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The Born–Oppenheimer approximation for a 1D 2+1 particle system with zero-range interactions

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ABSTRACT

We study the self-adjoint Hamiltonian that models the quantum dynamics of a one-dimensional three-body system consisting of a light particle interacting with two heavy ones through a zero-range force. For an attractive interaction we determine the behavior of the eigenvalues below the essential spectrum in the regime $\varepsilon \ll 1$, where ε is proportional to the square root of the mass ratio. We show that the n-th eigenvalue behaves as $E_n(\varepsilon) = -\alpha^2 + |\sigma_n|\alpha^2\varepsilon^{2/3} + O(\varepsilon)$, where α is a negative constant that explicitly relates to the physical parameters and σ_n is either the n-th extremum or the n-th zero of the Airy function Ai, depending on the kind (respectively, bosons or fermions) of the two heavy particles. Additionally, we prove that the essential spectrum coincides with the half-line $[-\frac{\alpha^2}{4+\varepsilon^2}, +\infty)$.

I. INTRODUCTION

We consider two heavy particles (both bosons or fermions) of mass M and one light particle of mass m in one dimension. Let x_1 and x_2 denote the coordinates of the heavy particles, and x_3 denote the coordinate of the light particle. The Hilbert space of the system is $L^2_{\mathrm{b}}(\mathbb{R}^3)$ or $L^2_{\mathrm{f}}(\mathbb{R}^3)$, i.e., the subspace of square integrable functions that are either symmetric (if the heavy particles are bosons) or antisymmetric (if the heavy particles are fermions) under the exchange of the coordinates x_1 and x_2:

$$L^2_{\mathrm{b/f}}(\mathbb{R}^3) := \{\Psi \in L^2(\mathbb{R}^3) : \ \Psi(x_1,x_2,x_3) = (+)_{\mathrm{b/f}}\,\Psi(x_2,x_1,x_3)\},$$

where $(+)_{\mathrm{b}} := +$ and $(+)_{\mathrm{f}} := -$.

We assume that the heavy particles do not interact between themselves and that they interact with the light particle through a contact interaction. We introduce, heuristically, the Hamiltonian

$$H^{\mathrm{b/f}}_{2+1} := -\frac{\hbar^2}{2M}\,\partial^2_{x_1} - \frac{\hbar^2}{2M}\,\partial^2_{x_2} - \frac{\hbar^2}{2m}\,\partial^2_{x_3} + \beta\delta(x_3 - x_1) + \beta\delta(x_3 - x_2).$$

Here β is real valued and represents the interaction parameter, and $\delta(x_3 - x_1)$, $\delta(x_3 - x_2)$ denote the Dirac delta distributions supported on the coincidence planes of the heavy particles with the light particle:

$$\Pi_1 := \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 = x_1\} \qquad \text{and} \qquad \Pi_2 := \{(x_1,x_2,x_3) \in \mathbb{R}^3 |\, x_3 = x_2\}.$$

To proceed, we pass to units in which $\hbar = 1$ and define the Jacobi coordinates

$$x_{cm} = \frac{M(x_1 + x_2) + m x_3}{M_{tot}}, \qquad x = x_1 - x_2, \qquad y = x_3 - \frac{x_1 + x_2}{2};$$

$$M_{tot} = 2M + m, \qquad \mu = \frac{2Mm}{M_{tot}}.$$

In Jacobi coordinates, the Hamiltonian $H^{\mathrm{b/f}}_{2+1}$ acts in the Hilbert space of square-integrable functions Ψ such that $\Psi(x_{cm}, x, y) = (+)_{\mathrm{b/f}}\Psi(x_{cm}, -x, y)$ and takes the form (we abuse the notation and continue to denote the Hamiltonian by $H^{\mathrm{b/f}}_{2+1}$)

$$H^{\mathrm{b/f}}_{2+1} = -\frac{1}{2M_{tot}}\,\partial^2_{x_{cm}} - \frac{1}{M}\,\partial^2_x - \frac{1}{2\mu}\,\partial^2_y + \beta\delta(y - x/2) + \beta\delta(y + x/2).$$

Neglecting the coordinates of the center of mass and keeping the previous notation, we consider the Hilbert space

$$L^2_{\mathrm{b/f}}(\mathbb{R}^2) := \{\psi \in L^2(\mathbb{R}^2):\ \psi(x,y) = (+)_{\mathrm{b/f}}\,\psi(-x,y)\}$$

and the coincidence lines

$$\Pi_1 = \{(x,y) \in \mathbb{R}^2:\ y = x/2\} \qquad \text{and} \qquad \Pi_2 = \{(x,y) \in \mathbb{R}^2:\ y = -x/2\}. \tag{1.1}$$

Additionally, since we consider the regime $m/M \ll 1$, we define the rescaled Hamiltonian

$$H^{\mathrm{b/f}}_\varepsilon = -\varepsilon^2\partial^2_x - \partial^2_y + \alpha\delta(y - x/2) + \alpha\delta(y + x/2)$$

where the small parameter $\varepsilon \ll 1$ is defined by

$$\varepsilon^2 = \frac{2\mu}{M}$$

and α is kept fixed (independent of ε); we remark that $H^{\mathrm{b/f}}_\varepsilon$ is to be considered as an operator in the Hilbert space $L^2_{\mathrm{b/f}}(\mathbb{R}^2)$. At a heuristic level, the Hamiltonian $H^{\mathrm{b/f}}_\varepsilon$ is given by $H^{\mathrm{b/f}}_\varepsilon = 2\,\mu\big(H^{\mathrm{b/f}}_{2+1} - K_{cm}\big)$, where $K_{cm} = -\frac{1}{2M_{tot}}\,\partial^2_{x_{cm}}$ represents the kinetic energy of the center of mass and α relates to β through the formula $\alpha = 2\mu\beta$.

The operator $H^{\mathrm{b/f}}_\varepsilon$ can be characterized as the self-adjoint, bounded from below, operator associated to the closed and bounded from below quadratic form

$$\mathcal{B}^{\mathrm{b/f}}_\varepsilon : H^1(\mathbb{R}^2) \cap L^2_{\mathrm{b/f}}(\mathbb{R}^2) \to \mathbb{R}$$

$$\mathcal{B}^{\mathrm{b/f}}_\varepsilon(\psi) = \int_{\mathbb{R}^2} \varepsilon^2|\partial_x\psi(x,y)|^2 + |\partial_y\psi(x,y)|^2\mathrm{d}\mathbf{x} + 2\alpha\int_{\mathbb{R}} |\psi(s, s/2)|^2\mathrm{d}s,$$

see Proposition 2.14.

We point out that functions in the domain of $H^{\mathrm{b/f}}_\varepsilon$ have to satisfy a boundary condition on the coincidence planes Π_1 and Π_2 [as defined in Eq. (1.1)]; more precisely, they are regular outside Π_1 and Π_2, continuous on Π_1 and Π_2, and

$$\left(-\frac{\varepsilon^2}{2}\partial_x\psi + \partial_y\psi\right)(x, (x/2)^+) - \left(-\frac{\varepsilon^2}{2}\partial_x\psi + \partial_y\psi\right)(x, (x/2)^-) = \alpha\psi(x, x/2) \text{ for a.e. } x \in \mathbb{R}, \tag{1.2}$$

$(x/2)^\pm$ denoting the right (respectively, left) limit. A similar condition holds true on the coincidence plane Π_2 and is obtained as a consequence of the bosonic or fermionic symmetry. The left hand side in the previous equation represents (up to a normalization factor) the jump of the normal derivative relative to $\varepsilon^2\partial^2_x + \partial^2_y$ across Π_1, see also Remark 2.10.

Our main result is the following:

Theorem 1.1. *Let $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. Then,*

$$\begin{aligned} &\sigma(H^{\mathrm{b/f}}_\varepsilon) = \sigma_{ess}(H^{\mathrm{b/f}}_\varepsilon) = [0, +\infty) && \text{if} \quad \alpha \ge 0,\\ &\sigma(H^{\mathrm{b/f}}_\varepsilon) \subseteq [-\alpha^2, +\infty), \qquad \sigma_{ess}(H^{\mathrm{b/f}}_\varepsilon) = \left[-\frac{\alpha^2}{4+\varepsilon^2}, +\infty\right) && \text{if} \quad \alpha < 0. \end{aligned} \tag{1.3}$$

Moreover, if $\alpha < 0$, for any fixed integer $n \ge 0$ there exists $\varepsilon > 0$ sufficiently small such that $H^{b/f}_\varepsilon$ has at least $(n+1)$ simple isolated eigenvalues

$$-\alpha^2 < E^{\mathrm{b/f}}_{\varepsilon,0} < E^{\mathrm{b/f}}_{\varepsilon,1} < \ldots < E^{\mathrm{b/f}}_{\varepsilon,n} < -\frac{\alpha^2}{4+\varepsilon^2},$$

such that

$$E_{\varepsilon,k}^{\mathrm{b/f}} = -\alpha^2 + s_k^{\mathrm{b/f}}\alpha^2\varepsilon^{2/3} + O(\varepsilon), \qquad \textit{for all } k = 0,\dots,n,$$

where

$$s_k^{\mathrm{b}} = -\sigma_{2k}, \qquad s_k^{\mathrm{f}} = -\sigma_{2k+1},$$

and the interlacing negative numbers σ_k

$$\dots < \sigma_{2k+1} < \sigma_{2k} < \sigma_{2k-1} < \dots < \sigma_2 < \sigma_1 < \sigma_0 < 0,$$

are either the extrema or the zeros of the Airy function Ai, *i.e.*, $\mathrm{Ai}'(\sigma_{2k}) = 0$ *or* $\mathrm{Ai}(\sigma_{2k+1}) = 0$.

The first part of Theorem 1.1 [Eq. (1.3)] characterizes the bottom of the spectrum and the essential spectrum for arbitrary values of the mass ratio ε. The second part establishes the validity of the Born–Oppenheimer approximation and provides the asymptotic behavior of the isolated eigenvalues in the small mass ratio limit.

The Born–Oppenheimer approximation, introduced in Ref. 8 in the early years of quantum theory, was developed as a method for deriving the molecular structure from quantum mechanical principles. We do not attempt to provide an overview of the extensive mathematical literature on this topic. For an accessible introduction and references to both classical and recent developments, we refer the reader to the reviews (Refs. 24 and 27).

In the context of zero-range interactions, the study of the quantum systems of three or more non relativistic bosons interacting via contact interactions in dimension three is plagued by the so-called ultraviolet catastrophe, see Refs. 30, 31, and 43 This singular behavior also appears in systems consisting of two or more bosons or fermions interacting with a particle of a different nature (see, e.g., Ref. 21, and Refs. 7, 12–14, and 45 for the fermionic case). Overcoming this issue typically requires the introduction of nonlocal modifications and/or effective three-body interaction terms, see, e.g., Refs. 6, 19–22, and 37 and references therein. This problem affects also the study of a many particle system in three spatial dimensions within the Born–Oppenheimer framework, see, e.g., Ref. 22.

However, these difficulties do not arise whenever one considers particle systems in one spatial dimension, see, e.g., Refs. 5 and 23. Nevertheless, even in this case, the standard mathematical procedure to prove the validity of the Born–Oppenheimer approximation (as given, e.g., in Ref. 11) does not work, for this reason we use the general scheme developed in Ref. 28.

Let us remark that the eigenvalue expansion given in Theorem 1.1 conforms with the one obtained (for the bosonic case) by Akbas and Turgut in Ref. 2; their approach is more aligned with theoretical physics literature and does not present a rigorous proof of the validity of the Born–Oppenheimer approximation. In the same spirit a first insight into the two-dimensional case is provided in Ref. 3.

Related to our work is the study of the spectrum of the Laplacian in dimension two with δ-interactions supported on two crossing lines, see, e.g., Ref. 18, or on an almost straight line, see, e.g., Ref. 17; even though, by the nature of the problem, in these works there is no account of the fermionic or bosonic symmetry. We mention also Ref. 34, where the case of δ'-interactions supported on two crossing lines is considered.

After completing our work, Nicholas Raymond drew our attention to Ref. 16 addressing the analysis of the discrete spectrum for the Laplacian in dimension two with a δ-interaction supported on a broken line, similar to Refs. 17 and 18. The analysis in Ref. 16 also makes use of a dimensional reduction argument and is closely related to the study of the Born–Oppenheimer approximation; by using bounds on the quadratic form, it results in an asymptotic expansion for the eigenvalues similar to the one given in Theorem 1.1.

A. Outline of the proof of Theorem 1.1

We first define the Hamiltonian $H_\varepsilon^{\mathrm{b/f}}$ by means of standard tools from the theory of self-adjoint extensions of symmetric operators and provide a formula for its resolvent. This is done in Theorem 2.12, within the approach developed in Ref. 35. This allows a precise characterization of its essential spectrum which is also relevant for the proof of the second part of Theorem 1.1.

The Born–Oppenheimer approximation is based on the idea that the dynamics of the system factorizes in a fast dynamics, relative to the light particle, and a slow one, describing the evolution of the heavy particles subsystem. We fix the relative position x of the heavy particles and study the spectrum of the light particle Hamiltonian h_x associated to the quadratic form

$$b_x : H^1(\mathbb{R}) \to \mathbb{R}$$

$$b_x(u) = \int_{\mathbb{R}} |u'(y)|^2\,\mathrm{d}y + \alpha|u(x/2)|^2 + \alpha|u(-x/2)|^2.$$

Note that h_x is the Hamiltonian in dimension one with two delta interactions centered in $y = x/2$ and $y = -x/2$, see Ref. 4. Functions in the domain of h_x are regular outside the points $y = \pm x/2$, continuous in $y = \pm x/2$, and satisfy the boundary conditions

$$u'((x/2)^+) - u'((x/2)^-) = \alpha u(x/2) \qquad \text{and} \qquad u'((-x/2)^+) - u'((-x/2)^-) = \alpha u(-x/2).$$

For $\alpha < 0$, h_x has non empty discrete spectrum and the lowest eigenvalue, denoted by $-\lambda_0(x)$, and the corresponding normalized eigenfunction, ψ_x^{BO}, can be explicitly computed.

To extract an effective contribution from the light particle component, we use the projection on the eigenfunction ψ_x^{BO} (more precisely, the direct integral of a family of projections). This procedure produces an effective Hamiltonian for the heavy particles subsystem. We follow closely the very versatile and general approach developed in Ref. 28. By doing so we reproduce and rewrite, in a slightly different way, some key estimates from that paper, giving simpler expressions for the bounds (see Propositions 4.5 and 4.7, and Lemma 4.8 compared with Ref. 28, Theorems 1.1 and 2.5, and Proposition 2.6).

To relate the Hamiltonians $H_\varepsilon^{\mathrm{b/f}}$ and h_x, we observe that given any function $\phi \in H^1(\mathbb{R}^2)$, for a.e. $x \in \mathbb{R}$, its x-section $\phi_x(y) := \phi(x,y)$ belongs to $H^1(\mathbb{R})$. Moreover, there holds

$$\mathcal{B}_\varepsilon^{\mathrm{b/f}}(\phi) = \int_{\mathbb{R}^2} \varepsilon^2 |\partial_x \phi(x,y)|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}} b_x(\phi_x)\mathrm{d}x \qquad \forall \phi \in H^1_{\mathrm{b/f}}(\mathbb{R}^2).$$

As a result, for any arbitrary mass ratio ε and $\alpha < 0$ we infer $\sigma(H_\varepsilon^{\mathrm{b/f}}) \subset [-\alpha^2, +\infty)$, see Proposition 4.1.

To proceed further, we notice that ψ_x^{BO} can be regarded as a function of two variables, denoted by ψ^{BO}, with the obvious identification $\psi^{BO}(x,y) \equiv \psi_x^{BO}(y)$; we introduce the orthogonal projection

$$\mathcal{P} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \qquad \mathcal{P}\phi(x,y) := \psi^{BO}(x,y) f_\phi(x),$$

where

$$f_\phi(x) := \int_{\mathbb{R}} \psi^{BO}(x,y)\phi(x,y)\, \mathrm{d}y;$$

additionally we set $\mathcal{P}^\perp := 1 - \mathcal{P}$ and point out that $\mathcal{P}$ leaves invariant both $L^2_{\mathrm{b}}(\mathbb{R}^2)$ and $L^2_{\mathrm{f}}(\mathbb{R}^2)$.

It turns out that the quadratic form

$$D\left(\widehat{\mathcal{B}}_\varepsilon^{\mathrm{b/f}}\right) := H^1_{\mathrm{b/f}}(\mathbb{R}^2) \qquad \widehat{\mathcal{B}}_\varepsilon^{\mathrm{b/f}}(\phi) := \mathcal{B}_\varepsilon^{\mathrm{b/f}}(\mathcal{P}\phi) + \mathcal{B}_\varepsilon^{\mathrm{b/f}}(\mathcal{P}^\perp\phi)$$

is well-defined, closed and bounded from below (see Remarks 4.2 and 4.4), and so, it defines a self-adjoint operator $\widehat{H}_\varepsilon^{\mathrm{b/f}}$.

We provide estimates regarding the relations between the resolvent sets of $\widehat{H}_\varepsilon^{\mathrm{b/f}}$ and $H_\varepsilon^{\mathrm{b/f}}$ and the difference between their resolvents (see Lemma 4.8 for the detailed statements). Note that for technical reasons, in Secs. IV–VI we prefer to work with positive definite quantities. To this end, we shift quadratic forms and operators by α^2. Specifically, all operators denoted by H and $\mathscr{L}$ – regardless of superscripts or subscripts – satisfy the relation $\mathscr{L} = H + \alpha^2$. An analogous convention holds for quadratic forms: $\mathcal{Q} = \mathcal{B} + \alpha^2$ and $q = b + \alpha^2$. Translating the results back to the original setting is straightforward.

Next we need to identify the effective Hamiltonian for the heavy particles subsystem. The operator $\widehat{H}_\varepsilon^{\mathrm{b/f}}$ can be written as a direct sum with the following spectrum:

$$\widehat{H}_\varepsilon^{\mathrm{b/f}} = \widehat{H}_{\varepsilon,\mathcal{P}}^{\mathrm{b/f}} \oplus \widehat{H}_{\varepsilon,\mathcal{P}^\perp}^{\mathrm{b/f}}, \qquad \sigma(\widehat{H}_{\varepsilon,\mathcal{P}}^{\mathrm{b/f}}) \subseteq [-\alpha^2, +\infty), \qquad \sigma(\widehat{H}_{\varepsilon,\mathcal{P}^\perp}^{\mathrm{b/f}}) \subseteq [-\alpha^2/4, +\infty).$$

For the study of the eigenvalues at the bottom of the spectrum (near $-\alpha^2$), the most relevant operator is $\widehat{H}_{\varepsilon,\mathcal{P}}^{\mathrm{b/f}}$. We observe that, by means of a unitary map, the Hamiltonian $\widehat{H}_{\varepsilon,\mathcal{P}}^{\mathrm{b/f}}$ can be reduced to an effective one dimensional operator on the heavy particles subsystem

$$D(H_\varepsilon^{\mathrm{eff}\, b/f}) = H^2(\mathbb{R}) \cap L^2_{\mathrm{b/f}}(\mathbb{R}) \qquad H_\varepsilon^{\mathrm{eff}\, b/f} = -\varepsilon^2 \frac{d^2}{dx^2} - \lambda_0 + \varepsilon^2 R,$$

where $-\lambda_0(x)$ is the lowest eigenvalue of h_x and $\varepsilon^2 R(x) := \varepsilon^2 \int_{\mathbb{R}} |\partial_x \psi^{BO}(x,y)|^2 dy$ is a perturbative potential term (see Remark 5.1). We remark that unlike the smooth potential case, $\mathrm{ran}(\mathcal{P}|D(H_\varepsilon^{\mathrm{b/f}})) \notin D(H_\varepsilon^{\mathrm{b/f}})$, because $\mathcal{P}\phi$ does not satisfy the boundary condition (1.2). For this reason, contrarily to what is done in Ref. 28, it is not possible to identify $\widehat{H}_{\varepsilon,\mathcal{P}}^{\mathrm{b/f}}$ with the compression $\mathcal{P}H_\varepsilon^{\mathrm{b/f}}\mathcal{P}$, which is not well defined.

Finally, to conclude the proof of the second part of Theorem 1.1, we prove that for any fixed integer $n \geq 0$ there exists $\varepsilon > 0$ sufficiently small such that $H_\varepsilon^{\mathrm{eff\,b/f}}$ has at least $(n+1)$ simple isolated eigenvalues which satisfy

$$E_{\varepsilon,k}^{\mathrm{eff\,b/f}} = -\alpha^2 + s_k^{\mathrm{b/f}} \alpha^2 \varepsilon^{2/3} + O(\varepsilon), \qquad \text{for all } k = 0, \ldots, n,$$

where $s_k^{\mathrm{b/f}}$ are the same numbers as in Theorem 1.1. This result follows immediately from Theorem 5.9 shifting the spectrum by the constant α^2. The proof of Theorem 5.9 is based on the seminal paper Ref. 40. We remark that with respect to the case studied in Ref. 40, where the potential is smooth, in our analysis the potential term $-\lambda_0$ is only piecewise smooth, it is continuous, but not differentiable at $x = 0$. Additionally, $-\lambda_0$ is linear around $x = 0$, rather than quadratic, as in the smooth case. For this reason the eigenvalue expansion involves zeros and extrema of the Airy function, a result already pointed out in Refs. 2 and 16.

The paper is organized as follows. In Sec. II, we characterize the Hamiltonian $H_\varepsilon^{\mathrm{b/f}}$ and its essential spectrum. In Sec. III, we study the Hamiltonian of the light particle, denoted by h_x, and we give a complete description of its spectrum. In Sec. IV, we carry out the dimensional reduction following the approach of Ref. 28. In Sec. V, we analyze the effective Hamiltonian $\mathscr{L}_\varepsilon^{\mathrm{eff,b/f}} := H_\varepsilon^{\mathrm{eff,b/f}} + \alpha^2$ and its eigenvalues below its essential spectrum. Finally, Sec. VI contains the proof of Theorem 1.1.

B. Notation

- $\mathbf{x} = (x, y) \in \mathbb{R}^2$.
- $\mathbf{k} = (k, p) \in \mathbb{R}^2$.
- We denote by $\mathscr{F} f$ or $\hat{f}$ the Fourier transform of f, defined as:

$$\hat{\psi}(\mathbf{k}) := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\mathbf{k}\cdot\mathbf{x}}\psi(\mathbf{x})\mathrm{d}\mathbf{x},$$

or

$$\hat{\xi}(p) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ipy}\xi(y)\mathrm{d}y.$$

The inverse Fourier transform is denoted by $\mathscr{F}^{-1} f$ or $\check{f}$.
- The $L^2(\mathbb{R}^n)$-norm and scalar product are denoted simply by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively; norm and scalar product in different Hilbert spaces are denoted by an appropriate subscript.
- The symbol $\natural$ denotes either b (bosonic) or f (fermionic), with the corresponding value $(+)_\natural$ defined as $(+)_\mathrm{b} = +$ and $(+)_\mathrm{f} = -$.
- $H^\nu(\mathbb{R}^n)$, $\nu \in \mathbb{R}$, denotes the usual Sobolev space of order ν; $H^\nu_\natural(\mathbb{R}^2) := H^\nu(\mathbb{R}^2) \cap L^2_\natural(\mathbb{R}^2)$, $\nu > 0$.
- $H^{\nu-} := \cap_{a<\nu} H^a$.
- $\langle\cdot,\cdot\rangle_{\mp\nu,\pm\nu}$ denotes the (anti-linear with respect to the first variable) $H^{\pm\nu}(\mathbb{R}^n)$-$H^{\mp\nu}(\mathbb{R}^n)$ duality pairing extending the scalar product in $L^2(\mathbb{R}^n)$.
- $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions.
- $\mathscr{B}(X, Y)$ denotes the Banach space of bounded linear operators between the two Hilbert spaces X and Y; we use the shorthand notation $\mathscr{B}(X) = \mathscr{B}(X, X)$. The operator norm is denoted by $\|\cdot\|$.
- $D(L)$ denotes the domain of the linear operator L; $\ker(L)$, $\mathrm{ran}(L)$ denote its kernel and range respectively.
- $\rho(L)$, $\sigma(L)$ denote the resolvent and the spectra of L.
- $\sigma_{ac}(L)$, $\sigma_{sc}(L)$, $\sigma_{ess}(L)$, $\sigma_p(L)$, $\sigma_d(L)$ denote the absolutely continuous, singular continuous, essential, point and discrete spectra of L.
- $L|V$ denotes the restriction of the linear operator L to the linear subspace $V \subset D(L)$.
- If $\mathcal{Q}$ is a sesquilinear form in a Hilbert space X, we use the same notation for the associated quadratic form $\mathcal{Q}(\psi) = \mathcal{Q}(\psi, \psi)$.
- $C_0^\infty(\mathbb{R}^n)$ denotes the set of smooth and compactly supported functions from $\mathbb{R}^n$ to $\mathbb{C}$.
- $\lambda_+ := \max\{0, \lambda\}$.

II. $H_\varepsilon^\natural$ As A Self-Adjoint Extension

In this Section we start working in the usual Hilbert space $L^2(\mathbb{R}^2)$. Our first aim is to construct the resolvent of a one-parameter family of self-adjoint extensions of the restriction of the free Hamiltonian

$$D(H_\varepsilon^0) := H^2(\mathbb{R}^2), \qquad H_\varepsilon^0 := -\varepsilon^2\partial_x^2 - \partial_y^2,$$

to the subspace of functions vanishing on the contact subset $\Pi = \Pi_1 \cup \Pi_2$. This family models interacting Hamiltonians describing zero-range interactions between the particle 1 and 3, and between the particle 2 and 3, without taking into account either bosonic or fermionic symmetry. Successively, in Sec. II B, we compress such self-adjoint Hamiltonians onto the subspace

$$L^2_\natural(\mathbb{R}^2) := \{\psi \in L^2(\mathbb{R}^2) : \ \psi(x, y) = (+)_\natural\, \psi(-x, y)\},$$

where $\natural = \mathrm{b}$ in the bosonic case, $\natural = \mathrm{f}$ in the fermionic case, and $(+)_\mathrm{b} = +$, $(+)_\mathrm{f} = -$. This procedure gives Hamiltonians modeling the same zero-range interactions with the additional constraint that particles 1 and 2 are either bosons or fermions. Such Hamiltonians correspond to sesquilinear forms in $L^2_\natural(\mathbb{R}^2)$, with domain $H^1_\natural(\mathbb{R}^2) = H^1(\mathbb{R}^2) \cap L^2_\natural(\mathbb{R}^2)$, of the kind

$$\mathcal{B}_\varepsilon^\natural(\varphi, \psi) := \int_{\mathbb{R}^2} \varepsilon^2\partial_x\overline{\varphi}(x, y)\partial_x\psi(x, y) + \partial_y\overline{\varphi}(x, y)\partial_y\psi(x, y)\mathrm{d}\mathbf{x} + 2\alpha\int_{\mathbb{R}} \overline{\varphi}(s, s/2)\,\psi(s, s/2)\mathrm{d}s.$$

A. Building a resolvent

For any $\phi \in \mathcal{S}(\mathbb{R}^2)$ we define τ_1 (resp. τ_2) as the trace of ϕ on Π_1 (resp. Π_2); explicitly,

$$(\tau_1\phi)(s) := \phi(s, s/2), \qquad (\tau_2\phi)(s) := \phi(-s, s/2), \qquad s \in \mathbb{R}.$$

The maps τ_1 and τ_2 have unique extensions to bounded and surjective linear operators (which we denote by the same symbols)

$$\tau_1 : H^\nu(\mathbb{R}^2) \to H^{\nu-1/2}(\mathbb{R}), \qquad \tau_2 : H^\nu(\mathbb{R}^2) \to H^{\nu-1/2}(\mathbb{R})$$

for any $\nu > 1/2$. Note that $C_0^\infty(\mathbb{R}) \subset \operatorname{ran}(\tau_j)$ and $C_0^\infty(\mathbb{R}^2\backslash\Pi_j) \subset \ker(\tau_j)$ are both dense in $L^2(\mathbb{R}^2)$, $j = 1, 2$.

We define the bounded linear operator

$$\mathcal{T} : H^\nu(\mathbb{R}^2) \to H^{\nu-1/2}(\mathbb{R}) \oplus H^{\nu-1/2}(\mathbb{R}), \qquad \mathcal{T}\phi := \tau_1\phi \oplus \tau_2\phi, \qquad \nu > 1/2.$$

Note that $C_0^\infty(\mathbb{R}\backslash\{0\}) \times C_0^\infty(\mathbb{R}\backslash\{0\}) \subset \operatorname{ran}(\mathcal{T})$ is dense in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and $C_0^\infty(\mathbb{R}^2\backslash\Pi) \subset \ker(\mathcal{T})$ is dense in $L^2(\mathbb{R}^2)$.

Remark 2.1. The linear operator $\mathcal{T}$ is not surjective whenever $\nu > 1$; indeed,

$$\operatorname{ran}(\mathcal{T}) \subseteq \left\{\xi_1 \oplus \xi_2 \in H^{\nu-1/2}(\mathbb{R}) \oplus H^{\nu-1/2}(\mathbb{R}) : \xi_1(0) = \xi_2(0)\right\} \subsetneq H^{\nu-1/2}(\mathbb{R}) \oplus H^{\nu-1/2}(\mathbb{R}).$$

Therefore, our successive construction of a self-adjoint extension of the symmetric operator $H_\varepsilon^0|\ker(\mathcal{T})$ does not embed into the framework of standard boundary triples theory. In particular, this prevents the use of the spectral results given, e.g., in Theorem 3.3 in Ref. 9. We provide analogous results adapted to our framework in Lemma 2.17 below.

Recalling that the resolvent set $\rho(H_\varepsilon^0)$ of the self-adjoint operator H_ε^0, is $\mathbb{C}\backslash[0, +\infty)$, for any $z \in \mathbb{C}\backslash[0, +\infty)$ we define the resolvent operator

$$R_\varepsilon^0(z) := (H_\varepsilon^0 - z)^{-1}.$$

Obviously

$$R_\varepsilon^0(z) : H^\nu(\mathbb{R}^2) \to H^{\nu+2}(\mathbb{R}^2), \qquad \nu \geq 0,$$

is a continuous bijection for all $z \in \mathbb{C}\backslash[0, +\infty)$ and it extends to a continuous bijection (which we denote by the same symbol),

$$R_\varepsilon^0(z) : H^\nu(\mathbb{R}^2) \to H^{\nu+2}(\mathbb{R}^2), \qquad \nu < 0.$$

For any $z \in \mathbb{C}\backslash[0, +\infty)$ we define the bounded operator

$$\breve{\mathbb{G}}_\varepsilon(z) : H^\nu(\mathbb{R}^2) \to H^{\nu+3/2}(\mathbb{R}) \oplus H^{\nu+3/2}(\mathbb{R}), \qquad \breve{\mathbb{G}}_\varepsilon(z) := \mathcal{T}R_\varepsilon^0(z), \qquad \nu > -3/2.$$

One has

$$\breve{\mathbb{G}}_\varepsilon(z)\psi = \breve{G}_{1,\varepsilon}(z)\psi \oplus \breve{G}_{2,\varepsilon}(z)\psi; \qquad \breve{G}_{j,\varepsilon}(z) : H^\nu(\mathbb{R}^2) \to H^{\nu+3/2}(\mathbb{R}), \qquad \breve{G}_{j,\varepsilon}(z) = \tau_j R_\varepsilon^0(z), \quad j = 1, 2.$$

Note that $\operatorname{ran}(\breve{G}_{j,\varepsilon}(z)) = H^{\nu+3/2}(\mathbb{R})$; however, by Remark 2.1, $\operatorname{ran}(\breve{\mathbb{G}}_\varepsilon(z)) \subsetneq H^{\nu+3/2}(\mathbb{R}) \oplus H^{\nu+3/2}(\mathbb{R})$ whenever $\nu > -1$.

We define the bounded operator

$$\mathbb{G}_\varepsilon(z) := \breve{\mathbb{G}}_\varepsilon(\bar{z})^* : H^\nu(\mathbb{R}) \oplus H^\nu(\mathbb{R}) \to H^{\nu+3/2}(\mathbb{R}^2), \qquad \nu < 0,$$

where the adjoint is defined in terms of the $H^{-\nu}(\mathbb{R}^d)$-$H^\nu(\mathbb{R}^d)$ duality (taken to be anti-linear with respect to the first variable) which extends the $L^2(\mathbb{R}^d)$ scalar product. We remark that $\mathbb{G}_\varepsilon(z)$ is also represented as

$$\mathbb{G}_\varepsilon(z)(\xi_1 \oplus \xi_2) = G_{1,\varepsilon}(z)\xi_1 + G_{2,\varepsilon}(z)\xi_2; \qquad G_{j,\varepsilon}(z) : H^\nu(\mathbb{R}) \to H^{\nu+3/2}(\mathbb{R}^2), \qquad G_{j,\varepsilon}(z) = \breve{G}_{j,\varepsilon}(\bar{z})^*.$$

In particular, for all $z \in \mathbb{C}\backslash[0, +\infty)$ and for all $\nu < 3/2$ there holds

$$G_{j,\varepsilon}(z) \in \mathscr{B}(L^2(\mathbb{R}), H^\nu(\mathbb{R}^2)), \qquad \mathbb{G}_\varepsilon(z) \in \mathscr{B}(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), H^\nu(\mathbb{R}^2)),$$

and so

$$\operatorname{ran}(G_{j,\varepsilon}(z)|L^2(\mathbb{R})) \subseteq H^{3/2-}(\mathbb{R}^2), \qquad \operatorname{ran}(\mathbb{G}_\varepsilon(z)|L^2(\mathbb{R}) \oplus L^2(\mathbb{R})) \subseteq H^{3/2-}(\mathbb{R}^2), \tag{2.1}$$

where $H^{3/2-}(\mathbb{R}^2) := \cap_{\nu<3/2} H^\nu(\mathbb{R}^2)$.

Remark 2.2. Since $\ker(\mathcal{T})$ is dense in $L^2(\mathbb{R}^2)$, there follows that

$$\operatorname{ran}(\mathbb{G}_\varepsilon(z)|L^2(\mathbb{R}) \oplus L^2(\mathbb{R})) \cap D(H_\varepsilon^0) = \{0\}, \tag{2.2}$$

see Remark 2.8, 2.9, and Theorem 2.1 in Ref. 35. Indeed, suppose that (2.2) is false. Then there exist $Q \in (L^2(\mathbb{R})\backslash\{0\}) \oplus (L^2(\mathbb{R})\backslash\{0\})$ and $\phi \in L^2(\mathbb{R}^2)\backslash\{0\}$, such that $R^0_\varepsilon(z)\phi = \mathbb{G}_\varepsilon(z)Q$. Hence, for all $\psi \in L^2(\mathbb{R}^2)$ one would have

$$\langle \phi, R^0_\varepsilon(\bar{z})\psi\rangle_{L^2(\mathbb{R}^2)} = \langle Q, \mathcal{T} R^0_\varepsilon(\bar{z})\psi\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}.$$

Since, $R^0_\varepsilon(\bar{z})$ maps $L^2(\mathbb{R}^2)$ onto $H^2(\mathbb{R}^2)$ and the set $\{f \in H^2(\mathbb{R}^2) : \mathcal{T} f = 0\}$ is dense in $L^2(\mathbb{R}^2)$, there follows $\phi = 0$, leading to a contradiction.

Remark 2.3. We note the following expressions for the integral kernels of the relevant operators in Fourier transform:

$$R^0_\varepsilon(\mathbf{x}, \mathbf{x}'; z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk};$$

$$\breve{G}_{1,\varepsilon}(s, \mathbf{x}'; z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i(k+\frac{p}{2})s}e^{-i\mathbf{k}\cdot\mathbf{x}'}}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk}; \qquad \breve{G}_{2,\varepsilon}(s, \mathbf{x}'; z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i(-k+\frac{p}{2})s}e^{-i\mathbf{k}\cdot\mathbf{x}'}}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk};$$

$$G_{1,\varepsilon}(\mathbf{x}, s'; z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i(k+\frac{p}{2})s'}}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk}; \qquad G_{2,\varepsilon}(\mathbf{x}, s'; z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i(-k+\frac{p}{2})s'}}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk}.$$

By the properties of the operators τ_1, τ_2 and $\mathcal{T}$, and by the mapping properties of the operators $\mathbb{G}_\varepsilon(z)$ and $G_{j,\varepsilon}(z)$ there follows that the following operators are well defined and bounded for any $z \in \mathbb{C}\backslash[0, +\infty)$:

$$M_{\ell j,\varepsilon}(z) : H^\nu(\mathbb{R}) \to H^{\nu+1}(\mathbb{R}), \qquad M_{\ell j,\varepsilon}(z) := \tau_\ell G_{j,\varepsilon}(z) \qquad \ell, j = 1, 2; \qquad \nu > -1.$$

and

$$\mathbb{M}_\varepsilon(z) : H^\nu(\mathbb{R}) \oplus H^\nu(\mathbb{R}) \to H^{\nu+1}(\mathbb{R}) \oplus H^{\nu+1}(\mathbb{R}), \qquad \mathbb{M}_\varepsilon(z) := \mathcal{T}\mathbb{G}_\varepsilon(z), \qquad \nu > -1.$$

In particular, $M_{\ell j,\varepsilon}(z) \in \mathscr{B}(L^2(\mathbb{R}))$ and $\mathbb{M}_\varepsilon(z) \in \mathscr{B}(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}))$.

$\mathbb{M}_\varepsilon(z)$ can be represented as the block operator matrix

$$\mathbb{M}_\varepsilon(z) = \begin{bmatrix} M_{11,\varepsilon}(z) & M_{12,\varepsilon}(z) \\ M_{21,\varepsilon}(z) & M_{22,\varepsilon}(z)\end{bmatrix}.$$

Lemma 2.4. For any $z, w \in \mathbb{C}\backslash[0, +\infty)$ there holds

$$\breve{\mathbb{G}}_\varepsilon(z) - \breve{\mathbb{G}}_\varepsilon(w) = (z - w)\breve{\mathbb{G}}_\varepsilon(z)R^0_\varepsilon(w) = (z - w)\breve{\mathbb{G}}_\varepsilon(w)R^0_\varepsilon(z), \tag{2.3}$$

$$\mathbb{G}_\varepsilon(z) - \mathbb{G}_\varepsilon(w) = (z - w)R^0_\varepsilon(w)\mathbb{G}_\varepsilon(z) = (z - w)R^0_\varepsilon(z)\mathbb{G}_\varepsilon(w), \tag{2.4}$$

$$\mathbb{M}_\varepsilon(z) - \mathbb{M}_\varepsilon(w) = (z - w)\breve{\mathbb{G}}_\varepsilon(w)\mathbb{G}_\varepsilon(z) = (z - w)\breve{\mathbb{G}}_\varepsilon(z)\mathbb{G}_\varepsilon(w). \tag{2.5}$$

Proof. The relations (2.3) follow by applying $\mathcal{T}$ to the resolvent identity

$$R^0_\varepsilon(z) - R^0_\varepsilon(w) = (z - w)R^0_\varepsilon(z)R^0_\varepsilon(w) = (z - w)R^0_\varepsilon(w)R^0_\varepsilon(z). \tag{2.6}$$

The relations (2.4) follow by evaluating (2.3) in $\bar{z}$ and $\bar{w}$ and taking the adjoint. Finally, by applying $\mathcal{T}$ to (2.4), one obtains (2.5). □

From now on, the operators $\mathbb{G}_\varepsilon(z)$, $\breve{\mathbb{G}}_\varepsilon(z)$ $\mathbb{M}_\varepsilon(z)$ are to be intended as bounded operators acting from L^2-spaces to L^2-spaces and with the previously stated regularity properties whenever restricted to smaller subspaces.

We note the following expressions for the integral kernels of the operators $M_{\ell j,\varepsilon}(z)$ in Fourier transform (as usual we denote the integral kernels and the operators by the same symbol):

$$M_{11,\varepsilon}(s, s'; z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i(k+\frac{p}{2})(s-s')}}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk}; \qquad M_{12,\varepsilon}(s, s'; z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i(k+\frac{p}{2})s}e^{-i(-k+\frac{p}{2})s'}}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk};$$

$$M_{21,\varepsilon}(s, s'; z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i(-k+\frac{p}{2})s}e^{-i(k+\frac{p}{2})s'}}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk}; \qquad M_{22,\varepsilon}(s, s'; z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i(-k+\frac{p}{2})(s-s')}}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk}.$$

Indeed, by the changing variable $k \to -k$, it is easy to convince oneself that $M_{11,\varepsilon}(s,s';\, z) = M_{22,\varepsilon}(s,s';\, z)$ and $M_{12,\varepsilon}(s,s';\, z) = M_{21,\varepsilon}(s,s';\, z)$, for this reason we introduce the notation

$$M_{d,\varepsilon}(s,s';z) := M_{11,\varepsilon}(s,s';z) = M_{22,\varepsilon}(s,s';z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i(k+\frac{p}{2})(s-s')}}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk};$$

$$M_{od,\varepsilon}(s,s';z) := M_{12,\varepsilon}(s,s';z) = M_{21,\varepsilon}(s,s';z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{i(k+\frac{p}{2})s}e^{-i(-k+\frac{p}{2})s'}}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk}.$$

The suffixes "d" and "od" stand for "*diagonal*" and "*off-diagonal*" respectively. With this notation one has

$$\mathbb{M}_\varepsilon(z) = \begin{bmatrix} M_{d,\varepsilon}(z) & M_{od,\varepsilon}(z) \\ M_{od,\varepsilon}(z) & M_{d,\varepsilon}(z) \end{bmatrix}.$$

We point out the following identities, $z \in \mathbb{C}\backslash[0,+\infty)$:

$$\langle \xi, M_{d,\varepsilon}(z)\xi\rangle = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{|\hat{\xi}(k+p/2)|^2}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk} = \langle M_{d,\varepsilon}(\bar{z})\xi,\xi\rangle, \tag{2.7}$$

$$\langle \eta, M_{od,\varepsilon}(z)\xi\rangle = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\overline{\hat{\eta}(k+p/2)}\hat{\xi}(-k+p/2)}{\varepsilon^2 k^2 + p^2 - z}\mathbf{dk} = \langle M_{od,\varepsilon}(\bar{z})\eta,\xi\rangle. \tag{2.8}$$

Since for all $\Xi, \widetilde{\Xi} \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ there holds

$$\langle \widetilde{\Xi}, \mathbb{M}_\varepsilon(z)\Xi\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R})} = \langle \tilde{\xi}_1, M_{d,\varepsilon}(z)\xi_1\rangle + \langle \tilde{\xi}_2, M_{d,\varepsilon}(z)\xi_2\rangle + \langle \tilde{\xi}_2, M_{od,\varepsilon}(z)\xi_1\rangle + \langle \tilde{\xi}_1, M_{od,\varepsilon}(z)\xi_2\rangle, \tag{2.9}$$

we deduce that

$$\mathbb{M}_\varepsilon(z) = \mathbb{M}_\varepsilon(\bar{z})^* \qquad z \in \mathbb{C}\backslash[0,+\infty) \tag{2.10}$$

[we remark that this property follows by construction since $\mathbb{M}_\varepsilon(z) = \mathcal{T}(\mathcal{T}R^0_\varepsilon(\bar{z}))^* = \mathcal{T}R^0_\varepsilon(z)\mathcal{T}^*$].

Note that the bounded operator

$$\frac{1}{\alpha} + \mathbb{M}_\varepsilon(z) : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \qquad \alpha \in \mathbb{R}\backslash\{0\},\ z \in \mathbb{C}\backslash[0,+\infty)$$

enjoys the same properties (2.5) and (2.10) as $\mathbb{M}_\varepsilon(z)$.

In the forthcoming Proposition 2.6 we will prove that for λ large enough $\frac{1}{\alpha} + \mathbb{M}_\varepsilon(-\lambda)$ is invertible with bounded inverse. In the proof we will use a well known result recalled in the following remark.

Remark 2.5. Let $\mathscr{H}$ be a Hilbert space and $T : D(T) \subset \mathscr{H} \to \mathscr{H}$ be a self-adjoint operator. If there exists a positive constant c such that

$$|\langle u, Tu\rangle_{\mathscr{H}}| \geq c\|u\|^2_{\mathscr{H}} \qquad \forall u \in D(T), \tag{2.11}$$

then, T is injective, surjective, and has (self-adjoint) inverse bounded by $1/c$. To see that this indeed the case, note that, by the Cauchy–Schwarz inequality, Eq. (2.11) implies

$$\|Tu\|_{\mathscr{H}} \geq c\|u\|_{\mathscr{H}} \qquad \forall u \in D(T). \tag{2.12}$$

So T is injective, which in turn implies $\overline{\operatorname{ran}(T)} = \mathscr{H}$. Hence, for any $v \in \mathscr{H}$, there exists a sequence $\{u_n\} \in D(T)$ such that $\|Tu_n - v\|_{\mathscr{H}} \to 0$. By the lower bound (2.12), there follows that $\{u_n\}$ is a Cauchy sequence in $\mathscr{H}$ and so it converges to $u_v \in \mathscr{H}$. Since T is closed, $u_v \in D(T)$ and $Tu_v = v$. Hence, T is also surjective, and has inverse bounded by $1/c$.

Proposition 2.6. Let $\alpha \in \mathbb{R}\backslash\{0\}$ and $\varepsilon > 0$, then there exists $\lambda_{\alpha,\varepsilon} > 0$ such that for all $\lambda > \lambda_{\alpha,\varepsilon}$ the operator $\frac{1}{\alpha} + \mathbb{M}_\varepsilon(-\lambda)$ is invertible in $L^2(\mathbb{R})$ with bounded inverse.

Proof. By Remark 2.5, it is enough to prove that there exists $\lambda_{\alpha,\varepsilon}$ such that for all $\lambda > \lambda_{\alpha,\varepsilon}$ there holds

$$\left|\left\langle \Xi, \left(\frac{1}{\alpha} + \mathbb{M}_\varepsilon(-\lambda)\right)\Xi\right\rangle\right| \geq c_\lambda\|\Xi\|^2 \qquad \forall \Xi \in L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}),$$

for some positive constant c_λ.

We point out the following identities [see Eqs. (2.7) and (2.8)]:

$$\langle \xi, M_{d,\varepsilon}(-\lambda)\xi\rangle = \int_{\mathbb{R}} |\hat{\xi}(\nu)|^2 \int_{\mathbb{R}} \frac{1}{\pi} \frac{1}{\varepsilon^2 k^2 + 4(\nu-k)^2 + \lambda} \mathrm{d}k\, \mathrm{d}\nu = \int_{\mathbb{R}} \frac{|\hat{\xi}(\nu)|^2}{\sqrt{4\varepsilon^2\nu^2 + (4+\varepsilon^2)\lambda}} \mathrm{d}\nu;$$

$$\langle \eta, M_{od,\varepsilon}(-\lambda)\xi\rangle = \frac{2}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\overline{\hat{\eta}(\nu)}\hat{\xi}(\nu')}{(4+\varepsilon^2)(\nu^2+\nu'^2) + 2(4-\varepsilon^2)\nu\nu' + 4\lambda} \mathrm{d}\nu\, \mathrm{d}\nu'.$$

Hence, in Fourier transform, $M_{d,\varepsilon}(-\lambda)$ is the multiplication operator for

$$\hat{M}_{d,\varepsilon}(\nu;-\lambda) = \frac{1}{\sqrt{4\varepsilon^2\nu^2 + (4+\varepsilon^2)\lambda}} \tag{2.13}$$

and $M_{od,\varepsilon}(-\lambda)$ is the operator with integral kernel

$$\hat{M}_{od,\varepsilon}(\nu,\nu';-\lambda) = \frac{2}{\pi}\frac{1}{(4+\varepsilon^2)(\nu^2+\nu'^2) + 2(4-\varepsilon^2)\nu\nu' + 4\lambda}.$$

Both $M_{d,\varepsilon}(-\lambda)$ and $M_{od,\varepsilon}(-\lambda)$ are bounded. Indeed,

$$\|M_{d,\varepsilon}(-\lambda)\| \le \frac{1}{\sqrt{(4+\varepsilon^2)\lambda}};$$

and $M_{od,\varepsilon}(-\lambda)$ is a Hilbert–Schmidt operator, hence it is also compact and bounded. To see that this is indeed the case notice that $(4+\varepsilon^2)(\nu^2 + \nu'^2) + 2(4-\varepsilon^2)\nu\nu' \ge 2\min(\varepsilon^2,4)(\nu^2+\nu'^2)$, hence

$$\int_{\mathbb{R}^2} |\hat{M}_{od,\varepsilon}(\nu,\nu';-\lambda)|^2\, \mathrm{d}\nu \mathrm{d}\nu' \le \frac{4}{\pi^2}\left(\int_{\mathbb{R}} \frac{1}{2\min(\varepsilon^2,4)\nu^2 + 4\lambda}\mathrm{d}\nu\right)^2 = \frac{1}{2\min(\varepsilon^2,4)\lambda}.$$

So that

$$\|M_{od,\varepsilon}(-\lambda)\| \le \|M_{od,\varepsilon}(-\lambda)\|_{HS} \le \sqrt{\frac{1}{2\min(\varepsilon^2,4)\lambda}}.$$

By Eq. (2.9) there follows that there exists $C_\varepsilon > 0$ such that $\left|(\Xi, \mathbb{M}_\varepsilon(-\lambda)\Xi)_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R})}\right| \le C_\varepsilon \|\Xi\|^2/\sqrt{\lambda}$. The latter bound gives

$$\left|\left\langle \Xi, \left(\frac{1}{\alpha} + \mathbb{M}_\varepsilon(-\lambda)\right)\Xi\right\rangle\right| \ge \left(\frac{1}{|\alpha|} - \frac{C_\varepsilon}{\sqrt{\lambda}}\right)\|\Xi\|^2 \ge c_\lambda \|\Xi\|^2$$

for $\lambda > \lambda_{\alpha,\varepsilon} = (C_\varepsilon\alpha)^2$, which concludes the proof. □

Proposition 2.6 leads to the following (for notational simplicity, here and below we avoid to explicitly indicate the α-dependence; notice that the case $\alpha = 0$ gives the free Hamiltonian)

Theorem 2.7. *Let $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $z \in \mathbb{C}\backslash[0,+\infty)$. Then the linear operator in $L^2(\mathbb{R}^2)$ defined by*

$$D(\widetilde{H}_\varepsilon) := \left\{\phi \in H^{3/2-}(\mathbb{R}^2) : \phi + \alpha\,\mathbb{G}_\varepsilon(z)\mathcal{T}\phi \in H^2(\mathbb{R}^2)\right\},$$

$$(\widetilde{H}_\varepsilon - z)\phi := (H^0_\varepsilon - z)(\phi + \alpha\,\mathbb{G}_\varepsilon(z)\mathcal{T}\phi) \tag{2.14}$$

is a z-independent, bounded-from-below, self-adjoint extension of the symmetric operator $H^0_\varepsilon|\ker(\mathcal{T})$.

Furthermore, the operator $1 + \alpha\,\mathbb{M}_\varepsilon(z) : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ has a bounded inverse for all $z \in \rho(\widetilde{H}_\varepsilon) \cap \mathbb{C}\backslash[0,+\infty)$ and the resolvent of $\widetilde{H}_\varepsilon$ is given by

$$\widetilde{R}_\varepsilon(z) = R^0_\varepsilon(z) - \alpha\,\mathbb{G}_\varepsilon(z)(1 + \alpha\,\mathbb{M}_\varepsilon(z))^{-1}\breve{\mathbb{G}}_\varepsilon(z) \qquad z \in \rho(\widetilde{H}_\varepsilon) \cap \mathbb{C}\backslash[0,+\infty).$$

Proof. The case $\alpha = 0$ is trivial. Let $\alpha \neq 0$. By Theorem 2.1 in Ref. 35, the bounded linear operator

$$\widetilde{R}_\varepsilon(-\lambda) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \qquad \lambda > \lambda_{\alpha,\varepsilon}$$

$$\widetilde{R}_\varepsilon(-\lambda) := R^0_\varepsilon(-\lambda) - \mathbb{G}_\varepsilon(-\lambda)\left(\frac{1}{\alpha} + \mathbb{M}_\varepsilon(-\lambda)\right)^{-1}\breve{\mathbb{G}}_\varepsilon(-\lambda) \tag{2.15}$$

is the resolvent of the self-adjoint extension $\widetilde{H}_\varepsilon \geq -\lambda_{\alpha,\varepsilon}$ of $H^0_\varepsilon|\ker(\mathcal{T})$ defined by

$$\widetilde{D} := \left\{\phi \in L^2(\mathbb{R}^2):\ \phi = \phi_{-\lambda} - \mathbb{G}_\varepsilon(-\lambda)\left(\frac{1}{\alpha} + \mathbb{M}_\varepsilon(-\lambda)\right)^{-1}\mathcal{T}\phi_{-\lambda},\ \phi_{-\lambda} \in D(H^0_\varepsilon)\right\};$$
$$\widetilde{H}_\varepsilon : \widetilde{D} \subseteq L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \qquad (\widetilde{H}_\varepsilon + \lambda)\phi := (H^0_\varepsilon + \lambda)\phi_{-\lambda}. \tag{2.16}$$

The definition of $\widetilde{H}_\varepsilon$ is independent of λ; for any fixed λ, the decomposition of ϕ in $D(\widetilde{H}_\varepsilon)$ is unique. For the reader's convenience, we sketch the proof of Theorem 2.1 in Ref. 35, referring to Ref. 35 for the details. By the properties (2.6) and (2.3)–(2.5) (see page 115 in Ref. 35) there follows that $\widetilde{R}_\varepsilon(-\lambda)$ satisfies the relation

$$\widetilde{R}_\varepsilon(-\lambda) - \widetilde{R}_\varepsilon(-\mu) = (\mu - \lambda)\widetilde{R}_\varepsilon(-\lambda)\widetilde{R}_\varepsilon(-\mu) \qquad \lambda, \mu > \lambda_{\alpha,\varepsilon}. \tag{2.17}$$

Hence, it is a pseudo-resolvent. $\widetilde{R}_\varepsilon(-\lambda)$ is the resolvent of a (closed) operator if and only if it is injective (see Theorem 4.10 in Ref. 41). $\widetilde{R}_\varepsilon(-\lambda)$ is indeed injective, since $\widetilde{R}_\varepsilon(-\lambda)\psi = 0$ would imply

$$R^0_\varepsilon(-\lambda)\psi = \mathbb{G}_\varepsilon(-\lambda)Q$$

with $Q = \left(\frac{1}{\alpha} + \mathbb{M}_\varepsilon(-\lambda)\right)^{-1}\breve{\mathbb{G}}_\varepsilon(-\lambda)\psi$, but this implies $\psi = 0$ by Remark 2.2. By the self-adjointness of the operator $\widetilde{R}_\varepsilon(-\lambda)$ there follows that $\widetilde{D} := \operatorname{ran}\big(\widetilde{R}_\varepsilon(-\lambda)\big)$ is dense, and independent of λ because of the resolvent identity (2.17); moreover, the closed operator $\widetilde{H}_\varepsilon = \widetilde{R}_\varepsilon(-\lambda)^{-1} - \lambda$ is self-adjoint, because $\widetilde{R}_\varepsilon(-\lambda)^* = \widetilde{R}_\varepsilon(-\lambda)$. By Remark 2.2 the decomposition of ϕ in $\widetilde{D}$ is unique. If $\phi \in D(H^0_\varepsilon) \cap \ker(\mathcal{T})$ one has $\phi = \phi_{-\lambda}$ and $\widetilde{H}_\varepsilon\phi = H_{0,\varepsilon}\phi$ by Eq. (2.16).

Then, by Theorem 2.19 in Ref. 10, the resolvent formula (2.15) extends to all $z \in \rho(\widetilde{H}_\varepsilon) \cap \mathbb{C}\backslash[0, +\infty)$ and

$$\widetilde{D} = \left\{\phi \in L^2(\mathbb{R}^2):\ \phi = \phi_z - \mathbb{G}_\varepsilon(z)\left(\frac{1}{\alpha} + \mathbb{M}_\varepsilon(z)\right)^{-1}\mathcal{T}\phi_z,\ \phi_z \in D(H^0_\varepsilon)\right\},$$

$$(\widetilde{H}_\varepsilon - z)\phi = (H^0_\varepsilon - z)\phi_z.$$

The definition of $\widetilde{H}_\varepsilon$ is independent of z and, for any fixed z, the decomposition of ϕ in $\widetilde{D}$ is unique. Notice that $\widetilde{D} \subseteq H^{3/2-}(\mathbb{R}^2)$ by the mapping properties of $\mathbb{G}_\varepsilon(z)$ [see (2.1)].

Let us now show that $\widetilde{D} = D(\widetilde{H}_\varepsilon)$. Since, by (2.4), $\operatorname{ran}(\mathbb{G}_\varepsilon(z) - \mathbb{G}_\varepsilon(w)) \subseteq D(H^0_\varepsilon) = H^2(\mathbb{R}^2)$, it is enough to prove $\widetilde{D} = D(\widetilde{H}_\varepsilon)$ whenever the z appearing in the definition of $D(\widetilde{H}_\varepsilon)$ belongs to $\rho(\widetilde{H}_\varepsilon) \cap \mathbb{C}\backslash[0, +\infty)$. Given $\phi \in \widetilde{D}$, one has $\mathcal{T}\phi = \mathcal{T}\phi_z - \alpha\,\mathbb{M}_\varepsilon(z)(1 + \alpha\,\mathbb{M}_\varepsilon(z))^{-1}\mathcal{T}\phi_z$, which entails $\mathcal{T}\phi = (1 + \alpha\,\mathbb{M}_\varepsilon(z))^{-1}\mathcal{T}\phi_z$. Hence, $\phi_z = \phi + \alpha\,\mathbb{G}_\varepsilon(z)\mathcal{T}\phi$; this gives $\widetilde{D} \subseteq D(\widetilde{H}_\varepsilon)$. Vice versa, given $\phi \in D(\widetilde{H}_\varepsilon)$ and defining $\phi_z \in H^2(\mathbb{R}^2)$ by $\phi_z := \phi + \alpha\,\mathbb{G}_\varepsilon(z)\mathcal{T}\phi$, one has $\mathcal{T}\phi + \alpha\,\mathbb{M}_\varepsilon(z)\mathcal{T}\phi = \mathcal{T}\phi_z$, which gives $\mathcal{T}\phi = (1 + \alpha\,\mathbb{M}_\varepsilon(z))^{-1}\mathcal{T}\phi_z$. Hence, $\phi = \phi_z - \alpha\mathbb{G}_\varepsilon(z)(1 + \alpha\,\mathbb{M}_\varepsilon(z))^{-1}\mathcal{T}\phi_z$; this entails $D(\widetilde{H}_\varepsilon) \subseteq \widetilde{D}$. Therefore, $\widetilde{D} = D(\widetilde{H}_\varepsilon)$. Furthermore, the previous calculations also give

$$(\widetilde{H}_\varepsilon - z)\phi = (H^0_\varepsilon - z)(\phi + \alpha\,\mathbb{G}_\varepsilon(z)\mathcal{T}\phi), \qquad z \in \rho(\widetilde{H}_\varepsilon) \cap \mathbb{C}\backslash[0, +\infty). \tag{2.18}$$

To conclude, let us show that (2.18) holds true even whenever $z \in \mathbb{C}\backslash[0, +\infty)$. Given any $w \in \rho(\widetilde{H}_\varepsilon) \cap \mathbb{C}\backslash[0, +\infty)$, one has, by (2.18) and (2.4),

$$\begin{aligned}(\widetilde{H}_\varepsilon - z)\phi &= (\widetilde{H}_\varepsilon - w)\phi + (w - z)\phi = (H^0_\varepsilon - w)(\phi + \alpha\,\mathbb{G}_\varepsilon(w)\mathcal{T}\phi) + (w - z)\phi\\ &= (H^0_\varepsilon - w)(\phi + \alpha\,\mathbb{G}_\varepsilon(z)\mathcal{T}\phi) + \alpha(H^0_\varepsilon - w)(\mathbb{G}_\varepsilon(w) - \mathbb{G}_\varepsilon(z))\mathcal{T}\phi + (w - z)\phi\\ &= (H^0_\varepsilon - w)(\phi + \alpha\,\mathbb{G}_\varepsilon(z)\mathcal{T}\phi) + (w - z)\alpha\,\mathbb{G}_\varepsilon(z)\mathcal{T}\phi + (w - z)\phi\\ &= (H^0_\varepsilon - z)(\phi + \alpha\,\mathbb{G}_\varepsilon(z)\mathcal{T}\phi).\end{aligned}$$

□

Remark 2.8. Introducing the adjoint $\mathcal{T}^* : H^{-\nu+1/2}(\mathbb{R}) \oplus H^{-\nu+1/2}(\mathbb{R}) \to H^{-\nu}(\mathbb{R}^2)$, $\nu > 1/2$, which provides, whenever $\xi_1 \oplus \xi_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ the tempered distribution $\mathcal{T}^*\xi_1 \oplus \xi_2 \in \mathcal{S}'(\mathbb{R}^2)$ acting on a test function $f \in \mathcal{S}(\mathbb{R}^2)$ as

$$(\mathcal{T}^*\xi_1 \oplus \xi_2)f = \int_{\mathbb{R}} \xi_1(s) f(s, s/2)\,ds + \int_{\mathbb{R}} \xi_2(s) f(-s, s/2)\,ds,$$

one has $\mathbb{G}_\varepsilon(z) = R^0_\varepsilon(z)\mathcal{T}^*$. The latter entails the distributional identity

$$(-\varepsilon^2\partial_x^2 - \partial_y^2 - z)\mathbb{G}_\varepsilon(z)\xi_1 \oplus \xi_2 = \mathcal{T}^*\xi_1 \oplus \xi_2 \tag{2.19}$$

and so, by (2.14), for any $\phi \in D(\widetilde{H}_\varepsilon)$ one gets

$$\widetilde{H}_\varepsilon \phi = \left(-\varepsilon^2 \partial_x^2 - \partial_y^2\right)\phi + \alpha \mathcal{T}^* \mathcal{T}\phi. \tag{2.20}$$

Notice that, unless $\phi \in \ker(\mathcal{T})$, neither of the two tempered distributions on the right hand side of the latter equation is in $L^2(\mathbb{R}^2)$ but their sum is. Moreover, since $\text{supp}\,(\mathcal{T}^*\xi_1 \oplus \xi_2) = \Pi$, one has

$$\widetilde{H}_\varepsilon \phi(\mathbf{x}) = \left(-\varepsilon^2 \partial_x^2 - \partial_y^2\right)\phi(\mathbf{x}) \quad \text{for a.e. } \mathbf{x} \text{ in } \mathbb{R}^2 \backslash \Pi.$$

Proposition 2.9. Let

$$\widetilde{\mathcal{B}}_\varepsilon : H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \subseteq L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \to \mathbb{C}$$

be the sesquilinear form

$$\widetilde{\mathcal{B}}_\varepsilon(\varphi, \phi) := \varepsilon^2 \langle \partial_x \varphi, \partial_x \phi \rangle + \langle \partial_y \varphi, \partial_y \phi \rangle + \alpha(\langle \tau_1 \varphi, \tau_1 \phi \rangle + \langle \tau_2 \varphi, \tau_2 \phi \rangle).$$

Then

$$\widetilde{\mathcal{B}}_\varepsilon(\varphi, \phi) = \langle \varphi, \widetilde{H}_\varepsilon \phi \rangle \qquad \forall\, \varphi \in H^1(\mathbb{R}^2),\ \forall \phi \in D(\widetilde{H}_\varepsilon).$$

Proof. For the sesquilinear form $\mathcal{B}_\varepsilon^0$ corresponding to H_ε^0, there holds

$$\mathcal{B}_\varepsilon^0(\varphi, \phi) = \langle \varphi, H_\varepsilon^0 \phi \rangle \qquad \forall\, \varphi \in H^1(\mathbb{R}^2),\ \forall \phi \in H^2(\mathbb{R}^2)$$

and

$$\mathcal{B}_\varepsilon^0(\varphi, \phi) = \langle (-\varepsilon^2 \partial_x^2 - \partial_y^2)\varphi, \phi \rangle_{-1,+1} = \langle \varphi, (-\varepsilon^2 \partial_x^2 - \partial_y^2)\phi \rangle_{+1,-1} \qquad \forall\, \psi, \phi \in H^1(\mathbb{R}^2),$$

where $\langle \cdot, \cdot \rangle_{\mp\nu,\pm\nu}$ denotes the (anti-linear with respect to the first variable) $H^{\pm\nu}(\mathbb{R}^2)$-$H^{\mp\nu}(\mathbb{R}^2)$ duality extending the scalar product in $L^2(\mathbb{R}^2)$.

Then, defining $\phi_z := \phi + \alpha \mathbb{G}_\varepsilon(z)\mathcal{T}\phi$, by (2.19) and since both ϕ and $\mathbb{G}_\varepsilon(z)\mathcal{T}\phi$ belong to $H^{3/2-}(\mathbb{R}^2) \subset H^1(\mathbb{R}^2)$, one gets

$$\begin{aligned}
\langle \varphi, \widetilde{H}_\varepsilon \phi \rangle &= \langle \varphi, (\widetilde{H}_\varepsilon - z)\phi \rangle + z\langle \varphi, \phi \rangle = \left(\varphi, (H_\varepsilon^0 - z)\phi_z\right) + z\langle \varphi, \phi \rangle \\
&= (\mathcal{B}_\varepsilon^0 - z)(\varphi, \phi_z) + z\langle \varphi, \phi \rangle = \mathcal{B}_\varepsilon^0(\varphi, \phi) + \alpha(\mathcal{B}_\varepsilon^0 - z)(\varphi, \mathbb{G}_\varepsilon(z)\mathcal{T}\phi) \\
&= \mathcal{B}_\varepsilon^0(\varphi, \phi) + \alpha\langle \varphi, (-\varepsilon^2 \partial_x^2 - \partial_y^2 - z)\mathbb{G}_\varepsilon(z)\mathcal{T}\phi \rangle_{+1,-1} \\
&= \mathcal{B}_\varepsilon^0(\varphi, \phi) + \alpha\langle \varphi, \mathcal{T}^*\mathcal{T}\phi \rangle_{+1,-1} = \mathcal{B}_\varepsilon^0(\varphi, \phi) + \alpha(\langle \tau_1 \varphi, \tau_1 \phi \rangle + \langle \tau_2 \varphi, \tau_2 \phi \rangle) \\
&= \widetilde{\mathcal{B}}_\varepsilon(\varphi, \phi).
\end{aligned}$$

□

Remark 2.10. Since $G_{j,\varepsilon}(z)$ corresponds to the single-layer operator for the elliptic operator $\varepsilon^2 \partial_x^2 + \partial_y^2 + z$ relative to the line Π_j, by the jump and regularity properties of the single-layer potentials one gets an alternative characterization of the self-adjointness domain of $\widetilde{H}_\varepsilon$ in terms of boundary conditions:

$$D(\widetilde{H}_\varepsilon) = \left\{\phi \in H^{3/2-}(\mathbb{R}^2) : \phi = \phi_1 + \phi_2, \phi_j \in H^2(\mathbb{R}^2 \backslash \Pi_j), [\tau'_{j,\varepsilon}]\phi_j = \alpha \tau_j \phi, \quad j = 1, 2\right\},$$

where $[\tau'_{j,\varepsilon}]\phi_j$ denotes the jump across Π_j of the normal derivative relative to $\varepsilon^2 \partial_x^2 + \partial_y^2$. The boundary condition on Π_1 is written explicitly in Eq. (1.2).

B. The compression of $\widetilde{H}_{\varepsilon,\alpha}$ onto $L^2_\natural(\mathbb{R}^2)$

In this section we introduce the bosonic and the fermionic symmetries. Recall that the symbol $\natural$ denotes either b (bosonic) or f (fermionic), with the corresponding value $(+)_\natural$ defined as $(+)_\mathrm{b} = +$ and $(+)_\mathrm{f} = -$.

At first, notice that

$$R_\varepsilon^0(z) : L^2_\natural(\mathbb{R}^2) \to H^2_\natural(\mathbb{R}^2) \qquad \forall z \in \mathbb{C} \backslash [0, +\infty). \tag{2.21}$$

We define the operator

$$S : H^\nu(\mathbb{R}^2) \to H^\nu(\mathbb{R}^2), \quad \nu \geq 0, \qquad (S\psi)(x, y) := \psi(-x, y), \tag{2.22}$$

and the orthogonal projector

$$\frac{1(+)_\natural S}{2} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \qquad \mathrm{ran}\left(\frac{1(+)_\natural S}{2}\right) = L^2_\natural(\mathbb{R}^2). \tag{2.23}$$

$L^2_\natural(\mathbb{R}^2)$ enjoys the property

$$S\psi = (+)_\natural \psi \qquad \forall \psi \in L^2_\natural(\mathbb{R}^2).$$

One has

$$\tau_1 S = \tau_2, \qquad \tau_2 S = \tau_1,$$

$$\breve{G}_{2,\varepsilon}(z) = \breve{G}_{1,\varepsilon}(z)S, \qquad G_{2,\varepsilon}(z) = SG_{1,\varepsilon}(z).$$

Noticing that

$$\tau_1\psi = (+)_\natural \tau_2\psi \qquad \forall \psi \in H^2_\natural(\mathbb{R}^2),$$

from now on we pose

$$\tau \equiv \tau_1$$

and introduce the bounded operators

$$\breve{G}_\varepsilon(z) : L^2_\natural(\mathbb{R}^2) \to L^2(\mathbb{R}), \qquad \breve{G}_\varepsilon(z) := \tau R^0_\varepsilon(z) \equiv \breve{G}_{1,\varepsilon}(z)|L^2_\natural(\mathbb{R}^2),$$

$$G_\varepsilon(z) : L^2(\mathbb{R}) \to L^2_\natural(\mathbb{R}^2), \qquad G_\varepsilon(z) := \breve{G}_\varepsilon(\bar{z})^* \equiv \frac{1(+)_\natural S}{2}\, G_{1,\varepsilon}(z),$$

$$M_\varepsilon(z) : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad M_\varepsilon(z) := \tau G_\varepsilon(z).$$

Remark 2.11. To avoid a too heavy indexing, we avoided to use the symbol $\natural$ to distinguish the bosonic and fermionic versions of the operators $\breve{G}_\varepsilon(z)$, $G_\varepsilon(z)$ and $M_\varepsilon(z)$. We trust that the reader will not be confused by this abuse of notation.

With such definitions one has

$$\begin{aligned} M_\varepsilon(z) &= \tau_1\, \frac{1(+)_\natural S}{2}\, G_{1,\varepsilon}(z) = \frac{1}{2}\,(\tau_1 G_{1,\varepsilon}(z)(+)_\natural \tau_1 S G_{1,\varepsilon}(z)) = \frac{1}{2}\,(\tau_1 G_{1,\varepsilon}(z)(+)_\natural \tau_1 G_{2,\varepsilon}(z)) \\ &= \frac{1}{2}\,(M_{d,\varepsilon}(z)(+)_\natural M_{od,\varepsilon}(z)). \end{aligned}$$

Then, by such relations, one gets

$$\breve{\mathbb{G}}_\varepsilon(z)\psi = \breve{G}_\varepsilon(z)\psi \oplus ((+)_\natural \breve{G}_\varepsilon(z)\psi), \qquad \forall \psi \in L^2_\natural(\mathbb{R}^2),$$

$$\mathbb{G}_\varepsilon(z)\xi \oplus ((+)_\natural \xi) = G_{1,\varepsilon}(z)\xi(+)_\natural G_{2,\varepsilon}(z)\xi = (1(+)_\natural S)G_{1,\varepsilon}(z)\xi = 2\, G_\varepsilon(z)\xi,$$

$$\mathbb{M}_\varepsilon(z)\xi \oplus ((+)_\natural \xi) = 2(M_\varepsilon(z)\xi \oplus ((+)_\natural M_\varepsilon(z)\xi)),$$

$$(1 + \alpha\,\mathbb{M}_\varepsilon(z))^{-1}\xi \oplus ((+)_\natural \xi) = (1 + 2\alpha\, M_\varepsilon(z))^{-1}\xi \oplus \left((+)_\natural (1 + 2\alpha\, M_\varepsilon(z))^{-1}\xi\right).$$

Therefore, for all $\psi \in L^2_\natural(\mathbb{R}^2)$ and for all $z \in \rho(\widetilde{H}_\varepsilon) \cap \mathbb{C}\backslash[0, +\infty)$ one obtains

$$\begin{aligned} \widetilde{R}_\varepsilon(z)\psi &= R^0_\varepsilon(z)\psi - \alpha\,\mathbb{G}_\varepsilon(z)(1 + \alpha\,\mathbb{M}_\varepsilon(z))^{-1}\breve{\mathbb{G}}_\varepsilon(z)\psi \\ &= R^0_\varepsilon(z)\psi - 2\alpha\, G_\varepsilon(z)(1 + 2\alpha\, M_\varepsilon(z))^{-1}\breve{G}_\varepsilon(z)\psi. \end{aligned} \tag{2.24}$$

Hence, by (2.21), $R_\varepsilon^\natural(z) := \widetilde{R}_\varepsilon(z)|L^2_\natural(\mathbb{R}^2)$ is a pseudo resolvent in $L^2_\natural(\mathbb{R}^2)$; furthermore, by the analogous properties for $\widetilde{R}_\varepsilon(z)$, one has that $R_\varepsilon^\natural(z)$ is injective and $(R_\varepsilon^\natural(z))^* = R_\varepsilon^\natural(\bar{z})$. Therefore, $R_\varepsilon^\natural(z)$ is the resolvent of a self-adjoint operator $\widehat{H}_\varepsilon^\natural$ in $L^2_\natural(\mathbb{R}^2)$ which, by construction, is given by the compression of $\widetilde{H}_\varepsilon$ onto $L^2_\natural(\mathbb{R}^2)$, i.e.,

$$\widehat{H}_\varepsilon^\natural = \frac{1(+)_\natural S}{2}\,\widetilde{H}_\varepsilon\,\frac{1(+)_\natural S}{2} \equiv \widetilde{H}_\varepsilon|D(\widetilde{H}_\varepsilon)\cap L^2_\natural(\mathbb{R}^2).$$

By Theorem 2.19 in Ref. 10, the resolvent formula (2.24) extends to all $z \in \mathbb{C}\backslash[0,+\infty)\cap\rho(\widehat{H}_\varepsilon^\natural)$.

Theorem 2.12. *Let $\alpha\in\mathbb{R}$, $\varepsilon > 0$ and $z\in\mathbb{C}\backslash[0,+\infty)$. Then the linear operator in $L^2_\natural(\mathbb{R}^2)$ defined by*

$$D(H_\varepsilon^\natural) := \left\{\psi \in H_\natural^{3/2-}(\mathbb{R}^2) : \psi + 2\alpha G_\varepsilon(z)\tau\psi \in H^2_\natural(\mathbb{R}^2)\right\},$$

$$(H_\varepsilon^\natural - z)\psi := (H_\varepsilon^0 - z)(\psi + 2\alpha G_\varepsilon(z)\tau\psi)$$

is a z-independent, bounded-from-below, self-adjoint extension of the symmetric operator $H_\varepsilon^0|\ker(\tau|H^2_\natural(\mathbb{R}^2))$.

Furthermore, the operator $1+2\alpha M_\varepsilon(z) : L^2(\mathbb{R})\to L^2(\mathbb{R})$ has a bounded inverse for all $z\in\rho(H_\varepsilon^\natural)\cap\mathbb{C}\backslash[0,+\infty)$ and the resolvent of $H_\varepsilon^\natural$ is given by

$$R_\varepsilon^\natural(z) = R_\varepsilon^0(z) - 2\alpha G_\varepsilon(z)(1+2\alpha M_\varepsilon(z))^{-1}\breve{G}_\varepsilon(z) \qquad z\in\rho(H_\varepsilon^\natural)\cap\mathbb{C}\backslash[0,+\infty). \tag{2.25}$$

By Remark 2.8, one gets

Remark 2.13. Introducing the adjoint $\tau^* : H^{-\nu+1/2}(\mathbb{R})\to H^{-\nu}(\mathbb{R}^2)$, $\nu > 1/2$, which provides, whenever $\xi\in L^2(\mathbb{R})$, the tempered distribution $\tau^*\xi\in\mathcal{S}'(\mathbb{R}^2)$ acting on a test function $f\in\mathcal{S}(\mathbb{R}^2)$ as

$$(\tau^*\xi)f = \int_{\mathbb{R}} \xi(s)\, f(s,s/2)\,ds,$$

one has $G_\varepsilon(z) = R_\varepsilon^0(z)\tau^*$. The latter entails the distributional identity

$$\left(-\varepsilon^2\partial_x^2 - \partial_y^2 - z\right)G_\varepsilon(z)\xi = \tau^*\xi$$

and so, by (2.20) and by $\mathcal{T}^*\xi\oplus\xi = 2\tau^*\xi$, one gets

$$H_\varepsilon^\natural\psi = \left(-\varepsilon^2\partial_x^2 - \partial_y^2\right)\psi + 2\alpha\,\tau^*\tau\psi.$$

By Proposition 2.9 there follows

Proposition 2.14. Let

$$\mathcal{B}_\varepsilon^\natural : H^1_\natural(\mathbb{R}^2)\times H^1_\natural(\mathbb{R}^2) \subseteq L^2_\natural(\mathbb{R}^2)\times L^2_\natural(\mathbb{R}^2)\to\mathbb{C} \tag{2.26}$$

be the sesquilinear form

$$\mathcal{B}_\varepsilon^\natural(\varphi,\psi) := \varepsilon^2\langle\partial_x\varphi,\partial_x\psi\rangle + \langle\partial_y\varphi,\partial_y\psi\rangle + 2\alpha\langle\tau\varphi,\tau\psi\rangle. \tag{2.27}$$

Then

$$\mathcal{B}_\varepsilon^\natural(\varphi,\psi) = \langle\varphi, H_\varepsilon^\natural\psi\rangle \qquad \forall\,\varphi\in H^1_\natural(\mathbb{R}^2),\ \forall\,\psi\in D(H_\varepsilon^\natural).$$

C. The spectrum of $H_\varepsilon^\natural$

The next proposition allows to build functions in $D(H_\varepsilon^\natural)$ given $\xi\in L^2(\mathbb{R})$. It will be helpful in the characterization of the spectrum of $H_\varepsilon^\natural$, see Lemma 2.17 below. We premise the following result, whose proof is the same as that of Lemma 2.4, since $\breve{G}_\varepsilon(z) = \tau R_\varepsilon^0(z)$, $G_\varepsilon(z) = (\tau R_\varepsilon^0(\bar{z}))^*$ and $M_\varepsilon(z) = \tau G_\varepsilon(z)$.

Lemma 2.15. For any $z, w\in\mathbb{C}\backslash[0,+\infty)$, there holds

$$\breve{G}_\varepsilon(z) - \breve{G}_\varepsilon(w) = (z-w)\breve{G}_\varepsilon(z)R_\varepsilon^0(w) = (z-w)\breve{G}_\varepsilon(w)R_\varepsilon^0(z),$$

$$G_\varepsilon(z) - G_\varepsilon(w) = (z-w)R_\varepsilon^0(w)G_\varepsilon(z) = (z-w)R_\varepsilon^0(z)G_\varepsilon(w), \tag{2.28}$$

$$M_\varepsilon(z) - M_\varepsilon(w) = (z-w)\check{G}_\varepsilon(w)G_\varepsilon(z) = (z-w)\check{G}_\varepsilon(z)G_\varepsilon(w). \tag{2.29}$$

Proposition 2.16. (i) For any $w, z \in \mathbb{C}\backslash[0,+\infty)$ and $\psi \in D(H_\varepsilon^\natural)$, defining

$$\psi_z := \psi + 2\alpha G_\varepsilon(z)\tau\psi,$$

one has

$$(H_\varepsilon^\natural - w)\psi = (H_\varepsilon^0 - w)\psi_z - (z-w)2\alpha G_\varepsilon(z)\tau\psi.$$

(ii) For any $w \in \mathbb{C}\backslash[0,+\infty)$, $z \in \rho(H_\varepsilon^\natural) \cap \mathbb{C}\backslash[0,+\infty)$ and $\xi \in L^2(\mathbb{R})$, defining

$$\phi_{w,\xi} := R_\varepsilon^\natural(z)G_\varepsilon(w)\xi,$$

one has

$$(H_\varepsilon^\natural - w)\phi_{w,\xi} = G_\varepsilon(z)(1+2\alpha M_\varepsilon(z))^{-1}(1+2\alpha M_\varepsilon(w))\xi.$$

Proof. (i) By the action of $H_\varepsilon^\natural$ on its domain, one has

$$\begin{aligned}(H_\varepsilon^\natural - w)\psi &= (H_\varepsilon^0 - z)(\psi + 2\alpha G_\varepsilon(z)\tau\psi) + (z-w)\psi\\ &= (H_\varepsilon^0 - z)\psi_z + (z-w)(\psi_z - 2\alpha G_\varepsilon(z)\tau\psi)\\ &= (H_\varepsilon^0 - w)\psi_z - (z-w)2\alpha G_\varepsilon(z)\tau\psi.\end{aligned}$$

(ii) By (2.25), (2.29), and (2.28), one gets

$$\begin{aligned}&(H_\varepsilon^\natural - w)\phi_{w,\xi} = (H_\varepsilon^\natural - z)\phi_{w,\xi} + (z-w)\phi_{w,\xi} = G_\varepsilon(w)\xi + (z-w)\phi_{w,\xi}\\ &= G_\varepsilon(w)\xi + (z-w)R_\varepsilon^0(z)G_\varepsilon(w)\xi - (z-w)2\alpha G_\varepsilon(z)(1+2\alpha M_\varepsilon(z))^{-1}\check{G}_\varepsilon(z)G_\varepsilon(w)\xi\\ &= G_\varepsilon(z)\xi - G_\varepsilon(z)(1+2\alpha M_\varepsilon(z))^{-1}(1+2\alpha M_\varepsilon(z) - 1 - 2\alpha M_\varepsilon(w))\xi\\ &= G_\varepsilon(z)(1+2\alpha M_\varepsilon(z))^{-1}(1+2\alpha M_\varepsilon(w))\xi.\end{aligned}$$

□

Lemma 2.17. Let $\lambda > 0$. Then,

(i) *$-\lambda \in \sigma_p(H_\varepsilon^\natural)$ if and only if $0 \in \sigma_p(1+2\alpha M_\varepsilon(-\lambda))$;*
(ii) *$-\lambda \in \sigma_{ess}(H_\varepsilon^\natural)$ if and only if $0 \in \sigma_{ess}(1+2\alpha M_\varepsilon(-\lambda))$.*

Proof. At first, let us notice that $-\lambda \in \rho(H_\varepsilon^0)$.

(i) Let $\psi \in D(H_\varepsilon^\natural)$ be an eigenvector of $H_\varepsilon^\natural$ with eigenvalue $-\lambda$. By Proposition 2.16(i),

$$0 = (H_\varepsilon^\natural + \lambda)\psi = (H_\varepsilon^0 + \lambda)(\psi + 2\alpha G_\varepsilon(z)\tau\psi) - (z+\lambda)2\alpha G_\varepsilon(z)\tau\psi, \qquad \forall z \in \mathbb{C}\backslash[0,+\infty).$$

This gives

$$\psi + 2\alpha G_\varepsilon(-\lambda)\tau\psi = 2\alpha(G_\varepsilon(-\lambda) - G_\varepsilon(z))\tau\psi - 2\alpha(z+\lambda)R_\varepsilon^0(-\lambda)G_\varepsilon(z)\tau\psi.$$

Applying τ to both sides and taking into account (2.29), one gets

$$(1+2\alpha M_\varepsilon(-\lambda))\tau\psi = 2\alpha(M_\varepsilon(-\lambda) - M_\varepsilon(z))\tau\psi - 2\alpha(z+\lambda)\check{G}_\varepsilon(-\lambda)G_\varepsilon(z)\tau\psi = 0.$$

On the other hand, assume that there exists $\xi \in L^2(\mathbb{R})$ such that $(1+2\alpha M_\varepsilon(-\lambda))\xi = 0$. Then, by Proposition 2.16(ii), $\phi_{-\lambda,\xi} = R_\varepsilon^\natural(z)G_\varepsilon(-\lambda)\xi$, $z \in \rho(H_\varepsilon^\natural) \cap \mathbb{C}\backslash[0,+\infty)$, is an eigenfunction with eigenvalue $-\lambda$.

(ii) Let $\{\psi_n\}$ be a singular Weyl sequence for $(H_\varepsilon^\natural, -\lambda)$, i.e., $\psi_n \in D(H_\varepsilon^\natural)$, $\|\psi_n\| = 1$, $\psi_n \rightharpoonup 0$, and $\|(H_\varepsilon^\natural + \lambda)\psi_n\| \to 0$. Arguing as above, one gets, for any $z \in \mathbb{C}\backslash[0,+\infty)$,

$$\begin{aligned}&\check{G}_\varepsilon(-\lambda)(H_\varepsilon^\natural + \lambda)\psi_n = \tau R_\varepsilon^0(-\lambda)\big((H_\varepsilon^0 + \lambda)(\psi_n + 2\alpha G_\varepsilon(z)\tau\psi_n) + (z+\lambda)2\alpha G_\varepsilon(z)\tau\psi_n\big)\\ &= \tau(\psi_n + 2\alpha G_\varepsilon(z)\tau\psi_n) + (z+\lambda)2\alpha\check{G}_\varepsilon(-\lambda)G_\varepsilon(z)\tau\psi_n\end{aligned}$$

$$= (1 + 2\alpha M_\varepsilon(-\lambda))\tau\psi_n - 2\alpha(M_\varepsilon(-\lambda) - M_\varepsilon(z))\tau\psi_n + 2\alpha(z+\lambda)\check{G}_\varepsilon(-\lambda)G_\varepsilon(z)\tau\psi_n$$
$$= (1 + 2\alpha M_\varepsilon(-\lambda))\tau\psi_n$$

and so $(1 + 2\alpha M_\varepsilon(-\lambda))\tau\psi_n \to 0$. Let us now show that $\tau\psi_n \rightharpoonup 0$.

Defining $\psi_{n,z} := \psi_n + 2\alpha G_\varepsilon(z)\tau\psi_n \in H^2_\natural(\mathbb{R}^2)$, one has, for any $f \in H^{-2}(\mathbb{R}^2)$ and for any $z \in \mathbb{C}\backslash[0,+\infty)$,

$$\langle (H_\varepsilon^\natural + \lambda)\psi_n, R_\varepsilon^0(\bar{z})f\rangle = \langle (H_\varepsilon^0 - z)\psi_{n,z}, R_\varepsilon^0(\bar{z})f\rangle + (\lambda - z)\langle\psi_n, R_\varepsilon^0(\bar{z})f\rangle$$
$$= \langle\psi_{n,z}, f\rangle_{+2,-2} + (\lambda - z)\langle\psi_n, R_\varepsilon^0(\bar{z})f\rangle.$$

This gives $\psi_{n,z} \rightharpoonup 0$ in $H^2(\mathbb{R}^2)$ and so, by $\tau \in \mathscr{B}(H^2(\mathbb{R}^2), L^2(\mathbb{R}))$ and by $\tau\psi_n = (1 + 2\alpha M_\varepsilon(z))^{-1}\tau\psi_{n,z}$, one gets $\tau\psi_n \rightharpoonup 0$ in $L^2(\mathbb{R})$. To get a singular Weyl sequence for $(1 + 2\alpha M_\varepsilon(-\lambda), 0)$ we need to normalize $\tau\psi_n$, so we need to show that $\|\tau\psi_n\|$ does not converge to zero. Let us assume that this is not the case. Then, by Proposition 2.16(i), there follows $\|(H_\varepsilon^0 + \lambda)\psi_{n,z}\| \to 0$, and, by $\|\psi_n\| = 1$, one would have $\|\psi_{n,z}\| \to 1$, but this is impossible because, by taking as singular Weyl sequence $\psi_{n,z}/\|\psi_{n,z}\|$, it would imply $-\lambda \in \sigma_{ess}(H_\varepsilon^0)$.

To conclude, we consider a singular Weyl sequence $\{\xi_n\}$, $\|\xi_n\| = 1$, $\xi_n \rightharpoonup 0$, and $\|(1 + 2\alpha M_\varepsilon(-\lambda))\xi_n\| \to 0$. By Proposition 2.16(ii), the sequence $\{\phi_n\}$, $\phi_n := R_\varepsilon^\natural(z)G_\varepsilon(-\lambda)\xi_n$, $z \in \rho(H_\varepsilon^\natural) \cap \mathbb{C}\backslash[0,+\infty)$ is such that $\phi_n \rightharpoonup 0$ and $\|(H_\varepsilon^\natural + \lambda)\phi_n\| \to 0$. We are left to prove that $\|\phi_n\|$ does not converge to zero. Let us assume that this is not the case. By $R_\varepsilon^0(z) = R_\varepsilon^\natural(z) + 2\alpha G_\varepsilon(z)\tau R_\varepsilon^0(z)$, there follows $\phi_n + 2\alpha G_\varepsilon(z)\tau\phi_n = R_\varepsilon^0(z)G_\varepsilon(-\lambda)\xi_n$. Thus,

$$(H_\varepsilon^\natural + \lambda)\phi_n = (H_\varepsilon^0 - z)(\phi_n + 2\alpha G_\varepsilon(z)\tau\phi_n) + (z+\lambda)\phi_n = G_\varepsilon(-\lambda)\xi_n + (z+\lambda)\phi_n,$$

and so $G_\varepsilon(-\lambda)\xi_n \to 0$. This contradicts the bounded invertibility of $1 + 2\alpha M_\varepsilon(z)$, which is equivalent to the existence of $c_z > 0$ such that $\|(1 + 2\alpha M_\varepsilon(z))\xi_n\| \geq c_z\|\xi_n\| = c_z$. Indeed, by (2.29), one obtains

$$0 < \|(1 + 2\alpha M_\varepsilon(z))\xi_n\| \leq \|(1 + 2\alpha M_\varepsilon(-\lambda))\xi_n\| + 2|\alpha|\|(M_\varepsilon(z) - M_\varepsilon(-\lambda))\xi_n\|$$
$$= \|(1 + 2\alpha M_\varepsilon(-\lambda))\xi_n\| + 2|\alpha|\,|z+\lambda|\|\check{G}_\varepsilon(z)G_\varepsilon(-\lambda)\xi_n\|.$$

□

Lemma 2.18. For all $\varepsilon > 0$ and $\alpha \in \mathbb{R}$, $[0,+\infty) \subseteq \sigma_{ess}(H_\varepsilon^\natural)$.

Proof. The idea is to construct a singular Weyl sequence supported away from the coincidence lines, which also serves as a singular sequence for the free Hamiltonian. Let χ be a $C_0^\infty(\mathbb{R}^2)$, spherically symmetric function, such that $\chi(\mathbf{x}) = 0$ for all $|\mathbf{x}| \geq 1$ and $\|\chi\| = 1$.

For $n \geq 1$, define a sequence of functions:

$$\chi_n(\mathbf{x}) := \frac{\sqrt{2}}{n}\chi\left(\frac{\mathbf{x} - \mathbf{x}_n}{n}\right), \qquad \mathbf{x}_n = (3n, 0).$$

Note that for any $n \geq 1$, the support of $\chi_n(\mathbf{x})$ lies in an open disk, which we denote by $\mathcal{D}_n$, centered at $\mathbf{x}_n$ with radius $R = n$. In particular, $\chi_n(\mathbf{x})$ vanishes on the coincidence lines $y = \pm x/2$. Next, for $n \geq 1$, we define

$$\psi_n(\mathbf{x}) := e^{-i\mathbf{k}\cdot\mathbf{x}}\chi_n(\mathbf{x}) \qquad \mathbf{k} = (\varepsilon^{-1}\sqrt{\lambda/2}, \sqrt{\lambda/2}),$$

and

$$\psi_n^\natural := \frac{1(+)_\natural S}{2}\,\psi_n,$$

with the orthogonal projection $\frac{1(+)_\natural S}{2}$ defined in Eqs. (2.22) and (2.23).

Note that for any $n \geq 1$, the support of ψ_n lies in the open disk D_n, while the support of $S\psi_n$ lies in an open disk centered at $\mathbf{x}_n = (-3n, 0)$ with radius $R = n$. The supports of ψ_n and $S\psi_n$ are disjoint and both functions vanish on the coincidence lines $y = \pm x/2$. By construction, $\|\psi_n^\natural\| = 1$, furthermore, $\psi_n^\natural \in D(H_\varepsilon^\natural) \cap D(H_\varepsilon^0)$ and $H_\varepsilon^\natural\psi_n^\natural = H_\varepsilon^0\psi_n^\natural$, and we have [we use the notation $\nabla_\varepsilon\phi = (\varepsilon\partial_x\phi, \partial_y\phi)$]:

$$\|(H_\varepsilon^\natural - \lambda)\psi_n^\natural\| = \|(H_\varepsilon^0 - \lambda)\psi_n^\natural\|$$
$$\leq \|(H_\varepsilon^0 - \lambda)\psi_n\| = \left\|2(\nabla_\varepsilon\chi_n)\cdot(\nabla_\varepsilon e^{-i\mathbf{k}\cdot\mathbf{x}}) + (H_\varepsilon^0\chi_n)e^{-i\mathbf{k}\cdot\mathbf{x}}\right\|$$
$$\leq 2\sqrt{\lambda}\|\nabla_\varepsilon\chi_n\| + \|H_\varepsilon^0\chi_n\|$$
$$\leq c_\varepsilon\left(\frac{\sqrt{\lambda}}{n} + \frac{1}{n^2}\right).$$

In the last inequality, we used the fact that $\chi \in C_0^\infty(\mathbb{R}^2)$, so its gradient and Laplacian are bounded. Therefore,

$$\lim_{n\to\infty} \|(H_\varepsilon^\natural - \lambda)\psi_n^\natural\| = 0.$$

Finally, for any $\phi \in L^2_\natural(\mathbb{R}^2)$ one has, by the Cauchy–Schwarz inequality and by the dominated convergence theorem,

$$\left|\langle \phi, \psi_n^\natural\rangle\right|^2 = |\langle \phi, \psi_n\rangle|^2 \le \frac{2}{n^2}\,\mathrm{Area}(D_n)\int_{\mathbb{R}^2}\left|\phi(\mathbf{x})\chi\left(\frac{\mathbf{x}-\mathbf{x}_n}{n}\right)\right|^2 d\mathbf{x} \to 0.$$

Therefore $\psi_n \rightharpoonup 0$ and this concludes the proof. □

We conclude this section with a result that is a part of Theorem 1.1.

Theorem 2.19. *For all $\varepsilon > 0$*

$$\sigma_{ess}(H_\varepsilon^\natural) = \begin{cases} \sigma(H_\varepsilon^\natural) = [0,+\infty) & \alpha \ge 0 \\ \left[-\dfrac{\alpha^2}{4+\varepsilon^2}, +\infty\right) & \alpha < 0. \end{cases}$$

Proof. By Lemma 2.18, $[0,+\infty) \subseteq \sigma_{ess}(H_\varepsilon^\natural)$ for any $\alpha \in \mathbb{R}$.

If $\alpha \ge 0$ the quadratic form associated to $H_\varepsilon^\natural$ is positive definite, hence $\sigma(H_\varepsilon^\natural) \subseteq [0,+\infty) \subseteq \sigma_{ess}(H_\varepsilon^\natural) \subseteq \sigma(H_\varepsilon^\natural)$ which concludes the proof in the case $\alpha \ge 0$.

Let $\alpha < 0$ and $\lambda > 0$. Since $M_{od,\varepsilon}(-\lambda)$ is compact (see the proof of Proposition 2.6), by the Weyl criterion (see, e.g., Sec. XIII 4, Lemma 3, in Ref. 36), there follows

$$\sigma_{ess}(1 + 2\alpha M_\varepsilon(-\lambda)) = \sigma_{ess}(1 + \alpha\,(M_{d,\varepsilon}(-\lambda)(+)_\natural M_{od,\varepsilon}(-\lambda))) = \sigma_{ess}(1 + \alpha M_{d,\varepsilon}(-\lambda)).$$

By (2.13), $1 + \alpha M_{d,\varepsilon}(-\lambda)$ is unitarily equivalent to the multiplication operator $\widehat{M}_{\lambda,\varepsilon}$ corresponding to the function

$$f_{\lambda,\varepsilon}(s) = 1 + \frac{\alpha}{\sqrt{4\varepsilon^2 s^2 + (4+\varepsilon^2)\lambda}}.$$

Hence,

$$\sigma_{ess}(1 + 2\alpha M_\varepsilon(-\lambda)) = \sigma_{ess}(\widehat{M}_{\lambda,\varepsilon}) = \sigma(\widehat{M}_{\lambda,\varepsilon}) = \mathrm{range}(f_{\lambda,\varepsilon}) = \left[1 - \frac{|\alpha|}{\sqrt{(4+\varepsilon^2)\lambda}}, 1\right].$$

By Lemma 2.17 there follows that, given $\lambda > 0$, $-\lambda \in \sigma_{ess}(H_\varepsilon^\natural)$ if and only if $0 \in \left[1 - \frac{|\alpha|}{\sqrt{(4+\varepsilon^2)\lambda}}, 1\right]$. This gives $\sigma_{ess}(H_\varepsilon^\natural) = \left[-\frac{\alpha^2}{4+\varepsilon^2}, 0\right) \cup [0,+\infty)$. □

III. THE HAMILTONIAN FOR THE LIGHT PARTICLE

Following Sec. II 2 1 in Ref. 4 (see also Sec. 8.5 in Ref. 42), we introduce the self-adjoint Hamiltonian h_x modeling a delta-interaction of a 1D quantum particle with two centers placed at points $-x/2$ and $x/2$. The operator h_x depends on a real parameter α associated to the strength of the interaction; for notational simplicity, we prefer not to explicitly indicate such a dependency. One has

$$D(h_x) := \{u \in H^2(\mathbb{R}\backslash\{\pm x/2\}) \cap H^1(\mathbb{R}) : [u'](\pm x/2) = \alpha u(\pm x/2)\}, \tag{3.1}$$

$$h_x u(y) = -u''(y) \qquad \text{for a.e. } y \in \mathbb{R}\backslash\{\pm x/2\}, \tag{3.2}$$

where $[u'](y)$ denotes the jump of the derivative u' across y.

Furthermore, the sesquilinear form associated to h_x is given by

$$b_x : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathbb{C} \tag{3.3}$$

$$b_x(u,v) := \langle u', v'\rangle + \alpha\bar{u}(x/2)v(x/2) + \alpha\bar{u}(-x/2)v(-x/2). \tag{3.4}$$

A. The spectrum of h_x

For any $z \in \mathbb{C}\backslash[0,+\infty)$ we denote by $r^0(z)$ the resolvent of the self-adjoint Laplacian in $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$; its integral kernel is

$$r^0(y-y';z) = i\,\frac{e^{i\sqrt{z}\,|y-y'|}}{2\sqrt{z}} \qquad z \in \mathbb{C}\backslash[0,+\infty), \quad \operatorname{Im}\sqrt{z} > 0.$$

Moreover, for any $z \in \mathbb{C}\backslash[0,+\infty)$ we set:

$$\breve{g}_x(z) : L^2(\mathbb{R}) \to \mathbb{C}^2, \qquad \breve{g}_x(z)u := \big((r^0(z)u)(x/2), (r^0(z)u)(-x/2)\big);$$

$$g_x(z) : \mathbb{C}^2 \to L^2(\mathbb{R}), \qquad g_x(z) := \breve{g}_x(\bar{z})^*,$$

$$(g_x(z)(\zeta_1,\zeta_2))(y) = r^0(y-x/2;z)\zeta_1 + r^0(y+x/2;z)\zeta_2,$$

and

$$m_x(z) : \mathbb{C}^2 \to \mathbb{C}^2 \qquad m_x(z) := \frac{i}{2\sqrt{z}}\begin{bmatrix} 1 & e^{i\sqrt{z}\,|x|} \\ e^{i\sqrt{z}\,|x|} & 1 \end{bmatrix}.$$

For all $z \in \mathbb{C}\backslash[0,+\infty)$ and for all $\alpha \in \mathbb{R}$ such that $\det(1+\alpha\, m_x(z)) \neq 0$, the resolvent of h_x is given by

$$(h_x - z)^{-1} = r^0(z) - \alpha g_x(z)(1+\alpha m_x(z))^{-1}\breve{g}_x(z).$$

The values $z = -\lambda < 0$, such that $\det(1+\alpha m_x(z)) = 0$ equivalently, such that

$$\left(\alpha + 2\sqrt{\lambda}\right)^2 = \alpha^2 e^{-2\sqrt{\lambda}\,|x|}, \tag{3.5}$$

correspond to the eigenvalues of h_x. Obviously, (3.5) has no solution for $\alpha \geq 0$. For $\alpha < 0$, Eq. (3.5) admits one or two solutions according to the values of α and x. We summarize the spectral properties of h_x in the following

Lemma 3.1.
(i) $\sigma_{ac}(h_x) = \sigma_{ess}(h_x) = [0,+\infty)$;

(ii) $\sigma_{sc}(h_x) = \emptyset$;

(iii)

$$\sigma_p(h_x) = \sigma_d(h_x) = \begin{cases} \emptyset & \alpha \geq 0 \\ \{-\lambda_0(x)\} & \alpha < 0,\ |x| \leq 2/|\alpha| \\ \{-\lambda_0(x), -\lambda_1(x)\} & \alpha < 0,\ |x| > 2/|\alpha|. \end{cases}$$

To better describe the behavior of the eigenvalues with respect to x, we introduce the Lambert W-function, defined as the solution of the equation

$$W(x)e^{W(x)} = x.$$

As consequence of such a definition, one gets the following

Lemma 3.2. Let $\alpha < 0$. The functions λ_0 and λ_1 are *represented as*

$$\lambda_0 : \mathbb{R} \to (0,+\infty), \qquad \lambda_0(x) = \left(\frac{W\left(\frac{|\alpha||x|}{2}\, e^{-\frac{|\alpha||x|}{2}}\right)}{|x|} + \frac{|\alpha|}{2}\right)^2,$$

$$\lambda_1 : \mathbb{R}\backslash[-2/|\alpha|, 2/|\alpha|] \to (0,+\infty), \qquad \lambda_1(x) = \left(\frac{W\left(-\frac{|\alpha||x|}{2}\, e^{-\frac{|\alpha||x|}{2}}\right)}{|x|} + \frac{|\alpha|}{2}\right)^2;$$

(i) *both λ_0 and λ_1 are even functions;*
(ii) $\alpha^2/4 < \lambda_0(x) \leq \alpha^2$, $0 < \lambda_1(x) < \alpha^2/4$;
(iii) $\lambda_0(0) = \alpha^2$, $\lim_{x\to(\pm 2/|\alpha|)_\pm}\lambda_1(x) = 0$, $\lim_{x\to\pm\infty}\lambda_0(x) = \lim_{x\to\pm\infty}\lambda_1(x) = \alpha^2/4$;
(iv) *λ_0 is strictly decreasing on $(0,+\infty)$, λ_1 is strictly increasing on $(2/|\alpha|,+\infty)$;*

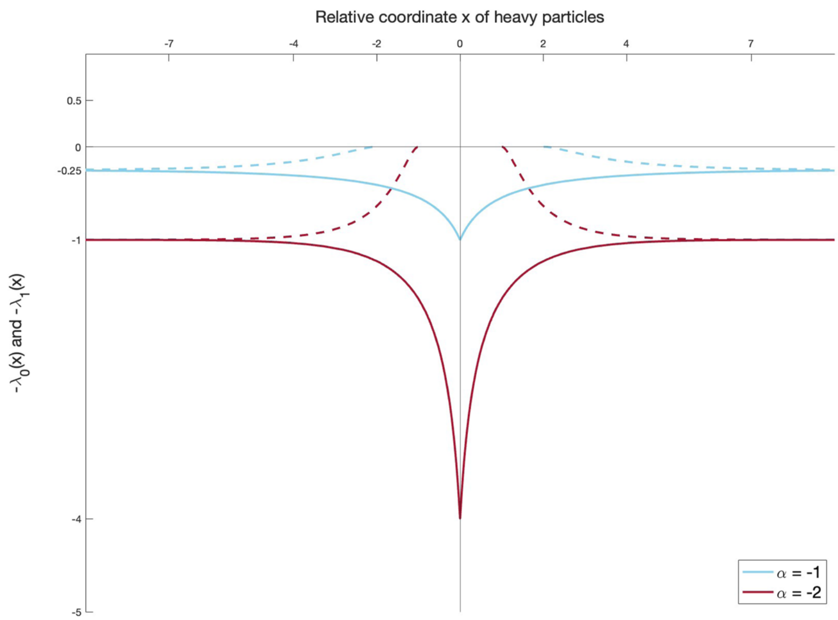

FIG. 1. Plot of the eigenvalues of the light-particle Hamiltonian as functions of x: $-\lambda_0$ is represented by the solid line; $-\lambda_1$ by the dashed line.

(v) $\lambda_0 \in C^\infty(\mathbb{R}\backslash\{0\})$, $\lambda_1 \in C^\infty(\mathbb{R}\backslash[-2/|\alpha|, 2/|\alpha|\,])$.

Remark 3.3. Even though $\lambda_1(x) = 0$ is a solution of (3.5), zero is not an eigenvalue of the Hamiltonian h_x. To prove that this is indeed the case, assume on the contrary that there is eigenfunction u_0 such that $h_x u_0 = 0$. This means that on $\mathbb{R}\backslash\{\pm\frac{x}{2}\}$, we have $-u_0'' = 0$. Any function in $D(h_x)$ that satisfies such an equation must be equal to zero on $(-\infty, \frac{x}{2}]$ and $[\frac{x}{2}, +\infty)$. Also, it has to be linear on $[-\frac{x}{2}, \frac{x}{2}]$, i.e., $u_0(x) = Ax + B$ for some constants A and B. By imposing the continuity and the boundary conditions required in $D(h_x)$ there follows that it must be $A = B = 0$.

By Lemma 3.2, there follows that $-\lambda_0$ reaches its minimum for $x = 0$,

$$\min_{x\in\mathbb{R}} (-\lambda_0(x)) = -\lambda_0(0) = -\alpha^2,$$

and $-\lambda_1$ reaches its infimum for $|x| \to +\infty$,

$$\lim_{|x|\to+\infty} -\lambda_1(x) = -\alpha^2/4.$$

We compare the eigenvalue $-\lambda_0(x)$ and $-\lambda_1(x)$ for the values $\alpha = -1, -2$ in Fig. 1.

By Lemmata 3.1 and 3.2 there follows the following theorem

Theorem 3.4. *For all $\alpha \geq 0$ and $x \in \mathbb{R}$, the quadratic form b_x and the associated self-adjoint operator h_x are non-negative. For all $\alpha < 0$ and $x \in \mathbb{R}$, $-\alpha^2$ is a lower bound for the quadratic form b_x and for the associated self-adjoint operator h_x.*

For later use, we introduce the normalized eigenfunction corresponding to the lowest eigenvalue of h_x (see, e.g., Refs. 4 and 2):

$$\psi_x^{BO}(y) := N(x)\Big(e^{-\sqrt{\lambda_0(x)}\,|x/2-y|} + e^{-\sqrt{\lambda_0(x)}\,|x/2+y|}\Big), \tag{3.6}$$

where $N(x)$ is the normalization constant given by

$$N(x) := \left(\frac{\sqrt{\lambda_0(x)}}{2\Big(1 + e^{-\sqrt{\lambda_0(x)}\,|x|}\Big(1 + \sqrt{\lambda_0(x)}\,|x|\Big)\Big)}\right)^{1/2}. \tag{3.7}$$

Further,

$$P_x\phi := \langle \psi_x^{BO}, \phi\rangle\, \psi_x^{BO} \tag{3.8}$$

denotes the corresponding spectral projection. We point out that $N(x)$ and ψ_x^{BO} are even in the parameter x.

IV. REDUCTION TO AN EFFECTIVE HAMILTONIAN FOR THE HEAVY PARTICLES SUBSYSTEM

We provide a dimensional reduction of $H_\varepsilon^\natural$ following the abstract scheme in Ref. 28. At first, we need to introduce the convenient notion of section of a function: given any $\phi \in L^2(\mathbb{R}^2)$, its x-section is defined by $\phi_x(y) := \phi(x,y)$. One has that, for a.e. $x \in \mathbb{R}$, ϕ_x is measurable and square integrable; furthermore, the function $x \mapsto \|\phi_x\|_{L^2(\mathbb{R})}$ is square integrable and

$$\|\phi\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}} \|\phi_x\|^2_{L^2(\mathbb{R})} dx.$$

Note that, by abuse of notation, we used the same symbol to denote an element in $L^2(\mathbb{R})$ and any of its representatives.

Similarly but less obviously (see, e.g., Sec. 5.6 in Ref. 29), for a.e. $x \in \mathbb{R}$, the x-section ϕ_x of any $\phi \in H^1(\mathbb{R}^2)$ is an absolutely continuous function having a square integrable derivative, the function $x \mapsto \|\phi_x\|_{H^1(\mathbb{R})}$ is square integrable and

$$\|\phi\|^2_{H^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\partial_x \phi(x,y)|^2 d\mathbf{x} + \int_{\mathbb{R}} \|\phi_x\|^2_{H^1(\mathbb{R})} dx.$$

Then, we consider the closed, lower bounded sesquilinear form b_x defined in (3.3) and (3.4) and the associated lower bounded self-adjoint operator h_x in $L^2(\mathbb{R})$ defined in (3.1) and (3.2).

Recalling the definition of $\mathcal{B}_\varepsilon^\natural$ in (2.26) and (2.27), by the discussion above, $b_x(\varphi_x, \psi_x)$ is well defined for any pair of functions $(\varphi, \psi) \in H^1_\natural(\mathbb{R}^2) \times H^1_\natural(\mathbb{R}^2)$ and

$$\mathcal{B}_\varepsilon^\natural(\varphi,\psi) = \int_{\mathbb{R}^2} \varepsilon^2 \partial_x \overline{\varphi}(x,y) \partial_x \psi(x,y) d\mathbf{x} + \int_{\mathbb{R}} b_x(\varphi_x, \psi_x) dx. \tag{4.1}$$

This identification, together with Theorem 3.4 gives the following result which completes Theorem 2.19.

Proposition 4.1. For any $\varepsilon > 0$ and $\alpha < 0$, $\sigma(H_\varepsilon^{b/f}) \subset [-\alpha^2, +\infty)$.

To proceed, we define the orthogonal projection $\mathcal{P}$ in $L^2(\mathbb{R}^2)$ associated to the projection P_x defined in Eq. (3.8):

$$\mathcal{P} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \qquad (\mathcal{P}\phi)_x := P_x \phi_x.$$

Additionally we set

$$\mathcal{P}^\perp := 1 - \mathcal{P}.$$

For later convenience, sometimes we regard ψ_x^{BO} as a function of two variables, denoted by ψ^{BO}, with the obvious identification $\psi^{BO}(x,y) \equiv \psi_x^{BO}(y)$. With this notation, one has

$$\mathcal{P} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \qquad \mathcal{P}\phi(x,y) := \psi^{BO}(x,y) f_\phi(x),$$

where

$$f_\phi(x) := \int_{\mathbb{R}} \psi^{BO}(x,y)\phi(x,y)\, dy = \langle \psi_x^{BO}, \phi_x\rangle.$$

Let us point out the inequality

$$\|f_\phi\|^2_{L^2(\mathbb{R})} \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\psi^{BO}(x,y)|\, |\phi(x,y)|\, dy \right)^2 dx \le \int_{\mathbb{R}} \|\psi_x^{BO}\|^2_{L^2(\mathbb{R})} \|\phi_x\|^2_{L^2(\mathbb{R})}\, dx = \|\phi\|^2_{L^2(\mathbb{R}^2)}.$$

Remark 4.2. Notice that, since ψ^{BO} is an even function with respect to the x variable, f_ϕ is an even function whenever $\phi \in L^2_{\mathrm{b}}(\mathbb{R}^2)$ and is an odd function whenever $\phi \in L^2_{\mathrm{f}}(\mathbb{R}^2)$. Hence, $\mathcal{P}$ is an orthogonal projector in $L^2_\natural(\mathbb{R}^2)$ as well.

Lemma 4.3. (i) $\mathcal{P}$ is a bounded operator in $H^1(\mathbb{R}^2)$; (ii) the commutator $[\partial_x, \mathcal{P}] : H^1(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ extends to a bounded operator in $L^2(\mathbb{R}^2)$ and

$$\|[\partial_x, \mathcal{P}]\| \le \delta,$$

with

$$\delta := 2\left(\sup_{x\in\mathbb{R}} \int |\partial_x \psi^{BO}(x,y)|^2 \mathrm{d}y\right)^{1/2}$$

and the bound $\delta \le 4|\alpha|$.

Proof. In the course of the proof, for the sake of brevity, we will often omit the dependence on x in $N(x)$ and $\lambda_0(x)$. From the formula (3.6), one gets

$$\partial_y \psi^{BO}(x,y) = N\sqrt{\lambda_0}\left(\mathrm{sgn}(x/2-y)e^{-\sqrt{\lambda_0}\,|x/2-y|} - \mathrm{sgn}\,(x/2+y)e^{-\sqrt{\lambda_0}\,|x/2+y|}\right),$$

and so

$$\begin{aligned}\int_{\mathbb{R}} \left|\partial_y \psi^{BO}(x,y)\right|^2 \mathrm{d}y &= 2N^2\sqrt{\lambda_0}\left(1 + e^{-\sqrt{\lambda_0}\,|x|} - \sqrt{\lambda_0}\,|x|e^{-\sqrt{\lambda_0}\,|x|}\right)\\ &= \lambda_0\,\frac{1 + e^{-\sqrt{\lambda_0}\,|x|} - \sqrt{\lambda_0}\,|x|e^{-\sqrt{\lambda_0}\,|x|}}{1 + e^{-\sqrt{\lambda_0}\,|x|} + \sqrt{\lambda_0}\,|x|e^{-\sqrt{\lambda_0}\,|x|}}\\ &\le \lambda_0 \le \alpha^2.\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^2} |\partial_y \mathcal{P}\phi(x,y)|^2 \mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^2} \left|\partial_y \psi^{BO}(x,y)\right|^2 |f_\phi(x)|^2 \mathrm{d}x\,\mathrm{d}y\\ &\le \left(\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} \left|\partial_y \psi^{BO}(x,y)\right|^2 dy\right) \|f_\phi\|^2_{L^2(\mathbb{R})}\\ &\le \alpha^2 \|\phi\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Defining

$$\tilde f_\phi(x) := \int_{\mathbb{R}} \left(\partial_x \psi^{BO}(x,y)\right)\phi(x,y)\mathrm{d}y,$$

one has

$$\|\tilde f_\phi\| \le \frac{\delta}{2}\|\phi\|.$$

Since

$$\partial_x \mathcal{P}\phi = \left(\partial_x \psi^{BO}\right) f_\phi + \psi^{BO}\tilde f_\phi + \psi^{BO} f_{\partial_x \phi},$$

there holds

$$\|\partial_x \mathcal{P}\phi\| \le \left\|\left(\partial_x \psi^{BO}\right) f_\phi\right\| + \|\psi^{BO}\tilde f_\phi\| + \|\psi^{BO} f_{\partial_x\phi}\| \le \frac{\delta}{2}\|f_\phi\| + \|\tilde f_\phi\| + \|\partial_x\phi\| \le (1+\delta)\|\phi\|_{H^1(\mathbb{R}^2)}.$$

In a similar way, one obtains

$$[\partial_x, \mathcal{P}]\phi = \left(\partial_x \psi^{BO}\right) f_\phi + \psi^{BO}\tilde f_\phi,$$

and

$$\|[\partial_x, \mathcal{P}]\phi\| \le \delta\|\phi\|.$$

We are left to prove the bound on δ. From the formula (3.6), one can explicitly compute $\partial_x \psi^{BO}$:

$$\begin{aligned}\partial_x \psi^{BO}(x,y) =& \underbrace{\frac{N'}{N}\psi^{BO}}_{\varphi_1^{BO}} - \underbrace{\frac{N\lambda_0'}{2\sqrt{\lambda_0}}\left(|x/2-y|e^{-\sqrt{\lambda_0}|x/2-y|} + |x/2+y|e^{-\sqrt{\lambda_0}|x/2+y|}\right)}_{\varphi_2^{BO}}\\ &- \underbrace{\frac{N\sqrt{\lambda_0}}{2}\left(\mathrm{sgn}\,(x/2-y)e^{-\sqrt{\lambda_0}|x/2-y|} + \mathrm{sgn}\,(x/2+y)e^{-\sqrt{\lambda_0}|x/2+y|}\right)}_{\varphi_3^{BO}}.\end{aligned}$$

Computing the norm of φ_1^{BO} one obtains

$$\|\varphi_1^{BO}\|^2 = \left(\frac{N'}{N}\right)^2.$$

From Eq. (3.7), by a straightforward calculation there follows

$$\frac{N'}{N} = \frac{(\sqrt{\lambda_0})'}{2\sqrt{\lambda_0}} + \frac{\frac{(\sqrt{\lambda_0})'}{\sqrt{\lambda_0}}\left(\lambda_0 x^2 + \frac{\lambda_0^{3/2}}{(\sqrt{\lambda_0})'}x\right)}{2\left(\sqrt{\lambda_0}|x| + e^{\sqrt{\lambda_0}|x|} + 1\right)}.$$

We notice that

$$\lambda_0(x) = \frac{\alpha^2}{4}v^2(|\alpha|x/2) \qquad x \ge 0$$

with

$$v(x) = \frac{W(xe^{-x})}{x} + 1.$$

Hence,

$$\left|\frac{(\sqrt{\lambda_0})'}{2\sqrt{\lambda_0}}\right| = \frac{|\alpha|}{4}\left|\frac{v'(ax/2)}{v(ax/2)}\right|.$$

From the identity $W(x)e^{W(x)} = x$, $x \ge 0$, we infer

$$v(x) = e^{-xv(x)} + 1.$$

Hence,

$$\frac{v'(x)}{v(x)} = -\frac{1}{e^{xv(x)} + x}$$

and $|v'(x)/v(x)| \le \frac{1}{e^x + x} \le 1$. We deduce the bound:

$$\left|\frac{(\sqrt{\lambda_0})'}{2\sqrt{\lambda_0}}\right| \le \frac{|\alpha|}{4}. \tag{4.2}$$

Recalling also that $\sqrt{\lambda_0} \le |\alpha|$, we infer

$$\left|\frac{\frac{(\sqrt{\lambda_0})'}{\sqrt{\lambda_0}}\left(\lambda_0 x^2 + \frac{\lambda_0^{3/2}}{(\sqrt{\lambda_0})'}x\right)}{2\left(\sqrt{\lambda_0}|x| + e^{\sqrt{\lambda_0}|x|} + 1\right)}\right| \le \frac{\frac{|\alpha|}{2}\lambda_0 x^2 + |\alpha|\sqrt{\lambda_0}|x|}{2\left(\sqrt{\lambda_0}|x| + e^{\sqrt{\lambda_0}|x|} + 1\right)} \le \frac{|\alpha|}{4},$$

where we used the inequality $\frac{\frac{1}{2}s^2 + s}{2(s + e^s + 1)} \le \frac{1}{4}$, $s \ge 0$. We deduce that

$$\left|\frac{N'}{N}\right| \le \frac{|\alpha|}{2} \qquad \text{and} \qquad \|\varphi_1^{BO}\|^2_{L^2(\mathbb{R},dy)} = \left(\frac{N'}{N}\right)^2 \le \frac{\alpha^2}{4}.$$

Next we focus attention on φ_2^{BO}.

$$\begin{aligned}\|\varphi_2^{BO}\|^2 &= \frac{(N\lambda_0')^2}{4(\lambda_0)^{5/2}}\left(1 + \frac{(\sqrt{\lambda_0}|x|)^3}{3}e^{-\sqrt{\lambda_0}|x|} + (1 + \sqrt{\lambda_0}|x|)e^{-\sqrt{\lambda_0}|x|}\right)\\ &= \frac{(\lambda_0')^2}{2(\lambda_0)^2}\,\frac{1 + \frac{(\sqrt{\lambda_0}|x|)^3}{3}e^{-\sqrt{\lambda_0}|x|} + (1 + \sqrt{\lambda_0}|x|)e^{-\sqrt{\lambda_0}|x|}}{1 + e^{-\sqrt{\lambda_0}|x|} + \sqrt{\lambda_0}|x|e^{-\sqrt{\lambda_0}|x|}} \le \frac{(\lambda_0')^2}{(\lambda_0)^2} \le \alpha^2,\end{aligned}$$

where in the latter step we used the bound $|\lambda_0'/\lambda_0| \le |\alpha|$ that can be easily obtained from the bound (4.2).

We conclude with a bound on $\|\varphi_3^{BO}\|$.

$$\|\varphi_3^{BO}\|^2 = \frac{N^2\sqrt{\lambda_0}}{2}\left(1 - e^{-\sqrt{\lambda_0}|x|} + \sqrt{\lambda_0}|x|e^{-\sqrt{\lambda_0}|x|}\right)$$
$$= \frac{\lambda_0}{4}\frac{1 - e^{-\sqrt{\lambda_0}|x|} + \sqrt{\lambda_0}|x|e^{-\sqrt{\lambda_0}|x|}}{1 + e^{-\sqrt{\lambda_0}|x|} + \sqrt{\lambda_0}|x|e^{-\sqrt{\lambda_0}|x|}} \le \frac{\lambda_0}{4} \le \frac{\alpha^2}{4}.$$

Summing up we obtain $\|\partial_x\psi^{BO}\| \le \sum_{j=1}^3 \|\varphi_j^{BO}\| \le 2|\alpha|$ and $\delta \le 4|\alpha|$. □

Remark 4.4. By Remark 4.2 and by point (i) in Lemma 4.3, $\mathcal{P}$ is a bounded operator in $H^1_\natural(\mathbb{R}^2)$.

By Proposition 4.1 there follows that $-\alpha^2$ is a lower bound for the quadratic form $\mathcal{B}_\varepsilon^\natural$ and for the associated self-adjoint operator $H_\varepsilon^\natural$. Preferring to work with non negative quadratic forms and operators, we define the sesquilinear form

$$q_x(u,v) := b_x(u,v) + \alpha^2$$

with corresponding self-adjoint, non-negative operator $h_x + \alpha^2$. Then, we define the sesquilinear form $\mathcal{Q}_\varepsilon^\natural$ with $D(\mathcal{Q}_\varepsilon^\natural) = H^1_\natural(\mathbb{R}^2)$ by

$$\mathcal{Q}_\varepsilon^\natural(\varphi,\psi) := \mathcal{B}_\varepsilon^\natural(\varphi,\psi) + \alpha^2(\varphi,\psi)_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2}\varepsilon^2\partial_x\overline{\varphi}(x,y)\partial_x\psi(x,y)\mathrm{d}\mathbf{x} + \int_{\mathbb{R}} q_x(\varphi_x,\psi_x)\mathrm{d}x. \tag{4.3}$$

The corresponding non-negative self-adjoint operator is

$$\mathscr{L}_\varepsilon^\natural := H_\varepsilon^\natural + \alpha^2.$$

By Remarks 4.2 and 4.4, the following sesquilinear form in $L^2_\natural(\mathbb{R}^2)$ is well defined:

$$D(\widehat{\mathcal{Q}}_\varepsilon^\natural) = H^1_\natural(\mathbb{R}^2) \qquad \widehat{\mathcal{Q}}_\varepsilon^\natural(\phi,\psi) := \mathcal{Q}_\varepsilon^\natural(\mathcal{P}\phi,\mathcal{P}\psi) + \mathcal{Q}_\varepsilon^\natural(\mathcal{P}^\perp\phi,\mathcal{P}^\perp\psi).$$

Proposition 4.5. For all $\phi,\psi \in H^1_\natural(\mathbb{R}^2)$ there holds

$$\left|\mathcal{Q}_\varepsilon^\natural(\phi,\psi) - \widehat{\mathcal{Q}}_\varepsilon^\natural(\phi,\psi)\right| \le 2\varepsilon\delta\left(\sqrt{\mathcal{Q}_\varepsilon^\natural(\phi)}\|\psi\| + \|\phi\|\sqrt{\widehat{\mathcal{Q}}_\varepsilon^\natural(\psi)}\right).$$

Proof. By the definition of $\widehat{\mathcal{Q}}_\varepsilon^\natural$ there follows

$$\mathcal{Q}_\varepsilon^\natural(\phi,\psi) - \widehat{\mathcal{Q}}_\varepsilon^\natural(\phi,\psi) = \mathcal{Q}_\varepsilon^\natural(\mathcal{P}\phi,\mathcal{P}^\perp\psi) + \mathcal{Q}_\varepsilon^\natural(\mathcal{P}^\perp\phi,\mathcal{P}\psi).$$

Defining $S_\varepsilon\phi := -i\varepsilon\partial_x$, since $q_x(\mathcal{P}_x u, \mathcal{P}_x^\perp u) = q_x(\mathcal{P}_x^\perp u, \mathcal{P}_x u) = 0$, by Definition 4.3, there follows

$$\begin{aligned}\mathcal{Q}_\varepsilon^\natural(\phi,\psi) - \widehat{\mathcal{Q}}_\varepsilon^\natural(\phi,\psi) =& \langle S_\varepsilon\mathcal{P}\phi, S_\varepsilon\mathcal{P}^\perp\psi\rangle + \langle S_\varepsilon\mathcal{P}^\perp\phi, S_\varepsilon\mathcal{P}\psi\rangle\\ =& \langle S_\varepsilon\mathcal{P}\phi, S_\varepsilon(1-\mathcal{P})\psi\rangle + \langle S_\varepsilon(1-\mathcal{P})\phi, S_\varepsilon\mathcal{P}\psi\rangle\\ =& \langle S_\varepsilon\mathcal{P}\phi, S_\varepsilon\psi\rangle + \langle S_\varepsilon\phi, S_\varepsilon\mathcal{P}\psi\rangle - 2\langle S_\varepsilon\mathcal{P}\phi, S_\varepsilon\mathcal{P}\psi\rangle\\ =& \langle[S_\varepsilon,\mathcal{P}]\phi, S_\varepsilon\psi\rangle + \langle\mathcal{P}S_\varepsilon\phi, S_\varepsilon\psi\rangle\\ &+ \langle S_\varepsilon\phi, S_\varepsilon\mathcal{P}\psi\rangle - 2\langle[S_\varepsilon,\mathcal{P}]\phi, S_\varepsilon\mathcal{P}\psi\rangle - 2\langle\mathcal{P}S_\varepsilon\phi, S_\varepsilon\mathcal{P}\psi\rangle.\end{aligned}$$

Note that

$$\begin{aligned}-2\langle\mathcal{P}S_\varepsilon\phi, S_\varepsilon\mathcal{P}\psi\rangle =& -2\langle S_\varepsilon\phi, \mathcal{P}S_\varepsilon\mathcal{P}\psi\rangle\\ =& -2\langle S_\varepsilon\phi, [\mathcal{P},S_\varepsilon]\mathcal{P}\psi\rangle - 2\langle S_\varepsilon\phi, S_\varepsilon\mathcal{P}^2\psi\rangle\\ =& 2\langle S_\varepsilon\phi, [S_\varepsilon,\mathcal{P}]\mathcal{P}\psi\rangle - 2\langle S_\varepsilon\phi, S_\varepsilon\mathcal{P}\psi\rangle,\end{aligned}$$

and

$$\langle\mathcal{P}S_\varepsilon\phi, S_\varepsilon\psi\rangle = \langle S_\varepsilon\phi, \mathcal{P}S_\varepsilon\psi\rangle = -\langle S_\varepsilon\phi, [S_\varepsilon,\mathcal{P}]\psi\rangle + \langle S_\varepsilon\phi, S_\varepsilon\mathcal{P}\psi\rangle.$$

Hence,

$$\begin{aligned}\mathcal{Q}_\varepsilon^\natural(\phi,\psi)-\widehat{\mathcal{Q}}_\varepsilon^\natural(\phi,\psi)=&\langle[S_\varepsilon,\mathcal{P}]\phi,S_\varepsilon\psi\rangle+(-\langle S_\varepsilon\phi,[S_\varepsilon,\mathcal{P}]\psi\rangle+\langle S_\varepsilon\phi,S_\varepsilon\mathcal{P}\psi\rangle)\\&+\langle S_\varepsilon\phi,S_\varepsilon\mathcal{P}\psi\rangle-2\langle[S_\varepsilon,\mathcal{P}]\phi,S_\varepsilon\mathcal{P}\psi\rangle+(2\langle S_\varepsilon\phi,[S_\varepsilon,\mathcal{P}]\mathcal{P}\psi\rangle-2\langle S_\varepsilon\phi,S_\varepsilon\mathcal{P}\psi\rangle)\\=&\langle[S_\varepsilon,\mathcal{P}]\phi,S_\varepsilon\psi\rangle-\langle S_\varepsilon\phi,[S_\varepsilon,\mathcal{P}]\psi\rangle-2\langle[S_\varepsilon,\mathcal{P}]\phi,S_\varepsilon\mathcal{P}\psi\rangle+2\langle S_\varepsilon\phi,[S_\varepsilon,\mathcal{P}]\mathcal{P}\psi\rangle\\=&\langle[S_\varepsilon,\mathcal{P}]\phi,S_\varepsilon\mathcal{P}^\perp\psi\rangle-\langle S_\varepsilon\phi,[S_\varepsilon,\mathcal{P}]\mathcal{P}^\perp\psi\rangle-\langle[S_\varepsilon,\mathcal{P}]\phi,S_\varepsilon\mathcal{P}\psi\rangle+\langle S_\varepsilon\phi,[S_\varepsilon,\mathcal{P}]\mathcal{P}\psi\rangle.\end{aligned}$$

By Lemma 4.3, this gives

$$\begin{aligned}&\left|\mathcal{Q}_\varepsilon^\natural(\phi,\psi)-\widehat{\mathcal{Q}}_\varepsilon^\natural(\phi,\psi)\right|\\&\le\varepsilon\delta\big(\|\varepsilon\partial_x\phi\|\big(\|\mathcal{P}\psi\|+\|\mathcal{P}^\perp\psi\|\big)+\|\phi\|\big(\|\varepsilon\partial_x\mathcal{P}\psi\|+\|\varepsilon\partial_x\mathcal{P}^\perp\psi\|\big)\big).\end{aligned}$$

By the obvious inequality $a+b\le 2\sqrt{a^2+b^2}$, there follows

$$\begin{aligned}&\left|\mathcal{Q}_\varepsilon^\natural(\phi,\psi)-\widehat{\mathcal{Q}}_\varepsilon^\natural(\phi,\psi)\right|\\&\le 2\varepsilon\delta\left(\|\varepsilon\partial_x\phi\|\sqrt{\|\mathcal{P}\psi\|^2+\|\mathcal{P}^\perp\psi\|^2}+\|\phi\|\sqrt{\|\varepsilon\partial_x\mathcal{P}\psi\|^2+\|\varepsilon\partial_x\mathcal{P}^\perp\psi\|^2}\right)\\&\le 2\varepsilon\delta\left(\sqrt{\mathcal{Q}_\varepsilon^\natural(\phi)}\|\psi\|+\|\phi\|\sqrt{\widehat{\mathcal{Q}}_\varepsilon^\natural(\psi)}\right),\end{aligned}$$

where in the latter inequality we used the fact that the quadratic form q_x is non-negative. □

In the next Lemma we recall several results from Ref. 28.

Lemma 4.6 (Lemmata 2.2 and 2.3 in Ref. 28). The quadratic form $\widehat{\mathcal{Q}}_\varepsilon^\natural$ *is closed, non-negative, and determines a unique self-adjoint, non-negative operator* $\widehat{\mathscr{L}}_\varepsilon^\natural$ *in* $L^2_\natural(\mathbb{R}^2)$. *Furthermore,*

(i)

$$\widehat{\mathscr{L}}_\varepsilon^\natural=\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}}^\natural\oplus\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}^\perp}^\natural,\qquad\sigma(\widehat{\mathscr{L}}_\varepsilon^\natural)=\sigma(\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}}^\natural)\cup\sigma(\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}^\perp}^\natural),$$

where

$$\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}}^\natural:D(\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}}^\natural)\subset\operatorname{ran}(\mathcal{P}|L^2_\natural(\mathbb{R}^2))\to\operatorname{ran}(\mathcal{P}|L^2_\natural(\mathbb{R}^2)),$$

$$\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}^\perp}^\natural:D(\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}^\perp}^\natural)\subset\operatorname{ran}(\mathcal{P}^\perp|L^2_\natural(\mathbb{R}^2))\to\operatorname{ran}(\mathcal{P}^\perp|L^2_\natural(\mathbb{R}^2));$$

(ii)

$$\langle\psi,\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}}^\natural\psi\rangle=\mathcal{Q}_\varepsilon^\natural(\mathcal{P}\psi,\mathcal{P}\psi)\ge 0,\qquad\forall\psi\in D(\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}}^\natural),$$

$$\langle\psi,\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}^\perp}^\natural\psi\rangle=\mathcal{Q}_\varepsilon^\natural(\mathcal{P}^\perp\psi,\mathcal{P}^\perp\psi)\ge\inf_{x\in\mathbb{R}}(-\lambda_1(x))+\alpha^2=\frac{3}{4}\alpha^2,\qquad\forall\psi\in D(\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}^\perp}^\natural).$$

Proposition 4.7. For all $\lambda\in\mathbb{R}$, $\phi\in D(\mathscr{L}_\varepsilon^\natural)$, *and* $\psi\in D(\widehat{\mathscr{L}}_\varepsilon^\natural)$, *there holds*

$$\begin{aligned}&\left|\langle\phi,(\widehat{\mathscr{L}}_\varepsilon^\natural-\lambda)\psi\rangle-\langle(\mathscr{L}_\varepsilon^\natural-\lambda)\phi,\psi\rangle\right|\\&\le 2\varepsilon\delta\left(\|\psi\|\sqrt{\big(\|(\mathscr{L}_\varepsilon^\natural-\lambda)\phi\|+\lambda_+\|\phi\|\big)\|\phi\|}+\|\phi\|\sqrt{\big(\|(\widehat{\mathscr{L}}_\varepsilon^\natural-\lambda)\psi\|+\lambda_+\|\psi\|\big)\|\psi\|}\right).\end{aligned}$$

Proof. For all $\phi\in D(\mathscr{L}_\varepsilon^\natural)$, there holds

$$\mathcal{Q}_\varepsilon^\natural(\phi)=\langle\phi,(\mathscr{L}_\varepsilon^\natural-\lambda)\phi\rangle+\lambda\|\phi\|^2.$$

Hence,

$$\mathcal{Q}_\varepsilon^\natural(\phi)\le\big(\|(\mathscr{L}_\varepsilon^\natural-\lambda)\phi\|+\lambda_+\|\phi\|\big)\|\phi\|.$$

Similarly, for all $\psi\in D(\widehat{\mathscr{L}}_\varepsilon^\natural)$,

$$\widehat{\mathcal{Q}}_\varepsilon^\natural(\psi)\le\big(\|(\widehat{\mathscr{L}}_\varepsilon^\natural-\lambda)\psi\|+\lambda_+\|\psi\|\big)\|\psi\|.$$

For all $\phi \in D(\mathscr{L}_\varepsilon^\natural)$ and $\psi \in D(\widehat{\mathscr{L}}_\varepsilon^\natural)$, there holds

$$\langle \phi, (\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)\psi \rangle - \langle (\mathscr{L}_\varepsilon^\natural - \lambda)\phi, \psi \rangle = \langle \phi, \widehat{\mathscr{L}}_\varepsilon^\natural \psi \rangle - \langle \mathscr{L}_\varepsilon^\natural \phi, \psi \rangle = \widehat{\mathcal{Q}}_\varepsilon^\natural(\phi, \psi) - \mathcal{Q}_\varepsilon^\natural(\phi, \psi).$$

Hence,

$$\begin{aligned}&\left| \langle \phi, (\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)\psi \rangle - \langle (\mathscr{L}_\varepsilon^\natural - \lambda)\phi, \psi \rangle \right| \\ &\le 2\varepsilon\delta \left(\|\psi\| \sqrt{\left(\|(\mathscr{L}_\varepsilon^\natural - \lambda)\phi\| + \lambda_+ \|\phi\| \right) \|\phi\|} + \|\phi\| \sqrt{\left(\|(\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)\psi\| + \lambda_+ \|\psi\| \right) \|\psi\|} \right).\end{aligned}$$

□

Lemma 4.8. (i) Defining

$$d_\varepsilon \equiv d_\varepsilon(\lambda) := \operatorname{dist}(\lambda, \sigma(\mathscr{L}_\varepsilon^\natural)), \qquad \textit{and} \qquad \widehat{d}_\varepsilon \equiv \widehat{d}_\varepsilon(\lambda) := \operatorname{dist}(\lambda, \sigma(\widehat{\mathscr{L}}_\varepsilon^\natural)),$$

one has

$$\left\{ \lambda \in [0, +\infty) \cap \rho(\widehat{\mathscr{L}}_\varepsilon^\natural) : \frac{4\varepsilon\delta}{\sqrt{\widehat{d}_\varepsilon(\lambda)}} \sqrt{1 + \frac{\lambda}{\widehat{d}_\varepsilon(\lambda)}} < \frac{1}{2} \right\} \subseteq \left\{ \lambda \in \rho(\mathscr{L}_\varepsilon^\natural) : d_\varepsilon(\lambda) \ge \frac{\widehat{d}_\varepsilon(\lambda)}{32} \right\}.$$

(ii) For any $\lambda \in \rho(\mathscr{L}_\varepsilon^\natural) \cap \rho(\widehat{\mathscr{L}}_\varepsilon^\natural)$, one has

$$\|(\mathscr{L}_\varepsilon^\natural - \lambda)^{-1} - (\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\| \le 2\varepsilon\delta \left(\frac{1}{\widehat{d}_\varepsilon} \sqrt{\frac{1}{d_\varepsilon}\left(1 + \frac{\lambda_+}{d_\varepsilon}\right)} + \frac{1}{d_\varepsilon} \sqrt{\frac{1}{\widehat{d}_\varepsilon}\left(1 + \frac{\lambda_+}{\widehat{d}_\varepsilon}\right)} \right).$$

Proof. In Proposition 4.7 set $\psi = (\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\phi$, then

$$\begin{aligned}&\left| \|\phi\|^2 - \langle (\mathscr{L}_\varepsilon^\natural - \lambda)\phi, (\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\phi \rangle \right| \\ &\le 2\varepsilon\delta \Big(\|(\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\phi\| \sqrt{\left(\|(\mathscr{L}_\varepsilon^\natural - \lambda)\phi\| + \lambda_+\|\phi\| \right)\|\phi\|} \\ &\qquad + \|\phi\| \sqrt{\left(\|\phi\| + \lambda_+ \|(\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\phi\| \right) \|(\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\phi\|} \Big) \\ &\le 2\varepsilon\delta\|\phi\| \left(\frac{1}{\widehat{d}_\varepsilon} \sqrt{\left(\|(\mathscr{L}_\varepsilon^\natural - \lambda)\phi\| + \lambda_+\|\phi\| \right)\|\phi\|} + \|\phi\| \sqrt{\frac{1}{\widehat{d}_\varepsilon}\left(1 + \frac{\lambda_+}{\widehat{d}_\varepsilon}\right)} \right) \\ &\le 2\varepsilon\delta\|\phi\| \left(\frac{1}{\widehat{d}_\varepsilon} \sqrt{\|(\mathscr{L}_\varepsilon^\natural - \lambda)\phi\|\|\phi\|} + \frac{\|\phi\|}{\widehat{d}_\varepsilon}\sqrt{\lambda_+} + \|\phi\| \sqrt{\frac{1}{\widehat{d}_\varepsilon}\left(1 + \frac{\lambda_+}{\widehat{d}_\varepsilon}\right)} \right) \\ &\le 2\varepsilon\delta\|\phi\| \left(\frac{1}{\widehat{d}_\varepsilon} \sqrt{\|(\mathscr{L}_\varepsilon^\natural - \lambda)\phi\|\|\phi\|} + 2\|\phi\| \sqrt{\frac{1}{\widehat{d}_\varepsilon}\left(1 + \frac{\lambda_+}{\widehat{d}_\varepsilon}\right)} \right).\end{aligned}$$

The latter inequality gives

$$\|\phi\|^2 \le \left| \langle (\mathscr{L}_\varepsilon^\natural - \lambda)\phi, (\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\phi \rangle \right| + 2\varepsilon\delta\|\phi\| \left(\frac{1}{\widehat{d}_\varepsilon} \sqrt{\|(\mathscr{L}_\varepsilon^\natural - \lambda)\phi\|\|\phi\|} + 2\|\phi\| \sqrt{\frac{1}{\widehat{d}_\varepsilon}\left(1 + \frac{\lambda_+}{\widehat{d}_\varepsilon}\right)} \right).$$

From which there follows,

$$\|\phi\| \le \frac{1}{\widehat{d}_\varepsilon}\|(\mathscr{L}_\varepsilon^\natural - \lambda)\phi\| + 2\varepsilon\delta \left(\frac{1}{\widehat{d}_\varepsilon} \sqrt{\|(\mathscr{L}_\varepsilon^\natural - \lambda)\phi\|\|\phi\|} + 2\|\phi\| \sqrt{\frac{1}{\widehat{d}_\varepsilon}\left(1 + \frac{\lambda_+}{\widehat{d}_\varepsilon}\right)} \right).$$

Defining

$$s := \sqrt{\frac{\|(\mathscr{L}_\varepsilon^\natural - \lambda)\phi\|}{\widehat{d}_\varepsilon \|\phi\|}},$$

one has

$$\left(s + \frac{\varepsilon\delta}{\sqrt{\widehat{d}_\varepsilon}}\right)^2 - \frac{\varepsilon^2\delta^2}{\widehat{d}_\varepsilon} \geq 1 - \frac{4\varepsilon\delta}{\sqrt{\widehat{d}_\varepsilon}}\sqrt{1 + \frac{\lambda_+}{\widehat{d}_\varepsilon}}.$$

If

$$\frac{4\varepsilon\delta}{\sqrt{\widehat{d}_\varepsilon}}\sqrt{1 + \frac{\lambda_+}{\widehat{d}_\varepsilon}} < \frac{1}{2},$$

then the following inequality must be satisfied

$$s \geq -\frac{\varepsilon\delta}{\sqrt{\widehat{d}_\varepsilon}} + \sqrt{\frac{\varepsilon^2\delta^2}{\widehat{d}_\varepsilon} + \frac{1}{2}} \geq \frac{1}{4}\frac{1}{\sqrt{1 + \frac{\varepsilon^2\delta^2}{\widehat{d}_\varepsilon}}}.$$

Hence,

$$\|(\mathscr{L}_\varepsilon^\natural - \lambda)\phi\| \geq \frac{\widehat{d}_\varepsilon}{16}\frac{1}{1 + \frac{\varepsilon^2\delta^2}{\widehat{d}_\varepsilon}}\|\phi\| \geq \frac{\widehat{d}_\varepsilon}{32}\|\phi\| > 0$$

which implies $\lambda \in \rho(\mathscr{L}_\varepsilon^\natural)$ and the bound on d_ε. We remark that if $\lambda < 0$, the statement is trivial since $(-\infty, 0) \subset \rho(\mathscr{L}_\varepsilon^\natural)$.

To prove the bound on the resolvents difference, let $\tilde{\chi}, \chi \in L^2_\natural(\mathbb{R}^2)$,

$$\phi = (\mathscr{L}_\varepsilon^\natural - \lambda)^{-1}\tilde{\chi} \qquad \text{and} \qquad \psi = (\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\chi,$$

and apply Proposition 4.7 to obtain

$$\begin{aligned}
&\left|\left\langle \tilde{\chi}, \left((\mathscr{L}_\varepsilon^\natural - \lambda)^{-1} - (\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\right)\chi\right\rangle\right| \\
&\leq 2\varepsilon\delta\Bigg(\|(\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\chi\|\sqrt{\left(\|\tilde{\chi}\| + \lambda_+\|(\mathscr{L}_\varepsilon^\natural - \lambda)^{-1}\tilde{\chi}\|\right)\|(\mathscr{L}_\varepsilon^\natural - \lambda)^{-1}\tilde{\chi}\|} \\
&\qquad + \|(\mathscr{L}_\varepsilon^\natural - \lambda)^{-1}\tilde{\chi}\|\sqrt{\left(\|\chi\| + \lambda_+\|(\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\chi\|\right)\|(\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\chi\|}\ \Bigg) \\
&\leq 2\varepsilon\delta\left(\frac{1}{\widehat{d}_\varepsilon}\sqrt{\frac{1}{d_\varepsilon}\left(1 + \frac{\lambda_+}{d_\varepsilon}\right)} + \frac{1}{d_\varepsilon}\sqrt{\frac{1}{\widehat{d}_\varepsilon}\left(1 + \frac{\lambda_+}{\widehat{d}_\varepsilon}\right)}\ \right)\|\chi\|\,\|\tilde{\chi}\|.
\end{aligned}$$

To conclude, take

$$\tilde{\chi} = \left((\mathscr{L}_\varepsilon^\natural - \lambda)^{-1} - (\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\right)\chi$$

so that

$$\left\|\left((\mathscr{L}_\varepsilon^\natural - \lambda)^{-1} - (\widehat{\mathscr{L}}_\varepsilon^\natural - \lambda)^{-1}\right)\chi\right\| \leq 2\varepsilon\delta\left(\frac{1}{\widehat{d}_\varepsilon}\sqrt{\frac{1}{d_\varepsilon}\left(1 + \frac{\lambda_+}{d_\varepsilon}\right)} + \frac{1}{d_\varepsilon}\sqrt{\frac{1}{\widehat{d}_\varepsilon}\left(1 + \frac{\lambda_+}{\widehat{d}_\varepsilon}\right)}\ \right)\|\chi\|.$$

□

V. THE EFFECTIVE HAMILTONIAN

We begin by a rewriting of the quadratic form $\mathcal{Q}_\varepsilon^\natural(\mathcal{P}\varphi, \mathcal{P}\psi)$ associated to the operator $\widehat{\mathscr{L}}_{\varepsilon,\mathcal{P}}^\natural$ given in Lemma 4.6, i.e.,

$$\mathcal{Q}_\varepsilon^\natural(\mathcal{P}\varphi, \mathcal{P}\psi) = \int_{\mathbb{R}^2} \varepsilon^2 \partial_x(\overline{\psi^{BO} f_\varphi})\partial_x(\psi^{BO} f_\psi)\mathrm{d}\mathbf{x} + \int_{\mathbb{R}} q_x(\psi_x^{BO}, \psi_x^{BO})\overline{f_\varphi} f_\psi\,\mathrm{d}x.$$

We point out the identity

$$\int_{\mathbb{R}^2} \partial_x(\overline{\psi^{BO} f_\varphi})\partial_x(\psi^{BO} f_\psi)\mathrm{d}\mathbf{x} = \int_{\mathbb{R}} \overline{f'_\varphi} f'_\psi \,\mathrm{d}x + \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\partial_x \psi^{BO}|^2 dy \right) \overline{f_\varphi} f_\psi \,\mathrm{d}x,$$

where we used $\int_{\mathbb{R}} \psi^{BO} \partial_x \psi^{BO} \mathrm{d}y = 0$, which is a consequence of the normalization condition $\|\psi_x^{BO}\| = 1$. Also we notice that

$$q_x(\psi_x^{BO}, \psi_x^{BO}) = -\lambda_0(x) + \alpha^2.$$

Hence,

$$\mathcal{Q}_\varepsilon^\natural(\mathcal{P}\varphi, \mathcal{P}\psi) = \int_{\mathbb{R}} \left(\varepsilon^2 \overline{f'_\varphi} f'_\psi + \left(V + \varepsilon^2 R\right) \overline{f_\varphi} f_\psi \right) \mathrm{d}x$$

with

$$V(x) := -\lambda_0(x) + \alpha^2 = -\left(\frac{W\left(\frac{|\alpha||x|}{2} e^{\frac{-|\alpha||x|}{2}} \right)}{|x|} + \frac{|\alpha|}{2} \right)^2 + \alpha^2,$$

and

$$R(x) := \int_{\mathbb{R}} |\partial_x \psi^{BO}(x,y)|^2 \mathrm{d}y.$$

Remark 5.1. Let us define

$$L^2_\natural(\mathbb{R}) := \{ f \in L^2(\mathbb{R}) : f(x) = (+)_\natural f(-x) \}, \qquad H^1_\natural(\mathbb{R}) := H^1(\mathbb{R}) \cap L^2_\natural(\mathbb{R})$$

and

$$U_{BO} : \mathrm{ran}(\mathcal{P}|L^2_\natural(\mathbb{R}^2)) \to L^2_\natural(\mathbb{R}), \qquad U_{BO}(\psi^{BO} f_\phi) := f_\phi.$$

By $\|\psi_x^{BO}\| = 1$,

$$\langle \psi^{BO} f_\phi, \psi^{BO} f_\psi \rangle = \int_{\mathbb{R}} \langle \psi_x^{BO}, \psi_x^{BO} \rangle \overline{f}_\phi(x) f_\psi(x)\, dx = \langle f_\phi, f_\psi \rangle,$$

i.e., U_{BO} preserves the scalar product. It is a bijection with inverse $U_{BO}^{-1} f = \psi^{BO} f$; hence, U_{BO} is unitary and

$$\mathrm{ran}(\mathcal{P}|L^2_\natural(\mathbb{R}^2)) \simeq L^2_\natural(\mathbb{R}).$$

By the previous Remark, $\mathcal{Q}_\varepsilon^\natural(\mathcal{P}\cdot, \mathcal{P}\cdot)$ identifies with the sesquilinear form in $L^2_\natural(\mathbb{R})$ defined by

$$D(\mathrm{Q}_\varepsilon^{\mathrm{eff}\natural}) := H^1_\natural(\mathbb{R}) \times H^1_\natural(\mathbb{R}), \qquad \mathrm{Q}_\varepsilon^{\mathrm{eff}\natural}(f,g) := \int_{\mathbb{R}} \varepsilon^2 \overline{f'} g' + \left(V + \varepsilon^2 R\right) \overline{f} g \,\mathrm{d}x.$$

Since $V + \varepsilon^2 R$ is a bounded potential, the associated self-adjoint Hamiltonian in $L^2_\natural(\mathbb{R})$ is given by

$$D(\mathscr{L}_\varepsilon^{\mathrm{eff}\natural}) = H^2_\natural(\mathbb{R}) := H^2(\mathbb{R}) \cap L^2_\natural(\mathbb{R}) \qquad \mathscr{L}_\varepsilon^{\mathrm{eff}\natural} = -\varepsilon^2 \frac{d^2}{dx^2} + V + \varepsilon^2 R.$$

Now, we analyze the spectrum of

$$\mathscr{L}_\varepsilon^{\mathrm{eff}\natural} - \varepsilon^2 R = -\varepsilon^2 \frac{d^2}{dx^2} + V.$$

We note that:

(i) V is even on $\mathbb{R}$;
(ii) $0 \le V(x) < \frac{3}{4}\alpha^2$;
(iii) $V(0) = 0$, and $\lim_{x \to \pm\infty} V(x) = \frac{3}{4}\alpha^2$;
(iv) V is strictly increasing on $(0, +\infty)$;

Furthermore, by

$$(W(ye^{-y}) + y)^2 - 4y^2 = -8y^3 + O(y^4), \qquad y \ll 1,$$

there follows

$$V(x) = |\alpha|^3|x| + O(x^2), \qquad |x| \ll 1. \tag{5.1}$$

To proceed, we set $\Lambda := \varepsilon^{-1}$ and define

$$\mathrm{L}_\Lambda^\natural := \Lambda^2 \mathscr{L}_{1/\Lambda}^{\mathrm{eff}\natural} - R \equiv -\frac{d^2}{dx^2} + \Lambda^2 V. \tag{5.2}$$

Denoting by L_Λ the self-adjoint operator in $L^2(\mathbb{R})$ defined by the same differential expression as $\mathrm{L}_\Lambda^\natural$, since $(V - \frac{3}{4}\alpha^2)$ is bounded and vanishes as $|x| \to +\infty$ one has (see, e.g., Theorem 3.8.2 in Ref. 38)

$$\sigma_{ess}(\mathrm{L}_\Lambda^\natural) \subseteq \sigma_{ess}(\mathrm{L}_\Lambda) = \sigma_{ess}\left(-\frac{d^2}{dx^2} + \frac{3}{4}\alpha^2\Lambda^2\right) = \left[\frac{3}{4}\alpha^2\Lambda^2, +\infty\right).$$

Since L_Λ is in the limit point case at both ends $\pm\infty$, its eigenvalues are simple (see, e.g., Sec. 10 in Ref. 44); hence the eigenvalues of $\mathrm{L}_\Lambda^\natural$ are simple as well.

We denote by $\ell_{\Lambda,n}^\natural$, $n = 0, 1, 2, \ldots$, the eigenvalues of $\mathrm{L}_\Lambda^\natural$ below the essential spectrum, numbered in increasing order

$$0 < \ell_{\Lambda,0}^\natural < \ell_{\Lambda,1}^\natural < \cdots < \ell_{\Lambda,n}^\natural < \cdots.$$

The behavior as $\Lambda \gg 1$ of such eigenvalues is provided in the following

Theorem 5.2. *For any fixed integer $n \geq 0$ and sufficiently large Λ, $\mathrm{L}_\Lambda^\natural$ has at least $n + 1$ simple isolated eigenvalues and*

$$\ell_{\Lambda,n}^\natural = e_n^\natural \Lambda^{4/3} + O(\Lambda), \qquad e_n^{\mathrm{b}} := e_{2n}, \quad e_n^{\mathrm{f}} := e_{2n+1},$$

where e_k denotes the $(k+1)$-th eigenvalue of the closure K^1 of the essentially self-adjoint operator

$$D(\dot{\mathrm{K}}^1) = C_0^\infty(\mathbb{R}), \qquad \dot{\mathrm{K}}^1 := -\frac{d^2}{dx^2} + |\alpha|^3|x|.$$

We prove this theorem by Barry Simon's approach presented in Ref. 40, where the author considered the case with a smooth, bounded from below potential. We adapt his strategy to our situation, where the potential is not differentiable at the origin. The proof of Theorem 5.2 follows from the combination of Lemma 5.7 and 5.8.

For later use, we recall several properties of the eigenvalues and eigenfunctions of K^1. We refer to Sec. 6.10 in Ref. 39 for the details (in particular, see Eqs. 6.10.9, 6.10.10, 6.9.1, and 6.9.2 in Ref. 39; in the notation there, $\sigma_{2n} = \overline{\overline{\sigma}}_{n+1}$ and $\sigma_{2n+1} = \overline{\sigma}_{n+1}$).

Lemma 5.3. The solutions $\phi_k \in L^2(\mathbb{R})$ of the eigenvalue equations

$$\mathrm{K}^1\phi_k = e_k\phi_k, \qquad k = 0, 1, 2, \ldots.$$

can be written in terms of the Airy function Ai *as*

$$\phi_{2n}(x) = C_{2n}\mathrm{Ai}\left(|\alpha||x| + \sigma_{2k}\right), \qquad n = 0, 1, 2, \ldots, \tag{5.3}$$

$$\phi_{2n+1}(x) = C_{2n+1}\mathrm{sgn}(x)\mathrm{Ai}\left(|\alpha||x| + \sigma_{2n+1}\right), \qquad n = 0, 1, 2, \ldots, \tag{5.4}$$

where the C_k's are normalization constants, the σ_{2n}'s and σ_{2n+1}'s interlace,

$$\ldots < \sigma_{2n+1} < \sigma_{2n} < \ldots < \sigma_1 < \sigma_0 < 0,$$

and are the extrema and the zeros of the Airy function respectively [i.e., $\mathrm{Ai}'(\sigma_{2n}) = 0$ and $\mathrm{Ai}(\sigma_{2n+1}) = 0$]. The eigenvalues

$$0 < e_0 < e_1 < e_2 < \ldots < e_k < \ldots.$$

are given by

$$e_k = |\sigma_k|\,\alpha^2.$$

Remark 5.4. Note that $L^2(\mathbb{R}) = L^2_{\rm b}(\mathbb{R}) \oplus L^2_{\rm f}(\mathbb{R})$, this is equivalent to decomposing any function in $L^2(\mathbb{R})$ in the sum of its even and odd parts. Hence, one has

$$\mathrm{K}^1 = \mathrm{K}^{1\mathrm{b}} \oplus \mathrm{K}^{1\mathrm{f}}.$$

By Lemma 5.3, $\{e^{\rm b}_n, \phi^{\rm b}_n\} := \{e_{2n}, \phi_{2n}\}$ and $\{e^{\rm f}_n, \phi^{\rm f}_n\} := \{e_{2n+1}, \phi_{2n+1}\}$ are the eigensystems of $\mathrm{K}^{1\mathrm{b}}$ and $\mathrm{K}^{1\mathrm{f}}$ respectively.

Remark 5.5. For $x \gg 1$ the Airy function Ai behaves as (see, e.g., Sec. 10.4 in Ref. 1)

$$\mathrm{Ai}\,(x) = \frac{1}{2\sqrt{\pi}x^{1/4}}\; e^{-\frac{2}{3}x^{3/2}}\Big(1 + O(x^{-3/2})\Big). \tag{5.5}$$

Remark 5.6. By Ref. 25, one has the bounds

$$-\left(\frac{3\pi}{8}\,(4n+3) + \frac{3}{2}\arctan\frac{5}{18\pi(4n+3)}\right)^{2/3} \le \sigma_{2n+1} \le -\left(\frac{3\pi}{8}\,(4n+3)\right)^{2/3}.$$

Then, by $\sigma_{2n+1} < \sigma_{2n} < \sigma_{2(n-1)+1}$, one obtains

$$-\left(\frac{3\pi}{8}\,(4n+3) + \frac{3}{2}\arctan\frac{5}{18\pi(4n+3)}\right)^{2/3} \le \sigma_{2n} \le -\left(\frac{3\pi}{8}\,(4n-1)\right)^{2/3}.$$

As a first step toward the proof of Theorem 5.2, we define an auxiliary self-adjoint operator $\mathrm{L}^{1\natural}_\Lambda$ to compare with $\mathrm{L}^{\natural}_\Lambda$; it is defined as

$$\mathrm{L}^{1\natural}_\Lambda := \Lambda^{4/3} U_\Lambda \mathrm{K}^{1\natural} U_\Lambda^{-1}, \tag{5.6}$$

where the unitary operator U_Λ in $L^2_\natural(\mathbb{R})$ is defined by $(U_\Lambda f)(x) := \Lambda^{1/3} f(\Lambda^{2/3}x)$. For any $\psi \in C^\infty_0(\mathbb{R}) \cap L^2_\natural(\mathbb{R})$ one has

$$\mathrm{L}^{1\natural}_\Lambda \psi(x) = -\psi''(x) + \Lambda^2|\alpha|^3|x|\psi(x).$$

Notice that, by (5.6), the spectrum of $\Lambda^{-4/3}\mathrm{L}^{1\natural}_\Lambda$ is independent of Λ; by Lemma 5.3 and Remark 5.4, the eigensystem of $\mathrm{L}^{1\natural}_\Lambda$ is given by $\{\Lambda^{4/3}e^\natural_n, U_\Lambda\phi^\natural_n\}$.

We introduce a cut-off function on a suitable scale of Λ. Let $j \in C^\infty_0(\mathbb{R}) \cap L^2(\mathbb{R})$ with $0 \le j \le 1$, even, and $j(x) = 1$ for $|x| \le 1$ and $j(x) = 0$ for $|x| \ge 2$. Define:

$$\mathrm{J}_1(x) := j(\Lambda^{1/2}x), \tag{5.7}$$

and let

$$\mathrm{J}_0(x) := \sqrt{1 - \mathrm{J}_1^2(x)} \qquad (\mathrm{J}_0^2 + \mathrm{J}_1^2 = 1).$$

We point out that both J_0 and J_1 depend on Λ, to simplify the notation we omit this dependence. We also remark that in the definition of J_1, see Eq. (5.7), any exponent between 1/3 and 2/3 would work.

We first point out that, by the Taylor expansion of V, taking into account the fact that J_1 is supported on $|x| \le 2/\Lambda^{1/2}$, there follows that $\mathrm{J}_1\big(\mathrm{L}^\natural_\Lambda - \mathrm{L}^{1\natural}_\Lambda\big)\mathrm{J}_1 = \mathrm{J}_1\big(\Lambda^2\big(V - |\alpha|^3|x|\big)\big)\mathrm{J}_1$ is a bounded operator in $L^2_\natural(\mathbb{R})$ and its norm satisfies the bound

$$\|\mathrm{J}_1\big(\mathrm{L}^\natural_\Lambda - \mathrm{L}^{1\natural}_\Lambda\big)\mathrm{J}_1\| = \|\mathrm{J}_1\big(\Lambda^2\big(V - |\alpha|^3|x|\big)\big)\mathrm{J}_1\| \le C\|\mathrm{J}_1\big(\Lambda^2x^2\big)\mathrm{J}_1\| = O(\Lambda). \tag{5.8}$$

A. Lower bound

To establish a lower bound for $\ell^\natural_{\Lambda,n}$, we use the IMS (Ismagilov, Morgan, Morgan–Simon, see Refs. 26, 32, and 33) localization technique (see, for instance, Lemma 3.1 in Ref. 40 or Chap. 11 in Ref. 15):

Lemma 5.7. Suppose that $\mathrm{L}^\natural_\Lambda$ has $n+1$ eigenvalues below the essential spectrum. Then

$$\ell^\natural_{\Lambda,n} \ge \Lambda^{4/3}e^\natural_n + O(\Lambda) \qquad \Lambda \gg 1. \tag{5.9}$$

Proof. Denote $P^{1\natural}_\Lambda$ the orthogonal projection of the n-dimensional subspace spanned by the first n eigenvectors of $L^{1\natural}_\Lambda$. Our goal is to prove

$$L^{\natural}_\Lambda \geq e^{\natural}_n \Lambda^{4/3} + F^{\natural}_1 + O(\Lambda), \tag{5.10}$$

where $F^{\natural}_1$ is the symmetric operator

$$F^{\natural}_1 := J_1 (L^{1\natural}_\Lambda - e_n \Lambda^{4/3}) P^{1\natural}_\Lambda J_1.$$

We remark that $F^{\natural}_1$ has finite rank at most equal to n. As usual, inequalities of the form of (5.10) have to be understood in the sense $\langle \psi, L^{\natural}_\Lambda \psi\rangle \geq e^{\natural}_n \Lambda^{4/3} \|\psi\|^2 + \langle \psi, F^{\natural}_1 \psi\rangle + O(\Lambda)$ for all $\psi \in D(L^{\natural}_\Lambda)$.

Inequality (5.10) implies (5.9). This is easily seen by noticing that one can choose a normalized vector ψ in the span of the first $n+1$ eigenvectors of $L^{\natural}_\Lambda$ such that $\psi \in \ker F^{\natural}_1$. For such ψ we obtain

$$\ell^{\natural}_{\Lambda,n} \geq \langle \psi, L^{\natural}_\Lambda \psi\rangle \geq e^{\natural}_n \Lambda^{4/3} + O(\Lambda)$$

which implies (5.9).

Let us first recall the IMS localization. Since J_0 and J_1 are smooth and ${J_0}^2 + {J_1}^2 = 1$, one has

$$L^{\natural}_\Lambda = \sum_{i=0}^{1} J_i L^{\natural}_\Lambda J_i - \sum_{i=0}^{1} \left({J_i}'\right)^2.$$

Therefore, we can re-write $L^{\natural}_\Lambda$ as

$$L^{\natural}_\Lambda = J_0 L^{\natural}_\Lambda J_0 + J_1 L^{1\natural}_\Lambda J_1 + J_1 \left(L^{\natural}_\Lambda - L^{1\natural}_\Lambda\right) J_1 - \sum_{i=0}^{1} \left({J_i}'\right)^2. \tag{5.11}$$

From the definition of J_0 and J_1, there follows the bound on the operator norm:

$$\left\| \sum_{i=0}^{1} \left({J_i}'\right)^2 \right\| = O(\Lambda). \tag{5.12}$$

By the definition of the orthogonal projection $P^{1\natural}_\Lambda$, there follows

$$L^{1\natural}_\Lambda = L^{1\natural}_\Lambda P^{1\natural}_\Lambda + L^{1\natural}_\Lambda \left(I - P^{1\natural}_\Lambda\right) = F^{\natural} + e^{\natural}_n \Lambda^{4/3} P^{1\natural}_\Lambda + L^{1\natural}_\Lambda \left(I - P^{1\natural}_\Lambda\right) \geq F^{\natural} + e^{\natural}_n \Lambda^{4/3},$$

with

$$F^{\natural} := (L^{1\natural}_\Lambda - e^{\natural}_n \Lambda^{4/3}) P^{1\natural}_\Lambda.$$

Hence,

$$J_1 L^{1\natural}_\Lambda J_1 \geq F^{\natural}_1 + e^{\natural}_n \Lambda^{4/3} {J_1}^2. \tag{5.13}$$

The latter inequality can be understood as an inequality for $\langle \psi, J_1 L^{1\natural}_\Lambda J_1 \psi\rangle$ for all $\psi \in D(L^{\natural}_\Lambda)$, since $\psi \in D(L^{\natural}_\Lambda)$ implies $J_1 \psi \in D(L^{1\natural}_\Lambda)$.

Also, we need a control on the support of J_0. Since V is even and increasing for $x > 0$, by the expansion (5.1), one gets

$$V(x) \geq V(1/\Lambda^{1/2}) \geq \frac{c}{\Lambda^{1/2}} \qquad \text{on supp}\,(J_0),$$

for some positive constant c and Λ large enough. Hence, since $-\frac{d^2}{dx^2}$ is a positive definite operator, one has

$$J_0 L^{\natural}_\Lambda J_0 \geq c\Lambda^{3/2} (J_0)^2 \geq e^{\natural}_n \Lambda^{4/3} (J_0)^2, \tag{5.14}$$

for Λ large enough. Taking into account (5.11), together with (5.8) and (5.12)–(5.14), the inequality (5.10) follows □

B. Upper bound

To obtain the upper bound, we use the Rayleigh-Ritz variational principle. In this approach, we take the scaled eigenfunctions of $K^{1\natural}$ as trial wave functions for $L^{1\natural}_\Lambda$.

Lemma 5.8. For any fixed integer $n \geq 0$ and sufficiently large Λ, $L^{\natural}_\Lambda$ has at least $n+1$ simple isolated eigenvalues and

$$\ell^{\natural}_{\Lambda,n} \leq \Lambda^{4/3} e^{\natural}_n + O(\Lambda). \tag{5.15}$$

Proof. Recall that the functions $\phi_n^\natural$'s in (5.3) and (5.4) are the orthonormal eigenfunctions of $\mathrm{K}^{1\natural}$ corresponding to the eigenvalues $e_n^\natural$. Therefore, the $U_\Lambda \phi_n^\natural$'s are the orthonormal eigenfunctions of $\mathrm{L}_\Lambda^{1\natural}$, that is

$$\mathrm{L}_\Lambda^{1\natural}\big(U_\Lambda \phi_n^\natural\big) = \Lambda^{4/3} e_n^\natural \big(U_\Lambda \phi_n^\natural\big).$$

Defining

$$\psi_n^\natural := \mathrm{J}_1 U_\Lambda \phi_n^\natural$$

and taking into account the asymptotic behavior of the Airy functions, see Eq. (5.5), one has

$$\langle \psi_n^\natural, \psi_m^\natural\rangle = \delta_{mn} + O(e^{-c\Lambda^{1/4}}), \tag{5.16}$$

for some positive constant c. Furthermore, by

$$\mathrm{J}_1 \mathrm{L}_\Lambda^{1\natural} \mathrm{J}_1 = \frac{1}{2}\big(\mathrm{J}_1^2 \mathrm{L}_\Lambda^{1\natural} + \mathrm{L}_\Lambda^{1\natural}\mathrm{J}_1^2\big) + (\mathrm{J}_1')^2,$$

by (5.16) and (5.12), one obtains

$$\begin{aligned}\langle \psi_n^\natural, \mathrm{L}_\Lambda^{1\natural}\psi_m^\natural\rangle &= \Lambda^{4/3}\left(\frac{e_n^\natural + e_m^\natural}{2}\right)\langle \psi_n^\natural, \psi_m^\natural\rangle + \Big(U_\Lambda\phi_n^\natural, (\mathrm{J}_1')^2 U_\Lambda \phi_m^\natural\Big)\\ &= \Lambda^{4/3} e_n^\natural \delta_{nm} + O(\Lambda).\end{aligned} \tag{5.17}$$

Therefore, by (5.8),

$$\langle \psi_n^\natural, \mathrm{L}_\Lambda^\natural \psi_m^\natural\rangle = \langle \psi_n^\natural, \mathrm{L}_\Lambda^{1\natural}\psi_m^\natural\rangle + O(\Lambda) = \Lambda^{4/3} e_n^\natural \delta_{mn} + O(\Lambda).$$

Let us fix $n \geq 0$. We want to apply the Rayleigh–Ritz variational method, however, since the vectors $\{\psi_i^\natural\}_{i=0}^n$ are not orthonormal [but almost orthonormal, see Eq. (5.16)], we proceed as follows. Noticing that the $\psi_i^\natural$'s are linearly independent, by the Gram–Schmidt algorithm we construct a set of orthonormal eigenfunctions $\{\tilde\psi_i^\natural\}_{i=0}^n$ which, by construction, still satisfy Eq. (5.17), that is

$$\langle \tilde\psi_n^\natural, \mathrm{L}_\Lambda^{1\natural}\tilde\psi_m^\natural\rangle = \Lambda^{4/3} e_n^\natural \delta_{nm} + O(\Lambda). \tag{5.18}$$

Then, a direct application of the Rayleigh–Ritz variational method (see e.g. Th. XIII.3 in Ref. 36), taking as trial space the span of $\{\tilde\psi_i^\natural\}_{i=1}^n$, gives the upper bound $\ell_{\Lambda,n}^\natural \leq \Lambda^{4/3} e_n^\natural + O(\Lambda)$. The fact that $\sigma_{ess}(\mathrm{L}_\Lambda^\natural) \subseteq [\frac{3}{4}\alpha^2\Lambda^2, +\infty)$ and Eq. (5.18) guarantee that there are at least $n+1$ eigenvalues. □

Our main result on the eigenvalues of $\mathscr{L}_\varepsilon^{\mathrm{eff}\natural}$ is given in the following

Theorem 5.9. *For any fixed integer $n \geq 0$ and $\varepsilon > 0$ sufficiently small, the $\mathscr{L}_\varepsilon^{\mathrm{eff}\natural}$ has at least $n+1$ simple isolated eigenvalues. The $(n+1)$-th eigenvalue is given by*

$$\mathscr{E}_{\varepsilon,n}^{\mathrm{eff}\natural} = s_n^\natural \alpha^2 \varepsilon^{2/3} + O(\varepsilon), \tag{5.19}$$

where $s_n^b := |\sigma_{2n}|$ and $s_n^f := |\sigma_{2n+1}|$ and the negative numbers σ_k are defined in Lemma 5.3.

Proof. Denote by $\tilde\ell_{\Lambda,n}^\natural$ the eigenvalues of the self-adjoint operator [compare the following definition with Eq. (5.2)]

$$\widetilde{\mathrm{L}}_\Lambda^\natural := \Lambda^2 \mathscr{L}_{1/\Lambda}^{\mathrm{eff}\natural} = \mathrm{L}_\Lambda^\natural + R.$$

The eigenvalues $\tilde\ell_{\Lambda,n}^\natural$ satisfy the lower bound (5.9), because R is positive. Moreover, the $\tilde\ell_{\Lambda,n}^\natural$'s satisfy the upper bound (5.15), because in the Rayleigh–Ritz variational approach [see Eq. (5.18)] R gives a contrubution of order 1. Hence, for any fixed integer $n \geq 0$, $\widetilde{\mathrm{L}}_\Lambda^\natural$ has at least $n+1$ simple isolated eigenvalues, and $\tilde\ell_{\Lambda,n}^\natural = \Lambda^{4/3} e_n^\natural + O(\Lambda)$. Noticing that $\mathscr{L}_\varepsilon^{\mathrm{eff}\natural} = \varepsilon^2 \widetilde{\mathrm{L}}_{1/\varepsilon}^\natural$ we obtain the expansion (5.19). □

VI. PROOF OF THEOREM 1.1

The result in Eq. (1.3) about the essential spectrum is part of Theorem 2.19. The lower bound on the spectrum in the same equation follows immediately from Theorem 3.4 and Eq. (4.1).

Fix $n \geq 0$. By Theorem 5.9, one can take ε so small that $\mathscr{L}_\varepsilon^{\mathrm{eff}\natural}$ has $n+1$ eigenvalues, $\mathscr{E}_{\varepsilon,0}^{\mathrm{eff}\natural}, \mathscr{E}_{\varepsilon,1}^{\mathrm{eff}\natural}, \dots, \mathscr{E}_{\varepsilon,n}^{\mathrm{eff}\natural}$. These are also eigenvalues for $\mathscr{L}_{\mathcal{P},\varepsilon}^\natural$, as a matter of fact they are the lowest eigenvalues (ordered in increasing order) of $\widehat{\mathscr{L}}_{\mathcal{P},\varepsilon}^\natural$, and of $\widehat{\mathscr{L}}_\varepsilon^\natural$, see Lemma 4.6. Choose $c_n > 0$ so that $\mathscr{E}_{\varepsilon,n-1}^{\mathrm{eff}\natural} < \mu_{\varepsilon,n}^\natural < \mathscr{E}_{\varepsilon,n}^{\mathrm{eff}\natural}$ and $\widehat{d}_\varepsilon(\mu_{\varepsilon,n}^\natural) > c_n\varepsilon^{2/3}$, where $\mu_{\varepsilon,n}^\natural := \mathscr{E}_{\varepsilon,n}^{\mathrm{eff}\natural} - c_n\varepsilon^{2/3}$. Since $(\mu_{\varepsilon,n}^\natural - \mathscr{E}_{\varepsilon,n}^{\mathrm{eff}\natural})^{-1}$ is the lowest eigenvalue of $(\mu_{\varepsilon,n}^\natural - \widehat{\mathscr{L}}_\varepsilon^\natural)^{-1}$,

$$(\mu^\natural_{\varepsilon,n} - \mathscr{E}^{\text{eff}\natural}_{\varepsilon,n})^{-1} = \min_{\psi \in L^2_\natural(\mathbb{R}^2), \|\psi\|=1} \langle \psi, (\mu^\natural_{\varepsilon,n} - \widetilde{\mathscr{L}}^\natural_\varepsilon)^{-1}\psi\rangle.$$

By Lemma 4.8 (i), possibly for smaller ε, there holds $\mu^\natural_{\varepsilon,n} \in \rho(\mathscr{L}^\natural_\varepsilon)$ and $d_\varepsilon(\mu^\natural_{\varepsilon,n}) > c'_n\varepsilon^{2/3}$. Let us define $\mathscr{E}^\natural_{\varepsilon,n}$ through the relation

$$(\mu^\natural_{\varepsilon,n} - \mathscr{E}^\natural_{\varepsilon,n})^{-1} = \inf_{\psi \in L^2_\natural(\mathbb{R}^2), \|\psi\|=1} \langle \psi, (\mu^\natural_{\varepsilon,n} - \mathscr{L}^\natural_\varepsilon)^{-1}\psi\rangle.$$

By the spectral mapping theorem, either $\mathscr{E}^\natural_{\varepsilon,n}$ is an eigenvalue of $\mathscr{L}^\natural_\varepsilon$ or $\mathscr{E}^\natural_{\varepsilon,n} = \inf \sigma_{ess}(\mathscr{L}^\natural_\varepsilon)$, where, by Theorem 2.19, $\inf \sigma_{ess}(\mathscr{L}^\natural_\varepsilon) = \frac{3+\varepsilon^2}{4+\varepsilon^2}\,\alpha^2$.

By the bound on the resolvent difference in Lemma 4.8 (ii), there exists $c''_n > 0$ such that

$$\left|(\mathscr{E}^\natural_{\varepsilon,n} - \mu^\natural_{\varepsilon,n})^{-1} - (\mathscr{E}^{\text{eff}\natural}_{\varepsilon,n} - \mu^\natural_{\varepsilon,n})^{-1}\right| \le c''_n.$$

Therefore,

$$\begin{aligned} |\mathscr{E}^\natural_{\varepsilon,n} - \mathscr{E}^{\text{eff}\natural}_{\varepsilon,n}| &\le c''_n\,|\mathscr{E}^\natural_{\varepsilon,n} - \mu^\natural_{\varepsilon,n}|\,|\mathscr{E}^{\text{eff}\natural}_{\varepsilon,n} - \mu^\natural_{\varepsilon,n}| \le c''_n c_n \varepsilon^{2/3}|\mathscr{E}^\natural_{\varepsilon,n} - \mathscr{E}^{\text{eff}\natural}_{\varepsilon,n} + c_n\varepsilon^{2/3}| \\ &\le c''_n c_n\varepsilon^{2/3}|\mathscr{E}^\natural_{\varepsilon,n} - \mathscr{E}^{\text{eff}\natural}_{\varepsilon,n}| + c''_n c_n^2\varepsilon^{4/3}. \end{aligned}$$

This inequality and Theorem 5.9 give

$$\mathscr{E}^\natural_{\varepsilon,n} = \mathscr{E}^{\text{eff}\natural}_{\varepsilon,n} + O(\varepsilon^{4/3}) = s^\natural_n\alpha^2\varepsilon^{2/3} + O(\varepsilon).$$

This shows that $\mathscr{E}^\natural_{\varepsilon,n} < \inf\sigma_{ess}(\mathscr{L}^\natural_\varepsilon)$ whenever ε is sufficiently small; hence, $\mathscr{E}^\natural_{\varepsilon,n}$ is an eigenvalue below the essential spectrum of $\mathscr{L}^\natural_\varepsilon$. One can repeat the argument above for any $k = 0, \dots, n$, and, by the inequalities

$$\mathscr{E}^{\text{eff}\natural}_{\varepsilon,k} - \tilde{c}_k\varepsilon^{4/3} \le \mathscr{E}^\natural_{\varepsilon,k} \le \mathscr{E}^{\text{eff}\natural}_{\varepsilon,k} + \tilde{c}_k\varepsilon^{4/3} \qquad k = 0,\dots,n$$

$$\mathscr{E}^{\text{eff}\natural}_{\varepsilon,k-1} + \tilde{c}_{k-1}\varepsilon^{4/3} \le \mathscr{E}^{\text{eff}\natural}_{\varepsilon,k} - \tilde{c}_k\varepsilon^{4/3} \qquad k = 1,\dots,n$$

which hold true for ε small enough, there follows that $\mathscr{E}^\natural_{\varepsilon,n}$ is the $(n+1)$-th eigenvalue. The proof is then concluded by $\sigma_d(H^\natural_\varepsilon) = \sigma_d(\mathscr{L}^\natural_\varepsilon) - \alpha^2$.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Claudio Cacciapuoti: Conceptualization (equal); Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal).
Andrea Posilicano: Conceptualization (equal); Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal).
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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] Abramowitz, M. and Stegun, I. A., *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, Volume No. 55 of National Bureau of Standards Applied Mathematics Series* (U.S. Government Printing Office, Washington, D.C., 1964); For sale by the Superintendent of Documents.

[2] Akbas, H. and Turgut, O. T., "Born–Oppenheimer approximation for a singular system," J. Math. Phys. **59**, 012107 (2018).
[3] Akbas, H. and Turgut, O. T., "Born–Oppenheimer approximation for a simple renormalizable system," in *Mathematical Challenges of Zero-Range Physics*, edited by Michelangeli, A. (Springer INdAM Series, 2021), Vol. 42.
[4] Albeverio, S., Gesztesy, F., Høegh-Krohn, R., and Holden, H., in *Solvable Models in Quantum Mechanics*, 2nd ed. (AMS Chelsea Publishing, Providence, RI, 2005); With an appendix by Pavel Exner.
[5] Basti, G., Cacciapuoti, C., Finco, D., and Teta, A., "The three-body problem in dimension one: From short-range to contact interactions," J. Math. Phys. **59**, 072104 (2018).
[6] Basti, G., Cacciapuoti, C., Finco, D., and Teta, A., "Three-body Hamiltonian with regularized zero-range interactions in dimension three," Ann. Henri Poincaré **24**, 223–276 (2023).
[7] Becker, S., Michelangeli, A., and Ottolini, A., "Spectral analysis of the 2 + 1 fermionic trimer with contact interactions," Math. Phys., Anal. Geom. **21**(4), 35 (2018).
[8] Born, M. and Oppenheimer, R., "Zur Quantentheorie der Molekeln," Ann. Phys. **389**, 457–484 (1927).
[9] Brüning, J., Geyler, V., and Pankrashkin, K., "Spectra of self-adjoint extensions and applications to solvable Schrödinger operators," Rev. Math. Phys. **20**, 1–70 (2008).
[10] Cacciapuoti, C., Fermi, D., and Posilicano, A., "On inverses of Kreĭn's $\mathscr{Q}$-functions," Rend. Mat. Appl. **39**(2), 229–240 (2018).
[11] Combes, J. M., Duclos, P., and Seiler, R., "The Born-Oppenheimer approximation," in *Rigorous Atomic and Molecular Physics Proceedings*, edited by Wightman, V. (Plenum, New York, 1981), Vol. 1980, p. 185.
[12] Correggi, M., Dell'Antonio, G., Finco, D., Michelangeli, A., and Teta, A., "Stability for a system of N fermions plus a different particle with zero-range interactions," Rev. Math. Phys. **24**(07), 1250017 (2012).
[13] Correggi, M., Dell'Antonio, G., Finco, D., Michelangeli, A., and Teta, A., "A class of Hamiltonians for a three-particle fermionic system at unitarity," Math. Phys., Anal. Geom. **18**(1), 32 (2015).
[14] Correggi, M., Finco, D., and Teta, A., "Energy lower bound for the unitary $N + 1$ fermionic model," Europhys. Lett. **111**(1), 10003 (2015).
[15] Cycon, H. L., Froese, R. G., Kirsch, W., and Simon, B., "Schrödinger operators with application to quantum mechanics and global geometry," in *Texts and Monographs in Physics*, study edition (Springer-Verlag, Berlin, 1987).
[16] Duchêne, V. and Raymond, N., "Spectral asymptotics of a broken δ-interaction," J. Phys. A: Math. Theor. **47**(15), 155203 (2014).
[17] Exner, P. and Ichinose, T., "Geometrically induced spectrum in curved leaky wires," J. Phys. A: Math. Gen. **34**(7), 1439 (2001).
[18] Exner, P. and Němcová, K., "Leaky quantum graphs: Approximations by point-interaction Hamiltonians," J. Phys. A: Math. Theor. **36**(40), 10173 (2003).
[19] Ferretti, D. and Teta, A., "Some remarks on the regularized Hamiltonian for three bosons with contact interactions," in *Quantum Mathematics I, INdAM 2022. Springer INdAM Series*, edited by Correggi, M. and Falconi, M. (Springer, Singapore, 2023), Vol. 57.
[20] Ferretti, D. and Teta, A., "Hamiltonian for a Bose gas with contact interactions," Rev. Math. Phys. **38**(2), 2550020 (2026).
[21] Ferretti, D. and Teta, A., "Hamiltonians for quantum systems with contact interactions," in *Operator Theory*, edited by D. Alpay, I. Sabadini, and F. Colombo (Springer, Basel, 2025), pp. 1–24.
[22] Figari, R., Saberbaghi, H., and Teta, A., "On a family of finitely many point interaction Hamiltonians free of ultraviolet pathologies," J. Phys. A: Math. Theor. **57**(5), 055303 (2024).
[23] Griesemer, M., Hofacker, M., and Linden, U., "From short-range to contact interactions in the 1D Bose gas," Math. Phys., Anal. Geom. **23**, 19 (2020).
[24] Hagedorn, G. and Joye, A., "Mathematical analysis of Born–Oppenheimer approximations," in *Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday*, *Proceedings of Symposia in Pure Mathematics* (AMS, Providence, RI, 2007), Vol. 76, pp. 203–226.
[25] Hethcote, H. W., "Bounds for zeros of some special functions," Proc. Am. Math. Soc. **25**, 72–74 (1970).
[26] Ismagilov, R. S., "Conditions for the semiboundedness and discreteness of the spectrum in the case of one-dimensional differential operators," Dokl. Akad. Nauk SSSR **140**, 33–36 (1961).
[27] Jecko, T., "On the mathematical treatment of the Born–Oppenheimer approximation," J. Math. Phys. **55**(5), 053504 (2014).
[28] Krejčiřík, D., Raymond, N., Royer, J., and Siegl, P., "Reduction of dimension as a consequence of norm-resolvent convergence and applications," Mathematika **64**(2), 406–429 (2018).
[29] Kufner, A., John, O., and Fucik, S., *Function Spaces* (Noordhoff International Publishing, 1977).
[30] Minlos, R. A. and Faddeev, L., "On the point interaction for a three-particle system in quantum mechanics," Sov. Phys. Dokl. **6**(12), 1072–1074 (1962).
[31] Minlos, R. A. and Faddeev, L., "Comment on the problem of three particles with point interactions," Sov. Phys. JETP. **14**(6), 1315–1316 (1962).
[32] Morgan, J. D., "Schrödinger operators whose potentials have separated singularities," J. Oper. Theory **1**(1), 109–115 (1979).
[33] Morgan, J. D. and Simon, B., "Behavior of molecular potential energy curves for large nuclear separations.," Int. J. Quantum Chem. **17**, 1143–1166 (1980).
[34] Pankrashkin, K. and Vogel, M., "On Schrödinger operators with δ'-potentials supported on star graphs," J. Phys. A: Math. Theor. **55**(29), 295201 (2022).
[35] Posilicano, A., "A Kreĭn-like formula for singular perturbations of self-adjoint operators and applications," J. Funct. Anal. **183**(1), 109–147 (2001).
[36] Reed, M. and Simon, B., in *Analysis of Operators*, *Volume 4 of Methods of Modern Mathematical Physics* (Academic Press, New York, 1978).
[37] Saberbaghi, H., "Efimov effect in the Born–Oppenheimer picture," arXiv:2511.15973 [math-ph], 36 (2025).
[38] Schechter, M., *Operator Methods in Quantum Mechanics* (Dover Publications, Inc., Mineola, NY, 2002); Reprint of the 1981 original.
[39] Schwinger, J., *Quantum Mechanics. Symbolism of Atomic Measurements* (Springer-Verlag, Berlin, 2001).
[40] Simon, B., "Semiclassical analysis of low lying eigenvalues. I. Nondegenerate minima: Asymptotic expansions," Ann. Inst. Henri Poincaré Sect. A (N.S.) **38**(3), 295–308 (1983); "Semiclassical analysis of low lying eigenvalues. I. Nondegenerate minima: Asymptotic expansions," **40**(2), 224 (1984).
[41] Stone, M. H., *Linear Transformations in Hilbert Space* (American Mathematical Society, New York, 1932).
[42] Teta, A., *A Mathematical Primer on Quantum Mechanics*, *Unitext for Physics* (Springer, Cham, 2018).
[43] Thomas, L. E., "Multiparticle Schrödinger Hamiltonians with point interactions," Phys. Rev. D **30**, 1233 (1984).
[44] Wiedmann, J., *Spectral Theory of Ordinary Differential Operators*, *Lecture Notes in Mathematics* (Springer-Verlag, Berlin, 1987), Vol. 1258.
[45] Yoshitomi, K., "Finiteness of the discrete spectrum in a three-body system with point interaction," Math. Slovaca **67**(4), 1031–1042 (2017).