\imath}}\mathcal{Z}^{\bar{\jmath}}\mathcal{Z}^{\bar{k}} \cdot \frac{1}{2}K_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}_{\bar{\imath}}^N\overline{\mathcal{V}}_{\bar{\jmath}}^P\overline{\mathcal{V}}_{\bar{k}}^Q$$

$$+ 8\mathcal{Z}^{\bar{\imath}}\mathcal{Z}^{\bar{\jmath}}\mathcal{Z}^{\bar{k}}\mathcal{Z}^{\bar{l}} \cdot \frac{1}{2}\Omega_{MNPQ}\overline{\mathcal{V}}_{\bar{\imath}}^M\overline{\mathcal{V}}_{\bar{\jmath}}^N\overline{\mathcal{V}}_{\bar{k}}^P\overline{\mathcal{V}}_{\bar{l}}^Q. \tag{5.7}$$

The vanishing of each term of the r.h.s. (5.7) of eq. (5.5) can be proved without performing any computation,[10] as follows. Through the expressions (3.8)–(3.22), the five two-centered invariants $\mathbf{I}_2$, $\mathbf{I}_1$, $\mathbf{I}_0$, $\mathbf{I}_{-1}$ and $\mathbf{I}_{-2}$ are quartic *homogeneous* polynomials in the respective variables:

$$\mathbf{I}_2 := \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q = \mathbf{I}_2\left(\mathcal{Z},\mathcal{Z}_i,\overline{\mathcal{Z}},\overline{\mathcal{Z}}_{\bar{\imath}}\right)\Big|_{\mathcal{Z}:=\langle\mathcal{Q},\mathcal{V}\rangle,\ \mathcal{Z}_i:=\langle\mathcal{Q},\mathcal{V}_i\rangle}; \tag{5.8}$$

$$\mathbf{I}_1 := \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{G}^Q$$
$$= \mathbf{I}_1\left(\mathcal{Z},\mathcal{Z}_i,\overline{\mathcal{Z}},\overline{\mathcal{Z}}_{\bar{\imath}},\mathcal{L},\mathcal{L}_i,\overline{\mathcal{L}},\overline{\mathcal{L}}_{\bar{\imath}},\right)\Big|_{\mathcal{Z}:=\langle\mathcal{Q},\mathcal{V}\rangle,\ \mathcal{Z}_i:=\langle\mathcal{Q},\mathcal{V}_i\rangle,\ \mathcal{L}:=\langle\mathcal{G},\mathcal{V}\rangle,\ \mathcal{L}_i:=\langle\mathcal{G},\mathcal{V}_i\rangle}; \tag{5.9}$$

$$\mathbf{I}_0 := \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{G}^P\mathcal{G}^Q$$
$$= \mathbf{I}_0\left(\mathcal{Z},\mathcal{Z}_i,\overline{\mathcal{Z}},\overline{\mathcal{Z}}_{\bar{\imath}},\mathcal{L},\mathcal{L}_i,\overline{\mathcal{L}},\overline{\mathcal{L}}_{\bar{\imath}},\right)\Big|_{\mathcal{Z}:=\langle\mathcal{Q},\mathcal{V}\rangle,\ \mathcal{Z}_i:=\langle\mathcal{Q},\mathcal{V}_i\rangle,\ \mathcal{L}:=\langle\mathcal{G},\mathcal{V}\rangle,\ \mathcal{L}_i:=\langle\mathcal{G},\mathcal{V}_i\rangle}; \tag{5.10}$$

$$\mathbf{I}_{-1} := \frac{1}{2}K_{MNPQ}\mathcal{Q}^M\mathcal{G}^N\mathcal{G}^P\mathcal{G}^Q = \mathbf{I}_1|_{\mathcal{Z}\leftrightarrow\mathcal{L},\ \mathcal{Z}_i\leftrightarrow\mathcal{L}_i}; \tag{5.11}$$

$$\mathbf{I}_{-2} := \frac{1}{2}K_{MNPQ}\mathcal{G}^M\mathcal{G}^N\mathcal{G}^P\mathcal{G}^Q = \mathbf{I}_2|_{\mathcal{Z}\to\mathcal{L},\ \mathcal{Z}_i\to\mathcal{L}_i}. \tag{5.12}$$

Thus, one can proceed and evaluate

$$\overline{\frac{1}{2}K_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}^N\overline{\mathcal{V}}^P\overline{\mathcal{V}}^Q} = \frac{1}{2}K_{MNPQ}\mathcal{V}^M\mathcal{V}^N\mathcal{V}^P\mathcal{V}^Q \overset{(5.8)}{=} \mathbf{I}_2\left(\mathbb{Y},\mathbb{Y}_i,\overline{\mathbb{Y}},\overline{\mathbb{Y}}_{\bar{\imath}}\right) = 0; \tag{5.13}$$

$$\overline{\frac{1}{2}K_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}^N\overline{\mathcal{V}}^P\overline{\mathcal{V}}_{\bar{\imath}}^Q} = \frac{1}{2}K_{MNPQ}\mathcal{V}^M\mathcal{V}^N\mathcal{V}^P\mathcal{V}_i^Q \overset{(5.9)}{=}$$
$$= \mathbf{I}_1\left(\mathbb{Y},\mathbb{Y}_j,\overline{\mathbb{Y}},\overline{\mathbb{Y}}_{\bar{\jmath}},\mathbb{X}_i,\mathbb{X}_{ij},\overline{\mathbb{X}}_{\bar{\imath}},\overline{\mathbb{X}}_{\bar{\imath}\bar{\jmath}},\right) = 0; \tag{5.14}$$

$$\overline{\frac{1}{2}K_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}^N\overline{\mathcal{V}}_{\bar{\imath}}^P\overline{\mathcal{V}}_{\bar{\jmath}}^Q} = \frac{1}{2}K_{MNPQ}\mathcal{V}^M\mathcal{V}^N\mathcal{V}_i^P\mathcal{V}_j^Q \overset{(5.10)}{=}$$
$$= \mathbf{I}_0\left(\mathbb{Y},\mathbb{Y}_k,\overline{\mathbb{Y}},\overline{\mathbb{Y}}_{\bar{k}},\mathbb{X}_{\mathbf{a}},\mathbb{X}_{\mathbf{a}k},\overline{\mathbb{X}}_{\bar{\mathbf{a}}},\overline{\mathbb{X}}_{\bar{\mathbf{a}}\bar{k}}\right) = 0; \tag{5.15}$$

$$\overline{\frac{1}{2}K_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}_{\bar{\imath}}^N\overline{\mathcal{V}}_{\bar{\jmath}}^P\overline{\mathcal{V}}_{\bar{k}}^Q} = \frac{1}{2}K_{MNPQ}\overline{\mathcal{V}}^M\mathcal{V}_i^N\mathcal{V}_j^P\mathcal{V}_k^Q \overset{(5.11)}{=}$$
$$= \mathbf{I}_1\left(\mathbb{Y},\mathbb{Y}_l,\overline{\mathbb{Y}},\overline{\mathbb{Y}}_{\bar{l}},\mathbb{X}_{\mathbf{b}},\mathbb{X}_{\mathbf{b}l},\overline{\mathbb{X}}_{\bar{\mathbf{b}}},\overline{\mathbb{X}}_{\bar{\mathbf{b}}\bar{l}}\right)\Big|_{\mathbb{Y}\leftrightarrow\mathbb{X}_{\mathbf{b}},\ \mathbb{Y}_l\leftrightarrow\mathbb{X}_{\mathbf{b}l}} = 0; \tag{5.16}$$

$$\overline{\frac{1}{2}K_{MNPQ}\overline{\mathcal{V}}_{\bar{\imath}}^M\overline{\mathcal{V}}_{\bar{\jmath}}^N\overline{\mathcal{V}}_{\bar{k}}^P\overline{\mathcal{V}}_{\bar{l}}^Q} = \frac{1}{2}K_{MNPQ}\mathcal{V}_i^M\mathcal{V}_j^N\mathcal{V}_k^P\mathcal{V}_l^Q \overset{(5.12)}{=}$$
$$= \mathbf{I}_2\left(\mathbb{Y},\mathbb{Y}_m,\overline{\mathbb{Y}},\overline{\mathbb{Y}}_{\overline{m}}\right)\Big|_{\mathbb{Y}\to\mathbb{X}_{\mathbf{c}},\ \mathbb{Y}_m\to\mathbb{X}_{\mathbf{c}m}} = 0, \tag{5.17}$$

---

[10]An explicit computation is presented in appendix A.

where $\mathbf{a} = i, j$, $\mathbf{b} = i, j, k$, and $\mathbf{c} = i, j, k, l$. Crucially, in the last step of eqs. (5.13)–(5.17) the homogeneity of the (suitable n-polarizations, with[11] $n = 1, 2, 3, 4$, of the) polynomials $\mathbf{I}_2$, $\mathbf{I}_1$, $\mathbf{I}_0$, $\mathbf{I}_{-1}$ and $\mathbf{I}_{-2}$ has been used, implying the vanishing of (5.13)–(5.17), because

$$\mathbb{Y} := \langle \mathcal{V}, \mathcal{V} \rangle = 0; \tag{5.18}$$

$$\mathbb{Y}_m := \langle \mathcal{V}, U_m \rangle = 0; \tag{5.19}$$

$$\mathbb{X}_{\mathbf{c}} := \langle U_{\mathbf{c}}, \mathcal{V} \rangle = 0; \tag{5.20}$$

$$\mathbb{X}_{\mathbf{c}m} := \langle U_{\mathbf{c}}, U_m \rangle = 0, \tag{5.21}$$

as a consequence of the identities (4.14)–(4.15).

Then, one can re-consider the equation (5.5),

$$(5.5) \Leftrightarrow \mathbf{I}_2 - 6S^2 \mathbf{I}_0 + S^4 \mathbf{I}_{-2} + 4\mathbf{i}\kappa\, S \left(\mathbf{I}_1 - S^2 \mathbf{I}_{-1} \right) = 0, \tag{5.22}$$

and obtain two relations between the quartic 2-polarizations and the (square of) BPS entropy, namely

$$\mathbf{I}_2 - 6S^2 \mathbf{I}_0 + S^4 \mathbf{I}_{-2} = 0 \Leftrightarrow S^2 \equiv S_\pm^2 = \frac{3\mathbf{I}_0}{\mathbf{I}_{-2}} \pm \frac{\sqrt{36\mathbf{I}_0^2 - 4\mathbf{I}_2\mathbf{I}_{-2}}}{2\mathbf{I}_{-2}} \tag{5.23}$$

and

$$\mathbf{I}_1 - S^2 \mathbf{I}_{-1} = 0 \Leftrightarrow S^2 = \frac{\mathbf{I}_1}{\mathbf{I}_{-1}}. \tag{5.24}$$

In turn, the consistency of such two expressions yield a polynomial cubic constraint among the quartic 2-polarizations for BPS black holes:

$$\pm \sqrt{36\mathbf{I}_0^2 - 4\mathbf{I}_2\mathbf{I}_{-2}} = 2\mathbf{I}_{-2} \left(\frac{\mathbf{I}_1}{\mathbf{I}_{-1}} - \frac{3\mathbf{I}_0}{\mathbf{I}_{-2}} \right); \tag{5.25}$$

$$\Downarrow$$

$$9\mathbf{I}_0^2\mathbf{I}_{-1}^2 - \mathbf{I}_2\mathbf{I}_{-1}^2\mathbf{I}_{-2} = (\mathbf{I}_1\mathbf{I}_{-2} - 3\mathbf{I}_0\mathbf{I}_{-1})^2 = \mathbf{I}_1^2\mathbf{I}_{-2}^2 + 9\mathbf{I}_0^2\mathbf{I}_{-1}^2 - 6\mathbf{I}_1\mathbf{I}_0\mathbf{I}_{-1}\mathbf{I}_{-2}; \tag{5.26}$$

$$\Updownarrow$$

$$-\mathbf{I}_2\mathbf{I}_{-1}^2\mathbf{I}_{-2} = \mathbf{I}_1^2\mathbf{I}_{-2}^2 - 6\mathbf{I}_1\mathbf{I}_0\mathbf{I}_{-1}\mathbf{I}_{-2}; \tag{5.27}$$

$$\Updownarrow_{\mathbf{I}_{-2} \neq 0}$$

$$\mathbf{I}_2\mathbf{I}_{-1}^2 - 6\mathbf{I}_1\mathbf{I}_0\mathbf{I}_{-1} + \mathbf{I}_1^2\mathbf{I}_{-2} = 0. \tag{5.28}$$

To make contact with literature, by setting $\mathbf{I}_2$, $\mathbf{I}_1$, $-6\mathbf{I}_0$, $-\mathbf{I}_{-1}$ and $\mathbf{I}_{-2}$ respectively equal to a_0, a_2, a_4, a_6 and a_8, formulæ (5.23), (5.24) and (5.28) respectively match eqs. (3.26), (3.27) and (3.28) of [52], and moreover formulæ (3.8)–(3.22) accomplish the task mentioned below eq. (3.28) therein.

Consequently, for BPS black holes (in models with *symmetric* vector multiplets' scalar manifolds M_v), as far as the evaluation of the 2-polarizations of the quartic duality invariant I_4 and their relation with BPS black hole entropy are concerned, three possibilities may arise:

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[11] Rigorously speaking, only the (1- and) 2-polarizations of the quartic structure I_4 are explicitly known. Nevertheless, this is immaterial for the reasoning made here, because only the homogeneity (of degree 4) matters.

5.1 General case

$$\mathbf{I}_1 \neq 0 \overset{(5.24)}{\Leftrightarrow} \mathbf{I}_{-1} \neq 0. \tag{5.29}$$

Both the expressions (5.23) and (5.24) hold true, with the constraint (5.28).

5.2 Vanishing of $\mathbf{I}_2$, $\mathbf{I}_{-2}$ or $\mathbf{I}_0$

$$\mathbf{I}_2 = 0 \overset{(5.28)}{\Leftrightarrow} \mathbf{I}_{-2} = 0 \overset{(5.28)}{\Leftrightarrow} \mathbf{I}_0 = 0. \tag{5.30}$$

In this case (5.23) is meaningless, and only the expression (5.24) holds true.

5.3 Vanishing of $\mathbf{I}_1$ or $\mathbf{I}_{-1}$

$$\mathbf{I}_1 = 0 \overset{(5.24)}{\Leftrightarrow} \mathbf{I}_{-1} = 0. \tag{5.31}$$

In this case (5.24) is meaningless, and only the expression (5.23) holds true.

5.3.1 A noteworthy BPS sub-class

A remarkable sub-class of BPS critical points, satisfying (4.2)–(4.3), is characterized by the further condition

$$C_{ijk}\overline{\mathcal{L}}^j\overline{\mathcal{L}}^k = 0, \ \forall i, \tag{5.32}$$

which implies also

$$C_{ijk}\overline{\mathcal{Z}}^j\overline{\mathcal{Z}}^k = -S^2 C_{ijk}\overline{\mathcal{L}}^j\overline{\mathcal{L}}^k = 0, \ \forall i. \tag{5.33}$$

Thus, when M_v is symmetric, at the BPS critical points which further satisfy (5.32)–(5.33), the 2-polarizations (3.8)–(3.22) of the quartic duality invariant I_4 read as follows:[12]

$$\mathbf{I}_2 = \left(|\mathcal{Z}|^2 - |\mathcal{Z}_i|^2\right)^2 = S^4\left(|\mathcal{L}|^2 - |\mathcal{L}_i|^2\right)^2; \tag{5.34}$$

$$\mathbf{I}_1 = 0; \tag{5.35}$$

$$\mathbf{I}_0 = \frac{1}{3}S^2\left(|\mathcal{L}|^2 - |\mathcal{L}_i|^2\right)^2; \tag{5.36}$$

$$\mathbf{I}_{-1} = 0; \tag{5.37}$$

$$\mathbf{I}_{-2} = \left(|\mathcal{L}|^2 - |\mathcal{L}_i|^2\right)^2, \tag{5.38}$$

yielding the relation

$$\mathbf{I}_0^2 = \frac{\mathbf{I}_2\mathbf{I}_{-2}}{9} \Rightarrow 3\mathbf{I}_0 = \sqrt{\mathbf{I}_2\mathbf{I}_{-2}}, \tag{5.39}$$

as well as the simple expression of BPS entropy,

$$S^4 = \frac{\mathbf{I}_2}{\mathbf{I}_{-2}}. \tag{5.40}$$

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[12]Even if it does not belong to the class of symmetric manifolds M_v's with cubic holomorphic prepotential (3.1), the class of the so-called *minimally coupled* models of $\mathcal{N} = 2$, $D = 4$ supergravity have $\overline{\mathbb{CP}}^n$, and thus symmetric, scalar manifolds [36], and the corresponding quartic structure is *non-primitive*, because it is reducible in terms of a quadratic symmetric invariant structure (cf. e.g. [62], as well the treatment of sections 3 and 4 of [61]). In this class of models the condition (5.32) holds globally (and not only at BPS attractors) because $C_{ijk} = 0$ globally. The BPS sub-class under consideration indeed encompasses *all* BPS critical points in such models. For the case $n = 1$, see section 12.3.

Interestingly, (5.39) is the very condition of vanishing of the radicand in the square root in eq. (5.23), which indeed simplifies (removing the inherent "±" branching) down to

$$S^2 = \frac{3\mathbf{I}_0}{\mathbf{I}_{-2}} \pm \frac{\sqrt{36\mathbf{I}_0^2 - 4\mathbf{I}_2\mathbf{I}_{-2}}}{2\mathbf{I}_{-2}} = \sqrt{\frac{\mathbf{I}_2}{\mathbf{I}_{-2}}}, \tag{5.41}$$

which is nothing but (5.40). In fact, the BPS sub-class under consideration, defined by (5.39), satisfies (5.31).

6 Effective black hole potential formalism

So far, we have been considering only BPS attractors; a generalization of the whole treatment to encompass all classes of extremal BH attractors, including the non-BPS ones,[13] can be achieved by exploiting the so-called *effective black hole potential* formalism. Indeed, regardless of the specific data of the projective special Kähler geometry of the vector multiplets's scalar manifolds as well as from the quaternionic Kähler geometry of the hypermultiplets' scalar manifolds, from the treatment of [30], then extended to Abelian gaugings of hypermultiplets in [31] and made manifestly symplectic-invariant in [32], the near-horizon attractor dynamics of the equations of motion is known to be governed by an *effective black hole potential* function[14] V_{eff}, whose critical points can be related to $\mathcal{Q}$'s (2.1) supporting extremal black hole solutions in the U(1) FI gauging of $\mathcal{N} = 2$, $D = 4$ supergravity specified by $\mathcal{G}$ (2.2). As specified at the start of this paper, we will not be considering the coupling to hypermultiplets. As resulting from eqs. (D.19)–(D.20), which we report here for simplicity's sake, the Bekenstein-Hawking [64, 65] black hole entropy S (in units of π, as always understood) is expressed by[15]

$$V_{\text{eff}} := \frac{1 - \kappa\sqrt{\kappa^2 - 4VV_{BH}}}{2V} \tag{6.1}$$

$$S = \kappa V_{\text{eff}} = \frac{\kappa - \sqrt{\kappa^2 - 4VV_{BH}}}{2V}, \tag{6.2}$$

where the (manifestly symplectic) effective black hole potential V_{BH} in the ungauged case [10] and the manifestly symplectic-invariant gauge potential V [28] are defined by

$$V_{BH} := |\mathcal{Z}|^2 + |\mathcal{Z}_i|^2 = -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\mathcal{N})\mathcal{Q}; \tag{6.3}$$

$$V := -3|\mathcal{L}|^2 + |\mathcal{L}_i|^2 = \frac{1}{2}\mathcal{G}^T\mathcal{M}(\mathcal{F})\mathcal{G} - 2|\mathcal{L}|^2. \tag{6.4}$$

Note how V_{BH} is non-negative by definition, whereas V can have any sign; in particular, the critical points of V (evaluated at spatial infinity) define the cosmological constant Λ of

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[13]Non-BPS extremal BH solutions in supergravity with U(1) FI gaugings have been discussed in literature, for instance in [46], in which the attractor mechanism and the scalar flow have been described by a first order formalism exploiting a suitably defined fake superpotential.

[14]In the specific example of the magnetic *STU* model treated in section 12.2 and appendix C, the introduction of V_{eff} is recalled in appendix D.

[15]The evaluation at $\partial V_{\text{eff}} = 0$ (which corresponds to $\partial_i V_{\text{eff}} = \partial_u V_{\text{eff}} = 0 \ \forall i, \forall u$) is understood throughout.

the asymptotical geometry of the black hole solution. For $\kappa = 1$ (spherical horizon), the (at least) local minima of V_{eff} support extremal black holes, whereas for $\kappa = -1$ (hyperbolic horizon) the (at least) local maxima of V_{eff} support extremal black holes. Moreover, we will see below that for $\kappa = -1$ the effective potential V_{eff} does not pertain to the entropy itself, but rather to the entropy density. It is here worth pointing out the consistency condition for V_{eff} (and thus for S),

$$1 - 4V_{BH}V \geqslant 0. \tag{6.5}$$

6.1 $\kappa = 1$

This case has spherical near-horizon geometry S^2. The angular integral is finite,

$$\int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta = 4\pi, \tag{6.6}$$

and the Bekenstein-Hawking entropy-area formula holds,

$$\frac{S}{\pi} = \frac{A_{S^2}}{4\pi} = r_H^2 = V_{\text{eff}}|_{\partial V_{\text{eff}}=0}, \tag{6.7}$$

where $A_{S^2} = 4\pi r_H^2$ is the area of the event horizon surface S^2 of radius r_H, and $V_{\text{eff}}|_{\partial V_{\text{eff}}=0} > 0$ necessarily. Explicitly, it holds that

$$\frac{S}{\pi} = V_{\text{eff}} = \frac{1 - \sqrt{1 - 4VV_{BH}}}{2V} > 0 \text{ for } \begin{cases} V < 0; \\ \text{or} \\ V > 0 : 1 - 4VV_{BH} \geqslant 0. \end{cases} \tag{6.8}$$

Note that in this case the symplectic vector of charges has magnetic and electric components defined as (cf. (2.1))

$$\mathcal{Q} := \left(p^\Lambda, q_\Lambda\right)^T, \tag{6.9}$$

$$\text{where } p^\Lambda := \frac{1}{4\pi} \int_{S^2} F^\Lambda, \ q_\Lambda := \frac{1}{4\pi} \int_{S^2} G_\Lambda. \tag{6.10}$$

6.2 $\kappa = -1$

This case has hyperbolic near-horizon geometry H^2. The angular integral [31]

$$\mathbf{V} := \int_{H^2} \sinh\theta d\theta \wedge d\varphi \tag{6.11}$$

diverges, and thus, strictly speaking, the black hole entropy $\mathbb{S}$ is infinite. However, for $\kappa = -1$ one can define the *entropy density*

$$S := \frac{\mathbb{S}}{\mathbf{V}} = -V_{\text{eff}}|_{\partial V_{\text{eff}}=0}, \tag{6.12}$$

which is finite and positive, and being given by the opposite of the critical value of V_{eff}; thus, it must necessarily hold that $V_{\text{eff}}|_{\partial V_{\text{eff}}=0} < 0$. Explicitly,

$$S = -V_{\text{eff}} = \frac{-1 - \sqrt{1 - 4VV_{BH}}}{2V}, \tag{6.13}$$

which is not consistent for $V > 0$, because in this case it would entail a negative entropy density:

$$S = -\frac{\left(1 + \sqrt{1 - 4\,|V|\,V_{BH}}\right)}{2\,|V|} \overset{!}{<} 0. \tag{6.14}$$

Therefore, for $\kappa = -1$ the relation (6.2), or equivalently (6.13), is consistent only for $V < 0$, for which

$$S = \frac{\left(1 + \sqrt{1 + 4\,|V|\,V_{BH}}\right)}{2\,|V|} > 0. \tag{6.15}$$

This is in line with the observation below (5.38) of [31]. Note that in this case the symplectic vector of charges has magnetic and electric components defined as (cf. (2.1))

$$\mathcal{Q} := \left(p^\Lambda, q_\Lambda\right)^T, \tag{6.16}$$

where p^Λ and q_Λ are actually *charge densities*, defined by the following expressions (cf. (3.10)-(3.11) of)

$$p^\Lambda := \frac{1}{\mathbf{V}} \int_{H^2} F^\Lambda; \tag{6.17}$$

$$q_\Lambda := \frac{1}{\mathbf{V}} \int_{H^2} G_\Lambda. \tag{6.18}$$

7 General properties of V_{eff}

In $\mathcal{N} = 2$, $D = 4$ supergravity coupled to vector multiplets and with U(1) FI gauging, regardless of the specific data of the projective special Kähler geometry of the vector multiplets' scalar manifolds, the attractor flow in the near-horizon limit is governed by the critical points (respectively minima for $\kappa = 1$ and maxima for $\kappa = -1$) of the *effective black hole potential*[16] V_{eff} (6.1) [30, 31], such that the attractors are critical points of V_{eff}, satisfying [30]

$$\partial_i V_{\text{eff}} = \frac{2V^2 \partial_i V_{BH} - \left(2V_{BH}V + \kappa\sqrt{1 - 4V_{BH}V} - 1\right)\partial_i V}{2V^2\sqrt{1 - 4V_{BH}V}} = 0, \ \forall i, \tag{7.1}$$

where

$$\partial_i V_{BH} = 2\overline{\mathcal{Z}}\mathcal{Z}_i + \mathbf{i}C_{ijk}\overline{\mathcal{Z}}^j\overline{\mathcal{Z}}^k; \tag{7.2}$$

$$\partial_i V = -2\overline{\mathcal{L}}\mathcal{L}_i + \mathbf{i}C_{ijk}\overline{\mathcal{L}}^j\overline{\mathcal{L}}^k. \tag{7.3}$$

Note how V_{eff} can have any sign. Indeed, as recalled below (6.3)–(6.4), it holds that

$$V_{BH}|_{\partial V_{BH}=0} = S_{\text{ungauged}} > 0; \tag{7.4}$$

$$V|_{\partial V=0} =: \Lambda \gtrless 0, \tag{7.5}$$

where S_{ungauged} denotes the Bekenstein-Hawking entropy (in units of π, as understood throughout) of the extremal black hole in ungauged $\mathcal{N} = 2$, $D = 4$ Maxwell-Einstein

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[16]Note that, apart from the redefinitions $V_{BH\text{ them}} \to \frac{V_{BH\text{ us}}}{(8\pi)^2}$ and $V_{\text{them}} \to \frac{V_{\text{us}}}{2}$ (cf. the last footnote of appendix D), the V_{eff} defined in [31] is κ times the V_{eff} defined by (D.19) or (6.1).

supergravity. The Bekenstein-Hawking entropy S of static extremal black hole solutions (with spherical or hyperbolic near-horizon geometry, respectively corresponding to $\kappa = 1$ and $\kappa = -1$) in U(1) FI gauged $\mathcal{N} = 2$, $D = 4$ supergravity is given by (6.2), in which we recall once more that the evaluation at the horizon of the extremal black hole solution under consideration (corresponding to the evaluation at $\partial_i V_{\text{eff}} = 0$ (7.1)), will be understood.

Note that the ungauged limit [30]

$$\lim_{V \to 0} V_{\text{eff}}|_{\partial_i V_{\text{eff}} = 0} = V_{BH}|_{\partial_i V_{BH} = 0} = S_{\text{ungauged}} \tag{7.6}$$

exists only for $\kappa = 1$ [10]. Moreover, from the treatment given at the start of section 6 at the critical points of V_{eff} the following consistency conditions must hold:

$$\begin{cases} 1 - 4V_{BH}V \geqslant 0; \\ V \neq 0. \end{cases} \tag{7.7}$$

If such two conditions hold, then

$$\partial_i V_{\text{eff}} = 0 \Leftrightarrow 2V^2 \partial_i V_{BH} - \left(2V_{BH}V + \kappa\sqrt{1 - 4V_{BH}V} - 1 \right) \partial_i V = 0, \ \forall i. \tag{7.8}$$

Let us also notice that the saturation of the consistency bound (6.5) corresponds to

$$1 - 4V_{BH}V = 0 \Leftrightarrow V = \frac{1}{4V_{BH}}. \tag{7.9}$$

If such a saturation holds, the Bekenstein-Hawking black hole entropy S of the extremal black hole reads (manifestly specifying the evaluation at critical points of V_{eff}) reads

$$S|_{1-4V_{BH}V=0} = \left. \frac{\kappa}{2V} \right|_{\partial_i V_{\text{eff}} = 0} = 2\kappa \ V_{BH}|_{\partial_i V_{\text{eff}} = 0} , \tag{7.10}$$

which however (for $\kappa = 1$) is generally not the double of $S_{\text{ungauged}} = V_{BH}|_{\partial_i V_{BH} = 0}$ (cf. (7.6)), because for $\kappa = 1$ in general it holds that

$$V_{BH}|_{\partial_i V_{\text{eff}} = 0} \neq V_{BH}|_{\partial_i V_{BH} = 0} . \tag{7.11}$$

8 Generalization of BPS entropy formula (4.10) to $\kappa = \pm 1$

The BPS critical points generally satisfy the criticality conditions (7.1) as well as the BPS conditions (4.2)–(4.3). By virtue of the latter, at BPS critical points it holds that

$$V \stackrel{(6.4)}{=} -3|\mathcal{L}|^2 + |\mathcal{L}_i|^2 = \frac{1}{S^2} \left(-3|\mathcal{Z}|^2 + |\mathcal{Z}_i|^2 \right). \tag{8.1}$$

Therefore, for $\kappa = \pm 1$, by virtue of (6.2) and (6.1), the BPS Bekenstein-Hawking entropy S satisfies the following equation:[17]

$$S = \kappa V_{\text{eff}} = \frac{\kappa - \sqrt{1 - \frac{4}{S^2}\left(|\mathcal{Z}|^2 + |\mathcal{Z}_i|^2\right)\left(-3|\mathcal{Z}|^2 + |\mathcal{Z}_i|^2\right)}}{\frac{2}{S^2}\left(-3|\mathcal{Z}|^2 + |\mathcal{Z}_i|^2\right)};$$

$$\Updownarrow$$

$$\frac{2}{S}\left(-3|\mathcal{Z}|^2 + |\mathcal{Z}_i|^2\right) = \kappa - \sqrt{1 - \frac{4}{S^2}\left(|\mathcal{Z}|^2 + |\mathcal{Z}_i|^2\right)\left(-3|\mathcal{Z}|^2 + |\mathcal{Z}_i|^2\right)};$$

$$\Downarrow$$

$$1 + \frac{4}{S^2}\left(-3|\mathcal{Z}|^2 + |\mathcal{Z}_i|^2\right)^2 - \frac{4\kappa}{S}\left(-3|\mathcal{Z}|^2 + |\mathcal{Z}_i|^2\right) = 1 - \frac{4}{S^2}\left(|\mathcal{Z}|^2 + |\mathcal{Z}_i|^2\right)\left(-3|\mathcal{Z}|^2 + |\mathcal{Z}_i|^2\right);$$

$$\Updownarrow$$

$$S = 2\kappa\left(|\mathcal{Z}_i|^2 - |\mathcal{Z}|^2\right) = \kappa \mathcal{Q}^T \mathcal{M}(\mathcal{F})\mathcal{Q}, \tag{8.2}$$

with consistency conditions given by

$$|\mathcal{Z}|^2 - |\mathcal{Z}_i|^2 \lessgtr 0 \Leftrightarrow |\mathcal{L}|^2 - |\mathcal{L}_i|^2 \lessgtr 0, \text{ for } \kappa = \pm 1; \tag{8.3}$$

$$\Updownarrow$$

$$\mathcal{Q}^T \mathcal{M}(\mathcal{F})\mathcal{Q} \gtrless 0 \Leftrightarrow \mathcal{G}^T \mathcal{M}(\mathcal{F})\mathcal{G} \gtrless 0, \text{ for } \kappa = \pm 1. \tag{8.4}$$

As announced below eq. (4.20), eq. (8.2), which holds true regardless of the specific data of the projective special Kähler geometry of the vector multiplets' scalar manifold M_v, provides the generalization to $\kappa = \pm 1$ of the first term in the r.h.s. of (4.10) (which hold only for $\kappa = 1$). By collecting eqs. (4.20) and (8.2), one can thus write that for $\kappa = \pm 1$ the BPS entropy reads

$$S = 2\kappa\left(|\mathcal{Z}_i|^2 - |\mathcal{Z}|^2\right) = \frac{\kappa}{2\left(|\mathcal{L}_i|^2 - |\mathcal{L}|^2\right)}, \tag{8.5}$$

which thus generalizes (4.10) (obtained in [28] for $\kappa = -1$) to $\kappa = \pm 1$, and still implies (4.11).

Finally, let us remark that, by virtue of the BPS conditions (4.2)–(4.3), at the BPS critical points of V_{eff} the gradients of V_{BH} and V become proportional (of a factor $-S^2$), namely:

$$\partial_i V_{BH} = S^2\left(2\overline{\mathcal{L}}\mathcal{L}_i - \mathbf{i}C_{ijk}\overline{\mathcal{L}}^j\overline{\mathcal{L}}^k\right) = -S^2 \partial_i V; \tag{8.6}$$

$$\Updownarrow$$

$$S^2 = -\frac{\partial_i V_{BH}}{\partial_i V}, \ \forall i \text{ (no summation on } i). \tag{8.7}$$

Again, (8.6) and (8.7) hold for any BPS extremal black hole (with $\kappa = \pm 1$) in $\mathcal{N} = 2$, $D = 4$ U(1) FI gauged supergravity coupled to vector multiplets, regardless of their scalar manifold M_v.

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[17]The eveluation at the BPS conditions (4.2)–(4.3) will be henceforth understood.

9 Classification of critical points of V_{BH}

The critical points of V_{BH} pertain to extremal black hole attractors in the ungauged limit, and by construction of V_{BH} they are placed at the (unique) event horizon of the extremal black hole; from (7.2), they satisfy [66]

$$\partial_i V_{BH} = 2\overline{\mathcal{Z}}\mathcal{Z}_i + \mathbf{i}C_{ijk}\overline{\mathcal{Z}}^j\overline{\mathcal{Z}}^k = 0, \ \forall i. \tag{9.1}$$

After [45], the relation between $\mathbf{I}_2$ (3.8)–(3.10) and V_{BH} (6.3) at the critical points of V_{BH} itself reads

$$\mathbf{I}_2 = V_{BH}^2 - \frac{32}{3}\left|\mathcal{Z}\right|^2\left|\mathcal{Z}_i\right|^2. \tag{9.2}$$

From the treatment of [66] (see also [42] or [41] for a complete list of references), three classes of critical points of V_{BH} exist, namely:

V_{BH}.**I** $\mathcal{Z}_i = 0$ $\forall i$, and $\mathcal{Z} \neq 0$ (for $\kappa = 1$, corresponding to the would-be $\frac{1}{2}$-BPS critical points in the ungauged limit), yielding

$$V_{BH} = \left|\mathcal{Z}\right|^2; \tag{9.3}$$

$$\mathbf{I}_2 = V_{BH}^2 = \left|\mathcal{Z}\right|^4. \tag{9.4}$$

V_{BH}.**II** $\mathcal{Z} = 0$, and $C_{ijk}\overline{\mathcal{Z}}^j\overline{\mathcal{Z}}^k = 0$ $\forall i$ (for $\kappa = 1$, corresponding to the would-be non-BPS $Z = 0$ critical points in the ungauged limit), yielding

$$V_{BH} = \left|\mathcal{Z}_i\right|^2; \tag{9.5}$$

$$\mathbf{I}_2 = V_{BH}^2 = \left|\mathcal{Z}_i\right|^4. \tag{9.6}$$

V_{BH}.**III** $\mathcal{Z} \neq 0$ and $\mathcal{Z}_i \neq 0$, such that (9.1) holds true (for $\kappa = 1$, corresponding to the would-be non-BPS $Z \neq 0$ critical points in the ungauged limit), yielding (cfr. sections 4-6 of [54] and refs. therein, and [56])

$$\left|\mathcal{Z}_i\right|^2 = 3\left|\mathcal{Z}\right|^2 + \Delta_{\mathcal{Z}}, \tag{9.7}$$

and thus

$$V_{BH} = 4\left|\mathcal{Z}\right|^2 + \Delta_{\mathcal{Z}}; \tag{9.8}$$

$$\mathbf{I}_2 = -16\left|\mathcal{Z}\right|^4 + \Delta_{\mathcal{Z}}^2 - \frac{8}{3}\Delta_{\mathcal{Z}}\left|\mathcal{Z}\right|^2, \tag{9.9}$$

where $\Delta_{\mathcal{Z}}$ is defined as[18]

$$\Delta_{\mathcal{Z}} := -\frac{1}{4}\frac{\left(D_m\overline{D}_{(\bar{\imath}}\overline{C}_{\bar{\jmath}\bar{k}\bar{l})}\right)\overline{\mathcal{Z}}^m\mathcal{Z}^{\bar{\imath}}\mathcal{Z}^{\bar{\jmath}}\mathcal{Z}^{\bar{k}}\mathcal{Z}^{\bar{l}}}{\overline{N}_3(\mathcal{Z})}, \tag{9.10}$$

where N_3 is the cubic form related to the tensor C_{ijk} of special geometry,

$$N_3(\overline{\mathcal{Z}}) \equiv N_3(\overline{\mathcal{Z}}, \overline{\mathcal{Z}}, \overline{\mathcal{Z}}) := C_{ijk}\overline{\mathcal{Z}}^i\overline{\mathcal{Z}}^j\overline{\mathcal{Z}}^k. \tag{9.11}$$

Note that $\Delta_{\mathcal{Z}} = 0$ (*at least* in) in symmetric scalar manifolds (because in those cases $D_m\overline{D}_{(\bar{\imath}}\overline{C}_{\bar{\jmath}\bar{k}\bar{l})} = 0$ identically).

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[18]Note that $\Delta_{\mathcal{Z}}$ (9.10) is generally complex, but at critical points of V_{BH} it is real, and it is such that $V_{BH}|_{\partial V_{BH}=0} \geqslant 0$.

10 Classification of critical points of V

In an analogous way, one can classify the critical points of the (manifestly symplectic invariant) potential V of the Abelian U(1) FI gauging in $\mathcal{N} = 2$, $D = 4$ supergravity. When placing the critical points of V at the spatial asymptotical background of the extremal black hole solution, they determine the type of flux vacua; in other words, the critical value of V at the asymptotical background determines the cosmological constant Λ. On the other hand, we will see in section 11 that the critical points of V placed at the event horizon of the extremal black hole will be relevant for the classification of the **class I** of critical points of V_{eff}.

From (7.3), the critical points of V satisfy

$$\partial_i V = -2\overline{\mathcal{L}}\mathcal{L}_i + \mathrm{i}C_{ijk}\overline{\mathcal{L}}^j\overline{\mathcal{L}}^k = 0, \ \forall i. \tag{10.1}$$

From the definition of $\mathbf{I}_{-2}$ (3.20)–(3.22) and V (6.4), at the critical points of V it holds that

$$\mathbf{I}_{-2} = V^2 + \frac{8}{3}\left|\mathcal{L}\right|^2 V = V\left(V + \frac{8}{3}\left|\mathcal{L}\right|^2\right) = \left(-3\left|\mathcal{L}\right|^2 + \left|\mathcal{L}_i\right|^2\right)\left(-\frac{1}{3}\left|\mathcal{L}\right|^2 + \left|\mathcal{L}_i\right|^2\right). \tag{10.2}$$

Three classes of critical points of V exist, namely

V.**I** $\mathcal{L}_i = 0 \ \forall i$, and $\mathcal{L} \neq 0$, yielding

$$V = -3\left|\mathcal{L}\right|^2; \tag{10.3}$$

$$\mathbf{I}_{-2} = \left|\mathcal{L}\right|^4. \tag{10.4}$$

If placed at spatial infinity, this class would correspond to supersymmetric anti-de Sitter (AdS$_4$) vacua ($\Lambda < 0$).

V.**II** $\mathcal{L} = 0$, and $C_{ijk}\overline{\mathcal{L}}^j\overline{\mathcal{L}}^k = 0 \ \forall i$, yielding

$$V = \left|\mathcal{L}_i\right|^2; \tag{10.5}$$

$$\mathbf{I}_{-2} = \left|\mathcal{L}_i\right|^4. \tag{10.6}$$

If placed at spatial infinity, this class would correspond to de Sitter (dS$_4$) vacua ($\Lambda > 0$).

V.**III** $\mathcal{L} \neq 0$ and $\mathcal{L}_i \neq 0$, such that (10.1) holds true. It can be proven that (see appendix B)

$$\left|\mathcal{L}_i\right|^2 = 3\left|\mathcal{L}\right|^2 + \Delta_{\mathcal{L}}, \tag{10.7}$$

and thus

$$V = \Delta_{\mathcal{L}}; \tag{10.8}$$

$$\mathbf{I}_{-2} = \Delta_{\mathcal{L}}^2 + \frac{8}{3}\left|\mathcal{L}\right|^2 \Delta_{\mathcal{L}}, \tag{10.9}$$

where, analogously to (9.10), $\Delta_{\mathcal{L}}$ is defined as[19]

$$\Delta_{\mathcal{L}} := -\frac{1}{4} \frac{\left(D_m \overline{D}_{(\bar{\imath}} \overline{C}_{\bar{\jmath}\bar{k}\bar{l})} \right) \overline{\mathcal{L}}^m \mathcal{L}^{\bar{\imath}} \mathcal{L}^{\bar{\jmath}} \mathcal{L}^{\bar{k}} \mathcal{L}^{\bar{l}}}{\overline{N}_3(\mathcal{L})}, \qquad (10.10)$$

where

$$N_3(\overline{\mathcal{L}}) \equiv N_3(\overline{\mathcal{L}}, \overline{\mathcal{L}}, \overline{\mathcal{L}}) := C_{ijk} \overline{\mathcal{L}}^i \overline{\mathcal{L}}^j \overline{\mathcal{L}}^k. \qquad (10.11)$$

Note that $\Delta_{\mathcal{L}} = 0$ (*at least* in) in symmetric scalar manifolds (because in those cases $D_m \overline{D}_{(\bar{\imath}} \overline{C}_{\bar{\jmath}\bar{k}\bar{l})} = 0$ identically). If placed at spatial infinity, this class would correspond to dS_4 vacua, Minkowski$_4$ vacua or AdS$_4$ vacua depending on whether $\Lambda \gtreqqless 0 \Leftrightarrow \Delta_{\mathcal{L}} \gtreqqless 0$.

Thus, it should be remarked that, (*at least*) for symmetric vector multiplets' scalar manifolds, *each* class of flux vacua is associated to *only one* class of critical points of V (placed at the asymptotical background):

$$
\begin{aligned}
\text{supersymmetric AdS}_4 \text{ vacua } (\Lambda < 0) &\Leftrightarrow \text{ class } V.\mathbf{I}; \\
\text{dS}_4 \text{ vacua} \qquad (\Lambda > 0) &\Leftrightarrow \text{ class } V.\mathbf{II}; \\
\text{Minkowski}_4 \text{ vacua} \qquad (\Lambda = 0) &\Leftrightarrow \text{ class } V.\mathbf{III}.
\end{aligned}
\qquad (10.12)
$$

11 Classification of critical points of V_{eff}

For $\kappa = \pm 1$, from (7.8), regardless of M_v, only two classes of critical points of V_{eff} exist, placed at the (unique) event horizon of the extremal black hole; namely:[20]

Class I corresponds to critical points of V_{eff} which are critical points of both V_{BH} and V, as well:

$$\left. \begin{array}{l} \partial_i V_{BH} = 0; \\ \partial_i V = 0; \end{array} \right\} \Rightarrow \partial_i V_{\text{eff}} = 0, \ \forall i. \qquad (11.1)$$

Again, the placement of the critical points of V_{BH} and V is at the (unique) event horizon of the extremal black hole. The nine sub-classes of **class I** will be listed and discussed below. Note that (11.1) is trivially consistent with (8.6); thus, we anticipate that the **class I** of critical points of V_{eff} includes various sub-classes admitting BPS critical points (namely, sub-classes **I.1**, **I.5** and **I.9**, at which (8.7) is meaningless, of course; see below).

Class II corresponds to critical points of V_{eff} which are *not* critical points of V_{BH} *nor* of V, with the gradients of V_{BH} and of V being proportional:

$$\partial_i V_{BH} = \frac{(2V_{BH}V + \kappa\sqrt{1 - 4V_{BH}V} - 1)}{2V^2} \partial_i V \overset{(6.1)}{=} \frac{(V_{BH} - V_{\text{eff}})}{V} \partial_i V \Rightarrow \partial_i V_{\text{eff}} = 0, \ \forall i. \qquad (11.2)$$

---

[19]Note that $\Delta_{\mathcal{L}}$ (10.10) is generally complex, but at critical points of V it is real.

[20]*A priori*, (7.8) would imply that a third class of critical points, characterized by $\partial_i V_{BH} = 0$ and $\partial_i V \neq 0$, but with $2V_{BH}V = 1 - \kappa\sqrt{1 - 4V_{BH}V}$, might exist. However, this class is not consistent with the equations of motion; see and eq. (5.35) of [31].

For what concerns the **BPS sector**, by exploiting the BPS conditions (4.2)–(4.3) within this class, and using (6.2), (6.1) and (8.7), one trivially obtains that the entropy of the BPS extremal black holes of **class II** satisfies the square of (6.2):

$$S^2 \overset{(8.7), \, \forall i}{=} -\frac{\partial_i V_{BH}}{\partial_i V} = \frac{\left(1 - 2V_{BH}V - \kappa\sqrt{1 - 4V_{BH}V}\right)}{2V^2} = V_{\text{eff}}^2. \qquad (11.3)$$

11.1 Class I

The **class I** of critical points splits into 9 sub-classes, given by the combinatorial product (denoted by "⊗") of classes of critical points of V_{BH} and V:

$$\underbrace{\begin{cases} V_{BH}.1 \\ V_{BH}.2 \\ V_{BH}.3 \end{cases}}_{\text{crit. pts of } V_{BH}} \otimes \underbrace{\begin{cases} V.1 \\ V.2 \\ V.3 \end{cases}}_{\text{crit. pts of } V} = 9 \text{ sub-classes of } \textbf{class I} \text{ of crit. pts of } V_{\text{eff}}. \qquad (11.4)$$

I.1. This sub-class is given by "$V_{BH}.1 \otimes V.1$", and thus it is characterized by all covariant derivatives of $\mathcal{Z}$ and $\mathcal{L}$ vanishing,

$$\forall i, \begin{cases} \mathcal{Z}_i = 0; \\ \mathcal{L}_i = 0, \end{cases} \qquad (11.5)$$

yielding

$$V_{BH} = |\mathcal{Z}|^2 > 0, \qquad \underset{\text{[if placed at spatial infinity: AdS}_4]}{V = -3\,|\mathcal{L}|^2 < 0;} \qquad (11.6)$$

$$S = \kappa V_{\text{eff}} = \frac{-\kappa + \sqrt{1 + 12|\mathcal{Z}|^2|\mathcal{L}|^2}}{6|\mathcal{L}|^2} > 0.$$

Ungauged limit.

$$\lim_{|\mathcal{L}| \to 0} S = \frac{-\kappa + 1 + 6\,|\mathcal{Z}|^2\,|\mathcal{L}|^2}{6\,|\mathcal{L}|^2} = \begin{cases} |\mathcal{Z}|^2 \ \ (\kappa = 1); \\[2mm] \frac{-1 + 3|\mathcal{Z}|^2|\mathcal{L}|^2}{3|\mathcal{L}|^2} \ \ (\kappa = -1); \end{cases} \qquad (11.7)$$

since the limit $|\mathcal{L}| \to 0$ corresponds to the limit $V \to 0^-$, from (7.6) one can conclude that only the $\kappa = 1$ case is allowed (in other words, the $\kappa = -1$ consistency condition $-1 + 3\,|\mathcal{Z}|^2\,|\mathcal{L}|^2 \geqslant 0$ never holds). When M_v is symmetric, the 2-polarizations of the quartic invariant (3.8)–(3.22) respectively read

$$\mathbf{I}_2 = |\mathcal{Z}|^4; \qquad (11.8)$$

$$\mathbf{I}_1 = |\mathcal{Z}|^2 \text{Re}\left(\mathcal{Z}\overline{\mathcal{L}}\right); \qquad (11.9)$$

$$\mathbf{I}_0 = \frac{1}{3}\,|\mathcal{Z}|^2\,|\mathcal{L}|^2 + \frac{2}{3}\text{Re}^2\left(\mathcal{Z}\overline{\mathcal{L}}\right); \qquad (11.10)$$

$$\mathbf{I}_{-1} = |\mathcal{L}|^2 \text{Re}\left(\mathcal{Z}\overline{\mathcal{L}}\right); \qquad (11.11)$$

$$\mathbf{I}_{-2} = |\mathcal{L}|^4. \qquad (11.12)$$

BPS sector. The BPS critical points of this sub-class further enjoy the following relations:

$$V_{BH} = S^2 \left|\mathcal{L}\right|^2 > 0, \quad V = -3\left|\mathcal{L}\right|^2 < 0 \tag{11.13}$$

and

$$S = \kappa V_{\text{eff}} = \frac{-\kappa + \sqrt{1 + 12S^2\left|\mathcal{L}\right|^4}}{6\left|\mathcal{L}\right|^2};$$

$$\Downarrow$$

$$12\left|\mathcal{L}\right|^2 S\left(2S\left|\mathcal{L}\right|^2 + \kappa\right) = 0 \overset{S\neq 0,\ \mathcal{L}\neq 0}{\Leftrightarrow} S = -\frac{\kappa}{2\left|\mathcal{L}\right|^2} = -2\kappa\left|\mathcal{Z}\right|^2, \tag{11.14}$$

which is also obtained from (8.2) by using (11.5). This expression is possible only for $\kappa = -1$: only extremal black holes with hyperbolic horizon topology can be BPS, within this sub-class. When M_v is symmetric, the 2-polarizations of the quartic invariant (11.8)–(11.12) respectively read

$$\mathbf{I}_2 = S^4\left|\mathcal{L}\right|^4; \tag{11.15}$$

$$\mathbf{I}_1 = 0; \tag{11.16}$$

$$\mathbf{I}_0 = \frac{S^2}{3}\left|\mathcal{L}\right|^4; \tag{11.17}$$

$$\mathbf{I}_{-1} = 0; \tag{11.18}$$

$$\mathbf{I}_{-2} = \left|\mathcal{L}\right|^4. \tag{11.19}$$

Summary. The sub-class **I.1** describes extremal black holes with AdS$_4$ asymptotics (*at least* in the doubly-extremal case). Both spherical and hyperbolic horizon geometries are allowed; however, the BPS subsector has only hyperbolic ($\kappa = -1$) near-horizon geometry. Thus, BPS doubly-extremal[21] black holes with spherical symmetry and AdS$_4$ asymptotics cannot exist, in this sub-class: the comment below (3.17) of [28], explaining the results of [67–69], is retrieved.

I.2. This sub-class is given by "$V_{BH}.1\otimes V.2$", and thus it is characterized by

$$\forall i, \begin{cases} \mathcal{Z}_i = 0; \\ \mathcal{L} = 0, \quad C_{ijk}\overline{\mathcal{L}}^j\overline{\mathcal{L}}^k = 0. \end{cases} \tag{11.20}$$

This forbids the existence of a BPS subsector, and moreover yields

$$V_{BH} = \left|\mathcal{Z}\right|^2 > 0, \quad \underset{\text{[if placed at spatial infinity: dS}_4\text{]}}{V = \left|\mathcal{L}_i\right|^2 > 0;} \tag{11.21}$$

$$S = \kappa V_{\text{eff}} = \frac{\kappa - \sqrt{1 - 4\left|\mathcal{Z}\right|^2\left|\mathcal{L}_i\right|^2}}{2\left|\mathcal{L}_i\right|^2} > 0,$$

---

[21]In the extremal but not doubly-extremal case, namely when the scalars are running, the asymptotics depends on whether the horizon attractor values of scalars and their values at spatial infinity (i.e., in the asymptotic background) belong to the same class of critical points of V, or not. In the former case, the asymptotics is still AdS$_4$ and the comment below (3.17) of [28] gets generalized to any extremal (BPS) black hole; in the latter case; the asymptotics will be Minkowski$_4$ or dS$_4$.

which forbids $\kappa = -1$ (i.e., hyperbolic near-horizon geometry). The consistency bound for this sub-class is

$$1 - 4\,|\mathcal{Z}|^2\,|\mathcal{L}_i|^2 \geqslant 0. \tag{11.22}$$

Saturation of consistency bound (11.22). When

$$2\,|\mathcal{L}_i|^2 = \frac{1}{2\,|\mathcal{Z}|^2}, \tag{11.23}$$

the bound (11.22) is saturated, and the entropy boils down to

$$S|_{1-4|\mathcal{Z}|^2|\mathcal{L}_i|^2=0} = \frac{\kappa}{2\,|\mathcal{L}_i|^2} = 2\kappa\,|\mathcal{Z}|^2\,, \tag{11.24}$$

which necessarily implies $\kappa = 1$.

Ungauged limit.

$$\lim_{|\mathcal{L}_i| \to 0} S = \frac{-\kappa + 1 + 2\,|\mathcal{Z}|^2\,|\mathcal{L}_i|^2}{2\,|\mathcal{L}_i|^2} = \begin{cases} |\mathcal{Z}|^2 \;\;(\kappa = 1)\,; \\[2mm] \frac{-1+|\mathcal{Z}|^2|\mathcal{L}_i|^2}{|\mathcal{L}_i|^2} \;\;(\kappa = -1)\,. \end{cases} \tag{11.25}$$

Again, $\kappa = -1$ cannot hold in the ungauged limit, because the entropy positivity condition $(-1 + |\mathcal{Z}|^2\,|\mathcal{L}_i|^2 \geqslant 0)$ is not consistent with (11.22): in the ungauged limit only a spherical horizon is allowed. When M_v is symmetric, the 2-polarizations of the quartic invariant (3.8)–(3.22) respectively read

$$\mathbf{I}_2 = |\mathcal{Z}|^4\,; \tag{11.26}$$

$$\mathbf{I}_1 = 0\,; \tag{11.27}$$

$$\mathbf{I}_0 = -\frac{1}{3}\,|\mathcal{Z}|^2\,|\mathcal{L}_i|^2\,; \tag{11.28}$$

$$\mathbf{I}_{-1} = 0\,; \tag{11.29}$$

$$\mathbf{I}_{-2} = |\mathcal{L}_i|^4\,. \tag{11.30}$$

Summary. The sub-class **I.2** describes non-supersymmetric extremal black holes with spherical near-horizon geometry and dS$_4$ asymptotics (in the doubly-extremal case, or when the classes of critical points of V - to which horizon scalars resp. asymptotic scalars belong - coincide; cf. footnote 17, which will be understood throughout), and characterized by the bound (11.22).

I.3. This sub-class is given by "$V_{BH}.1 \otimes V.3$", and thus it is characterized by

$$\forall i, \begin{cases} \mathcal{Z}_i = 0\,; \\[1mm] \mathcal{L}_i = \frac{\mathbf{i}}{2\mathcal{L}} C_{ijk} \overline{\mathcal{L}}^j \overline{\mathcal{L}}^k\,. \end{cases} \tag{11.31}$$

This forbids the existence of a BPS subsector, and moreover yields

$$V_{BH} = |\mathcal{Z}|^2 > 0, \qquad\qquad V = \Delta_{\mathcal{L}}\,;$$

$$S = \kappa V_{\text{eff}} = \frac{\kappa - \sqrt{1 - 4\,|\mathcal{Z}|^2\,\Delta_{\mathcal{L}}}}{2\Delta_{\mathcal{L}}}, \tag{11.32}$$

within the following consistency conditions:

$$\Delta_{\mathcal{L}} \leqslant \frac{1}{4|\mathcal{Z}|^2};\qquad\qquad\qquad (11.33)$$

$$\Delta_{\mathcal{L}} \neq 0.\qquad\qquad\qquad (11.34)$$

Eq. (11.34) implies that this sub-class does not exist when M_v is symmetric (and whenever $D_m \overline{D}_{(\bar{\imath}} \overline{C}_{\bar{\jmath}\bar{k}\bar{l})} = 0$; cf. discussion in section 10). Moreover, there is no an asymptotic Minkowski solution in this sub-class.[22]

Saturation of consistency bound (11.33).   When the bound (11.33) is saturated, the entropy boils down to a very simple expression, valid only for $\kappa = 1$,

$$S = \frac{1}{2V} = 2|\mathcal{Z}|^2,\qquad\qquad\qquad (11.35)$$

which represents a dS$_4$ extremal black hole with spherical symmetry.

Ungauged limit.

$$\lim_{\Delta_{\mathcal{L}} \to 0} S = \frac{\kappa - 1 + 2\,|\mathcal{Z}|^2\,\Delta_{\mathcal{L}}}{2\Delta_{\mathcal{L}}} = \begin{cases} |\mathcal{Z}|^2 \;\; (\kappa = 1)\,; \\[2mm] \frac{-1+|\mathcal{Z}|^2\Delta_{\mathcal{L}}}{\Delta_{\mathcal{L}}} \;\; (\kappa = -1)\,; \end{cases} \qquad (11.36)$$

in such a limit, the hyperbolic geometry would further constrain the attractor such that

$$\begin{cases} -1 + |\mathcal{Z}|^2\,\Delta_{\mathcal{L}} \geqslant 0; \\ \Delta_{\mathcal{L}} > 0; \end{cases} \text{ or } \begin{cases} -1 + |\mathcal{Z}|^2\,\Delta_{\mathcal{L}} \leqslant 0; \\ \Delta_{\mathcal{L}} < 0; \end{cases} \qquad (11.37)$$

however, again, the limit $\Delta_{\mathcal{L}} \to 0$ corresponds to the limit $V \to 0$, and thus, from (7.6), only the $\kappa = 1$ case is allowed, and therefore conditions (11.37) never hold. When M_v is symmetric, the 2-polarizations of the quartic invariant (3.8)–(3.22) respectively read

$$\mathbf{I}_2 = |\mathcal{Z}|^4\,;\qquad\qquad\qquad (11.38)$$

$$\mathbf{I}_1 = |\mathcal{Z}|^2 \operatorname{Re}\left(\mathcal{Z}\overline{\mathcal{L}}\right);\qquad\qquad\qquad (11.39)$$

$$\mathbf{I}_0 = -\frac{1}{3}\,|\mathcal{Z}|^2\left(2\,|\mathcal{L}|^2 + \Delta_{\mathcal{L}}\right) + \frac{2}{3}\operatorname{Re}^2\left(\mathcal{Z}\overline{\mathcal{L}}\right);\qquad (11.40)$$

$$\mathbf{I}_{-1} = -\left(2\,|\mathcal{L}|^2 + \Delta_{\mathcal{L}}\right)\operatorname{Re}\left(\mathcal{Z}\overline{\mathcal{L}}\right);\qquad\qquad (11.41)$$

$$\mathbf{I}_{-2} = \Delta_{\mathcal{L}}^2 + \frac{8}{3}\,|\mathcal{L}|^2\,\Delta_{\mathcal{L}}.\qquad\qquad\qquad (11.42)$$

Summary.   The sub-class **I.3** describes asymptotically non-flat and non-supersymmetric extremal black holes characterized by the bound (11.33), as well as by (11.34). The spatial asymptotics of the extremal black hole is controlled by the asymptotic, critical value of V: by assuming that such a value belongs to the class $V.3$, it corresponds to the asymptotical value of $\Delta_{\mathcal{L}}$. In turn, the evaluation of $\Delta_{\mathcal{L}}$ at the event horizon determines the near-horizon geometry: when $\Delta_{\mathcal{L}} > 0$, (11.33) must hold and only $\kappa = 1$ is allowed; on the other hand, when $\Delta_{\mathcal{L}} < 0$ (11.33) is automatically satisfied, and no restriction on the horizon geometry holds.

---

[22]This holds in the doubly-extremal case, or when the classes of critical points of V - to which horizon scalars resp. asymptotic scalars belong - coincide.

I.4. This sub-class is given by "$V_{BH}.2 \otimes V.1$", and thus it is characterized by

$$\forall i, \begin{cases} \mathcal{Z} = 0, \qquad C_{ijk}\overline{\mathcal{Z}}^i\overline{\mathcal{Z}}^j = 0; \\ \mathcal{L}_i = 0. \end{cases} \tag{11.43}$$

This forbids the existence of a BPS subsector, and moreover yields

$$V_{BH} = |\mathcal{Z}_i|^2 > 0, \qquad \underset{\text{[if placed at spatial infinity: AdS}_4]}{V = -3|\mathcal{L}|^2 < 0;} \tag{11.44}$$

$$S = \kappa V_{\text{eff}} = \frac{-\kappa + \sqrt{1 + 12|\mathcal{Z}_i|^2|\mathcal{L}|^2}}{6|\mathcal{L}|^2} > 0. $$

Ungauged limit.

$$\lim_{|\mathcal{L}| \to 0} S = \frac{-\kappa + 1 + 6|\mathcal{Z}_i|^2|\mathcal{L}|^2}{6|\mathcal{L}|^2} = \begin{cases} |\mathcal{Z}_i|^2 \quad (\kappa = 1); \\ \frac{-1 + 3|\mathcal{Z}_i|^2|\mathcal{L}|^2}{3|\mathcal{L}|^2} \quad (\kappa = -1), \end{cases} \tag{11.45}$$

in this limit, the hyperbolic geometry ($\kappa = -1$) would further constrain the attractor such that $-1 + 3|\mathcal{Z}_i|^2|\mathcal{L}|^2 \geqslant 0$; however, the limit $|\mathcal{L}| \to 0$ corresponds to the limit $V \to 0^-$, and thus, from (7.6), one can conclude that only the $\kappa = 1$ case is allowed (in other words, the $\kappa = -1$ consistency condition $-1 + 3|\mathcal{Z}_i|^2|\mathcal{L}|^2 \geqslant 0$ never holds). When M_v is symmetric, the 2-polarizations of the quartic invariant (3.8)–(3.22) respectively read

$$\mathbf{I}_2 = |\mathcal{Z}_i|^4; \tag{11.46}$$

$$\mathbf{I}_1 = 0; \tag{11.47}$$

$$\mathbf{I}_0 = -\frac{1}{3}|\mathcal{Z}_i|^2|\mathcal{L}|^2; \tag{11.48}$$

$$\mathbf{I}_{-1} = 0; \tag{11.49}$$

$$\mathbf{I}_{-2} = |\mathcal{L}|^4. \tag{11.50}$$

Summary. The sub-class **I.4** describes only asymptotically AdS$_4$ and non-supersymmetric extremal black holes, with no restriction on the near-horizon geometry (the observation done in footnote 18 holds true here, as well).

I.5. This sub-class is given by "$V_{BH}.2 \otimes V.2$", and thus it is characterized by

$$\forall i, \begin{cases} \mathcal{Z} = 0, \qquad C_{ijk}\bar{\mathcal{Z}}^j\bar{\mathcal{Z}}^k = 0; \\ \mathcal{L} = 0, \qquad C_{ijk}\bar{\mathcal{L}}^j\bar{\mathcal{L}}^k = 0, \end{cases} \tag{11.51}$$

yielding

$$V_{BH} = |\mathcal{Z}_i|^2 > 0, \qquad \underset{\text{[if placed at spatial infinity: dS}_4]}{V = |\mathcal{L}_i|^2 > 0;} \tag{11.52}$$

$$S = \kappa V_{\text{eff}} = \frac{\kappa - \sqrt{1 - 4|\mathcal{Z}_i|^2|\mathcal{L}_i|^2}}{2|\mathcal{L}_i|^2} > 0, $$

which does not allow for flat ($\kappa = 0$) or hyperbolic ($\kappa = -1$) near-horizon geometry, within the conditions:

$$1 - 4|\mathcal{Z}_i|^2|\mathcal{L}_i|^2 \geqslant 0; \tag{11.53}$$

$$|\mathcal{L}_i|^2 \neq 0. \tag{11.54}$$

Saturation of consistency bound (11.53). When (11.53) is saturated, the entropy boils down to a very simple expression, valid only for $\kappa = 1$,

$$S = \frac{\kappa}{2\left|\mathcal{L}_i\right|^2} = 2\kappa \left|\mathcal{Z}_i\right|^2, \tag{11.55}$$

which represents an extremal black hole with spherical horizon and dS$_4$ asymptotics (the observation done in footnote 18 holds true here, as well).

Ungauged limit.

$$\lim_{|\mathcal{L}_i| \to 0} S = \frac{\kappa - 1 + 2\left|\mathcal{Z}_i\right|^2 \left|\mathcal{L}_i\right|^2}{2\left|\mathcal{L}_i\right|^2} = \begin{cases} \left|\mathcal{Z}_i\right|^2 \ \ (\kappa = 1); \\[2mm] \frac{-1 + \left|\mathcal{Z}_i\right|^2 \left|\mathcal{L}_i\right|^2}{\left|\mathcal{L}_i\right|^2} \ \ (\kappa = -1). \end{cases} \tag{11.56}$$

Again, the case $\kappa = -1$ cannot hold, because the entropy positivity condition ($-1 + |\mathcal{Z}|^2 |\mathcal{L}_i|^2 \geqslant 0$) is not consistent with (11.53): in the ungauged limit only spherical horizon is allowed. When M_v is symmetric, the 2-polarizations of the quartic invariant (3.8)–(3.22) respectively read

$$\mathbf{I}_2 = \left|\mathcal{Z}_i\right|^4; \tag{11.57}$$

$$\mathbf{I}_1 = \left|\mathcal{Z}_i\right|^2 \operatorname{Re}\left(\mathcal{Z}^{\bar{\jmath}} \overline{\mathcal{L}}_{\bar{\jmath}}\right); \tag{11.58}$$

$$\mathbf{I}_0 = \frac{1}{3}\left|\mathcal{Z}_i\right|^2 \left|\mathcal{L}_i\right|^2 + \frac{2}{3}\operatorname{Re}^2\left(\mathcal{Z}^{\bar{\jmath}} \overline{\mathcal{L}}_{\bar{\jmath}}\right) - \frac{2}{3} g^{i\bar{\jmath}} C_{ikl} \overline{C}_{\overline{\jmath}\overline{m}\overline{n}} \overline{\mathcal{Z}}^k \overline{\mathcal{L}}^l \mathcal{Z}^{\overline{m}} \mathcal{L}^{\overline{n}}; \tag{11.59}$$

$$\mathbf{I}_{-1} = \left|\mathcal{L}_i\right|^2 \operatorname{Re}\left(\mathcal{Z}^{\bar{\jmath}} \overline{\mathcal{L}}_{\bar{\jmath}}\right); \tag{11.60}$$

$$\mathbf{I}_{-2} = \left|\mathcal{L}_i\right|^4. \tag{11.61}$$

BPS sector. The BPS critical points of this sub-class saturate the consistency condition (11.53); indeed, they enjoy the following relations:

$$V_{BH} = S^2 \left|\mathcal{L}_i\right|^2 > 0; \tag{11.62}$$

$$V = \left|\mathcal{L}_i\right|^2 > 0, \tag{11.63}$$

which imply (11.55), clearly valid only for a spherical horizon topology. When M_v is symmetric, the 2-polarizations of the quartic invariant (11.57)–(11.61) can be further simplified as follows:

$$\mathbf{I}_2 = S^4 \left|\mathcal{L}_i\right|^4; \tag{11.64}$$

$$\mathbf{I}_1 = 0; \tag{11.65}$$

$$\mathbf{I}_0 = -S^2 \left(\left|\mathcal{L}_i\right|^4 + \frac{2}{3}\mathcal{R}\left(\mathcal{L}\right)\right); \tag{11.66}$$

$$\mathbf{I}_{-1} = 0; \tag{11.67}$$

$$\mathbf{I}_{-2} = \left|\mathcal{L}_i\right|^4, \tag{11.68}$$

where $\mathcal{R}\left(\mathcal{L}\right)$ denotes the sectional curvature evaluated on $\mathcal{L}$'s,

$$\mathcal{R}\left(\mathcal{L}\right) \equiv \mathcal{R}\left(\overline{\mathcal{L}}, \mathcal{L}, \overline{\mathcal{L}}, \mathcal{L}\right) := R_{i\bar{\jmath}k\bar{l}} \overline{\mathcal{L}}^i \mathcal{L}^{\bar{\jmath}} \overline{\mathcal{L}}^k \mathcal{L}^{\bar{l}}, \tag{11.69}$$

with $R_{i\bar{\jmath}k\bar{l}}$ denoting the Riemann tensor of the vector multiplets' scalar manifold.

Summary. The sub-class describes only "large", asymptotically dS$_4$ extremal black holes characterized by the bound (11.53), as well as by (11.54), and having only spherical ($\kappa = 1$) near horizon geometry. At the horizon, such black holes can be supersymmetric ($\frac{1}{4}$-BPS).

I.6. This sub-class is given by "$V_{BH}.2 \otimes V.3$", and thus it is characterized by

$$\forall i, \begin{cases} \mathcal{Z} = 0, \qquad C_{ijk}\overline{\mathcal{Z}}^j \overline{\mathcal{Z}}^k = 0; \\ \mathcal{L}_i = \frac{\mathbf{i}}{2\overline{\mathcal{L}}} C_{ijk}\overline{\mathcal{L}}^j \overline{\mathcal{L}}^k. \end{cases} \tag{11.70}$$

This forbids the existence of a BPS subsector, and moreover yields

$$V_{BH} = |\mathcal{Z}_i|^2 > 0, \quad V = \Delta_\mathcal{L};$$

$$S = \kappa V_{\text{eff}} = \frac{\kappa - \sqrt{1 - 4|\mathcal{Z}_i|^2 \Delta_\mathcal{L}}}{2\Delta_\mathcal{L}}, \tag{11.71}$$

within the following conditions:

$$\Delta_\mathcal{L} \leqslant \frac{1}{4|\mathcal{Z}_i|^2}; \tag{11.72}$$

$$\Delta_\mathcal{L} \neq 0. \tag{11.73}$$

Thus, since $\Delta_\mathcal{L} \neq 0$, this class does not exist when M_v is symmetric (and whenever $D_m \overline{D}_{(\bar{\imath}} \overline{C}_{\bar{\jmath}\bar{k}\bar{l})} = 0$; cf. discussion in section 10). Moreover, there is no an asymptotic Minkowski solution in this sub-class (the observation done in footnote 18 holds true here, as well).

Saturation of consistency bound (11.72). When the bound (11.33) is saturated, the entropy boils down to a very simple expression, valid only for $\kappa = 1$,

$$S = \frac{\kappa}{2\Delta_\mathcal{L}} = 2|\mathcal{Z}_i|^2, \tag{11.74}$$

which represents an extremal black hole with spherical horizon and dS$_4$ asymptotics (again, the observation done in footnote 18 holds true here, as well).

Ungauged limit.

$$\lim_{\Delta_\mathcal{L} \to 0} S = \frac{\kappa - 1 + 2|\mathcal{Z}_i|^2 \Delta_\mathcal{L}}{2\Delta_\mathcal{L}} = \begin{cases} |\mathcal{Z}_i|^2 \ (\kappa = 1); \\ \frac{-1 + |\mathcal{Z}_i|^2 \Delta_\mathcal{L}}{\Delta_\mathcal{L}} \ (\kappa = -1). \end{cases} \tag{11.75}$$

A priori, the hyperbolic geometry further constrains the attractor such that

$$\begin{cases} -1 + |\mathcal{Z}|^2 \Delta_\mathcal{L} \geqslant 0; \\ \Delta_\mathcal{L} > 0; \end{cases} \text{or} \begin{cases} -1 + |\mathcal{Z}|^2 \Delta_\mathcal{L} \leqslant 0; \\ \Delta_\mathcal{L} < 0; \end{cases} \tag{11.76}$$

however, the limit $\Delta_\mathcal{L} \to 0$ corresponds to the limit $V \to 0$, and thus, from (7.6), one can conclude that the $\kappa = 1$ case is allowed (in other words, the $\kappa = -1$ consistency condition

$\frac{-1 + |\mathcal{Z}_i|^2 \Delta_{\mathcal{L}}}{\Delta_{\mathcal{L}}} \geqslant 0$ never holds). When M_v is symmetric, the 2-polarizations of the quartic invariant (3.8)–(3.22) respectively read

$$\mathbf{I}_2 = |\mathcal{Z}_i|^4; \tag{11.77}$$

$$\mathbf{I}_1 = |\mathcal{Z}_i|^2 \operatorname{Re}\left(\mathcal{Z}^{\bar{j}} \overline{\mathcal{L}}_{\bar{j}}\right); \tag{11.78}$$

$$\mathbf{I}_0 = -\frac{2}{3} |\mathcal{Z}_i|^2 |\mathcal{L}|^2 - \frac{2}{3} \mathcal{R}\left(\overline{\mathcal{Z}}, \mathcal{Z}, \overline{\mathcal{L}}, \mathcal{L}\right) + \frac{5}{3} \operatorname{Re}^2\left(\mathcal{Z}^{\bar{j}} \overline{\mathcal{L}}_{\bar{j}}\right) + \operatorname{Im}^2\left(\mathcal{Z}^{\bar{j}} \overline{\mathcal{L}}_{\bar{j}}\right); \tag{11.79}$$

$$\mathbf{I}_{-1} = -\left(4 |\mathcal{L}|^2 + \Delta_{\mathcal{L}}\right) \operatorname{Re}\left(\mathcal{Z}^{\bar{j}} \overline{\mathcal{L}}_{\bar{j}}\right) - \operatorname{Im}\left(\mathcal{L} \overline{N}_3(\mathcal{L}, \mathcal{L}, \mathcal{Z})\right) - \operatorname{Re}\left(\mathcal{R}\left(\overline{\mathcal{L}}, \mathcal{Z}, \overline{\mathcal{L}}, \mathcal{L}\right)\right); \tag{11.80}$$

$$\mathbf{I}_{-2} = \Delta_{\mathcal{L}}^2 + \frac{8}{3} |\mathcal{L}|^2 \Delta_{\mathcal{L}}, \tag{11.81}$$

where

$$\mathcal{R}\left(\overline{\mathcal{Z}}, \mathcal{Z}, \overline{\mathcal{L}}, \mathcal{L}\right) := R_{i\bar{j}k\bar{l}} \overline{\mathcal{Z}}^i \mathcal{Z}^{\bar{j}} \overline{\mathcal{L}}^k \mathcal{L}^{\bar{l}}; \tag{11.82}$$

$$\mathcal{R}\left(\overline{\mathcal{L}}, \mathcal{Z}, \overline{\mathcal{L}}, \mathcal{L}\right) := R_{i\bar{j}k\bar{l}} \overline{\mathcal{L}}^i \mathcal{Z}^{\bar{j}} \overline{\mathcal{L}}^k \mathcal{L}^{\bar{l}} \tag{11.83}$$

are suitable polarizations of the sectional curvature (11.69), and

$$\overline{N}_3(\mathcal{L}, \mathcal{L}, \mathcal{Z}) := \overline{C}_{\bar{i}\bar{j}\bar{k}} \mathcal{L}^{\bar{i}} \mathcal{L}^{\bar{j}} \mathcal{Z}^{\bar{k}}. \tag{11.84}$$

Summary. The sub-class **I.6** describes only "large", asymptotically non-flat and non-supersymmetric extremal black holes characterized by the bound (11.72), as well as by (11.73). The spatial asymptotics of the extremal black hole is controlled by the asymptotic, critical value of V: by assuming that such a value belongs to the class $V.3$, it corresponds to the asymptotical value of $\Delta_{\mathcal{L}}$. In turn, the evaluation of $\Delta_{\mathcal{L}}$ at the event horizon determines the near-horizon geometry: when $\Delta_{\mathcal{L}} > 0$, (11.72) must hold and only $\kappa = 1$ is allowed; on the other hand, when $\Delta_{\mathcal{L}} < 0$ (11.72) is automatically satisfied, and no restriction on the horizon geometry holds.

I.7. This sub-class is given by "$V_{BH}.3 \otimes V.1$", and thus it is characterized by

$$\forall i, \begin{cases} \mathcal{Z}_i = -\frac{\mathbf{i}}{2\overline{\mathcal{Z}}} C_{ijk} \overline{\mathcal{Z}}^j \overline{\mathcal{Z}}^k; \\ \mathcal{L}_i = 0. \end{cases} \tag{11.85}$$

This forbids the existence of a BPS subsector, and moreover yields

$$V_{BH} = 4 |\mathcal{Z}|^2 + \Delta_{\mathcal{Z}} > 0, \qquad V = -3 |\mathcal{L}|^2 < 0;$$
$$\text{[if placed at spatial infinity: AdS}_4] \tag{11.86}$$

$$S = \kappa V_{\text{eff}} = \frac{-\kappa + \sqrt{1 + 12\left(4|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)|\mathcal{L}|^2}}{6|\mathcal{L}|^2} > 0.$$

Ungauged limit.

$$\lim_{|\mathcal{L}| \to 0} S = \frac{-\kappa + 1 + 6\left(4 |\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right) |\mathcal{L}|^2}{6 |\mathcal{L}|^2} = \begin{cases} 4 |\mathcal{Z}|^2 + \Delta_{\mathcal{Z}} \ (\kappa = 1); \\ \\ \frac{-1 + 3\left(4|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)|\mathcal{L}|^2}{3|\mathcal{L}|^2} \ (\kappa = -1). \end{cases} \tag{11.87}$$

The hyperbolic geometry would further constrain the attractor such that $-1 + 3\left(4\left|\mathcal{Z}\right|^2 + \Delta_{\mathcal{Z}}\right)\left|\mathcal{L}\right|^2 \geqslant 0$; however, the limit $\left|\mathcal{L}\right| \to 0$ corresponds to the limit $V \to 0^-$, and thus, from (7.6), one can conclude that only the $\kappa = 1$ case is allowed (in other words, the $\kappa = -1$ consistency condition $-1 + 3\left(4\left|\mathcal{Z}\right|^2 + \Delta_{\mathcal{Z}}\right)\left|\mathcal{L}\right|^2 \geqslant 0$ never holds). When M_v is symmetric, the 2-polarizations of the quartic invariant (3.8)–(3.22) respectively read

$$\mathbf{I}_2 = -16\left|\mathcal{Z}\right|^4 + \Delta_{\mathcal{Z}}^2 - \frac{8}{3}\Delta_{\mathcal{Z}}\left|\mathcal{Z}\right|^2; \tag{11.88}$$

$$\mathbf{I}_1 = -2\left(\left|\mathcal{Z}\right|^2 + \frac{\Delta_{\mathcal{Z}}}{2}\right)\mathrm{Re}\left(\mathcal{Z}\overline{\mathcal{L}}\right) + 2\left(\left|\mathcal{Z}\right|^2 + \frac{\Delta_{\mathcal{Z}}}{3}\right)\mathrm{Re}\left(\mathcal{Z}\mathcal{L}\right); \tag{11.89}$$

$$\mathbf{I}_0 = -\frac{1}{3}\left(2\left|\mathcal{Z}\right|^2 + \Delta_{\mathcal{Z}}\right)\left|\mathcal{L}\right|^2 + \frac{2}{3}\mathrm{Re}^2\left(\mathcal{Z}\overline{\mathcal{L}}\right); \tag{11.90}$$

$$\mathbf{I}_{-1} = \left|\mathcal{L}\right|^2\mathrm{Re}\left(\mathcal{Z}\overline{\mathcal{L}}\right); \tag{11.91}$$

$$\mathbf{I}_{-2} = \left|\mathcal{L}\right|^4. \tag{11.92}$$

Summary. The sub-class **I.7** describes asymptotically AdS$_4$ and non-supersymmetric extremal black holes, with no restrictions on the near-horizon geometry (the observation done in footnote 18 holds true here, as well).

I.8. This sub-class is given by "$V_{BH}.3 \otimes V.2$" , and thus it is characterized by

$$\forall i, \begin{cases} \mathcal{Z}_i = -\frac{\mathbf{i}}{2\overline{\mathcal{Z}}}C_{ijk}\overline{\mathcal{Z}}^j\overline{\mathcal{Z}}^k; \\ \mathcal{L} = 0, \qquad C_{ijk}\overline{\mathcal{L}}^j\overline{\mathcal{L}}^k = 0. \end{cases} \tag{11.93}$$

This forbids the existence of a BPS subsector, and moreover yields

$$V_{BH} = 4\left|\mathcal{Z}\right|^2 + \Delta_{\mathcal{Z}} > 0, \qquad \underset{\substack{\text{[if placed at spatial infinity: dS}_4]}}{V = \left|\mathcal{L}_i\right|^2 > 0;}$$

$$S = \kappa V_{\mathrm{eff}} = \frac{\kappa - \sqrt{1 - 4\left(4\left|\mathcal{Z}\right|^2 + \Delta_{\mathcal{Z}}\right)\left|\mathcal{L}_i\right|^2}}{2\left|\mathcal{L}_i\right|^2} > 0, \tag{11.94}$$

which does not allow for flat or hyperbolic near-horizon geometry, within the condition

$$1 - 4\left(4\left|\mathcal{Z}\right|^2 + \Delta_{\mathcal{Z}}\right)\left|\mathcal{L}_i\right|^2 \geqslant 0. \tag{11.95}$$

Saturation of consistency bound (11.95). When (11.95) is saturated, the entropy boils down to a very simple expression, valid only for $\kappa = 1$,

$$S = \frac{1}{2\left|\mathcal{L}_i\right|^2} = 2\left(4\left|\mathcal{Z}\right|^2 + \Delta_{\mathcal{Z}}\right), \tag{11.96}$$

which represents an extremal black hole with spherical horizon and dS$_4$ asymptotics (the observation done in footnote 18 holds true here, as well).

Ungauged limit.

$$\lim_{|\mathcal{L}_i| \to 0} S = \frac{\kappa - 1 + 2\left(4\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)|\mathcal{L}_i|^2}{2\,|\mathcal{L}_i|^2} = \begin{cases} 4\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}} \ (\kappa = 1)\,; \\[1em] \frac{-1 + \left(4|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)|\mathcal{L}_i|^2}{|\mathcal{L}_i|^2} \ (\kappa = -1)\,. \end{cases} \tag{11.97}$$

Again, the hyperbolic geometry would further constrain the attractor such that $-1 + \left(4\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)|\mathcal{L}_i|^2 \geqslant 0$; however, the limit $|\mathcal{L}_i| \to 0$ corresponds to the limit $V \to 0^+$, and thus, from (7.6), one can conclude that only the $\kappa = 1$ case is allowed (in other words, the $\kappa = -1$ consistency condition $-1 + \left(4\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)|\mathcal{L}_i|^2 \geqslant 0$ never holds). When M_v is symmetric, the 2-polarizations of the quartic invariant (3.8)–(3.22) respectively read

$$\mathbf{I}_2 = -16\,|\mathcal{Z}|^4 + \Delta_{\mathcal{Z}}^2 - \frac{8}{3}\Delta_{\mathcal{Z}}\,|\mathcal{Z}|^2\,; \tag{11.98}$$

$$\mathbf{I}_1 = -\left(4\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)\mathrm{Re}\left(\mathcal{Z}^{\bar{\imath}}\overline{\mathcal{L}}_{\bar{\imath}}\right) - \mathrm{Re}\left(\mathcal{R}\left(\overline{\mathcal{Z}}, \mathcal{L}, \overline{\mathcal{Z}}, \mathcal{Z}\right)\right) + 2\mathrm{Re}\left(\mathcal{Z}^2 \mathcal{Z}^{\bar{\imath}}\overline{\mathcal{L}}_{\bar{\imath}}\right)\,; \tag{11.99}$$

$$\mathbf{I}_0 = -\frac{1}{3}\,|\mathcal{L}_i|^2\left(4\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right) - \frac{2}{3}\mathcal{R}\left(\overline{\mathcal{Z}}, \mathcal{Z}, \overline{\mathcal{L}}, \mathcal{L}\right) - \frac{2}{3}\mathrm{Im}^2\left(\mathcal{Z}^{\bar{\imath}}\overline{\mathcal{L}}_{\bar{\imath}}\right)\,; \tag{11.100}$$

$$\mathbf{I}_{-1} = |\mathcal{L}_i|^2\,\mathrm{Re}\left(\mathcal{Z}^{\bar{\imath}}\overline{\mathcal{L}}_{\bar{\imath}}\right)\,; \tag{11.101}$$

$$\mathbf{I}_{-2} = |\mathcal{L}_i|^4\,. \tag{11.102}$$

Summary. The sub-class **I.8** describes asymptotically dS$_4$, non-supersymmetric extremal black holes characterized by the bound (11.95), and having only spherical ($\kappa = 1$) near-horizon geometry (the observation done in footnote 18 holds true here, as well).

I.9. This sub-class is given by "$V_{BH}.3 \otimes V.3$", and thus it is characterized by

$$\forall i, \begin{cases} \mathcal{Z}_i = -\frac{\mathbf{i}}{2\overline{\mathcal{Z}}}C_{ijk}\overline{\mathcal{Z}}^j\overline{\mathcal{Z}}^k\,; \\[0.5em] \mathcal{L}_i = \frac{\mathbf{i}}{2\overline{\mathcal{L}}}C_{ijk}\overline{\mathcal{L}}^j\overline{\mathcal{L}}^k\,, \end{cases} \tag{11.103}$$

yielding

$$V_{BH} = 4\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}} > 0, \qquad\qquad V = \Delta_{\mathcal{L}}\,;$$

$$S = \kappa V_{\mathrm{eff}} = \frac{\kappa - \sqrt{1 - 4\left(4\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)\Delta_{\mathcal{L}}}}{2\Delta_{\mathcal{L}}} > 0, \tag{11.104}$$

with the following conditions:

$$1 - 4\left(4\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)\Delta_{\mathcal{L}} \geqslant 0\,; \tag{11.105}$$

$$\Delta_{\mathcal{L}} \neq 0\,. \tag{11.106}$$

Thus, since $\Delta_{\mathcal{L}} \neq 0$, this class does not exist when M_v is symmetric (and whenever $D_m\overline{D}_{(\bar{\imath}}\overline{C}_{\bar{\jmath}\bar{k}\bar{l})} = 0$; cf. discussion in section 10). Moreover, there is no an asymptotic Minkowski solution in this sub-class (the observation done in footnote 18 holds true here, as well).

Saturation of consistency bound (11.105). When (11.105) is saturated, the entropy boils down to a very simple expression, valid only for $\kappa = 1$,

$$S = \frac{1}{2\Delta_{\mathcal{L}}} = 2\left(4|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right), \tag{11.107}$$

which represents an extremal black hole with spherical horizon and dS_4 asymptotics (again, the observation done in footnote 18 holds true here, as well).

Ungauged limit.

$$\lim_{\Delta_{\mathcal{L}} \to 0} S = \frac{\kappa - 1 + 2\left(4|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)\Delta_{\mathcal{L}}}{2\Delta_{\mathcal{L}}} = \begin{cases} 4|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}} \;\; (\kappa = 1)\,; \\[2mm] \frac{-1 + \left(4|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)\Delta_{\mathcal{L}}}{\Delta_{\mathcal{L}}} \;\; (\kappa = -1)\,. \end{cases} \tag{11.108}$$

As in previous cases, the hyperbolic geometry would further constrains the attractor such that $\frac{-1 + \left(4|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)\Delta_{\mathcal{L}}}{\Delta_{\mathcal{L}}} \geqslant 0$; however, the limit $\Delta_{\mathcal{L}} \to 0$ corresponds to the limit $V \to 0$, and thus, from (7.6), one can conclude that only the $\kappa = 1$ case is allowed (in other words, the $\kappa = -1$ consistency condition $\frac{-1 + \left(4|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)\Delta_{\mathcal{L}}}{\Delta_{\mathcal{L}}} \geqslant 0$ never holds). When M_v is symmetric, the 2-polarizations of the quartic invariant (3.8)–(3.22) respectively read

$$\mathbf{I}_2 = -16|\mathcal{Z}|^4 + \Delta_{\mathcal{Z}}^2 - \frac{8}{3}\Delta_{\mathcal{Z}}|\mathcal{Z}|^2\,; \tag{11.109}$$

$$\mathbf{I}_1 = -\left(|\mathcal{Z}|^2 + \frac{\Delta_{\mathcal{Z}}}{2}\right)\left(\mathcal{Z}\overline{\mathcal{L}} + \overline{\mathcal{Z}}\mathcal{L} - \mathcal{Z}^{\bar{\jmath}}\overline{\mathcal{L}}_{\bar{\jmath}} - \overline{\mathcal{Z}}^j \mathcal{L}_j\right) +$$
$$+ \frac{2}{3}\mathrm{Re}\left[\left(3\mathcal{Z}^2 \mathcal{L}^{\bar{\imath}}\overline{\mathcal{Z}}_{\bar{\imath}} + \mathcal{Z}\mathcal{L}\left(3|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right)\right)\right] - 4|\mathcal{Z}|^2\mathrm{Re}\left(\mathcal{Z}_i\overline{\mathcal{L}}^i\right)\,; \tag{11.110}$$

$$\mathbf{I}_0 = \frac{1}{3}\left(2|\mathcal{Z}|^2 + \frac{\Delta_{\mathcal{Z}}}{2}\right)\left(2|\mathcal{L}|^2 + \frac{\Delta_{\mathcal{L}}}{2}\right) +$$
$$- \frac{4}{3}\mathrm{Re}\left[\mathcal{Z}\mathcal{L}\left(\mathcal{Z}^{\bar{\imath}}\overline{\mathcal{L}}_{\bar{\imath}} - \overline{\mathcal{Z}}_{\bar{\imath}}\mathcal{L}^{\bar{\imath}}\right)\right] +$$
$$- \frac{1}{6}\left[g^{i\bar{\jmath}}C_{ikl}\overline{C}_{\overline{\jmath mn}}4\overline{\mathcal{Z}}^k \mathcal{Z}^{\overline{m}}\overline{\mathcal{L}}^l \mathcal{L}^{\overline{n}}\right] + \frac{4}{3}\mathrm{Re}\left(\overline{\mathcal{Z}}\mathcal{L}\mathcal{Z}_i\overline{\mathcal{L}}^i\right)\,; \tag{11.111}$$

$$\mathbf{I}_{-1} = -\left(|\mathcal{L}|^2 + \frac{\Delta_{\mathcal{L}}}{2}\right)\left(\mathcal{Z}\overline{\mathcal{L}} + \overline{\mathcal{Z}}\mathcal{L} - \mathcal{Z}^{\bar{\jmath}}\overline{\mathcal{L}}_{\bar{\jmath}} - \overline{\mathcal{Z}}^j \mathcal{L}_j\right) +$$
$$- \frac{2}{3}\mathrm{Re}\left[\left(3\mathcal{L}^2 \mathcal{L}^{\bar{\imath}}\mathcal{Z}_{\bar{\imath}} + \mathcal{Z}\mathcal{L}\left(3|\mathcal{L}|^2 + \Delta_{\mathcal{L}}\right)\right)\right] - 4|\mathcal{L}|^2\mathrm{Re}\left(\mathcal{Z}_i\overline{\mathcal{L}}^i\right)\,; \tag{11.112}$$

$$\mathbf{I}_{-2} = \Delta_{\mathcal{L}}^2 + \frac{8}{3}|\mathcal{L}|^2 \Delta_{\mathcal{L}}. \tag{11.113}$$

BPS sector. At the BPS critical points of this sub-class, (4.3) and the definitions (9.10) and (10.10) yield

$$\Delta_{\mathcal{Z}} = S^2 \Delta_{\mathcal{L}}, \tag{11.114}$$

and thus

$$V_{BH} = S^2\left(4|\mathcal{L}|^2 + \Delta_{\mathcal{L}}\right) > 0, \tag{11.115}$$

which in turn implies

$$S = \frac{\kappa}{2\left(2|\mathcal{L}|^2 + \Delta_{\mathcal{L}}\right)} = 2\kappa\left(2|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}\right), \tag{11.116}$$

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which can be obtained by plugging (9.7) and (10.7) into (8.2). The expression (11.116) constrains the near-horizon geometry, depending on $\mathrm{sgn}(\Delta_{\mathcal{Z}}) \overset{(11.114)}{=} \mathrm{sgn}(\Delta_{\mathcal{L}})$. For $\Delta_{\mathcal{Z}} > 0$, only $\kappa = 1$ is allowed; for $\Delta_{\mathcal{Z}} < 0$, the bound (11.105) yields

$$4\,|\mathcal{Z}|^2 > 4\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}} \geqslant \frac{1}{4\Delta_{\mathcal{L}}}, \tag{11.117}$$

which in principle admits both signs of $2\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}}$: when $2\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}} > 0$, once again only spherical ($\kappa = 1$) near-horizon geometry is allowed; on the other hand, when $2\,|\mathcal{Z}|^2 + \Delta_{\mathcal{Z}} < 0$, only hyperbolic ($\kappa = -1$) near-horizon geometry is allowed. When M_v is symmetric, the 2-polarizations of the quartic invariant (11.109)–(11.113) can be further simplified as follows:

$$\mathbf{I}_2 = -16\,|\mathcal{Z}|^4 + \Delta_{\mathcal{Z}}^2 - \frac{8}{3}\Delta_{\mathcal{Z}}\,|\mathcal{Z}|^2; \tag{11.118}$$

$$\mathbf{I}_1 = -\frac{8}{3}\frac{\kappa}{S}|\mathcal{Z}_i|^2 \mathrm{Im}\mathcal{Z}^2; \tag{11.119}$$

$$\mathbf{I}_0 = \frac{1}{3S^2}\left(2|\mathcal{Z}|^2 + \frac{\Delta_{\mathcal{Z}}}{2}\right)^2 - \frac{8}{3}\frac{|\mathcal{Z}_i|^2}{S^2}\mathrm{Re}\mathcal{Z}^2 + \frac{2}{3}\frac{|\mathcal{Z}|^2|\mathcal{Z}_i|^2}{S^2}; \tag{11.120}$$

$$\mathbf{I}_{-1} = \frac{8}{3}\kappa S^3|\mathcal{Z}_i|^2 \mathrm{Im}\mathcal{Z}^2; \tag{11.121}$$

$$\mathbf{I}_{-2} = \frac{1}{S^2}\Delta_{\mathcal{Z}}^2 + \frac{8}{3}\,|\mathcal{Z}|^2\,\Delta_{\mathcal{Z}}. \tag{11.122}$$

Summary. The sub-class **I.9** describes asymptotically non-flat extremal black holes characterized by the bound (11.105), as well as by (11.106). The spatial asymptotics of the extremal black hole is controlled by the asymptotic, critical value of V: by assuming that such a value belongs to the class $V.3$, it corresponds to the asymptotical value of $\Delta_{\mathcal{L}}$. In turn, the evaluation of $\Delta_{\mathcal{L}}$ at the event horizon determines the near-horizon geometry: when $\Delta_{\mathcal{L}} > 0$, (11.105) must hold and only $\kappa = 1$ is allowed; on the other hand, when $\Delta_{\mathcal{L}} < 0$ (11.105) is automatically satisfied, and no restriction on the horizon geometry holds. Such black holes can be supersymmetric ($\frac{1}{4}$-BPS), *a priori* admitting both spherical and hyperbolic horizons, once again depending on $\mathrm{sgn}(\Delta_{\mathcal{Z}}) = \mathrm{sgn}(\Delta_{\mathcal{L}})$: when $\Delta_{\mathcal{L}} > 0$, only $\kappa = 1$ is allowed, whereas when $\Delta_{\mathcal{L}} < 0$, only $\kappa = -1$ is allowed.

11.2 Class II

The **class II** of critical points of V_{eff} is such that both $\partial_i V_{BH} \neq 0$ and $\partial_i V \neq 0$, but nevertheless $\partial_i V_{\mathrm{eff}} = 0$, because $\partial_i V_{BH}$ and $\partial_i V$ are suitably proportional, as given by (11.2). Note that the ungauged limit is ill-defined in this class, since it would imply $V_{\mathrm{eff}} \to V_{BH}$ (cfr. eq. (7.6)), but $\partial_i V_{\mathrm{eff}} = 0 \to \partial_i V_{BH} \neq 0$. As we will see below, the **class II** of critical points of V_{eff} splits into 15 sub-classes.

11.2.1 $\mathcal{Q}$-sector

Since (cfr. (9.1))

$$\partial_i V_{BH} = 2\overline{\mathcal{Z}}\mathcal{Z}_i + \mathbf{i}C_{ijk}\overline{\mathcal{Z}}^j\overline{\mathcal{Z}}^k, \tag{11.123}$$

one can compute[23]

$$|\partial_i V_{BH}|^2 \equiv g^{i\bar{j}}\partial_i V_{BH}\partial_{\bar{j}}V_{BH} = 2\,|\mathcal{Z}_i|^4 + 4\,|\mathcal{Z}|^2\,|\mathcal{Z}_i|^2 + \mathcal{R}\,(\mathcal{Z}) - 4\mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right)$$
$$= -\mathbf{I}_2 + V_{BH}^2 - \frac{16}{3}\mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) > 0. \tag{11.124}$$

From section 9, we recall that $\mathcal{Z}_i = 0$ is a sufficient condition for $\partial_i V_{BH} = 0$; thus, the critical points of V_{eff} of **class II** will be characterized by the condition

$$\mathcal{Z}_i \neq 0 \text{ for } at\ least \text{ some } i\text{'s.} \tag{11.125}$$

Thus, for a non-vanishing $\mathcal{Q}$, in the $\mathcal{Q}$-sector (flux sector) we can then recognize three sub-classes of critical points of V_{eff} of **class II**:

$\mathcal{Q}.\mathbf{1}$ $\mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) = 0$;

$\mathcal{Q}.\mathbf{2}$ $\mathbf{I}_2 = 0$;

$\mathcal{Q}.\mathbf{3}$ generic, with non-vanishing $\mathbf{I}_2$ and $\mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right)$.

11.2.2 $\mathcal{L}$-sector

Since (cfr. (10.1))

$$\partial_i V = -2\overline{\mathcal{L}}\mathcal{L}_i + \mathbf{i}C_{ijk}\overline{\mathcal{L}}^j\overline{\mathcal{L}}^k, \tag{11.126}$$

one can compute[24]

$$|\partial_i V|^2 \equiv g^{i\bar{j}}\partial_i V\partial_{\bar{j}}V = 2\,|\mathcal{L}_i|^4 + 4\,|\mathcal{L}|^2\,|\mathcal{L}_i|^2 + \mathcal{R}\,(\mathcal{L}) + 4\mathrm{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right)$$
$$= -\mathbf{I}_{-2} + V^2 + 16\,|\mathcal{L}|^4 - 8V\,|\mathcal{L}|^2 + \frac{8}{3}\mathrm{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right) > 0, \tag{11.127}$$

From section 10, we recall that $\mathcal{L}_i = 0$ is a sufficient condition for $\partial_i V = 0$; thus, the critical points of V_{eff} of **class II** will be characterized by the condition

$$\mathcal{L}_i \neq 0 \text{ for } at\ least \text{ some } i\text{'s.} \tag{11.128}$$

Thus, for a non-vanishing $\mathcal{L}$, in the $\mathcal{L}$-sector (gauging sector) we can then recognize five sub-classes of critical points of V_{eff} of **class II**:

$\mathcal{L}.\mathbf{1}$ $\mathrm{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right) = 0$;

$\mathcal{L}.\mathbf{2}$ $\mathbf{I}_{-2} = 0$;

$\mathcal{L}.\mathbf{3}$ $V = 0$;

$\mathcal{L}.\mathbf{4}$ $\mathcal{L} = 0$;

$\mathcal{L}.\mathbf{5}$ generic, with non-vanishing $\mathbf{I}_{-2}$, V, $\mathcal{L}$ and $\mathrm{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right)$.

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[23]Note that, since M_v has not been specified to be symmetric, the $\mathbf{I}_2$ in the second line of (11.124) may also depend on scalar fields coordinatizing M_v (which in (11.124) as well as in conditions $\mathcal{Q}$.1-$\mathcal{Q}$.3 are understood to be stabilized at the critical points of V_{eff}).

[24]Note that, since M_v has not been specified to be symmetric, the $\mathbf{I}_{-2}$ in the second line of (11.127) may also depend on scalar fields coordinatizing M_v (which in (11.127) as well as in conditions $\mathcal{L}$.1-$\mathcal{L}$.3 are understood to be stabilized at the critical points of V_{eff}).

11.2.3 General properties

On the other hand, from (11.2), at the **class II** of critical points of V_{eff} it holds that

$$\partial_i V_{BH} = \frac{(V_{BH} - V_{\text{eff}})}{V} \partial_i V, \ \forall i, \tag{11.129}$$

and thus[25]

$$|\partial_i V_{BH}|^2 = \frac{(V_{BH} - V_{\text{eff}})^2}{V^2} |\partial_i V|^2. \tag{11.130}$$

Further equivalent expressions, involving $\mathbf{I}_2$, $\mathbf{I}_{-2}$, V_{BH} and V can be obtained by plugging (11.124) and (11.127) into (11.130).

Moreover, one can compute:

$$g^{i\bar{j}} \partial_i V_{BH} \partial_{\bar{j}} V = 2 \left(\overline{\mathcal{Z}}_{\bar{i}} \mathcal{L}^{\bar{i}} \right)^2 - 4\mathcal{L} \overline{\mathcal{Z}} \mathcal{Z}_i \overline{\mathcal{L}}^i - 2\mathbf{i} \mathcal{L} N_3 \left(\overline{\mathcal{L}}, \overline{\mathcal{Z}}, \overline{\mathcal{Z}} \right)$$
$$- 2\mathbf{i} \overline{\mathcal{Z}} N_3 \left(\mathcal{Z}, \mathcal{L}, \mathcal{L} \right) + \mathcal{R} \left(\overline{\mathcal{Z}}, \mathcal{L}, \overline{\mathcal{Z}}, \mathcal{L} \right), \tag{11.131}$$

where $N_3 \left(\overline{\mathcal{L}}, \overline{\mathcal{Z}}, \overline{\mathcal{Z}} \right)$ and $\mathcal{R} \left(\overline{\mathcal{Z}}, \mathcal{L}, \overline{\mathcal{Z}}, \mathcal{L} \right)$ denote suitable polarizations of the cubic form associated to C_{ijk} and of the sectional curvature (11.69), respectively,

$$N_3 \left(\overline{\mathcal{L}}, \overline{\mathcal{Z}}, \overline{\mathcal{Z}} \right) := C_{ijk} \overline{\mathcal{L}}^i \overline{\mathcal{Z}}^j \overline{\mathcal{Z}}^k; \tag{11.132}$$

$$\mathcal{R} \left(\overline{\mathcal{Z}}, \mathcal{L}, \overline{\mathcal{Z}}, \mathcal{L} \right) := R_{i\bar{j}k\bar{l}} \overline{\mathcal{Z}}^i \mathcal{L}^{\bar{j}} \overline{\mathcal{Z}}^k \mathcal{L}^{\bar{l}}. \tag{11.133}$$

While (11.124) and (11.127) are manifestly real, (11.131) seems a complex quantity, but actually, it is a real one. Indeed, from (11.129), it follows that

$$g^{i\bar{j}} \partial_i V_{BH} \partial_{\bar{j}} V = \frac{(V_{BH} - V_{\text{eff}})}{V} |\partial_i V|^2, \tag{11.134}$$

which is a manifestly real quantity, thus implying that

$$0 = \text{Im} \left(g^{i\bar{j}} \partial_i V_{BH} \partial_{\bar{j}} V \right)$$
$$= \text{Im} \left[\begin{array}{c} 2 \left(\overline{\mathcal{Z}}_{\bar{i}} \mathcal{L}^{\bar{i}} \right)^2 - 4\mathcal{L} \overline{\mathcal{Z}} \mathcal{Z}_i \overline{\mathcal{L}}^i - 2\mathbf{i} \mathcal{L} N_3 \left(\overline{\mathcal{L}}, \overline{\mathcal{Z}}, \overline{\mathcal{Z}} \right) \\ -2\mathbf{i} \overline{\mathcal{Z}} N_3 \left(\mathcal{Z}, \mathcal{L}, \mathcal{L} \right) + \mathcal{R} \left(\overline{\mathcal{Z}}, \mathcal{L}, \overline{\mathcal{Z}}, \mathcal{L} \right) \end{array} \right]. \tag{11.135}$$

Recalling that, from (6.2), the critical values of κV_{eff} determine the Bekenstein-Hawking entropy S (in units of π), the relations (11.129) allows to obtain S at critical points of κV_{eff} (or, equivalently, of V_{eff}) of the **class II**, also in the non-supersymmetric case. Indeed, from (11.129) one obtains that ($\forall i$, no Einstein summation on dummy indices)

$$S = \kappa V_{BH} - \kappa V \frac{\partial_i V_{BH}}{\partial_i V}. \tag{11.136}$$

---

[25]Eq. (11.130) holds $\forall i$, and thus *a fortiori* when summed over i (namely, when $|\partial_i V_{BH}|^2$ and $|\partial_i V|^2$ are given by (11.124) resp. (11.127)).

In order to relate S to the quantities $|\partial_i V_{BH}|^2$ (11.124) and $|\partial_i V|^2$ (11.127), one can observe that (11.130) entails an inhomogeneous quadratic equation[26] in S,

$$V_{\text{eff}}^2 - 2V_{BH}V_{\text{eff}} + V_{BH}^2 - V^2 \frac{|\partial_i V_{BH}|^2}{|\partial_i V|^2} = 0, \tag{11.137}$$

whose solution reads

$$\kappa V_{\text{eff}\,\pm} = S_\pm = \kappa V_{BH} \pm \frac{1}{2}\sqrt{4V_{BH}^2 - 4\left(V_{BH}^2 - V^2 \frac{|\partial_i V_{BH}|^2}{|\partial_i V|^2}\right)}$$

$$= \kappa V_{BH} \pm \sqrt{V^2 \frac{|\partial_i V_{BH}|^2}{|\partial_i V|^2}} = \kappa V_{BH} \pm |V|\sqrt{\frac{|\partial_i V_{BH}|^2}{|\partial_i V|^2}}. \tag{11.138}$$

The sign of the first term in the r.h.s. of (11.138) is κ, whereas the sign of the second term is $\pm$. In order to maximize the entropy, the "+" branch should be chosen. By recalling (11.124) and (11.127), one thus obtains the following expression for the entropy S at the critical points of V_{eff} of **class II** (regardless of their BPS properties and of the symmetricity[27] of M_v):

$$S = \kappa V_{BH} + |V|\sqrt{\frac{2|\mathcal{Z}_i|^4 + 4|\mathcal{Z}|^2|\mathcal{Z}_i|^2 + \mathcal{R}(\mathcal{Z}) - 4\text{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right)}{2|\mathcal{L}_i|^4 + 4|\mathcal{L}|^2|\mathcal{L}_i|^2 + \mathcal{R}(\mathcal{L}) + 4\text{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right)}} \tag{11.139}$$

$$= \kappa V_{BH} + |V|\sqrt{\frac{-\mathbf{I}_2 + V_{BH}^2 - \frac{16}{3}\text{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right)}{-\mathbf{I}_{-2} + V^2 + 16|\mathcal{L}|^4 - 8V|\mathcal{L}|^2 + \frac{8}{3}\text{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right)}}, \tag{11.140}$$

which can thus be evaluated in the various sub-classes of **class II** (see below).

We should also observe that, for what concerns the **BPS sector**, one obtains nothing new. Indeed, the BPS conditions (4.2)–(4.3) plugged into (11.123) and (11.126) allow to elaborate (11.136) for the BPS entropy as follows:

$$S = \kappa V_{BH} + \kappa V S^2 \Leftrightarrow \kappa V S^2 - S + \kappa V_{BH} = 0. \tag{11.141}$$

Such inhomogeneous quadratic equation in S is consistent with $S = \kappa V_{\text{eff}}$ (namely, (6.2) at the BPS critical points of V_{eff}) by suitably choosing the "$\pm$" branching (in the determination of the roots of (11.141)) and the value of κ such that $\pm\kappa = -1$.

From previous treatment, it follows that the **class II** of critical points of V_{eff} splits into 15 sub-classes, given by the combinatorial product (denoted by "$\otimes$") of the possibilities in

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[26]Eqs. (11.137) and (11.138) hold $\forall i$, and also when $|\partial_i V_{BH}|^2$ and $|\partial_i V|^2$ are given by (11.124) resp. (11.127).

[27]Again, since M_v has not been specified to be symmetric, the $\mathbf{I}_2$ and $\mathbf{I}_{-2}$ in the second line of (11.140) may also depend on scalar fields coordinatizing M_v (which in (11.140) as well as in section 11.2 are understood to be stabilized at the critical points of V_{eff}).

the $\mathcal{Q}$- and $\mathcal{L}$- sectors (namely, in the flux sector and in the gauging sector):

$$
\begin{cases}
\mathcal{Q}.\mathbf{1} : \text{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) = 0; \\
\mathcal{Q}.\mathbf{2} : \mathbf{I}_2 = 0; \\
\mathcal{Q}.\mathbf{3} : \text{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) \neq 0, \ \mathbf{I}_2 \neq 0;
\end{cases}
$$
$$
\underbrace{\phantom{\mathcal{Q}.\mathbf{1} : \text{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) = 0;}}_{\mathcal{Q}\text{-sector}}
$$
$$
\otimes
\begin{cases}
\mathcal{L}.\mathbf{1} : \text{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right) = 0; \\
\mathcal{L}.\mathbf{2} : \mathbf{I}_{-2} = 0; \\
\mathcal{L}.\mathbf{3} : V = 0; \\
\mathcal{L}.\mathbf{4} : \mathcal{L} = 0; \\
\mathcal{L}.\mathbf{5} : \text{none of } \text{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right), \mathbf{I}_{-2}, V \text{ and } \mathcal{L} \text{ vanishing,}
\end{cases}
\tag{11.142}
$$
$$
\underbrace{\phantom{\mathcal{L}.\mathbf{5} : \text{none of } \text{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right)}}_{\mathcal{G}\text{-sector}}
$$

with the generic sub-class being given by the case[28] "**3**⊗**5**".

II.1. This sub-class is given by "$\mathcal{Q}.1\otimes\mathcal{L}.1$", and thus it is characterized by

$$
\text{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) = 0 = \text{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right);
\tag{11.143}
$$

therefore, from (11.140), its entropy reads

$$
S = \kappa V_{BH} + |V| \sqrt{\frac{-\mathbf{I}_2 + V_{BH}^2}{-\mathbf{I}_{-2} + V^2 + 16\,|\mathcal{L}|^4 - 8V\,|\mathcal{L}|^2}}.
\tag{11.144}
$$

II.2. This sub-class is given by "$\mathcal{Q}.1\otimes\mathcal{L}.2$", and thus it is characterized by

$$
\begin{cases}
\text{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) = 0; \\
\mathbf{I}_{-2} = 0;
\end{cases}
\tag{11.145}
$$

therefore, from (11.140), its entropy reads

$$
S = \kappa V_{BH} + |V| \sqrt{\frac{-\mathbf{I}_2 + V_{BH}^2}{V^2 + 16\,|\mathcal{L}|^4 - 8V\,|\mathcal{L}|^2 + \frac{8}{3}\text{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right)}}.
\tag{11.146}
$$

II.3. This sub-class is given by "$\mathcal{Q}.1\otimes\mathcal{L}.3$", and thus it is characterized by

$$
\begin{cases}
\text{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) = 0; \\
V = 0;
\end{cases}
\tag{11.147}
$$

therefore, from (11.140), its entropy reads

$$
S = \kappa V_{BH},
\tag{11.148}
$$

which is meaningful only for $\kappa = 1$, i.e. for spherical horizon. Despite the (assumed) non-vanishingness of the gauging vector $\mathcal{G}$, the extremal black holes of this sub-class have, formally, the same entropy and the same asymptotical behaviour of their counterparts in the ungauged limit; of course, such a similarity is only formal, because in general $V_{BH}|_{\partial V_{\text{eff}}=0} \neq V_{BH}|_{\partial V_{BH}=0}$, thus their entropy will generally be different.

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[28]Throughout the present treatment, the first number denotes the sub-class in the $\mathcal{Q}$-sector, whereas the second number denotes the sub-class in the $\mathcal{L}$-sector.

II.4. This sub-class is given by "$\mathcal{Q}.1 \otimes \mathcal{L}.4$" , and thus it is characterized by

$$
\begin{cases}
\mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) = 0; \\
\mathcal{L} = 0;
\end{cases}
\tag{11.149}
$$

therefore, from (11.140), its entropy reads

$$
S = \kappa V_{BH} + |\mathcal{L}_i|^2 \sqrt{\frac{-\mathbf{I}_2 + V_{BH}^2 - \frac{16}{3}\mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right)}{g^{i\bar{j}}C_{ikl}\overline{C}_{\overline{jmn}}\overline{\mathcal{L}}^k\overline{\mathcal{L}}^l\mathcal{L}^{\overline{m}}\mathcal{L}^{\overline{n}}}}.
\tag{11.150}
$$

II.5. This sub-class is given by "$\mathcal{Q}.1 \otimes \mathcal{L}.5$" , and thus it is characterized by

$$
\begin{cases}
\mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) = 0; \\
\text{none of } \mathrm{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right), \mathbf{I}_{-2}, V \text{ and } \mathcal{L} \text{ vanishing};
\end{cases}
\tag{11.151}
$$

therefore, from (11.140), its entropy reads

$$
S = \kappa V_{BH} + |V|\sqrt{\frac{-\mathbf{I}_2 + V_{BH}^2}{-\mathbf{I}_{-2} + V^2 + 16\,|\mathcal{L}|^4 - 8V\,|\mathcal{L}|^2 + \frac{8}{3}\mathrm{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right)}}.
\tag{11.152}
$$

No **BPS sector** is allowed in this sub-class.

II.6. This sub-class is given by "$\mathcal{Q}.2 \otimes \mathcal{L}.1$" , and thus it is characterized by

$$
\begin{cases}
\mathbf{I}_2 = 0; \\
\mathrm{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right) = 0;
\end{cases}
\tag{11.153}
$$

therefore, from (11.140), its entropy reads

$$
S = \kappa V_{BH} + |V|\sqrt{\frac{V_{BH}^2 - \frac{16}{3}\mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right)}{-\mathbf{I}_{-2} + V^2 + 16\,|\mathcal{L}|^4 - 8V\,|\mathcal{L}|^2}}.
\tag{11.154}
$$

II.7. This sub-class is given by "$\mathcal{Q}.2 \otimes \mathcal{L}.2$", and thus it is characterized by

$$
\begin{cases}
\mathbf{I}_2 = 0; \\
\mathbf{I}_{-2} = 0;
\end{cases}
\tag{11.155}
$$

therefore, from (11.140), its entropy reads

$$
S = \kappa V_{BH} + |V|\sqrt{\frac{V_{BH}^2 - \frac{16}{3}\mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right)}{V^2 + 16\,|\mathcal{L}|^4 - 8V\,|\mathcal{L}|^2 + \frac{8}{3}\mathrm{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right)}}.
\tag{11.156}
$$

II.8. This sub-class is given by "$\mathcal{Q}.2 \otimes \mathcal{L}.3$" , and thus it is characterized by

$$
\begin{cases}
\mathbf{I}_2 = 0; \\
V = 0,
\end{cases}
\tag{11.157}
$$

therefore, from (11.140), its entropy reads

$$
S = \kappa V_{BH},
\tag{11.158}
$$

which is meaningful only for $\kappa = 1$, i.e. for spherical horizon. Considerations analogous to the ones made for the sub-class **II.3**, hold here, as well.

II.9. This sub-class is given by "$\mathcal{Q}.2 \otimes \mathcal{L}.4$", and thus it is characterized by

$$\begin{cases} \mathbf{I}_2 = 0; \\ \mathcal{L} = 0; \end{cases} \tag{11.159}$$

therefore, from (11.140), its entropy reads

$$S = \kappa V_{BH} + |\mathcal{L}_i|^2 \sqrt{\frac{V_{BH}^2 - \frac{16}{3} \mathrm{Im}\left(\mathcal{Z} N_3(\overline{\mathcal{Z}}, \overline{\mathcal{Z}}, \overline{\mathcal{Z}})\right)}{g^{i\bar{j}} C_{ikl} \overline{C}_{\overline{jmn}} \overline{\mathcal{L}}^k \overline{\mathcal{L}}^l \mathcal{L}^{\overline{m}} \mathcal{L}^{\overline{n}}}}. \tag{11.160}$$

II.10. This sub-class is given by "$\mathcal{Q}.2 \otimes \mathcal{L}.5$", and thus it is characterized by

$$\begin{cases} \mathbf{I}_2 = 0; \\ \text{none of } \mathrm{Im}\left(\mathcal{L} N_3(\overline{\mathcal{L}}, \overline{\mathcal{L}}, \overline{\mathcal{L}})\right), \mathbf{I}_{-2}, V \text{ and } \mathcal{L} \text{ vanishing}; \end{cases} \tag{11.161}$$

therefore, from (11.140), its entropy reads

$$S = \kappa V_{BH} + |V| \sqrt{\frac{V_{BH}^2 - \frac{16}{3} \mathrm{Im}\left(\mathcal{Z} N_3(\overline{\mathcal{Z}}, \overline{\mathcal{Z}}, \overline{\mathcal{Z}})\right)}{-\mathbf{I}_{-2} + V^2 + 16\left|\mathcal{L}\right|^4 - 8V\left|\mathcal{L}\right|^2 + \frac{8}{3} \mathrm{Im}\left(\mathcal{L} N_3(\overline{\mathcal{L}}, \overline{\mathcal{L}}, \overline{\mathcal{L}})\right)}}. \tag{11.162}$$

No **BPS sector** is allowed in this sub-class.

II.11. This sub-class is given by "$\mathcal{Q}.3 \otimes \mathcal{L}.1$", and thus it is characterized by

$$\begin{cases} \mathrm{Im}\left(\mathcal{Z} N_3(\overline{\mathcal{Z}}, \overline{\mathcal{Z}}, \overline{\mathcal{Z}})\right) \neq 0, \ \mathbf{I}_2 \neq 0 \\ \mathrm{Im}\left(\mathcal{L} N_3(\overline{\mathcal{L}}, \overline{\mathcal{L}}, \overline{\mathcal{L}})\right) = 0; \end{cases} \tag{11.163}$$

therefore, from (11.140), its entropy reads

$$S = \kappa V_{BH} + |V| \sqrt{\frac{-\mathbf{I}_2 + V_{BH}^2 - \frac{16}{3} \mathrm{Im}\left(\mathcal{Z} N_3(\overline{\mathcal{Z}}, \overline{\mathcal{Z}}, \overline{\mathcal{Z}})\right)}{-\mathbf{I}_{-2} + V^2 + 16\left|\mathcal{L}\right|^4 - 8V\left|\mathcal{L}\right|^2}}. \tag{11.164}$$

No **BPS sector** is allowed in this sub-class.

II.12. This sub-class is given by "$\mathcal{Q}.3 \otimes \mathcal{L}.2$", and thus it is characterized by

$$\begin{cases} \mathrm{Im}\left(\mathcal{Z} N_3(\overline{\mathcal{Z}}, \overline{\mathcal{Z}}, \overline{\mathcal{Z}})\right) \neq 0, \ \mathbf{I}_2 \neq 0 \\ \mathbf{I}_{-2} = 0; \end{cases} \tag{11.165}$$

therefore, from (11.140), its entropy reads

$$S = \kappa V_{BH} + |V| \sqrt{\frac{-\mathbf{I}_2 + V_{BH}^2 - \frac{16}{3} \mathrm{Im}\left(\mathcal{Z} N_3(\overline{\mathcal{Z}}, \overline{\mathcal{Z}}, \overline{\mathcal{Z}})\right)}{V^2 + 16\left|\mathcal{L}\right|^4 - 8V\left|\mathcal{L}\right|^2 + \frac{8}{3} \mathrm{Im}\left(\mathcal{L} N_3(\overline{\mathcal{L}}, \overline{\mathcal{L}}, \overline{\mathcal{L}})\right)}}. \tag{11.166}$$

No **BPS sector** is allowed in this sub-class.

II.13. This sub-class is given by "$\mathcal{Q}.3\otimes\mathcal{L}.3$" , and thus it is characterized by

$$\begin{cases} \mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) \neq 0, \; \mathbf{I}_2 \neq 0 \\ V = 0; \end{cases} \tag{11.167}$$

therefore, from (11.140), its entropy reads

$$S = \kappa V_{BH}, \tag{11.168}$$

which is meaningful only for $\kappa = 1$, i.e. for spherical horizon. Similarly to sub-classes II.3 and II.8, despite the (assumed) non-vanishing of the gauging vector $\mathcal{G}$, the extremal black holes of this sub-class have, formally, the same entropy and the same asymptotical behaviour of their counterparts in the ungauged limit, Again, since $V_{\mathrm{eff}}|_{\partial V_{\mathrm{eff}}=0} \neq V_{BH}|_{\partial V_{BH}=0}$, their entropy will generally be different.

II.14. This sub-class is given by "$\mathcal{Q}.3\otimes\mathcal{L}.4$" , and thus it is characterized by

$$\begin{cases} \mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right) \neq 0, \; \mathbf{I}_2 \neq 0 \\ \mathcal{L} = 0; \end{cases} \tag{11.169}$$

therefore, from (11.140), its entropy reads

$$S = \kappa V_{BH} + |\mathcal{L}_i|^2 \sqrt{\frac{-\mathbf{I}_2 + V_{BH}^2 - \frac{16}{3}\mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right)}{g^{i\bar{j}}C_{ikl}\overline{C}_{\overline{jmn}}\overline{\mathcal{L}}^k\overline{\mathcal{L}}^l\mathcal{L}^m\mathcal{L}^n}}. \tag{11.170}$$

No **BPS sector** is allowed in this sub-class.

II.15. This sub-class is given by "$\mathcal{Q}.3\otimes\mathcal{L}.5$" , and thus it corresponds to the generic case, in which none of $\mathrm{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right)$, $\mathbf{I}_2$, $\mathrm{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right)$, $\mathbf{I}_{-2}$, V and $\mathcal{L}$ is vanishing. The entropy is thus given by the general expression (11.140).

12 Taxonomy

By way of example, in this section, we report the main features of some known solutions of static and extremal BHs to $\mathcal{N} = 2$, $D = 4$ supergravity coupled to vector multiplets (in the STU model, in the axion-dilaton $\overline{\mathbb{CP}}^1$ model, and in the T^3 model), with U(1) FI gaugings.[29] It is here worth remarking that a complete taxonomy of all known solutions goes beyond the aim of the present paper. Since the classification of each known solution requires a detailed treatment and a good deal of computations, we will report on it in a future work.

12.1 Electric *STU*

We start and take under consideration the electric *STU* model [28], defined by the holomorphic prepotential

$$F = \frac{X^1 X^2 X^3}{X^0}. \tag{12.1}$$

---

[29] In such a framework, non-extremal solutions have been discussed e.g. in [77, 78] and [46].

Introducing the usual coordinates

$$s := \frac{X^1}{X^0}, \qquad t := \frac{X^2}{X^0}, \qquad u := \frac{X^1}{X^0}, \tag{12.2}$$

the symplectic vector $\mathcal{V}$ can be written as

$$\mathcal{V} = e^{K/2} \left(1, \, s, \, t, \, u, \, -stu, \, tu, \, su, \, st \right)^T, \tag{12.3}$$

and the Kähler potential and the target space metric are

$$K = -\log\left(-8\operatorname{Im}s\operatorname{Im}t\operatorname{Im}u\right), \tag{12.4}$$

$$g_{s\bar{s}} = -\frac{1}{(s-\bar{s})^2}, \qquad g_{t\bar{t}} = -\frac{1}{(t-\bar{t})^2}, \qquad g_{u\bar{u}} = -\frac{1}{(u-\bar{u})^2}.$$

We make the following choices for the charges ($i = 1, 2, 3$)

$$\mathcal{G} = (\, 0, \, g^i, \, g_0, \, 0)^T, \qquad \mathcal{Q} = (\, p^0, \, 0, \, 0, \, q_i)^T, \tag{12.5}$$

with $g^i = g$ and $q_i = q$; then, the central charges and their derivatives are

$$\mathcal{Z} = -e^{K/2}\left[q\left(s+t+u\right)+p^0 stu\right]; \tag{12.6}$$

$$\mathcal{L} = e^{K/2}\left[-g_0 + g\left(tu + su + st\right)\right]; \tag{12.7}$$

$$\mathcal{Z}_s = -\frac{\mathbf{i}e^{K/2}}{2\operatorname{Im}s}\left[q\left(s+t+u\right)+p^0 stu\right] - e^{K/2}\left(q + p^0 tu\right); \tag{12.8}$$

$$\mathcal{Z}_t = -\frac{\mathbf{i}e^{K/2}}{2\operatorname{Im}t}\left[q\left(s+t+u\right)+p^0 stu\right] - e^{K/2}\left(q + p^0 su\right); \tag{12.9}$$

$$\mathcal{Z}_u = -\frac{\mathbf{i}e^{K/2}}{2\operatorname{Im}u}\left[q\left(s+t+u\right)+p^0 stu\right] - e^{K/2}\left(q + p^0 st\right); \tag{12.10}$$

$$\mathcal{L}_s = -\frac{\mathbf{i}e^{K/2}}{2\operatorname{Im}s}\left[-g_0 + g\left(tu + su + st\right)\right] + g e^{K/2}\left(u + t\right); \tag{12.11}$$

$$\mathcal{L}_t = -\frac{\mathbf{i}e^{K/2}}{2\operatorname{Im}t}\left[-g_0 + g\left(tu + su + st\right)\right] + g e^{K/2}\left(s + u\right); \tag{12.12}$$

$$\mathcal{L}_u = -\frac{\mathbf{i}e^{K/2}}{2\operatorname{Im}u}\left[-g_0 + g\left(tu + su + st\right)\right] + g e^{K/2}\left(s + t\right). \tag{12.13}$$

We now calculate the derivatives of the symplectic vector $\mathcal{V}$:

$$\mathcal{V}_s = \frac{\mathbf{i}}{2\operatorname{Im}s}\mathcal{V} + e^{K/2}\left(0, \, 1, \, 0, \, 0, \, -tu, \, 0, \, u, \, t\right)^T; \tag{12.14}$$

$$\mathcal{V}_t = \frac{\mathbf{i}}{2\operatorname{Im}t}\mathcal{V} + e^{K/2}\left(0, \, 0, \, 1, \, 0, \, -su, \, u, \, 0, \, s\right)^T : \tag{12.15}$$

$$\mathcal{V}_u = \frac{\mathbf{i}}{2\operatorname{Im}u}\mathcal{V} + e^{K/2}\left(0, \, 0, \, 0, \, 1, \, -st, \, t, \, s, \, 0\right)^T; \tag{12.16}$$

$$D_s\mathcal{V}_s = \frac{\mathbf{i}}{\operatorname{Im}s}\mathcal{V}_s; \tag{12.17}$$

$$D_u\mathcal{V}_s = \frac{\mathbf{i}}{2\operatorname{Im}u}\mathcal{V}_s + \frac{\mathbf{i}}{2\operatorname{Im}s}\left(\mathcal{V}_u - \frac{\mathbf{i}}{2\operatorname{Im}u}\mathcal{V}\right) + e^{K/2}\left(0, \, 0, \, 0, \, 0, \, -t, \, 0, \, 1, \, 0\right)^T; \tag{12.18}$$

$$D_t\mathcal{V}_s = \frac{\mathbf{i}}{2\operatorname{Im}t}\mathcal{V}_s + \frac{\mathbf{i}}{2\operatorname{Im}s}\left(\mathcal{V}_t - \frac{\mathbf{i}}{2\operatorname{Im}t}\mathcal{V}\right) + e^{K/2}\left(0, \, 0, \, 0, \, 0, \, -u, \, 0, \, 0, \, 1\right)^T. \tag{12.19}$$

Taking the symplectic products of these derivatives we can calculate the only non-zero element of C_{ijk},

$$\langle \mathcal{V}_t, D_u \mathcal{V}_s \rangle = C_{stu} = -1, \tag{12.20}$$

then the solution belongs to Class **II.15**: $= \mathcal{Q}$. **3**$\otimes\mathcal{L}$.**5**.

Next, we take all scalar fields equal $s = t = u = -iy$ and this yields to

$$K = -\log(8y^3), \qquad e^{K/2} = \frac{1}{\sqrt{8y^3}}, \qquad p^0 = \frac{1}{g_0}(-1 + 3gq), \tag{12.21}$$

while for the central charges we have

$$\mathcal{Z} = \mathbf{i}e^{K/2}y\left(3q + \frac{1}{g_0}(1 - 3gq)y^2\right), \qquad \mathcal{L} = -e^{K/2}(g_0 + 3gy^2). \tag{12.22}$$

Then, the non-vanishing 2-polarizations of the quartic invariant are given by the following expressions

$$\mathbf{I}_2 = -4p^0q^3, \qquad \mathbf{I}_{-2} = 4g_0g^3, \qquad \mathbf{I}_0 = -\frac{1}{6}(1 - 12gq + 24g^2q^2), \tag{12.23}$$

and the entropy of extremal ($\frac{1}{4}$-)BPS black holes reads

$$S = \sqrt{\frac{3\mathbf{I}_0}{\mathbf{I}_{-2}} + \frac{\sqrt{36\mathbf{I}_0^2 - 4\mathbf{I}_2\mathbf{I}_{-2}}}{2\mathbf{I}_{-2}}} = \frac{1}{4}\sqrt{\frac{1 + 2(1 - 4gq)\sqrt{1 - 16gq + 48g^2q^2} - 3(1 - 4gq)^2}{g_0g^3}}, \tag{12.24}$$

corresponding to (5.23) with the choice of the branch "+" (for entropy maximization). From the discussion in section 5, one can immediately observe that the BPS extremal black holes of this example satisfy the condition (5.31).

12.2 Magnetic *STU*

In the previous section, we have considered the electric *STU* model in the symplectic frame defined by (12.1), in which the quartic invariant reads (cf. (3.8)–(3.9))

$$\mathbf{I}_2 := I_4(\mathcal{Q}) = -(p^0q_0 + p^iq_i)^2 + 4q_0p^1p^2p^3 - 4p^0q_1q_2q_3 + 4(p^1p^2q_1q_2 + p^1p^3q_1q_3 + p^2p^3q_2q_3). \tag{12.25}$$

By performing a symplectic transformation defined by the following matrix [28]:

$$\mathcal{S} := \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{12.26}$$

one can switch to the so-called magnetic symplectic frame defined by[30]

$$F_{STU} := -2\mathbf{i}\sqrt{X^0 X^1 X^2 X^3}, \tag{12.27}$$

and

$$\mathcal{S}\left(p^0, p^i, q_0, q_i\right)^T =: \left(-p^0, -q_i, -q_0, p^i\right)^T; \tag{12.28}$$

$$\mathcal{S}\left(1, s, t, u, -stu, tu, su, st\right)^T =: -\left(1, tu, su, st, -stu, -s, -t, -u\right)^T, \tag{12.29}$$

and in which the quartic invariant reads

$$\mathbf{I}_2 = -(p^0 q_0 - p^i q_i)^2 + 4q_0 q_1 q_2 q_3 + 4p^0 p^1 p^2 p^3 + 4(p^1 p^2 q_1 q_2 + p^1 p^3 q_1 q_3 + p^2 p^3 q_2 q_3). \tag{12.30}$$

Now, we present a new non-BPS solution (with hyperbolic horizon) to this prepotential ($I = 1, 2, 3$). For details, see appendix C; note that only the scalar τ_1 is running, whereas τ_2 and τ_3 are frozen at their asymptotical values, which are critical points for V itself:

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + \left(r^2 - \Delta^2\right)\left(d\theta^2 + \sinh^2\theta d\phi^2\right); \tag{12.31}$$

$$A(r) := \left(\frac{(64)^2 a^2 + G\left(b^2 G + 4\left(r^2 - \Delta^2\right)\left(-\Delta^2 G + Gr^2 - 8\right)\right)}{32G\left(r^2 - \Delta^2\right)}\right); \tag{12.32}$$

$$\tau_1 = \sqrt{\frac{g_0 g_1}{g_2 g_3}}\,\tau\left(r\right), \qquad \tau_2 = \sqrt{\frac{g_0 g_2}{g_1 g_3}}, \qquad \tau_3 = \sqrt{\frac{g_0 g_3}{g_1 g_3}}, \tag{12.33}$$

$$\tau(r) := \frac{r - \Delta}{r + \Delta}; \tag{12.34}$$

$$F^I = (\pm)_I \frac{Gb}{64(r^2 - \Delta^2)}\begin{pmatrix}\frac{(r+\Delta)}{g_0(r-\Delta)} \\ \frac{(r+\Delta)}{g_1(r-\Delta)} \\ \frac{(r-\Delta)}{g_2(r+\Delta)} \\ \frac{(r-\Delta)}{g_3(r+\Delta)}\end{pmatrix} dt \wedge dr + (\pm)_I \frac{a}{g_I}\sinh\theta\, d\theta \wedge d\phi. \tag{12.35}$$

This solution represents a non-extremal black hole in AdS$_4$ with electric and magnetic charges

$$G = 64\sqrt{g_0 g_1 g_2 g_3}, \quad g_I > 0, \quad p^I = (\pm)_I \frac{a}{g_I}, \quad q_I = (\pm)_I b\, g_I, \quad (\text{no sum on } I), \tag{12.36}$$

where $(\pm)_I$ is a vector in which in each component one can choose between the values $\pm = \{+1, -1\}$. When the extremality condition

$$(64)^2 a^2 G^2 + b^2 G^4 \leqslant 64\, G^2 \tag{12.37}$$

is saturated, the unique event horizon is located at

$$r_H = \sqrt{\Delta^2 + \frac{4}{G}}. \tag{12.38}$$

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[30]This symplectic frame can be obtained from the $\mathcal{N} = 2$ truncation of the SO(8) gauged $\mathcal{N} = 8$ supergravity [34, 71].

The Bekenstein-Hawking entropy reads

$$\frac{S}{\pi} = \frac{A}{4} = \left(\frac{\sqrt{-(64)^2 a^2 - b^2 G^2 + 64}}{2G} + \frac{4}{G} \right) (g-1), \qquad (12.39)$$

where we compactified to a Riemann surface of genus g. The entropy in the non-BPS (and non-extremal) case still does not depend on the values of the scalars, but only on the values of the charges. This might seem quite unexpected, since the attractor mechanism in the non-extremal case would not work. In fact, there is no attractor mechanism, and the non-extremal (non-BPS) BH entropy would depend also on the asymptotical values of scalar fields, which however are stabilized in terms of the gauging parameters and of the BH charges in the asymptotical background (as critical points, actually local minima) of the gauge potential. Thus, the non-extremal BH entropy may be recast in an explicit form depending only on the BH charges and gauging parameters supporting the solution under consideration. The potentials read

$$V_{BH} = \left(\frac{64a^2}{G} + \frac{Gb^2}{64} \right) \left(\frac{1 + |\tau|^2}{\mathrm{Re}\,\tau} \right), \quad \partial_\tau V_{BH} = \frac{1}{2} \left(\frac{64a^2}{G} + \frac{Gb^2}{64} \right) \left(\frac{\bar{\tau}^2 - 1}{\mathrm{Re}^2 \tau} \right); \quad (12.40)$$

$$V = -\frac{G}{16} \left(4 + \frac{1 + |\tau|^2}{\mathrm{Re}\,\tau} \right), \qquad \partial_\tau V = \frac{G}{32} \left(\frac{1 - \bar{\tau}^2}{\mathrm{Re}^2 \tau} \right); \qquad (12.41)$$

$$\partial_{\tau_2} V = 0, \qquad \partial_{\tau_3} V = 0, \qquad (12.42)$$

while the effettive potential is defined by (D.19) with $\kappa = -1$.

Focusing on the extremal case, we can take the branch which allows the limit $b = 0$ which is supersymmetric, namely

$$a = \frac{1}{64} \sqrt{64 - b^2 G^2}; \qquad (12.43)$$

the entropy density reduces to the supersymmetric value (cf. (6.12))

$$S = \frac{\mathbb{S}}{\mathbf{V}} = \frac{4}{G} = \frac{1}{16\sqrt{g_0 g_1 g_2 g_3}}. \qquad (12.44)$$

At the unique event horizon, we have the following values

$$B|_H = \frac{4}{G}, \qquad \tau_1|_H = \sqrt{\frac{g_0 g_1}{g_2 g_3}} \frac{\sqrt{\Delta^2 G + 4} - \Delta G}{\sqrt{\Delta^2 G + 4} + \Delta G}, \qquad (12.45)$$

$$V_{\mathrm{eff}}|_H = \frac{4}{G}, \qquad \partial_\tau V_{\mathrm{eff}}|_H = 0, \qquad (12.46)$$

and we see that these configurations are extremizing the effective potential. It is here worth remarking that (12.44) yields

$$S^2 = \frac{16}{G^2} = -\frac{\partial_{\tau_1} V_{BH}|_H}{\partial_{\tau_1} V|_H}, \qquad (12.47)$$

consistent with the result (8.7) for BPS critical points of V_{eff}. Since $\partial_{\tau_1} V_{BH}|_H \neq 0$ and $\partial_{\tau_1} V|_H \neq 0$, one concludes that such BPS critical points of V_{eff} belong to **class II.15** , discussed in section 11.2.

It is here worth remarking a curious fact: by varying the value of the parameter Δ, one can switch between **class II.15** and class **I.1** of critical points of V_{eff}, respectively discussed in sections 11.1 and 11.2. In fact, a (continuous) deformation of one into the other can be achieved by suitably choosing the parametric dependence of the scalar fields τ_I's. By setting

$$\Delta = 0, \tag{12.48}$$

one obtains

$$\tau = 1; \tag{12.49}$$

$$\partial_I V_{BH} = 0, \tag{12.50}$$

$$\partial_I V = 0, \tag{12.51}$$

thus corresponding to the sub-class $\mathbf{1.I} := V_{BH}.1 \otimes V.1$, since for the extremal solution at the horizon holds that

$$V_{BH} = |\mathcal{Z}|^2 = \frac{2}{G}, \qquad V = -3|\mathcal{L}|^2 = -3\frac{G}{8}. \tag{12.52}$$

By considering the entropy formula (11.14), one consistently obtains the result (recall that $\kappa = -1$)

$$S = -2\kappa |\mathcal{Z}|^2 = \frac{4}{G}, \tag{12.53}$$

for any extremal black hole. This is a very interesting phenomenon, whose investigation in detail is left to future work; here, we confine ourselves to observe that the transition from $\Delta \neq 0$ to $\Delta = 0$ as specified by (12.48) corresponds to a transition among *different* classes of critical points of V_{eff} which, in a symmetric model like the *STU* model, should correspond to a transition among *different* duality orbits in the representation spaces $\mathcal{Q}$ and $\mathcal{G}$. Namely, we have transited from class **II.15** to class **I.1** by imposing (12.48); this cannot be achieved by a *U*-duality transformation, but rather through a symplectic finite transformation belonging to the pseudo-Riemannian coset $\text{Sp}(8, \mathbb{R})/\text{SL}(2, \mathbb{R})^3$.

With the choices

$$p^I = -\frac{1}{8g_I}, \qquad q_I = 0, \tag{12.54}$$

the extremal critical point becomes $(\frac{1}{4}\text{-})$BPS, and the corresponding BPS entropy enjoys the expression (5.41), thus belonging to the noteworthy BPS sub-class discussed in section 5.3.1. One can thus conclude that the BPS extremal black hole supported by (12.48) and (12.54), belonging to the **BPS sub-sector** of class **I.1** of critical points of V_{eff}, is characterized by (5.32), and provides an example in which (5.31), and thus (5.39), is satisfied.

Finally, by performing the further identifications

$$g_0, g_1 \rightarrow g_0/2 \qquad g_2, g_3 \rightarrow g_2/2, \qquad \ell = \frac{8}{G}.$$

we get the static solution to the axion-dilaton model [13, 63] $F = -\mathbf{i} X^0 X^1$, presented in the next section.

12.3 $\overline{\mathbb{CP}}^1$

Starting from the *STU* model, in order to obtain the *minimally coupled* model of $\mathcal{N} = 2$, $D = 4$ supergravity with $\overline{\mathbb{CP}}^1$ vector multiplet's scalar manifold in the symplectic frame defined by

$$F_{\overline{\mathbb{CP}}^1} := -\mathbf{i}X^0 X^1, \tag{12.55}$$

one needs to identify the contravariant symplectic sections as follows:

$$X^2 \Rightarrow X^0/\sqrt{2}, \tag{12.56}$$
$$X^0 \Rightarrow X^0/\sqrt{2}, \tag{12.57}$$
$$X^1 \Rightarrow X^1/\sqrt{2}, \tag{12.58}$$
$$X^3 \Rightarrow X^1/\sqrt{2}, \tag{12.59}$$

thus getting that the quartic invariant boils down to be the square of a quadratic invariant:

$$\mathbf{I}_2 = (q_0 q_1 + p^0 p^1)^2. \tag{12.60}$$

We now present the investigation of the attractor dynamics of the complex scalar field (axion-dilaton) within a subclass of extremal solutions previously found in [13] and [63], in presence of U(1) FI gauging. These correspond to the choices ($A = 0, 1$)

$$\mathcal{G} = (0, g_A)^T, \ \mathcal{Q} = (\kappa p^A, 0)^T. \tag{12.61}$$

The symplectic section can be parametrised in terms of the complex scalar field τ by choosing $X^0 = 1$, $X^1 = \tau$, so that the holomorphic symplectic section reads (cf. (2.16))

$$\mathfrak{H} = \left(1, \ \tau, \ -\mathbf{i}\tau, \ -\mathbf{i}\right)^T, \tag{12.62}$$

where τ coordinatizes $M_v \equiv \overline{\mathbb{CP}}^1$. The Kähler potential and the non-vanishing components of the metric of the scalar manifold are respectively

$$e^{-K} = 4\text{Re}\tau \qquad g_{\tau\bar{\tau}} = g_{\bar{\tau}\tau} = \partial_\tau \partial_{\bar{\tau}} K = (2\text{Re}\tau)^{-2}. \tag{12.63}$$

By recalling (2.16), the Kähler-covariantly holomorphic symplectic section reads

$$\mathcal{V} = \frac{1}{2\sqrt{\text{Re}\tau}} \mathfrak{H}. \tag{12.64}$$

Its derivative and the central charges respectively read

$$D_\tau \mathcal{V} \equiv \mathcal{V}_\tau = \frac{e^{K/2}}{2\tau} \left(-1, \ \tau, \ \mathbf{i}\tau, \ -\mathbf{i}\right)^T; \tag{12.65}$$

$$\mathcal{Z} := \langle \mathcal{Q}, \mathcal{V} \rangle = -\mathbf{i}e^{K/2}\kappa(p^0\tau + p^1), \tag{12.66}$$

$$\mathcal{L} := \langle \mathcal{G}, \mathcal{V} \rangle = -4g_0 g_1 e^{K/2}(p^0\tau + p^1), \tag{12.67}$$

and the derivative of the central charges is

$$D_\tau \mathcal{Z} = \langle \mathcal{Q}, \mathcal{V}_\tau \rangle = \mathbf{i}\,\kappa\frac{e^{K/2}}{2\tau}(p^1 - p^0\tau). \tag{12.68}$$

Since also this model is symmetric, one can compute the 2-polarizations of the quartic structure that, by virtue of (12.60), is *non-primitive* (namely, the square of a quadratic invariant structure; see footnotes 5 and 11):

$$\mathbf{I}_2 = (p^0 p^1)^2, \tag{12.69}$$

$$\mathbf{I}_1 = 0, \tag{12.70}$$

$$\mathbf{I}_0 = \frac{1}{3} p^0 p^1 g_0 g_1, \tag{12.71}$$

$$\mathbf{I}_{-1} = 0, \tag{12.72}$$

$$\mathbf{I}_{-2} = (g_0 g_1)^2. \tag{12.73}$$

In this noteworthy subclass, as discussed in section 5.3.1, from (5.41) the BPS extremal black hole entropy reads

$$S = \sqrt[4]{\frac{\mathbf{I}_2}{\mathbf{I}_{-2}}} \overset{(5.39)}{=} \sqrt[4]{9 \frac{\mathbf{I}_0^2}{\mathbf{I}_{-2}^2}} = \sqrt{3 \frac{\mathbf{I}_0}{\mathbf{I}_{-2}}} = \sqrt{\frac{p^0 p^1}{g_0 g_1}}. \tag{12.74}$$

We should recall that in [13] and [63] the following identifications were made:

$$p^0 = -\frac{1}{4g_0}, \qquad p^1 = -\frac{1}{4g_1}. \tag{12.75}$$

resulting into the entropy (12.74) to simplify down to

$$S = \frac{1}{4g_0 g_1}. \tag{12.76}$$

When the solution presents an hyperbolic horizon, S denotes the entropy density, and one can compactify to a Riemannian surface of genus g, and the identification with the above formalism is trivial, since

$$\mathbf{V} = 4\pi(g - 1). \tag{12.77}$$

12.4 T^3

Finally, we consider the solution in [72] for the model with prepotential

$$F = \frac{(X^1)^3}{X^0}, \tag{12.78}$$

with non vanishing FI U(1) gauging parameters $g_0 = g\xi_0$ and $g^1 = g\xi^1$. The BH solution has one magnetic charge p^0 and one electric charge q_1. The charges and the constants of the solution are

$$p^0 = \mp \frac{1}{g\xi_0} \left(\frac{1}{8} + \frac{8(g\xi^1\beta_1)^2}{3} \right), \qquad q_1 = \pm \frac{1}{g\xi^1} \left(\frac{3}{8} - \frac{8(g\xi^1\beta_1)^2}{3} \right), \tag{12.79}$$

$$\beta^0 = \frac{\xi^1\beta_1}{\xi_0}, \qquad \alpha^0 = \pm \frac{1}{4\xi_0}, \qquad \alpha_1 = \mp \frac{3}{4\xi^1}, \qquad c = 1 - \frac{32}{3}(g\xi^1\beta_1)^2. \tag{12.80}$$

Then, the central charges and their derivatives read

$$\mathcal{Z} = -e^{K/2}\left(3s\frac{\left(\frac{3}{8} - \frac{8}{3}(\beta_1 g^1)^2\right)}{g_1} - s^2\frac{\left(\frac{1}{8} + \frac{8}{3}(\beta_1 g^1)^2\right)}{g_0}\right), \tag{12.81}$$

$$\mathcal{L} = e^{K/2}\left(3g^1 s^2 - g^0\right), \tag{12.82}$$

$$D_s\mathcal{Z} = -i\frac{1}{2Ims}\mathcal{Z} - e^{K/2}\left(\frac{\left(\frac{3}{8} - \frac{8}{3}(\beta_1 g^1)^2\right)}{g_1} - s^2\frac{\left(\frac{1}{8} + \frac{8}{3}(\beta_1 g^1)^2\right)}{g_0}\right), \tag{12.83}$$

$$D_s\mathcal{L} = e^{K/2}(3g^1 s^2 - g_0), \tag{12.84}$$

where s is the only scalar field and K is the Kähler potential as usual. The $\mathbf{I}_{\pm 2}$ quartic invariants read

$$\mathbf{I}_2 = -4p^0 q_1^3, \qquad \mathbf{I}_{-2} = 4g_0(g^1)^3.$$

Thus, $\mathcal{L}$, $\mathbf{I}_{-2}$ and $\mathbf{I}_2$ are non-vanishing, and exploiting the above results it is straightforward to show that also $\text{Im}\left(\mathcal{L}N_3(\overline{\mathcal{L}},\overline{\mathcal{L}},\overline{\mathcal{L}})\right)$ and $\text{Im}\left(\mathcal{Z}N_3(\overline{\mathcal{Z}},\overline{\mathcal{Z}},\overline{\mathcal{Z}})\right)$ do not vanish, implying that this solution belongs to the class $\mathbf{II.15}:=\mathcal{Q}.\mathbf{3}\otimes\mathcal{L}.\mathbf{5}$.

13 Conclusion

In this paper, we have considered $\mathcal{N} = 2$, $D = 4$ supergravity coupled to Abelian vector multiplets with U(1) Fayet-Iliopoulos gaugings.

By exploiting the identities determining the structure of projective special Kähler geometry endowing the vector multiplets' scalar manifold in presence of electric and magnetic BH charges as well as of (generally dyonic) gauging parameters, we retrieved, extended and generalized various results on the expression of Bekenstein-Hawking entropy of asymptotically AdS$_4$ BPS BHs in gauged supergravity. In doing this, we made use of the quartic structure (and 2-polarizations thereof) characterizing the U-duality groups of type E_7 corresponding to symmetric scalar manifolds. Then, we have presented a complete classification of the critical points of the effective black hole potential V_{eff} which governs the attractor mechanism at the horizon of extremal BHs, relating - when possible - the resulting attractors to the critical points of the gauge potential V as well as of the effective black hole potential in the ungauged case, V_{BH}. In all cases, we have analyzed the existence of BPS sub-sectors and studied their features. Finally, we have inserted explicit known examples of asymptotically AdS$_4$ static extremal (BPS) BH in gauged supergravity in the aforementioned classification, and, as a by-product of our treatment, we also have provided a novel, static extremal BH solution to the STU model, with the dilaton interpolating between a hyperbolic horizon and AdS$_4$ at infinity.

The classification of the critical points of V_{eff} which we have provided in the present work will hopefully be instrumental in order to discover and explore new solutions of Maxwell-Einstein supergravity with non-vanishing gauge potential. Some directions for possible further developments also concern the extension to the planar case ($\kappa = 0$), the coupling of hypermultiplets (cf. e.g. [32, 73]), and the generalization to stationary solutions.

It is finally worth remarking that an almost uncharted territory is provided by non-Abelian gaugings of $\mathcal{N} = 2$ $D = 4$ supergravity, which just a few works (see e.g. [74–76]) have hitherto dared to investigate; the question whether in presence of non-Abelian gaugings an effective black hole potential formalism for the (covariant) attractor mechanism can be established, still remains unanswered.

Finally, it is worth mentioning that the few examples discussed in section 12 belong to two classes only. A quick procedure for the identification of a given solution into one class of our classification is not currently available; actually, a considerable deal of work and computations is needed in order to do so. While this is of course not an impossible task, it would nevertheless be helpful to develop some characterization theorems in order to simplify such an identification. Interestingly, such a characterization would likely also provide a strategy for the construction of explicit solutions in any given class, or, possibly otherwise, prove the emptiness of some classes.

A Computation of the r.h.s. of eq. (5.7)

In special Kähler geometry based on the *cubic* holomorphic prepotential (3.1), named *very special* geometry, the cubic form is defined as (cfr. e.g. [53, 54])

$$\mathbb{V} := -\frac{1}{3!} d_{klm} \mathrm{Im}\left(\frac{X^k}{X^0}\right) \mathrm{Im}\left(\frac{X^l}{X^0}\right) \mathrm{Im}\left(\frac{X^m}{X^0}\right), \tag{A.1}$$

and the scalar manifold of the corresponding minimal supergravity theory in $D = 5$ is defined as the hypersurface at $\mathbb{V} = 1$. In order to compute the contractions in the r.h.s. of (5.7), we have to recall some basic formulæ of very special Kähler geometry. From e.g. the treatment of [53], choosing the so-called 4D/5D special coordinates' symplectic frame and fixing the Kähler gauge such that $X^0 = 1$, with $\frac{X^i}{X^0} =: z^i = x^i - \mathbf{i}\mathbb{V}^{1/3}\hat{\lambda}^i$ (such that $\frac{1}{3!} d_{ijk}\hat{\lambda}^i\hat{\lambda}^j\hat{\lambda}^k = 1$, which is a way to rewrite (A.1)), one can define

$$\hat{\kappa}_{ij} := d_{ijk}\hat{\lambda}^k, \qquad \hat{\kappa}_i := d_{ijk}\hat{\lambda}^j\hat{\lambda}^k, \qquad \hat{\kappa} := d_{ijk}\hat{\lambda}^i\hat{\lambda}^j\hat{\lambda}^k = 6; \tag{A.2}$$

$$h_{ij} := d_{ijk}x^k, \qquad h_i := d_{ijk}x^j x^k, \qquad h := d_{ijk}x^i x^j x^k. \tag{A.3}$$

Then, the symplectic sections read

$$\mathcal{V}^M =: e^{K/2} \begin{pmatrix} X^0 \\ X^i \\ F_0 \\ F_i \end{pmatrix} = e^{K/2} \begin{pmatrix} 1 \\ z^i \\ -F \\ \frac{\partial F}{\partial X^i} \end{pmatrix}, \tag{A.4}$$

where

$$e^{K/2} = \frac{1}{2\sqrt{2}}\mathbb{V}^{-1/2}, \tag{A.5}$$

and

$$F = \frac{1}{3!}d_{ijk}z^i z^j z^k = \frac{1}{3!}d_{ijk}\left(x^i - \mathbf{i}\mathbb{V}^{1/3}\hat{\lambda}^i\right)\left(x^j - \mathbf{i}\mathbb{V}^{1/3}\hat{\lambda}^j\right)\left(x^k - \mathbf{i}\mathbb{V}^{1/3}\hat{\lambda}^k\right)$$

$$= \frac{1}{3!}d_{ijk}x^i x^j x^k - \frac{\mathbf{i}}{2}\mathbb{V}^{1/3}d_{ijk}x^i x^j \hat{\lambda}^k - \frac{1}{2}\mathbb{V}^{2/3}d_{ijk}x^i \hat{\lambda}^j \hat{\lambda}^k + \frac{\mathbf{i}}{3!}\mathbb{V}d_{ijk}\hat{\lambda}^i \hat{\lambda}^j \hat{\lambda}^k$$

$$= \frac{h}{6} - \frac{1}{2}\mathbb{V}^{2/3}\hat{\kappa}_i x^i + \mathbf{i}\frac{\mathbb{V}^{1/3}}{2}\left(-h_i\hat{\lambda}^i + 2\mathbb{V}^{2/3}\right)$$

$$= \frac{h}{6} - \frac{1}{2}\mathbb{V}^{2/3}h_{ij}\hat{\lambda}^i \hat{\lambda}^j + \mathbf{i}\frac{\mathbb{V}^{1/3}}{2}\left(-\hat{\kappa}_{ij}x^i x^j + 2\mathbb{V}^{2/3}\right), \tag{A.6}$$

and

$$F_i = \frac{\partial F}{\partial X^i} = \frac{1}{2}d_{ijk}z^j z^k = \frac{1}{2}d_{ijk}\left(x^j - \mathbf{i}\mathbb{V}^{1/3}\hat{\lambda}^j\right)\left(x^k - \mathbf{i}\mathbb{V}^{1/3}\hat{\lambda}^k\right)$$

$$= \frac{1}{2}d_{ijk}x^j x^k - \mathbf{i}\mathbb{V}^{1/3}d_{ijk}x^j \hat{\lambda}^k - \frac{1}{2}\mathbb{V}^{2/3}d_{ijk}\hat{\lambda}^j \hat{\lambda}^k =$$

$$= \frac{1}{2}h_i - \frac{1}{2}\mathbb{V}^{2/3}\hat{\kappa}_i - \mathbf{i}\mathbb{V}^{1/3}h_{ij}\hat{\lambda}^j = \frac{1}{2}h_i - \frac{1}{2}\mathbb{V}^{2/3}\hat{\kappa}_i - \mathbf{i}\mathbb{V}^{1/3}\hat{\kappa}_{ij}x^j, \tag{A.7}$$

such that

$$\mathcal{Z} := \langle \mathcal{Q}, \mathcal{V} \rangle = e^{K/2}\left(q_0 + z^i q_i - p^0 F_0 - p^i F_i\right) = e^{K/2}\left(q_0 + z^i q_i + p^0 F - p^i F_i\right). \tag{A.8}$$

On the other hand, the Kähler-covariant derivatives of the symplectic sections read

$$\mathcal{V}_i^M \equiv D_i \mathcal{V}^M = e^{K/2}\begin{pmatrix} \tilde{D}_i X^0 \\ \tilde{D}_i X^j \\ \tilde{D}_i F_0 \\ \tilde{D}_i F_j \end{pmatrix}, \tag{A.9}$$

$$\tilde{D}_i \equiv \partial_i K + \partial_i, \tag{A.10}$$

where

$$\tilde{D}_i X^0 = -\frac{\mathbf{i}}{4}\mathbb{V}^{-1/3}\hat{\kappa}_i; \tag{A.11}$$

$$\tilde{D}_i X^j = \delta_i^j - \frac{\mathbf{i}}{4}\mathbb{V}^{-1/3}\hat{\kappa}_i\left(x^j - \mathbf{i}\mathbb{V}^{1/3}\hat{\lambda}^j\right); \tag{A.12}$$

$$\tilde{D}_i F_0 = -\frac{h_i}{2} + \frac{1}{4}\mathbb{V}^{-1/3}\hat{\kappa}_i + \frac{1}{8}\hat{\kappa}_i\hat{\kappa}_{jk}x^j x^k$$

$$+ \mathbf{i}\mathbb{V}^{1/3}\left(\frac{1}{24}\mathbb{V}^{-2/3}h\hat{\kappa}_i + \hat{\kappa}_{ij}x^j - \frac{1}{8}\hat{\kappa}_i\hat{\kappa}_j x^j\right); \tag{A.13}$$

$$\tilde{D}_i F_j = h_{ij} - \frac{1}{4}\hat{\kappa}_i\hat{\kappa}_{jk}x^k + \mathbf{i}\left(\frac{1}{8}\mathbb{V}^{1/3}\hat{\kappa}_i\hat{\kappa}_j - \mathbb{V}^{1/3}\hat{\kappa}_{ij} - \frac{1}{8}\mathbb{V}^{-1/3}\hat{\kappa}_i h_j\right), \tag{A.14}$$

such that

$$\mathcal{Z}_i \equiv D_i \mathcal{Z} = \langle \mathcal{Q}, \mathcal{V}_i \rangle = e^{K/2}\left(q_0 D_i X^0 + q_j D_i X^j - p^0 D_i F_0 - p^j D_i F_j\right). \tag{A.15}$$

Thus, from (5.7) and recalling the identity (3.2), one can proceed and compute

$$e^{-2K}\frac{1}{2}\Omega_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}^N\overline{\mathcal{V}}^P\overline{\mathcal{V}}^Q$$

$$\overset{(3.8)-(3.9)}{=}-\left(\overline{X}^0\overline{F}_0+\overline{X}^i\overline{F}_i\right)^2+\frac{2}{3}\overline{F}_0d_{ijk}\overline{X}^i\overline{X}^j\overline{X}^k-\frac{2}{3}\overline{X}^0d^{ijk}\overline{F}_i\overline{F}_j\overline{F}_k+d_{ijk}d^{ilm}\overline{X}^j\overline{X}^k\overline{F}_l\overline{F}_m$$

$$=-\left(-\overline{F}+\frac{1}{2}d_{ijk}\overline{z}^i\overline{z}^j\overline{z}^k\right)^2-\frac{2}{3}\overline{F}d_{ijk}\overline{z}^i\overline{z}^j\overline{z}^k$$

$$-\frac{1}{12}d^{ijk}d_{ilm}d_{jnp}d_{krs}\overline{z}^l\overline{z}^m\overline{z}^n\overline{z}^p\overline{z}^r\overline{z}^s+\frac{1}{4}d_{ijk}d^{ilm}\overline{z}^j\overline{z}^k d_{lpq}d_{mrs}\overline{z}^p\overline{z}^q\overline{z}^r\overline{z}^s$$

$$=-4\overline{F}^2-4\overline{F}^2-4\overline{F}^2+12\overline{F}^2=0. \tag{A.16}$$

$$e^{-2K}\frac{1}{2}\Omega_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}^N\overline{\mathcal{V}}^P\overline{\mathcal{V}}_{\bar{i}}^Q$$

$$\overset{(3.11)-(3.12)}{=}-\frac{1}{2}\left[\left(\overline{X}^0\right)^2\left(D_{\bar{i}}\overline{F}_0\right)\overline{F}_0+\overline{X}^0\left(D_{\bar{i}}\overline{X}^0\right)\overline{F}_0^2\right]-\frac{1}{2}\left[\overline{X}^k\overline{F}_k\overline{X}^j D_{\bar{i}}\overline{F}_j+\overline{X}^k\overline{F}_k\left(D_{\bar{i}}\overline{X}^j\right)\overline{F}_j\right]$$

$$-\frac{1}{2}\left[\overline{X}^0\overline{F}_0\overline{X}^j D_{\bar{i}}\overline{F}_j+\overline{X}^0\overline{F}_0\left(D_{\bar{i}}\overline{X}^j\right)\overline{F}_j+\overline{X}^0\left(D_{\bar{i}}\overline{F}_0\right)\overline{X}^j\overline{F}_j+\left(D_{\bar{i}}\overline{X}^0\right)\overline{F}_0\overline{X}^j\overline{F}_j\right]$$

$$+\frac{1}{6}\left[\left(D_{\bar{i}}\overline{F}_0\right)d_{jkl}\overline{X}^j\overline{X}^k\overline{X}^l+3\overline{F}_0d_{jkl}\overline{X}^j\overline{X}^k D_{\bar{i}}\overline{X}^k\right]-\frac{1}{6}\left[\left(D_{\bar{i}}\overline{X}^0\right)d^{jkl}\overline{F}_j\overline{F}_k\overline{F}_l+3\overline{X}^0 d^{jkl}\overline{F}_j\overline{F}_k D_{\bar{i}}\overline{F}_l\right]$$

$$+\frac{1}{2}d_{njk}d^{nlm}\overline{X}^j\overline{X}^k\overline{F}_l D_{\bar{i}}\overline{F}_m+\frac{1}{2}d_{njk}d^{nlm}\overline{X}^j\left(D_{\bar{i}}\overline{X}^k\right)\overline{F}_l\overline{F}_m$$

$$=-\frac{1}{2}\left[-(D_{\bar{i}}\overline{F}_0)\overline{F}+\left(D_{\bar{i}}\overline{X}^0\right)\overline{F}^2\right]-\frac{1}{2}\left[\overline{z}^k\overline{F}_k\overline{z}^j D_{\bar{i}}\overline{F}_j+\overline{z}^k\overline{F}_k\left(D_{\bar{i}}\overline{X}^j\right)\frac{1}{2}d_{jkl}\overline{z}^k\overline{z}^l\right]$$

$$-\frac{1}{2}\left[-\overline{F}\overline{z}^j D_{\bar{i}}\overline{F}_j-\frac{1}{2}\overline{F}\left(D_{\bar{i}}\overline{X}^j\right)d_{jmn}\overline{z}^m\overline{z}^n+\frac{1}{2}(D_{\bar{i}}\overline{F}_0)\overline{z}^j d_{jmn}\overline{z}^m\overline{z}^n-\frac{1}{2}\left(D_{\bar{i}}\overline{X}^0\right)\overline{F}\overline{z}^j d_{jmn}\overline{z}^m\overline{z}^n\right]$$

$$+\frac{1}{6}\left[(D_{\bar{i}}\overline{F}_0)d_{jkl}\overline{z}^j\overline{z}^k\overline{z}^l-3\overline{F}d_{jkl}\overline{z}^j\overline{z}^k D_{\bar{i}}\overline{X}^k\right]$$

$$-\frac{1}{6}\left[\frac{1}{8}\left(D_{\bar{i}}\overline{X}^0\right)d^{jkl}d_{jmn}d_{kpq}d_{lrs}\overline{z}^m\overline{z}^n\overline{z}^p\overline{z}^q\overline{z}^r\overline{z}^s+\frac{3}{4}d^{jkl}d_{jmn}d_{kpq}\overline{z}^m\overline{z}^n\overline{z}^p\overline{z}^q D_{\bar{i}}\overline{F}_l\right]$$

$$+\frac{1}{4}d_{njk}d^{nlm}\overline{z}^j\overline{z}^k d_{lrs}\overline{z}^r\overline{z}^s D_{\bar{i}}\overline{F}_m+\frac{1}{8}d_{njk}d^{nlm}\overline{z}^j\left(D_{\bar{i}}\overline{X}^k\right)d_{lqs}d_{mrt}\overline{z}^q\overline{z}^s\overline{z}^r\overline{z}^t$$

$$=\frac{1}{2}(D_{\bar{i}}\overline{F}_0)\overline{F}-\frac{1}{2}\left(D_{\bar{i}}\overline{X}^0\right)\overline{F}^2-\frac{3}{2}\overline{F}\overline{z}^j D_{\bar{i}}\overline{F}_j-\frac{3}{4}\overline{F}\left(D_{\bar{i}}\overline{X}^j\right)d_{jkl}\overline{z}^k\overline{z}^l$$

$$+\frac{1}{2}\overline{F}\overline{z}^j D_{\bar{i}}\overline{F}_j+\frac{1}{4}\overline{F}\left(D_{\bar{i}}\overline{X}^j\right)d_{jmn}\overline{z}^m\overline{z}^n-\frac{3}{2}\overline{F}(D_{\bar{i}}\overline{F}_0)+\frac{3}{2}\left(D_{\bar{i}}\overline{X}^0\right)\overline{F}^2$$

$$+\overline{F}(D_{\bar{i}}\overline{F}_0)-\frac{1}{2}\overline{F}d_{jkl}\overline{z}^j\overline{z}^k D_{\bar{i}}\overline{X}^k$$

$$-\overline{F}^2\left(D_{\bar{i}}\overline{X}^0\right)-\overline{F}\overline{z}^m D_{\bar{i}}\overline{F}_m$$

$$+2\overline{F}\overline{z}^m D_{\bar{i}}\overline{F}_m+\overline{F}d_{njk}\overline{z}^n\overline{z}^j\left(D_{\bar{i}}\overline{X}^k\right)$$

$$=0. \tag{A.17}$$

Analogously, one can check that

$$e^{-2K}\frac{1}{2}\Omega_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}^N\overline{\mathcal{V}}_{\bar{i}}^P\overline{\mathcal{V}}_{\bar{j}}^Q=0. \tag{A.18}$$

$$e^{-2K}\frac{1}{2}\Omega_{MNPQ}\overline{\mathcal{V}}^M\overline{\mathcal{V}}_{\bar{i}}^N\overline{\mathcal{V}}_{\bar{j}}^P\overline{\mathcal{V}}_{\bar{k}}^Q=0. \tag{A.19}$$

$$e^{-2K}\frac{1}{2}\Omega_{MNPQ}\overline{\mathcal{V}}_{\bar{i}}^M\overline{\mathcal{V}}_{\bar{j}}^N\overline{\mathcal{V}}_{\bar{k}}^P\overline{\mathcal{V}}_{\bar{l}}^Q=0. \tag{A.20}$$

B Proof of (10.7)

At the class V.**III** of critical points of V (cf. section 10), it holds that

$$\mathcal{L}_i = \frac{\mathbf{i}}{2\overline{\mathcal{L}}} C_{ijk} \overline{\mathcal{L}}^j \overline{\mathcal{L}}^k, \tag{B.1}$$

which implies

$$2\overline{\mathcal{L}}\mathcal{L}_i = -\frac{\mathbf{i}}{4\mathcal{L}^2} C_{ijk} \overline{C}^j{}_{(\overline{mn}|} \overline{C}^k{}_{|\overline{pq})} \mathcal{L}^{\overline{m}} \mathcal{L}^{\overline{n}} \mathcal{L}^{\overline{p}} \mathcal{L}^{\overline{q}}, \tag{B.2}$$

and

$$|\mathcal{L}_i|^2 = \frac{\mathbf{i}}{2\overline{\mathcal{L}}} N_3(\overline{\mathcal{L}}) = -\frac{\mathbf{i}}{2\mathcal{L}} \overline{N_3(\overline{\mathcal{L}})} \equiv -\frac{\mathbf{i}}{2\mathcal{L}} \overline{N}_3(\mathcal{L}), \tag{B.3}$$

where we have defined (cf. (10.11))

$$N_3(\overline{\mathcal{L}}) \equiv N_3(\overline{\mathcal{L}}, \overline{\mathcal{L}}, \overline{\mathcal{L}}) := C_{ijk} \overline{\mathcal{L}}^i \overline{\mathcal{L}}^j \overline{\mathcal{L}}^k. \tag{B.4}$$

By using the special geometry identity (3.1.1.2.12) of [41], one obtains

$$2\overline{\mathcal{L}}\mathcal{L}_i = -\frac{\mathbf{i}}{3\mathcal{L}^2} \mathcal{L}_i \overline{N}_3(\mathcal{L}) - \frac{\mathbf{i}}{12\mathcal{L}^2} \left(D_i \overline{D}_{(\overline{i}} \overline{C}_{\overline{jk}\overline{l})} \right) \mathcal{L}^{\overline{i}} \mathcal{L}^{\overline{j}} \mathcal{L}^{\overline{k}} \mathcal{L}^{\overline{l}}. \tag{B.5}$$

Therefore, (B.3) and (B.5) yield to

$$\left(\overline{\mathcal{L}} - \frac{1}{3} \frac{|\mathcal{L}_i|^2}{\mathcal{L}} \right) |\mathcal{L}_j|^2 = -\frac{\mathbf{i}}{24\mathcal{L}^2} \left(D_m \overline{D}_{(\overline{i}} \overline{C}_{\overline{jk}\overline{l})} \right) \overline{\mathcal{L}}^m \mathcal{L}^{\overline{i}} \mathcal{L}^{\overline{j}} \mathcal{L}^{\overline{k}} \mathcal{L}^{\overline{l}}; \tag{B.6}$$

$$\Updownarrow$$

$$|\mathcal{L}|^2 - \frac{1}{3}|\mathcal{L}_i|^2 = -\frac{\mathbf{i} \left(D_m \overline{D}_{(\overline{i}} \overline{C}_{\overline{jk}\overline{l})} \right) \overline{\mathcal{L}}^m \mathcal{L}^{\overline{i}} \mathcal{L}^{\overline{j}} \mathcal{L}^{\overline{k}} \mathcal{L}^{\overline{l}}}{24\mathcal{L} |\mathcal{L}_j|^2} \overset{\text{(B.3)}}{=} \frac{\left(D_m \overline{D}_{(\overline{i}} \overline{C}_{\overline{jk}\overline{l})} \right) \overline{\mathcal{L}}^m \mathcal{L}^{\overline{i}} \mathcal{L}^{\overline{j}} \mathcal{L}^{\overline{k}} \mathcal{L}^{\overline{l}}}{12\overline{N}_3(\mathcal{L})}; \tag{B.7}$$

$$\Updownarrow$$

$$|\mathcal{L}_i|^2 = 3|\mathcal{L}|^2 - \frac{\left(D_m \overline{D}_{(\overline{i}} \overline{C}_{\overline{jk}\overline{l})} \right) \overline{\mathcal{L}}^m \mathcal{L}^{\overline{i}} \mathcal{L}^{\overline{j}} \mathcal{L}^{\overline{k}} \mathcal{L}^{\overline{l}}}{4\overline{N}_3(\mathcal{L})}, \tag{B.8}$$

which, by definition (10.10), gives eq. (10.7) ∎

C Details on the magnetic *STU*

We use the conventions of [12]. We consider $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to n Abelian vector multiplets. Apart from the vierbein e^a_μ, the bosonic field content includes the vectors A^I_μ enumerated by $I = 0, \ldots, n$, and the complex scalars z^α where $\alpha = 1, \ldots, n$. These scalars parametrize a special Kähler manifold M_v, i.e. an n-dimensional Hodge-Kähler manifold that is the base of a symplectic bundle, with the covariantly holomorphic sections

$$\mathcal{V} = \begin{pmatrix} X^I \\ F_I \end{pmatrix}, \qquad D_{\bar\alpha} \mathcal{V} = \partial_{\bar\alpha} \mathcal{V} - \frac{1}{2}(\partial_{\bar\alpha} K) \mathcal{V} = 0, \tag{C.1}$$

where K is the Kähler potential and D denotes the Kähler-covariant derivative. $\mathcal{V}$ obeys the symplectic constraint

$$\langle \mathcal{V}, \overline{\mathcal{V}} \rangle = X^I \overline{F}_I - F_I \overline{X}^I = \mathbf{i}. \tag{C.2}$$

To solve this condition, one defines

$$\mathcal{V} =: e^{\mathcal{K}(z,\bar{z})/2}\mathfrak{H}(z)\,, \tag{C.3}$$

where $\mathfrak{H}(z)$ is a holomorphic symplectic vector,

$$\mathfrak{H}(z) = \begin{pmatrix} X^I(z) \\ \frac{\partial}{\partial X^I}F(X)(z) \end{pmatrix}\,. \tag{C.4}$$

where F is an holomorphic function homogeneous of degree two, called the *prepotential*, whose existence is assumed in order to obtain the last expression. The Kähler potential is then (cf. (2.18))

$$e^{-K(z,\bar{z})} = \mathbf{i}\langle \mathfrak{H}, \overline{\mathfrak{H}}\rangle\,. \tag{C.5}$$

The matrix $\mathcal{N}_{IJ}$ determining the coupling between the scalars z^α and the vectors A^I_μ is defined by the relations

$$F_I = \mathcal{N}_{IJ}X^J\,, \qquad D_{\bar{\alpha}}\overline{F}_I = \mathcal{N}_{IJ}D_{\bar{\alpha}}\overline{X}^J\,. \tag{C.6}$$

The bosonic action reads

$$e^{-1}\mathcal{L}_{\text{bos}} = \frac{1}{2}R + \frac{1}{4}(\text{Im}\,\mathcal{N})_{IJ}F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{8}(\text{Re}\,\mathcal{N})_{IJ}\,e^{-1}\epsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}$$
$$-g_{\alpha\bar{\beta}}\partial_\mu z^\alpha \partial^\mu \overline{z}^{\bar{\beta}} - V\,, \tag{C.7}$$

with the scalar potential

$$V = -2g^2\xi_I\xi_J[(\text{Im}\,\mathcal{N})^{-1|IJ} + 8\overline{X}^I X^J]\,, \tag{C.8}$$

that results from U(1) FI gauging. Here, g denotes the gauge coupling and the ξ_I are FI gauging parameters. In what follows, we define $g_I \equiv g\xi_I$. The Einstein's equations of motion from (C.7) are given by

$$G_{\mu\nu} = T_{\mu\nu} = {}^{(0)}T_{\mu\nu} + {}^{(1)}T_{\mu\nu} - g_{\mu\nu}V\,, \tag{C.9}$$

$${}^{(0)}T_{\mu\nu} = 2g_{\alpha\bar{\beta}}\partial_{(\mu}z^\alpha\partial_{\nu)}\overline{z}^{\bar{\beta}} - g_{\mu\nu}g_{\alpha\bar{\beta}}\partial_\sigma z^\alpha\partial^\sigma\overline{z}^{\bar{\beta}}\,, \tag{C.10}$$

$${}^{(1)}T_{\mu\nu} = -(Im\mathcal{N})_{IJ}F^I_{\mu\sigma}F^{J\sigma}_\nu + g_{\mu\nu}\frac{1}{4}(Im\mathcal{N})_{IJ}F^I_{\sigma\rho}F^{J\sigma\rho}\,, \tag{C.11}$$

where we have made explicit the contribution form the spin 0 and the spin 1 part. We can rewrite the full system as

$$R_{\mu\nu} = -(\text{Im}\,\mathcal{N})_{IJ}F^I_{\mu\lambda}F^{J\,\lambda}_\nu + 2g_{\alpha\bar{\beta}}\partial_{(\mu}z^\alpha\partial_{\nu)}\overline{z}^{\bar{\beta}} + g_{\mu\nu}\left[\frac{1}{4}(\text{Im}\,\mathcal{N})_{IJ}F^I_{\rho\sigma}F^{J\rho\sigma} + V\right]\,, \tag{C.12}$$

$$0 = \nabla_\mu\left[(\text{Im}\,\mathcal{N})_{IJ}F^{J\mu\nu} - \frac{1}{2}(\text{Re}\,\mathcal{N})_{IJ}\,e^{-1}\epsilon^{\mu\nu\rho\sigma}F^J_{\rho\sigma}\right]\,, \tag{C.13}$$

$$0 = \frac{1}{4}\frac{\delta(\text{Im}\,\mathcal{N})_{IJ}}{\delta z^\alpha}F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{8}\frac{\delta(\text{Re}\,\mathcal{N})_{IJ}}{\delta z^\alpha}\,e^{-1}\epsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma} + \frac{\delta g_{\alpha\bar{\beta}}}{\delta\overline{z}^{\bar{\gamma}}}\partial_\lambda\overline{z}^{\bar{\gamma}}\partial^\lambda\overline{z}^{\bar{\beta}}$$
$$+g_{\alpha\bar{\beta}}\nabla_\lambda\nabla^\lambda\overline{z}^{\bar{\beta}} - \frac{\delta V}{\delta z^\alpha}\,, \tag{C.14}$$

which hold independently from the existence and the choice of a prepotential $F(X)$.

Defining the tensor

$$G_{I\mu\nu} := R_{IJ}F^I_{\mu\nu} + I_{IJ}\tilde{F}_{\mu\nu} \quad , \quad \tilde{F}^J_{\mu\nu} := \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}F^{J\rho\sigma}. \tag{C.15}$$

then eq. (C.13), the Bianchi identities and the charges can be written as

$$\epsilon^{\mu\nu\rho\sigma}\partial_\mu \begin{pmatrix} F^I_{\rho\sigma} \\ G_{I\rho\sigma} \end{pmatrix} = 0 \quad , \quad \frac{1}{4\pi}\int_{\Sigma_\infty} \begin{pmatrix} F^I \\ G_I \end{pmatrix} =: \begin{pmatrix} p^I \\ q_I \end{pmatrix}. \tag{C.16}$$

For the magnetic STU model, the prepotential is given by (12.27), and the symplectic section can be parametrised in terms of three complex scalar fields τ_1, τ_2 and τ_3 by choosing $X^0 = 1$, $X^1 = \tau_2\tau_3$, $X^2 = \tau_1\tau_3$, $X^3 = \tau_1\tau_2$, so that

$$\mathfrak{H} = \left(1,\ \tau_2\tau_3,\ \tau_1\tau_3,\ \tau_1\tau_2,\ -\mathbf{i}\tau_1\tau_2\tau_3,\ -\mathbf{i}\tau_1,\ -\mathbf{i}\tau_2,\ -\mathbf{i}\tau_3\right)^T, \tag{C.17}$$

where τ_α's coordinatize M_v. The Kähler potential and the non-vanishing components of the metric on the scalar manifold are respectively

$$e^{-K} = 8\mathrm{Re}\tau_1\mathrm{Re}\tau_2\mathrm{Re}\tau_3, \qquad g_{\alpha\bar{\alpha}} = g_{\bar{\alpha}\alpha} = \partial_\alpha\partial_{\bar{\alpha}}K = (\tau_\alpha + \bar{\tau}_{\bar{\alpha}})^{-2}. \tag{C.18}$$

In particular, we notice the relations

$$F_I = \frac{F}{2X^I}, \qquad \mathcal{N}_{IJ} = \frac{F}{2(X^I)^2}\delta_{IJ} \tag{C.19}$$

between the prepotential and the period matrix.

We have

$$\tau_1 = \sqrt{\frac{g_0g_1}{g_2g_3}}\tau(r) \equiv \sqrt{\frac{g_0g_1}{g_2g_3}}\left(f(r) + \mathbf{i}g(r)\right), \qquad \tau_2 = \sqrt{\frac{g_0g_2}{g_1g_3}}, \qquad \tau_3 = \sqrt{\frac{g_0g_3}{g_1g_2}}. \tag{C.20}$$

The symplectic section X^I and the period matrix $\mathcal{N}_{IJ}$ in terms of these scalar fields boil down to

$$X^I = \frac{1}{8}\sqrt{\frac{G}{8}}\frac{1}{\sqrt{\mathrm{Re}\tau}}\begin{pmatrix} \frac{1}{|g_0|} \\ \frac{1}{|g_1|} \\ \frac{\tau}{|g_2|} \\ \frac{\tau}{|g_3|} \end{pmatrix}, \qquad \mathcal{N}_{IJ} = -\mathbf{i}\frac{64}{G}\begin{pmatrix} g_0^2\tau & 0 & 0 & 0 \\ 0 & g_1^2\tau & 0 & 0 \\ 0 & 0 & \frac{g_2^2}{\tau} & 0 \\ 0 & 0 & 0 & \frac{g_3^2}{\tau} \end{pmatrix}, \tag{C.21}$$

and $\mathrm{Re}\tau > 0$, in order to guarantee the positive definiteness of the spin-1 kinetic terms of the action, namely the fact that $\mathrm{Im}\mathcal{N}_{IJ}$ is negative definite. Explicitly, the real and imaginary parts read

$$\mathrm{Im}\mathcal{N}_{IJ} = I_{IJ} = -\frac{64}{G}\mathrm{diag}\left(g_0^2\mathrm{Re}\tau,\ g_1^2\mathrm{Re}\tau,\ g_2^2\frac{\mathrm{Re}\tau}{|\tau|^2},\ g_3^2\frac{\mathrm{Re}\tau}{|\tau|^2}\right); \tag{C.22}$$

$$\mathrm{Re}\mathcal{N}_{IJ} = R_{IJ} = \frac{64}{G}\mathrm{diag}\left(g_0^2\mathrm{Im}\tau,\ g_1^2\mathrm{Im}\tau,\ -g_2^2\frac{\mathrm{Im}\tau}{|\tau|^2},\ -g_3^2\frac{\mathrm{Im}\tau}{|\tau|^2}\right). \tag{C.23}$$

Now, we employ the two following *Ansätze* for the metric and the electromagnetic field

$$ds^2 = -A(r)dt^2 + A(r)^{-1}dr^2 + B(r)d\Omega_k^2, \tag{C.24}$$

$$F_{tr}^I = \frac{(\text{Im}\mathcal{N})^{-1|IJ}}{B(r)}\left((\text{Re}\mathcal{N})_{JS}\, p^S - q_J\right), \qquad F_{\theta\phi}^I = p^I f_\kappa(\theta), \tag{C.25}$$

where $d\Omega_k^2 = d\theta^2 + f_\kappa^2(\theta)d\phi^2$ is the metric on the two-dimensional surfaces $\Sigma = \{S^2, H^2\}$ of constant scalar curvature $R = 2\kappa$, with $\kappa = \pm 1$, and (cf. e.g. (5.10) of [31] and refs. therein)

$$f_\kappa(\theta) = \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}\theta) = \begin{cases} \sin\theta & \kappa = 1\,, \\ \sinh\theta & \kappa = -1\,. \end{cases} \tag{C.26}$$

The stress tensor for the spin-1 part $T_{\mu\nu}^{(1)}$ can be computed as

$$^{(1)}T_0^0 = ^{(1)}T_1^1 = -^{(1)}T_2^2 = -^{(1)}T_3^3 = -\frac{1}{B^2}V_{BH}; \tag{C.27}$$

one can also check that

$$\frac{1}{4}\frac{\partial I_{IJ}}{\partial z^a}F_{\mu\nu}^I F^{J|\mu\nu} - \frac{1}{8}\frac{\partial R_{IJ}}{\partial z^a}F_{\mu\nu}^I {}^*F^{J|\mu\nu} = -\frac{1}{B^2}\frac{\partial V_{BH}}{\partial z^a}, \tag{C.28}$$

where we defined the so-called black hole (BH) potential [10, 43]

$$V_{BH} := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\mathcal{N})\mathcal{Q}. \tag{C.29}$$

One also have for $g_I > 0$

$$V = -\frac{G}{16}\left(4 + \frac{1 + |\tau|^2}{\text{Re}\tau}\right), \qquad \partial_\tau V = \frac{G}{32}\left(\frac{1 - \overline{\tau}^2}{\text{Re}^2\tau}\right). \tag{C.30}$$

The field Maxwell and Bianchi field equations (C.16) are satisfied, while the Einsteins equations of motion and the scalar field equation read[31]

$$R_{tt} = \frac{A}{2B}\left(A''B + A'B'\right) = \frac{A}{B^2}V_{BH} - AV; \tag{C.31}$$

$$R_{rr} = -\frac{1}{2AB^2}\left(A''^2 B^2 + A'B'B + 2AB''B - AB'^2\right) = -\frac{1}{AB^2}V_{BH} + \frac{1}{A}V + \frac{\tau'\overline{\tau}'}{2\text{Re}^2\tau}; \tag{C.32}$$

$$R_{\theta\theta} = -\frac{1}{2}\left(A'B' + AB''\right) + \kappa = \frac{1}{B}V_{BH} + BV; \tag{C.33}$$

$$0 = \frac{1}{B^2}\partial_\tau V_{BH} + \partial_\tau V - \frac{1}{B}\frac{(BA\overline{\tau}')'}{4\text{Re}^2\tau} + A\frac{\overline{\tau}'^2}{4\text{Re}^3\tau}. \tag{C.34}$$

The system can be rewritten as

$$\frac{(2B''B - B'^2)}{B^2} = -\frac{\tau'\overline{\tau}'}{\text{Re}^2\,\tau}; \tag{C.35}$$

$$A''B - AB'' = 4\frac{V_{BH}}{B} - 2\kappa; \tag{C.36}$$

$$(AB)'' = -4BV + 2\kappa; \tag{C.37}$$

$$0 = \frac{1}{B^2}\partial_\tau V_{BH} + \partial_\tau V - \frac{1}{B}\frac{(BA\overline{\tau}')'}{4\text{Re}^2\tau} + A\frac{\overline{\tau}'^2}{4\text{Re}^3\tau}. \tag{C.38}$$

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[31]The priming denotes differentiation with respect to the radial coordinate.

Assuming $B := r^2 - \Delta^2$ and hyperbolic symmetry $\kappa = -1$, equation (C.35) is solved by

$$f(r) = \frac{r - \Delta}{r + \Delta}, \qquad g(r) = 0. \tag{C.39}$$

With such a position, the equation (C.36) reads

$$\left(r^2 - \Delta^2\right) A''(r) - 2A(r) - \frac{\left(\Delta^2 + r^2\right)\left((64)^2 a^2 + b^2 G^2\right)}{8G(r^2 - \Delta^2)^2} - 2 = 0, \tag{C.40}$$

and the part of solution which is consistent with the remaining equations of motion is

$$A(r) = \frac{(64)^2 a^2 + b^2 G^2}{32G\left(r^2 - \Delta^2\right)} + c_1 \left(r^2 - \Delta^2\right) - \frac{r^2}{\Delta^2}, \tag{C.41}$$

where c_1 is a constant, that can be fixed using the remaining equations (C.37) and (C.38). In fact, those equations are satisfied if we require the following condition,

$$-8c_1 \Delta^2 + \Delta^2 G + 8 = 0. \tag{C.42}$$

D Near-horizon limit and V_{eff}

By considering the treatment of the magnetic STU model given in section 12.2 as well as in appendix C, we recall here the near-horizon limit of the equations of motion, which yields to the definition of the effective black hole potential V_{eff} in gauged supergravity . We start with the metric *Ansatz*

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + B(r)d\Omega_\kappa^2, \tag{D.1}$$

where

$$d\Omega_\kappa^2 = d\theta^2 + f_\kappa^2\left(\theta\right) d\varphi^2 \tag{D.2}$$

is the metric on the two-dimensional surface S^2 for $\kappa = 1$ or H^2 for $\kappa = -1$. Such a surface has constant scalar curvature $R = 2\kappa$, and $f_\kappa\left(\theta\right)$ is defined by (C.26). In the near-horizon limit, it must hold that

$$A(r) \to \frac{r_H^2}{r_A^2} \Rightarrow A'(r) \to \frac{2r_H}{r_A^2}, \; A''(r) \to \frac{2}{r_A^2}; \tag{D.3}$$

$$B(r) \to r_H^2 \Rightarrow B'(r) \to 0, \; B''(r) \to 0. \tag{D.4}$$

Thus:

1. the near-horizon limit of eq. (C.31) reads

$$\frac{A}{2B}\left(A''B + A'B'\right) = \frac{A}{B^2}V_{BH} - AV; \tag{D.5}$$

$$\Downarrow$$

$$\frac{r_H^2}{2r_A^2 r_H^2}\frac{2}{r_A^2}r_H^2 = \frac{r_H^2}{r_A^2 r_H^4}V_{BH} - \frac{r_H^2}{r_A^2}V \Leftrightarrow \frac{1}{r_A^2} = \frac{1}{r_H^4}V_{BH} - V, \tag{D.6}$$

which matches eq. (2.17) of [30], or eq. (5.33) of [31] (by setting $V_{BH\;\text{them}} \to \frac{V_{BH\;\text{us}}}{(8\pi)^2}$ and $V_{\text{them}} \to \frac{V_{\text{us}}}{2}$).

2. the near-horizon limit of eq. (C.32) reads

$$-\frac{1}{2AB^2}\left(A''B^2 + A'B'B + 2AB''B - AB'^2\right) = -\frac{1}{AB^2}V_{BH} + \frac{1}{A}V + \frac{\tau'\overline{\tau}'}{2\left(\text{Re}\tau\right)^2}; \quad \text{(D.7)}$$

$$\Downarrow$$

$$-\frac{1}{2}\frac{r_A^2}{r_H^2}\frac{1}{r_H^4}\frac{2r_H^4}{r_A^2} = -\frac{r_A^2}{r_H^6}V_{BH} + \frac{r_A^2}{r_H^2}V \Leftrightarrow \frac{1}{r_A^2} = \frac{1}{r_H^4}V_{BH} - V, \quad \text{(D.8)}$$

again matching eq. (2.17) of [30], or eq. (5.33) of [31].

3. the near-horizon limit of eq. (C.33) reads

$$-\frac{1}{2}\left(A'B' + AB''\right) + \kappa = \frac{1}{B}V_{BH} + BV; \quad \text{(D.9)}$$

$$\Downarrow$$

$$\kappa = \frac{1}{r_H^2}V_{BH} + r_H^2 V \Leftrightarrow \frac{\kappa}{r_H^2} = \frac{1}{r_H^4}V_{BH} + V, \quad \text{(D.10)}$$

which generalizes to $\kappa = \pm 1$ eq. (2.16) of [30] (to which it reduces by setting $\kappa = 1$), or eq. (5.34)[32] of [31].

4. the near-horizon limit of eq. (C.34) reads

$$0 = \frac{1}{B^2}\partial_\tau V_{BH} + \partial_\tau V - \frac{1}{B}\frac{(BA\overline{\tau}')'}{4\left(\text{Re}\tau\right)^2} + A\frac{(\overline{\tau}')^2}{4\left(\text{Re}\tau\right)^3}; \quad \text{(D.11)}$$

$$\Downarrow$$

$$0 = \frac{1}{B^2}\partial_\tau V_{BH} + \partial_\tau V \Leftrightarrow 0 = \frac{1}{r_H^4}\partial_\tau V_{BH} + \partial_\tau V, \quad \text{(D.12)}$$

which - by trivial extension to the generic case with many scalars - matches eq. (2.18) of [30], or eq. (5.35) of [31].

Thus, by solving (D.10), one obtains

$$\frac{\kappa}{r_H^2} = \frac{1}{r_H^4}V_{BH} + V \Leftrightarrow Vr_H^4 - \kappa r_H^2 + V_{BH} = 0 \Leftrightarrow r_{H,\pm}^2 = \frac{\kappa}{2V} \pm \frac{\sqrt{\kappa^2 - 4VV_{BH}}}{2V}. \quad \text{(D.13)}$$

On the other hand, from (D.6) or (D.8) one obtains

$$\frac{1}{r_A^2} = \frac{1}{r_{H,\pm}^4}V_{BH} - V \overset{\text{(D.13)}}{\Leftrightarrow} \frac{1}{r_A^2} = -2V + \frac{\kappa}{r_{H,\pm}^2} \Leftrightarrow r_{A,\pm}^2 = \frac{1}{\frac{\kappa}{r_{H,\pm}^2} - 2V} = \frac{r_{H,\pm}^2}{\kappa - 2Vr_{H,\pm}^2}, \quad \text{(D.14)}$$

such that

$$r_{A,\pm}^2 = \frac{r_{H,\pm}^2}{\kappa - 2Vr_{H,\pm}^2} = \mp\frac{r_{H,\pm}^2}{\sqrt{\kappa^2 - 4VV_{BH}}}, \quad \text{(D.15)}$$

where in the last step the result

$$\text{Eq. (D.13)} \Leftrightarrow \kappa - 2Vr_{H,\pm}^2 = \mp\sqrt{\kappa^2 - 4VV_{BH}} \quad \text{(D.16)}$$

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[32]Let us remark that (D.10) fixes a typo in eq. (5.34) of [31].

has been used. Since $r_A^2 > 0$, only $r_{A,-}^2$ and $r_{H,-}^2$ make sense:

$$r_H^2 = \frac{\kappa - \sqrt{\kappa^2 - 4VV_{BH}}}{2V}; \tag{D.17}$$

$$r_A^2 = \frac{r_H^2}{\sqrt{\kappa^2 - 4VV_{BH}}}. \tag{D.18}$$

Thus, one can define[33]

$$V_{\text{eff}} := \frac{1 - \kappa\sqrt{\kappa^2 - 4VV_{BH}}}{2V}, \tag{D.19}$$

such that for $\kappa = \pm 1$ it holds that

$$S = \kappa \, V_{\text{eff}}|_{\partial V_{\text{eff}} = 0}. \tag{D.20}$$

In fact,

$$\partial_i V_{\text{eff}} = 0 \tag{D.21}$$

$$\Updownarrow$$

$$\kappa^2 \partial_i V_{BH} + \frac{\left(\kappa^2 - 2\kappa^2 V_{BH} V - \kappa\sqrt{\kappa^2 - 4VV_{BH}}\right)}{2V^2} \partial_i V$$

$$\overset{(\text{D.19})}{=} \partial_i V_{BH} + \kappa^2 V_{\text{eff}}^2 \partial_i V = \partial_i V_{BH} + r_H^4 \partial_i V = 0, \tag{D.22}$$

which is satisfied by virtue of the trivial generalization of (D.12) to the generic case of many scalars, namely

$$\partial_\tau V_{BH} + r_H^4 \partial_\tau V = 0 \Rightarrow \partial_i V_{BH} + r_H^4 \partial_i V = 0, \ \forall i. \tag{D.23}$$

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[33]It should be noted that, apart from the renaming $V_{BH \text{ them}} \to \frac{V_{BH \text{ us}}}{(8\pi)^2}$ and $V_{\text{them}} \to \frac{V_{\text{us}}}{2}$, the effective potential V_{eff} defined by (D.19) is κ times the effective potential defined by eq. (5.39) of [31].

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