Three-dimensional coherence of light speckles: Theory

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(Received 31 July 2008; published 2 December 2008)

We provide a detailed analysis of the three-dimensional spatial coherence properties of light speckles, based on very general assumptions. We show that, while in the deep Fresnel region close to the source the longitudinal coherence of speckles is ruled by the laws of ordinary diffraction, on approach to the Fraunhofer zone the longitudinal coherence length tends to become infinite. We offer both a quantitative and a qualitative description of the emergence of these different behaviors.

DOI: 10.1103/PhysRevA.78.063806 PACS number(s): 42.25.Kb, 42.30.Ms

I. INTRODUCTION

Coherent light transmitted by a random diffuser or reflected from a rough surface produces a granular interference pattern, known as a *speckle pattern*. There exists by now a vast literature describing the two-dimensional (2D) transverse spatial coherence properties of speckle light (see, e.g., [1–4]). These studies have mainly concentrated on the Fresnel zone with respect to the source. Typically, the transverse coherence of the speckles in this region is described by means of the Van Cittert–Zernike (VCZ) theorem [5], derived under the assumption of complete spatial incoherence of the source, or by its generalized version [3,6], which accounts for partial incoherence of the source. More recently, a detailed study of the transverse spatial coherence in the deep Fresnel zone, close to the source (also called the "near-field zone"), was performed [7], accounting for the invariance of the spatial coherence with the distance from the source in this zone, which was observed for the first time by Giglio *et al.* [8,9].

In comparison, the longitudinal spatial coherence properties of light speckles have been the subject of rather few investigations [10–13], with no complete and consistent view emerging from these works. Besides its fundamental relevance, the longitudinal coherence of speckled light is important for several areas of speckle metrology, such as, for example, speckle photography, speckle interferometry, and speckle holography (see, e.g., Chap. 8 of [4] and references therein). More recently, it turned out to be a relevant issue also in novel optical imaging techniques such as dynamic speckle illumination microscopy [14,15], and in so-called thermal ghost imaging [16–18]. In both cases, the sample is illuminated with a time-varying speckle pattern whose longitudinal correlation length establishes the depth of view of the image formation and can explain, in the case of ghost imaging, the blurring or not of the image in the out-of focus condition [19,20].

The aim of this paper is to provide a general analysis of the three-dimensional (3D) coherence properties of light speckles, in the framework of the paraxial approximation, with no specific assumptions about the shape of the illuminating region or the nature of the random diffuser or the rough surface that originates the speckles. We shall offer a detailed and complete view of the problem, encompassing the deep Fresnel region close to the source, the Fresnel zone, and up to the Fraunhofer zone.

A very common picture emerging from the existing literature [10–12] is that of speckles as "jelly-bean-like" coherence volumes, with their lengths ruled by the diffraction law. In this work, we find deviations from this picture as the Fraunhofer distance from the source is approached, and we provide a quantitative analysis as well as qualitative explanation in terms of the wave front curvature of the light beam.

The paper is organized as follows. Section II introduces the problem by describing the relevant properties of the incoherent light source that we consider. Section III is devoted to the 3D coherence properties of speckles in the deep Fresnel region. Section IV reviews some aspects of the 2D transverse coherence properties of speckles, showing in particular the transition from the near-field speckles to speckles described by the VCZ theorem. Section V analyzes the 3D coherence of speckles in the VCZ zone (Fresnel and Frauhnofer zones), and shows the emergence of a quantitatively different behavior of the longitudinal coherence as the Fraunhofer zone is approached. A qualitative discussion that accounts for the emergence of the different behaviors of the longitudinal coherence is offered in Sec. VI. Section VII concludes and summarizes.

II. THE SOURCE

We consider the geometry schematically illustrated in Fig. 1, where speckled light is produced at a source plane $z=0$, by an incoherent light source (e.g., by means of a collimated laser illuminating a random diffuser), and the light distributions are measured on planes parallel to the source plane at various distances z. We assume that the light is characterized by a circular Gaussian statistics [6] in which the statistical field moments at any order can be expressed via the second-order moment (mutual coherence function)

$$G^{(1)}(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2) = \langle A^*(\mathbf{x}_1,z_1)A(\mathbf{x}_2,z_2)\rangle, \qquad (1)$$

where A indicates the electric field envelope, $\langle\cdot\rangle$ is a statistical average, $\mathbf{x}_1,\mathbf{x}_2$ are the 2D transverse coordinates, and z_1,z_2 are distances from the source plane (see Fig. 1). In the definition (1) we implicitly considered the fields at equal times, and omitted the time argument for brevity of

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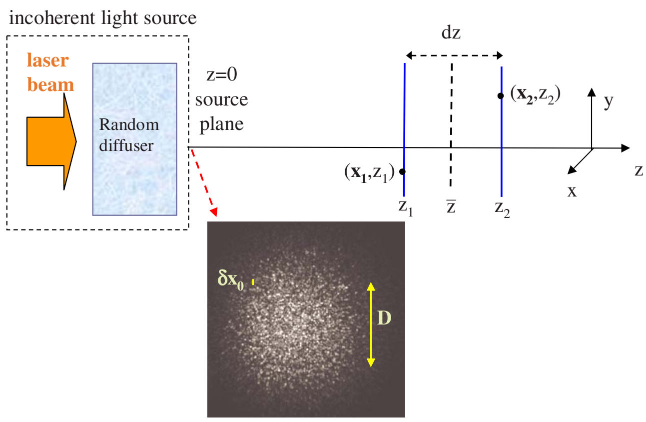

FIG. 1. (Color online) Schematic view of the problem. An incoherent light source produces at $z=0$ a speckle beam characterized by speckles of size δx_0 and by an overall diameter D. We shall investigate the full 3D coherence properties of the light at a generic distance $\bar{z}$ from the source plane, i.e., the mutual coherence function at two 3D points $(\mathbf{x}_1,z_1)$, $(\mathbf{x}_2,z_2)$.

notation.[1] In particular, a Siegert-like relation holds,

$$\langle I(\mathbf{x}_1,z_1)I(\mathbf{x}_2,z_2)\rangle = \langle I(\mathbf{x}_1,z_1)\rangle\langle I(\mathbf{x}_2,z_2)\rangle + |G^{(1)}(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2)|^2, \tag{2}$$

linking the intensity correlation function calculated at two 3D spatial points to the mutual coherence function $G^{(1)}$. In this way, the light speckles can be pictured as 3D coherence volumes, distributed both in the beam cross section and in the longitudinal direction. The 3D shape of these coherence volumes is completely characterized by the 3D mutual coherence function $G^{(1)}$; this function is precisely the object of the investigations in this work.

We assume that at the source plane $z=0$ the coherence function is known:

$$G^{(1)}(\mathbf{x}_1,0;\mathbf{x}_2,0) \coloneqq G_0^{(1)}(\mathbf{x}_1;\mathbf{x}_2). \tag{3}$$

Under very general assumptions on $G_0^{(1)}$, we shall study $G^{(1)}(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2)$:

(i) On a plane at distance $z_1=z_2=\bar{z}$ from the source plane. This study aims at characterizing the 2D transverse coherence properties of light as the distance $\bar{z}$ varies. This problem is a well-known one in the literature [1,3,6–8], and this part of our analysis will review known results in the framework of our formalism.

(ii) On two planes z_1 and z_2, separated by a distance $dz = z_2-z_1$ (see Fig. 1). This study will provide a full 3D characterization of the spatial coherence properties of light speckles, as a function both of the transverse separation $\mathbf{x}_2 - \mathbf{x}_1$ and of the longitudinal separation dz, which represents the very aim of our work.

Our main assumption is that at the source plane the coherence function has two well-separated scales of spatial variation: as a function of the distance $\mathbf{x}_2-\mathbf{x}_1$ it dies out on a "small" distance δx_0, the transverse size of the speckles at the source plane; as a function of the mean transverse cordinate $(\mathbf{x}_1+\mathbf{x}_2)/2$ it dies out on a "large" scale, the transverse size of the source D. In typical pseudothermal sources, the size of the speckles generated close to the scattering medium is determined by the scatterers, or by the surface roughness size, and can be as small as a few micrometers; D is determined by the diameter of the laser illuminating the medium or of the pinhole after it, and can be as large as several millimeters. If the two scales of variations are well separated, we can therefore assume that the coherence function factorizes,

$$G_0^{(1)}(\mathbf{x}_1,\mathbf{x}_2) = C_0(\mathbf{x}_2-\mathbf{x}_1)I_0\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}\right), \tag{4}$$

where I_0 is a real function representing the mean transverse intensity distribution at the source plane, varying on the large scale D (the source diameter). C_0 is a real function with the property $C_0(0)=1$, which dies out over the small scale δx_0 (the speckle transverse size). Additionally, in writing Eq. (4), we assumed that the light has on average a flat wave front at the source plane, which is, e.g., the case if the laser beam is focused on a thin scattering medium. Mathematically, this implies that G_0 is real. A source of this kind is known in literature as a *planar quasihomogeneous* source. The main results of our analysis will be derived without making any further assumption about the specific shapes of functions C_0 and I_0, and will thus be of very general validity.

III. 3D COHERENCE IN THE DEEP FRESNEL REGION

In order to describe the evolution of the coherence function when the field propagates over distances z_1,z_2, it is convenient first to go to the Fourier domain,

$$A(\mathbf{x},z) = \int \frac{d^2\mathbf{q}}{2\pi} e^{i\mathbf{q}\cdot\mathbf{x}}\tilde{A}(\mathbf{q},z) \tag{5}$$

and then insert the Fresnel propagator in Fourier space,

$$\tilde{A}(\mathbf{q},z) = \tilde{A}(\mathbf{q},0)e^{-i(q^2z/2k)}, \tag{6}$$

which describes free propagation over a distance z in the paraxial approximation ($k=2\pi/\lambda$ being the carrier wave number). By taking these two simple steps we readily obtain

$$G^{(1)}(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2) = \int \frac{d^2\mathbf{q}_1 d^2\mathbf{q}_2}{(2\pi)^2}[e^{-i\mathbf{q}_1\cdot\mathbf{x}_1}e^{i\mathbf{q}_2\cdot\mathbf{x}_2} \times e^{i(q_1^2z_1/2k)}e^{-i(q_2^2z_2/2k)}\langle\tilde{A}^*(\mathbf{q}_1,0)\tilde{A}(\mathbf{q}_2,0)\rangle]. \tag{7}$$

The function $\langle\tilde{A}^*(\mathbf{q}_1,0)\tilde{A}(\mathbf{q}_2,0)\rangle$ can be calculated from the definition of $G_0^{(1)}$ in Eq. (4) by using the inverse of the transformation (5). After some simple passages we get

[1]The assumption of equal time arguments in the definition of $G^{(1)}$ is appropriate for typical pseudothermal sources, for which the coherence time is usually much longer than both the measurement time and the time lag between two planes z_1 and z_2.

$$\langle\tilde{A}^*(\mathbf{q}_1,0)\tilde{A}(\mathbf{q}_2,0)\rangle = \int \frac{d^2\mathbf{x}_1 d^2\mathbf{x}_2}{(2\pi)^2} e^{i\mathbf{q}_1\cdot\mathbf{x}_1} e^{-i\mathbf{q}_2\cdot\mathbf{x}_2} G_0^{(1)}(\mathbf{x}_1,\mathbf{x}_2) = \tilde{C}_0\left(\frac{\mathbf{q}_1+\mathbf{q}_2}{2}\right)\tilde{I}_0(\mathbf{q}_2-\mathbf{q}_1), \tag{8}$$

where $\tilde{C}_0$ and $\tilde{I}_0$ denote the Fourier tranforms of the functions C_0 and I_0, respectively:

$$\tilde{C}_0(\mathbf{Q}) = \int \frac{d^2\mathbf{r}}{2\pi} e^{-i\mathbf{Q}\cdot\mathbf{r}} C_0(\mathbf{r}), \tag{9}$$

$$\tilde{I}_0(\mathbf{q}) = \int \frac{d^2\mathbf{X}}{2\pi} e^{-i\mathbf{q}\cdot\mathbf{X}} I_0(\mathbf{X}). \tag{10}$$

Note that $\tilde{C}_0$ as a function of $(\mathbf{q}_1+\mathbf{q}_2)/2$ now dies out over a "large" distance, on the order of $2\pi/\delta x_0$. In contrast, $\tilde{I}_0(\mathbf{q}_2-\mathbf{q}_1)$ has a "small" scale of variation, on the order of $2\pi/D$. For example, if both functions are Gaussians, $C_0(r) = \exp(-r^2/\delta x_0^2)$, $I_0(X) = I_0\exp(-X^2/D^2)$, their Fourier transforms are also Gaussians, with waists respectively $2/\delta x_0$ and $2/D$.

By inserting Eq. (8) in Eq. (7), and making the change of variables

$$(\mathbf{q}_1,\mathbf{q}_2) \to \left(\mathbf{Q} = \frac{\mathbf{q}_1+\mathbf{q}_2}{2}, \mathbf{q} = \mathbf{q}_2 - \mathbf{q}_1\right)$$

inside the integral on the right-hand side (RHS) of Eq. (7), we arrive at our first result:

$$G^{(1)}(\mathbf{x}_1,\bar{z}-dz/2;\mathbf{x}_2,\bar{z}+dz/2) = \int \frac{d^2\mathbf{Q}}{2\pi}[\tilde{C}_0(\mathbf{Q})e^{-iQ^2(dz/2k)}]e^{i\mathbf{Q}\cdot(\mathbf{x}_2-\mathbf{x}_1)} \times \int \frac{d^2\mathbf{q}}{2\pi}[\tilde{I}_0(\mathbf{q})e^{-iq^2(dz/8k)}]e^{i\mathbf{q}\cdot[(\mathbf{x}_2+\mathbf{x}_1)/2]}e^{-i\mathbf{Q}\cdot\mathbf{q}(\bar{z}/k)}. \tag{11}$$

So far no approximation has been made since the initial ansatz of factorability of the mutual coherence at the source plane [Eq. (4)]. By inspection of Eq. (11), we immediately notice that the integral at on the RHS factorizes into the product of two integrals if the factor $e^{-i\mathbf{Q}\cdot\mathbf{q}(\bar{z}/k)}$ is negligible for all values of $\mathbf{q}$ and $\mathbf{Q}$ in the domain of integration. Recalling that $\tilde{C}_0(\mathbf{Q})$ goes to zero for $\mathbf{Q}$ on the order of $2\pi/\delta x_0$ and $\tilde{I}_0(\mathbf{q})$ goes to zero for $\mathbf{q}$ on the order of $2\pi/D$, this condition amounts to requiring that $\frac{2\pi}{\delta x_0}\frac{2\pi}{D}\frac{\bar{z}}{k} \ll 2\pi$, which is equivalent to

$$\bar{z} \ll \frac{D\,\delta x_0}{\lambda} \stackrel{\text{def}}{=} z_{\text{VCZ}}. \tag{12}$$

Thus, in the zone defined by the condition (12), which following Cerbino [7] we shall call the *deep Fresnel zone*, Eq. (11) reduces to

$$G^{(1)}(\mathbf{x}_1,\bar{z}-dz/2;\mathbf{x}_2,\bar{z}+dz/2) \underset{\bar{z}\ll z_{\text{VCZ}}}{=} \int \frac{d^2\mathbf{Q}}{2\pi}[\tilde{C}_0(\mathbf{Q})e^{-iQ^2(dz/2k)}]e^{i\mathbf{Q}\cdot(\mathbf{x}_2-\mathbf{x}_1)} \times \int \frac{d^2\mathbf{q}}{2\pi}[\tilde{I}_0(\mathbf{q})e^{-iq^2(dz/8k)}]e^{i\mathbf{q}\cdot[(\mathbf{x}_2+\mathbf{x}_1)/2]}. \tag{13}$$

In particular, on the same plane $z_2=z_1$ $(dz=0)$,

$$G^{(1)}(\mathbf{x}_1,\bar{z};\mathbf{x}_2,\bar{z}) = C_0(\mathbf{x}_2-\mathbf{x}_1)I_0\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}\right) = G_0^{(1)}(\mathbf{x}_1,\mathbf{x}_2). \tag{14}$$

Thus we recover the same condition and result of [7,8]: in the deep Fresnel zone ($\bar{z} \ll z_{\text{VCZ}}$) the coherence function remains the same as in the source plane and the transverse spatial coherence properties of light do not change upon propagation. In particular, the transverse size of the speckles[2] δx_n remains the same as at the source plane,

$$\delta x_n = \delta x_0. \tag{15}$$

When we come to the longitudinal coherence observed between two different planes ($dz \neq 0$), Eq. (13) can be recast as

$$G^{(1)}\left(\mathbf{x}_1,\bar{z}-\frac{dz}{2};\mathbf{x}_2,\bar{z}+\frac{dz}{2}\right) = C_{dz}(\mathbf{x}_2-\mathbf{x}_1)I_{dz/4}\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}\right), \tag{16a}$$

where C_{dz} and I_{dz} indicate the functions C_0 and I_0 after propagation over a distance dz with the free Fresnel propagator,

$$C_{dz}(\mathbf{x}) = \int \frac{d^2\mathbf{Q}}{2\pi}\tilde{C}_0(\mathbf{Q})e^{-iQ^2(dz/2k)}e^{i\mathbf{Q}\cdot\mathbf{x}} = \int \frac{d^2\mathbf{x}'}{i\lambda dz}C_0(\mathbf{x}')e^{i|\mathbf{x}-\mathbf{x}'|^2(k/2dz)}, \tag{16b}$$

$$I_{dz/4}(\mathbf{x}) = \int \frac{d^2\mathbf{q}}{2\pi}\tilde{I}_0(\mathbf{q})e^{-iq^2[(dz/4)/2k]}e^{i\mathbf{q}\cdot\mathbf{x}} = \int \frac{d^2\mathbf{x}'}{i\lambda dz}I_0(\mathbf{x}')e^{i|\mathbf{x}-\mathbf{x}'|^2[k/(2dz/4)]} \approx I_0(\mathbf{x}). \tag{16c}$$

The last line is obtained by observing that $dz/4 < \bar{z} \ll z_{\text{VCZ}} \ll D^2/\lambda$, where D^2/λ is the typical diffraction length for a beam of waist D, so that $I_{dz/4}(\mathbf{x}) \approx I_0(\mathbf{x})$.

Equations (16a)–(16c) represents the first key result of this work: in the deep Fresnel region the coherence function between two planes separated by adistance dz evolves with dz following the free propagation of a light beam having $C_0(\mathbf{x})$ as its field distribution at $dz=0$. In this way, we can picture the 3D light speckles in the deep Fresnel zone as 3D

[2] In the following the subscript n will denote quantities in the deep Fresnel region $z \ll z_{\text{VCZ}}$ (near field), such as the transverse and longitudinal coherence lengths δx_n, δz_n. The subscript f will label quantities in the region of validity of the VCZ theorem (far field), $z \gg z_{\text{VCZ}}$.

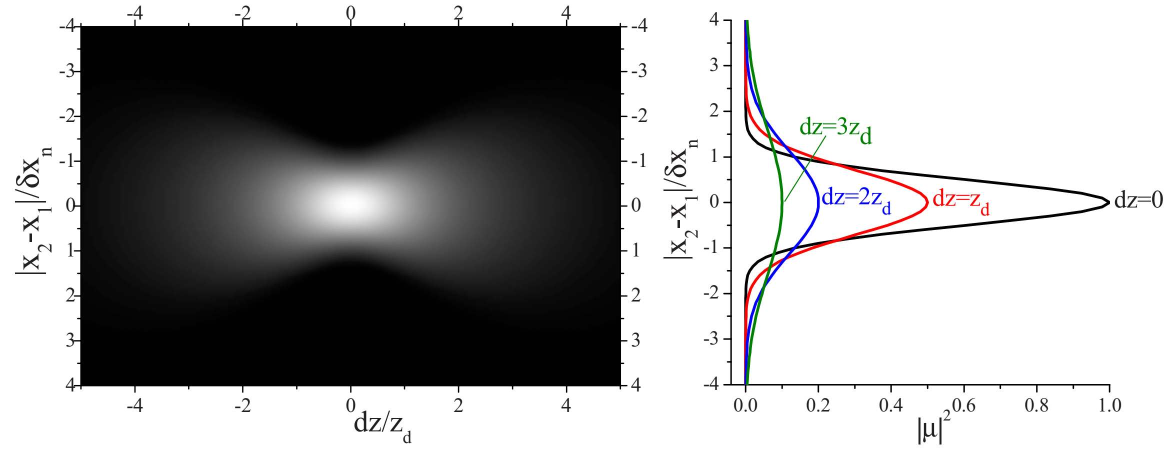

FIG. 2. (Color online) 3D Gaussian speckles in the deep Fresnel region. The gray scale image shows the degree of mutual coherence $|\mu|^2$ [Eq. (23)] at two 3D points $(\mathbf{x}_1,z_1)$, $(\mathbf{x}_2,z_2)$, plotted as a function of $|\mathbf{x}_2-\mathbf{x}_1|/\delta x_n$ (vertical axis) and $dz/z_d=(z_2-z_1)/z_d$ (horizontal axis). The graph on the right shows sections of the image along the radial coordinate at fixed dz.

coherence volumes whose shape reproduces the shape of a diffracting light beam having its focus in $dz=0$.

As a general result of this analysis, we can conclude that in the deep Fresnel zone the coherence decays with the longitudinal separation dz with the same law of ordinary diffraction. The *longitudinal coherence length* δz_n depends on the transverse coherence length δx_n at $dz=0$ as

$$\delta z_n \propto \frac{\delta x_n^2}{\lambda}. \tag{17}$$

Equation (17) relates the coherence length of speckles δz_n to the transverse size δx_n, as expected for ordinary diffracting beams. From Eq. (16b) another remarkable property follows:

$$\int d^2\mathbf{x}|C_{dz}(\mathbf{x})|^2 = \int d^2\mathbf{x}|C_0(\mathbf{x})|^2 = \text{const}. \tag{18}$$

This conservation law implies that, as the transverse peak $C_{dz}(0)$ decays with dz, the tranverse profile of the coherence function broadens with increasing dz. If we call $\delta x_n(dz)$ the transverse width of the coherence function between two planes at distance dz (not to be confused with the speckle transverse size, observed in the laboratory at $dz=0$), this increases with dz as $\delta x_n/|C_{dz}(0)|$. This is exactly what happens to a diffracting beam as propagation distance increases, and Eq. (18) is equivalent to beam power conservation.

Following Goodman [3], the quantity that best describes the coherence of a beam is the *degree of mutual coherence*

$$|\mu(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2)|^2 = \frac{|G^{(1)}(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2)|^2}{\langle I(\mathbf{x}_1,z_1)\rangle\langle I(\mathbf{x}_2,z_2)\rangle}. \tag{19}$$

With this definition the Siegert relation (2) takes the form

$$\langle I(\mathbf{x}_1,z_1)I(\mathbf{x}_2,z_2)\rangle = \langle I(\mathbf{x}_1,z_1)\rangle\langle I(\mathbf{x}_2,z_2)\rangle \times[1+|\mu(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2)|^2], \tag{20}$$

so that in a Hanbury-Brown and Twiss [21] kind of mesurement $|\mu|^2$ represents the visibility of the intensity autocorrelation. The assumption of well-separated scales of spatial variation for the intensity distribution and the correlation ensures that for small distances $|\mathbf{x}_2-\mathbf{x}_1|\approx\delta x_0$, $I_0[(\mathbf{x}_2+\mathbf{x}_1)/2]\simeq I_0(\mathbf{x}_1)\simeq I_0(\mathbf{x}_2)$, so that in the deep Fresnel region the remarkable result holds:

$$|\mu(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2)|^2 = |C_{dz}(\mathbf{x}_2-\mathbf{x}_1)|^2. \tag{21}$$

The degree of mutual coherence depends in this limit only on the differences $dz=z_2-z_1$, and $\mathbf{x}_2-\mathbf{x}_1$.

A simple example is that of Gaussian speckles. Assuming that, at the source plane, $C_0(\mathbf{x}_2-\mathbf{x}_1)=\exp(-|\mathbf{x}_2-\mathbf{x}_1|^2/\delta x_0^2)$, we get

$$G^{(1)}\left(\mathbf{x}_1,\bar{z}-\frac{dz}{2};\mathbf{x}_2,\bar{z}+\frac{dz}{2}\right) = I_0\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}\right)C_{dz}(\mathbf{x}_2-\mathbf{x}_1) = \frac{I_0[(\mathbf{x}_1+\mathbf{x}_2)/2]}{\sqrt{1+\frac{dz^2}{z_d^2}}}e^{-[|\mathbf{x}_2-\mathbf{x}_1|^2/\delta x_n^2(dz)][1+i(dz/z_d)]}e^{-i\tan^{-1}(dz/z_d)}, \tag{22a}$$

where

$$\delta x_n(dz) = \delta x_0\sqrt{1+\frac{dz^2}{z_d^2}}, \tag{22b}$$

and $z_d=\pi\delta x_0^2/\lambda$ is the diffraction length for Gaussian beams (Rayleigh range). Using again the approximation $I_0[(\mathbf{x}_2+\mathbf{x}_1)/2]\simeq I_0(\mathbf{x}_1)\simeq I_0(\mathbf{x}_2)$, we obtain

$$\left|\mu\left(\mathbf{x}_1,\bar{z}-\frac{dz}{2};\mathbf{x}_2,\bar{z}+\frac{dz}{2}\right)\right|^2 = \frac{1}{1+\frac{dz^2}{z_d^2}}e^{(-2|\mathbf{x}_2-\mathbf{x}_1|^2)/\delta x_n^2(dz)}. \tag{23}$$

Figure 2 displays the 3D degree of mutual coherence $|\mu|^2$ of a Gaussian speckle (gray scale image) in the plane $(dz,|\mathbf{x}_2-\mathbf{x}_1|)$. The right panel plots sections of $|\mu|^2$ at fixed dz, showing how the peak of the radial correlation function becomes less pronounced and wider with increasing dz.

IV. 2D TRANSVERSE COHERENCE: FROM DEEP FRESNEL SPECKLES TO VCZ THEOREM

This section will review, for the convenience of the reader, the transverse coherence properties of light at a generic distance z from the source plane. This will allow us to show the transition from speckles in the deep Fresnel zone, described in the previous section, to the VCZ speckles, whose properties are described by the classical Van Cittert–Zernike theorem. The results will be derived with the formalism outlined in our previus sections, and will turn out to be completely equivalent to those recently presented in [7], in the framework of a different formalism.

Let us come back to Eq. (11), which describes the 3D mutual coherence function under the general assumption of a quasihomogeneous source. Let us focus on the case $dz=0$, i.e., on the transverse coherence on a plane at a distance $z=\bar{z}$ from the source plane. By performing the integral over the variable $\mathbf{Q}$, Eq. (11) for $dz=0$ can be recast in the alternative and equivalent form

$$G^{(1)}(\mathbf{x}_1,z;\mathbf{x}_2,z) = \int \frac{d^2\mathbf{q}}{2\pi}\tilde{I}_0(\mathbf{q})C_0\left(\mathbf{x}_2-\mathbf{x}_1-\mathbf{q}\frac{z}{k}\right)e^{i\mathbf{q}\cdot[(\mathbf{x}_2+\mathbf{x}_1)/2]} = \left(\frac{k}{z}\right)^2\int\frac{d^2\boldsymbol{\xi}}{2\pi}\tilde{I}_0\left(\frac{k}{z}\boldsymbol{\xi}\right)C_0(\mathbf{x}_2-\mathbf{x}_1-\boldsymbol{\xi})e^{[i\boldsymbol{\xi}\cdot(\mathbf{x}_2+\mathbf{x}_1)/2](k/z)}, \tag{24}$$

where the last equality has been obtained by performing the change of integration variable $\mathbf{q}z/k\rightarrow\boldsymbol{\xi}$. In this way, the coherence function at a generic plane z takes the form of a convolution integral between (a) the Fourier tranform of the intensity distribution $\tilde{I}_0$, peaked at $\boldsymbol{\xi}=\mathbf{0}$, with width $\delta x_f(z)=z\lambda/D$, and (b) the transverse correlation function at the source plane C_0, peaked at $\boldsymbol{\xi}=\mathbf{x}_2-\mathbf{x}_1$, with width δx_0. Incidentally, we remark that, by following [7], we may define

$$J_z^{\mathrm{VCZ}}(\mathbf{X};\mathbf{r}) = \left(\frac{k}{z}\right)^2 e^{i(k/z)\mathbf{X}\cdot\mathbf{r}}\tilde{I}_0\left(\frac{k}{z}\mathbf{r}\right), \tag{25}$$

where $\mathbf{X}=(\mathbf{x}_2+\mathbf{x}_1)/2$ and $\mathbf{r}=\mathbf{x}_2-\mathbf{x}_1$. By using this definition, Eq. (24) takes the form

$$G^{(1)}(\mathbf{x}_1,z;\mathbf{x}_2,z) = \left[J_z^{\mathrm{VCZ}}\left(\frac{\mathbf{x}_2+\mathbf{x}_1}{2};\cdot\right)\otimes C_0(\cdot)\right](\mathbf{x}_2-\mathbf{x}_1), \tag{26}$$

which is exactly Eq. (9) of [7].

Let us come back to our Eq. (24): as in all the convolutions, the broader function "wins" over the other. There are two limiting behaviors:

(i) $\delta x_f(z)\ll\delta x_0$, equivalent to $z\ll z_{\mathrm{VCZ}}$ (deep Fresnel zone). C_0 varies much more slowly than $\tilde{I}_0$. In all the integration region, where $\tilde{I}_0\neq 0$, C_0 is approximately constant and equal to its value at $\boldsymbol{\xi}=0$, so that

$$G^{(1)}(\mathbf{x}_1,z;\mathbf{x}_2,z) \underset{z\ll z_{\mathrm{VCZ}}}{=} C_0(\mathbf{x}_2-\mathbf{x}_1)\left(\frac{k}{z}\right)^2 \times\int\frac{d^2\boldsymbol{\xi}}{2\pi}\tilde{I}_0\left(\frac{k}{z}\boldsymbol{\xi}\right)e^{i\boldsymbol{\xi}\cdot[(\mathbf{x}_2+\mathbf{x}_1)/2](k/z)} = C_0(\mathbf{x}_2-\mathbf{x}_1)I_0\left(\frac{\mathbf{x}_2+\mathbf{x}_1}{2}\right), \tag{27}$$

where the last line as been obtained by recognizing in the integral on the RHS, after a simple change of variables, the inverse Fourier transform of $\tilde{I}_0$. We retrieve the behavior described in the previous section: in the deep Fresnel zone the transverse coherence properties remain the same as at the source plane.

(ii) $\delta x_f(z)\gg\delta x_0$, equivalent to $z\gg z_{\mathrm{VCZ}}$ (VCZ zone). In this case $\tilde{I}_0$ has a slow variation compared to C_0, so that in all the integration region it can be considered constant and equal to its value at $\boldsymbol{\xi}=\mathbf{x}_2-\mathbf{x}_1$. Equation (24) thus becomes

$$G^{(1)}(\mathbf{x}_1,z;\mathbf{x}_2,z) \underset{z\gg z_{nf}}{=} \left(\frac{k}{z}\right)^2\tilde{I}_0\left((\mathbf{x}_2-\mathbf{x}_1)\frac{k}{z}\right) \times\int\frac{d^2\boldsymbol{\xi}}{2\pi}C_0(\mathbf{x}_2-\mathbf{x}_1-\boldsymbol{\xi})e^{i\boldsymbol{\xi}\cdot[(\mathbf{x}_2+\mathbf{x}_1)/2](k/z)}. \tag{28}$$

After some manipulation, the integral on the RHS is calculated, yielding

$$G^{(1)}(\mathbf{x}_1,z;\mathbf{x}_2,z) \underset{z\gg z_{nf}}{=} \left(\frac{k}{z}\right)^2 e^{i(x_2^2-x_1^2)(k/2z)}\tilde{C}_0\left(\frac{\mathbf{x}_2+\mathbf{x}_1}{2}\frac{k}{z}\right) \times\tilde{I}_0\left((\mathbf{x}_2-\mathbf{x}_1)\frac{k}{z}\right). \tag{29}$$

Equation (29) is nothing else than the generalized form of the Van Cittert–Zernike theorem (see [3]). In this equation, the term $\tilde{C}_0$ accounts for the overall profile of the beam, which in the VCZ region has a width on the order of $D_f=z\lambda/\delta x_0$. The term $\tilde{I}_0$ is the usual correlation term of Van–Cittert Zernike form: in the VCZ region the transverse correlation function is basically the Fourier tranform of the intensity distribution at the source plane, with a width that depends on the distance z, and is on the order of

$$\delta x_f = \delta x_f(z) = \frac{z\lambda}{D}. \tag{30}$$

V. 3D COHERENCE IN THE VCZ REGION

This section is devoted to the 3D coherence in the VCZ region, defined by $z\gg z_{\mathrm{VCZ}}$. In comparison with the behavior in the deep Fresnel region, some relevant differences will emerge, mainly due to the presence of a non-negligible wave front curvature of the overall beam.

In order to calculate the 3D coherence function in this region, it is convenient to write the field propagation in direct space:

$$A(\mathbf{x},z) = \int \frac{d^2\mathbf{x}'}{i\lambda z} A(\mathbf{x}',0) e^{i|\mathbf{x}-\mathbf{x}'|^2(k/2z)}. \tag{31}$$

This formalism is more convenient with respect to description in Fourier space, adopted in the deep Fresnel region [Eq. (6)], because in the VCZ region z is typically large with respect to the other lengths in play, so that the phase factor $\exp - iq^2 z/2k$ tends to have a fast variation, giving rise to singular behaviors. In contrast, the phase $|\mathbf{x}-\mathbf{x}'|^2 k/2z$ remains limited.

With this in mind, we insert the z propagation into the definition of $G^{(1)}$, and make use of the factorability ansatz (4):

$$G^{(1)}(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2) = \int \frac{d^2\mathbf{x}_1' d^2\mathbf{x}_2'}{\lambda^2 z_1 z_2}\Bigg[e^{-i|\mathbf{x}_1-\mathbf{x}_1'|^2(k/2z_1)} \times e^{i|\mathbf{x}_2-\mathbf{x}_2'|^2(k/2z_2)} C_0(\mathbf{x}_2'-\mathbf{x}_1') I_0\left(\frac{\mathbf{x}_1'+\mathbf{x}_2'}{2}\right)\Bigg]. \tag{32}$$

By using the condition $z_1,z_2 \gg z_{\mathrm{VCZ}} = D\delta x_0/\lambda$ which characterizes the VCZ region, the integral on the RHS of Eq. (32) can be simplified. After some steps that are described in detail in the Appendix, we arrive at the final result

$$G^{(1)}(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2) \underset{z_1 \gg z_{\mathrm{VCZ}}}{=} \frac{k^2}{z_1 z_2} e^{i(k/2)[(x_2^2/z_2)-(x_1^2/z_1)]} \times \tilde{C}_0\left[\frac{k}{2z_1}\left(\mathbf{x}_2\frac{z_1}{z_2}+\mathbf{x}_1\right)\right] \times \phi_{dz}\left[\frac{k}{z_1}\left(\mathbf{x}_2\frac{z_1}{z_2}-\mathbf{x}_1\right)\right], \tag{33a}$$

where

$$\phi_{dz}(\mathbf{q}) = \int \frac{d^2\mathbf{X}}{2\pi} e^{i(k/2)X^2[1/(z_1+dz)-(1/z_1)]} e^{-i\mathbf{X}\cdot\mathbf{q}} I_0(\mathbf{X}). \tag{33b}$$

Equation (33a) and (33b), which establishes the very general form of the 3D coherence in the VCZ zone, is the second key result of this work. This equation generalizes Eq. (4.76) of Goodman [12], as well as the results reported in [10] to the case of partial coherence of the source and to an arbitrary longitudinal separation dz (i.e., we do not use the approximation $dz \ll z_1$).

If in Eq (33a) and (33b) we put $dz=0$ we get

$$\phi_0\left(\frac{k}{z_1}(\mathbf{x}_2-\mathbf{x}_1)\right) = \tilde{I}_0\left(\frac{k}{z_1}(\mathbf{x}_2-\mathbf{x}_1)\right). \tag{34}$$

In this way we recover, as we must, the generalized Van Cittert–Zernike theorem expressed by Eq. (29) of the previous section.

Let us focus, for the sake of definiteness, on the case $dz > 0$ (i.e., $z_2 > z_1$). In comparison of Eq. (33a) and (33b) with Eq. (29), the following main differences appear.

(i) In the factor $1/z_1 z_2$ in place of $1/z_1^2$, which takes into account the attenuation of the light intensity because of beam broadening. This is a rather inessential factor, which disappears when the degree of coherence $|\mu|^2$ is considered.

(ii) In the function $\phi_{dz}[(k/z_1)(\mathbf{x}_2 z_1/z_2 - \mathbf{x}_1)]$ in place of $\tilde{I}_0[(k/z_1)(\mathbf{x}_2-\mathbf{x}_1)]$. First of all we notice that for $z_2 > z_1$ the correlation peak occurs for $\mathbf{x}_2 = \mathbf{x}_1 z_2/z_1$. This sort of "magnification" factor is again a consequence of the beam broadening at $z_2 > z_1$. Second, we notice the presence of the phase factor $\exp i(kX^2/2)[1/(z_1+dz)-1/z_1]$ inside the integral on the RHS of Eq. (33b). This factor suggests a decay of the longitudinal coherence with dz. We will come back in detail to this key point in the following, and show conditions where this decay actually does not take place.

For the sake of simplicity, we shall ignore the details of the intensity profile variation, that is, we take $\tilde{C}_0[(k/2z_1)(\mathbf{x}_2 z_1/z_2 + \mathbf{x}_1)] \simeq \tilde{C}_0[(k/z_1)\mathbf{x}_1] \simeq \tilde{C}_0[(k/z_2)\mathbf{x}_2]$ (actually, this is a quite good approximation when both $\mathbf{x}_1$ and $\mathbf{x}_2$ lie inside a transverse coherence area). By taking into account Eqs. (33a), (33b), and (29), we thus have

$$|\mu(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2)|^2 = \left| \frac{\phi_{dz}\left[\frac{k}{z_1}\left(\mathbf{x}_2\frac{z_1}{z_2}-\mathbf{x}_1\right)\right]}{\tilde{I}_0(0)} \right|^2. \tag{35}$$

A first general insight into the longitudinal coherence properties in the VCZ zone is obtained by observing that, under the condition

$$\frac{kX^2}{2}\left(\frac{1}{z_1+dz}-\frac{1}{z_1}\right) \ll \pi, \tag{36}$$

for all values of $\mathbf{X}$ for which $I_0(\mathbf{X}) \neq 0$, the phase factor inside the integral in Eq. (33b) is negligible and

$$\phi_{dz}\left[\frac{k}{z_1}\left(\mathbf{x}_2\frac{z_1}{z_2}-\mathbf{x}_1\right)\right] = \tilde{I}_0\left[\frac{k}{z_1}\left(\mathbf{x}_2\frac{z_1}{z_2}-\mathbf{x}_1\right)\right]. \tag{37}$$

In these conditions, by taking $\mathbf{x}_2 = \mathbf{x}_1 z_2/z_1$, that is, by following the radial broadening of the beam with increasing z_2, the degree of coherence is preserved: $|\mu(\mathbf{x}_1,z_1;\mathbf{x}_1 z_2/z_1,z_2)|^2 = 1$. Recalling that I_0 dies out over a distance D, condition (36) amounts to requiring

$$\frac{D^2 dz}{\lambda z_1(z_1+dz)} \ll 1. \tag{38}$$

This condition provides an approximate order of magnitude for the VCZ longitudinal coherence length δz_f: setting, e.g., δz_f as the value of dz at which the LHS of Eq. (38) is equal to unity, we obtain

$$\delta z_f = \frac{\lambda z_1^2}{D^2}\frac{1}{(1-\lambda z_1/D^2)} = \frac{\delta x_f^2(z_1)}{\lambda}\frac{1}{(1-z_1/z_{\mathrm{Fr}})}, \tag{39}$$

where

$$z_{\mathrm{Fr}} = \frac{D^2}{\lambda} \tag{40}$$

is the diffraction length (*Fraunhofer distance*) for a beam of size D, and $\delta x_f = \delta x_f(z_1) = z_1\lambda/D$ is the transverse coherence length (the transverse speckle size) at $z = z_1$.

First of all we notice that, for positive dz, Eq. (39) can be satisfied only for $z_1 < z_{\mathrm{Fr}}$. Second, we notice that only for $z_1 \ll z_{\mathrm{Fr}}$, that is, well away from the Fraunhofer zone, does the coherence function evolve longitudinally with dz following the ordinary laws of diffraction: $\delta z_f \simeq \delta x_f^2(z_1)/\lambda$. On the contrary, as z_1 approaches the Fraunhofer zone, the longitudinal coherence length approaches infinity; as a consequence *in the Fraunhofer zone the coherence volumes (the 3D speckles) become truly infinite in length* (see Fig. 5 for an example).

Incidentally, we remark that $\delta x_f^2(z_1)/\lambda = z_1 z_1/z_{\mathrm{Fr}}$, so that, in the Fraunhofer zone where $z_1 \geq z_{\mathrm{Fr}}$, the longitudinal decay of coherence takes place on distances on the order of z_1. Therefore, the approximation $dz \ll z_1$, commonly used in the literature [10–12], which leads to the jelly bean view of speckles, loses its validity as the Fraunhofer zone is entered.

A closer insight into the problem is gained by inspecting the case of a Gaussian intensity distribution: $I_0(\mathbf{X}) = I_0 \exp -X^2/D^2$. By performing the simple Gaussian integral involved in ϕ_{dz}, we obtain the result

$$|\mu(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2)|^2 = \frac{1}{1+\gamma^2(dz)} \exp - \frac{2|\mathbf{x}_2 z_1/z_2 - \mathbf{x}_1|^2}{\delta x_f^2(z_1;dz)}, \tag{41a}$$

where

$$\gamma(dz) = \frac{\pi D^2 dz}{\lambda z_1 z_2}, \tag{41b}$$

$$\delta x_f(z_1;dz) = \delta x_f(z_1)\sqrt{1+\gamma^2(dz)}, \tag{41c}$$

and

$$\delta x_f(z_1) = \frac{z_1 \lambda}{\pi D} \tag{41d}$$

is the tranverse coherence length of Gaussian speckles in the VCZ zone at $z = z_1$.

Clearly, both longitudinal and transverse coherence properties are preserved upon propagation along dz as long as $\gamma(dz) \ll 1$. This condition is closely similar to the condition (38) found in the general case. By setting

$$z_{\mathrm{Fr}} = \frac{\pi D^2}{\lambda}, \tag{42}$$

the Fraunhofer distance for a Gaussian beam of waist D, and

$$z_d = \frac{\pi \delta x_f^2(z_1)}{\lambda}, \tag{43}$$

the diffraction length for a Gaussian speckle of waist $\delta x_f(z_1)$, after some algebraic manipulation, Eq. (41b) can be recast as

$$\gamma(dz) = \frac{dz}{z_d} \frac{1}{1 + \frac{dz}{z_d}\frac{z_1}{z_{\mathrm{Fr}}}}. \tag{44}$$

Notice that the ordinary form of a diffracting Gaussian speckle, such as, for example, the one occurring in the deep Fresnel region, would involve a linear increase of γ with dz [see Eq. (23)]. However, in the VCZ region, γ increases linearly with dz only for $dz \ll z_d z_{\mathrm{Fr}}/z_1$. On the contrary, for $dz \gg z_d z_{\mathrm{Fr}}/z_1$, it saturates to the asymptotic value z_{Fr}/z_1. Focusing on the longitudinal coherence properties of Gaussian speckles, we therefore find for the transverse peak value of $|\mu|^2$

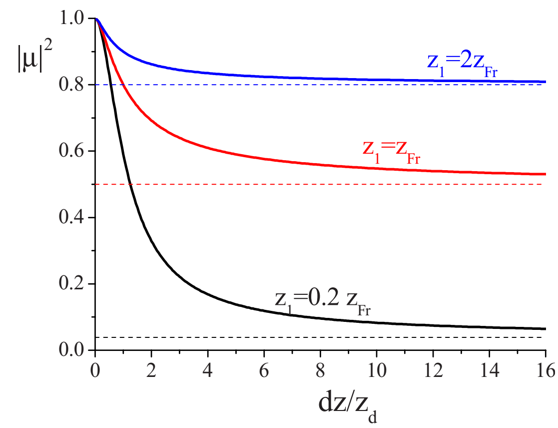

FIG. 3. (Color online) Longitudinal coherence of 3D Gaussian speckles in the VCZ region. The graph shows the degree of mutual coherence $|\mu|^2$, at its tranverse peak value, i.e., at the two points $(\mathbf{x}_1,z_1)$, $(\mathbf{x}_2 = \mathbf{x}_1 z_2/z_1, z_2)$, plotted as a function of dz/z_d. The three solid lines correspond to different values of the distance z_1, namely, $z_1/z_{\mathrm{Fr}} = 0.2, 1, 2$, respectively. The dashed lines are the asymptotic values for $dz \to \infty$.

$$\left|\mu\left(\mathbf{x}_1,z_1;\mathbf{x}_1\frac{z_2}{z_1},z_2\right)\right|^2 = \frac{1}{1+\gamma^2(dz)} \tag{45}$$

$$\to \begin{cases} \dfrac{1}{1+dz^2/z_d^2} & \text{for } dz \ll z_d \dfrac{z_{\mathrm{Fr}}}{z_1}, \quad (46) \\[2ex] \dfrac{1}{1+z_{\mathrm{Fr}}^2/z_1^2} & \text{for } dz \gg z_d \dfrac{z_{\mathrm{Fr}}}{z_1}. \quad (47) \end{cases}$$

Thus, in the zone $z_{\mathrm{VCZ}} \ll z_1 \ll z_{\mathrm{Fr}}$ the behavior of the transverse peak value follows basically a Lorenztian decay with dz, similarly to what happens in the deep Fresnel zone [Eq. (23)]. However, approaching the Fraunhofer zone, the decay is more and more inhibited, and the tranverse peak value stays close to its value at $dz = 0$; in this zone the longitudinal coherence length becomes substantially infinite. This behavior is illustrated in Fig. 3.

Concerning the transverse coherence properties of speckles, they follow a similar behavior: by using Eqs. (41d) and (44), we get the two limiting trends

$$\frac{\delta x_f(z_1;dz)}{\delta x_f(z_1)} \to \begin{cases} \sqrt{1+\dfrac{dz^2}{z_d^2}} & \text{for } dz \ll z_d \dfrac{z_{\mathrm{Fr}}}{z_1}, \quad (48) \\[2ex] \sqrt{1+\dfrac{z_{\mathrm{Fr}}^2}{z_1^2}} & \text{for } dz \gg z_d \dfrac{z_{\mathrm{Fr}}}{z_1}. \quad (49) \end{cases}$$

As a consequence, well inside the Fraunhofer zone, when the radial broadening of the beam is followed (that is, the transverse degree of coherence of the speckle is plotted as a function of $\mathbf{x}_2 z_1/z_2 - \mathbf{x}_1$), the transverse profile of the speckle remains substantially invariate upon propagation along dz.

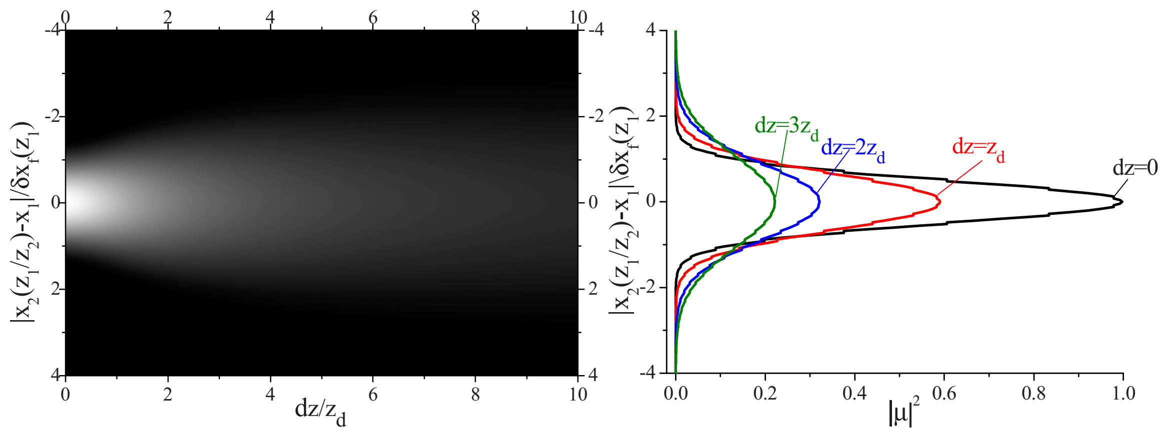

FIG. 4. (Color online) 3D Gaussian speckles in the VCZ region, far from the Fraunhofer region ($z_1=0.2z_{\rm Fr}$). The gray scale image shows the degree of mutual coherence $|\mu|^2$, calculated at two 3D points $(\mathbf{x}_1,z_1)$, $(\mathbf{x}_2,z_2)$, plotted as a function of $|\mathbf{x}_2z_1/z_2-\mathbf{x}_1|/\delta x_f(z_1)$ and $dz/z_d=(z_2-z_1)/z_d$. The right panel shows sections of the image along the radial coordinate at fixed dz.

Example of these results are given in Figs. 4 and 5, for $z_1=0.2z_{\rm Fr}$ (far from the Fraunhofer zone) and $z_{\rm Fr}$ (at the border of the Fraunhofer zone), respectively. The gray-scale images show the degree of coherence as a function of the longitudinal and transverse separations, and clearly evidence how, on approaching the Fraunhofer zone, the speckles become infinite in length.

VI. DISCUSSION

The different behaviors of the longitudinal coherence of speckles in the deep Fresnel and the Fraunhofer zones have a rather intuitive explanation. Qualitatively, we can picture the speckles as light spots with an average diameter δx, separated by an average distance δx (Fig. 6). In the deep Fresnel zone, and in the portion of the Fresnel zone well away from the Fraunhofer zone, the average beam wave front is basically flat. Speckles broaden only because of their diffraction, and their broadening rate is characterized by the diffraction angle $\theta_{\rm diff}\approx\lambda/\delta x$. Their longitudinal coherence length can be approximately evaluated as the distance at which light originating from two neighbor speckles starts to interfere: $\delta z\propto\delta x/\theta_{\rm diff}=\delta x^2/\lambda$ [Fig. 6(a)]. If one took pictures of the light distributions on two planes at distances larger than δz apart, they would appear different because interference of light originating from different speckles has taken place. We can thus conclude that in this region the longitudinal coherence properties are ruled by diffraction.

Strikingly different is the behavior of speckles in the Fraunhofer zone. As the distance z from the source increases, speckles grow in transverse size, so that their diffraction angle becomes more and more negligible: $\theta_{\rm diff}\approx\lambda/\delta x_f=D/z$. On the other hand, the wave front of the overall beam acquires a curvature. In the Fraunhofer region, the average beam wave front is approximately spherical, with a radius of curvature equal to the distance z from the source. Because of this curvature, two neighbor speckles of size $\delta x_f=z\lambda/D$ broaden and at the same time spread apart one from each other at a rate $\theta_{\rm curv}=\delta x_f/z=\lambda/D$ [Fig. 6(b)]. We have that

$$\theta_{\rm diff}<\theta_{\rm curv}\quad \text{for } z>z_{\rm Fr}=\frac{D^2}{\lambda}. \tag{50}$$

In the Fraunhofer zone where the condition (50) is satisfied, the broadening of speckles due to pure diffraction becomes

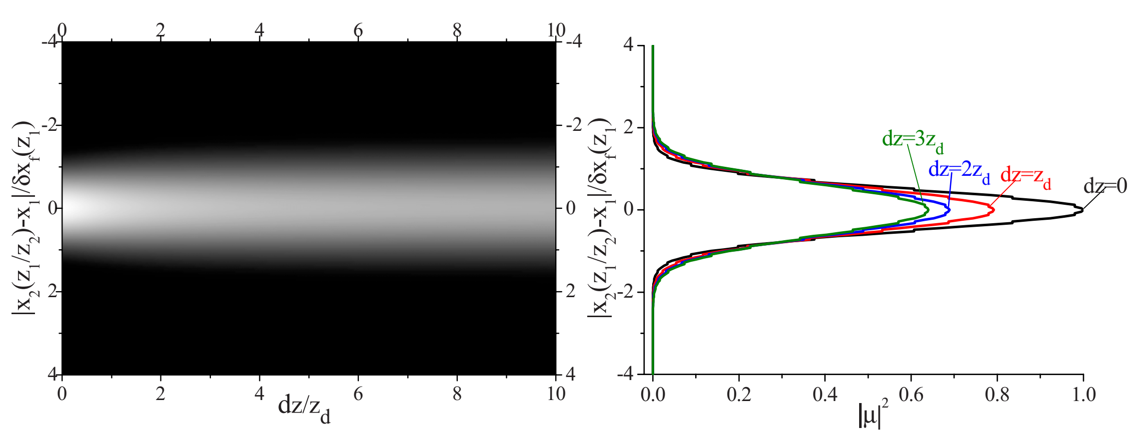

FIG. 5. (Color online) 3D Gaussian speckles in the VCZ region, close to the Fraunhofer zone ($z_1=z_{\rm Fr}$). The gray scale image shows the degree of mutual coherence $|\mu|^2$, calculated at two 3D points $(\mathbf{x}_1,z_1)$, $(\mathbf{x}_2,z_2)$, plotted as a function of $|\mathbf{x}_2z_1/z_2-\mathbf{x}_1|/\delta x_f$ and $dz/z_d=(z_2-z_1)/z_d$. The right panel shows sections of the image along the radial coordinate at fixed dz.

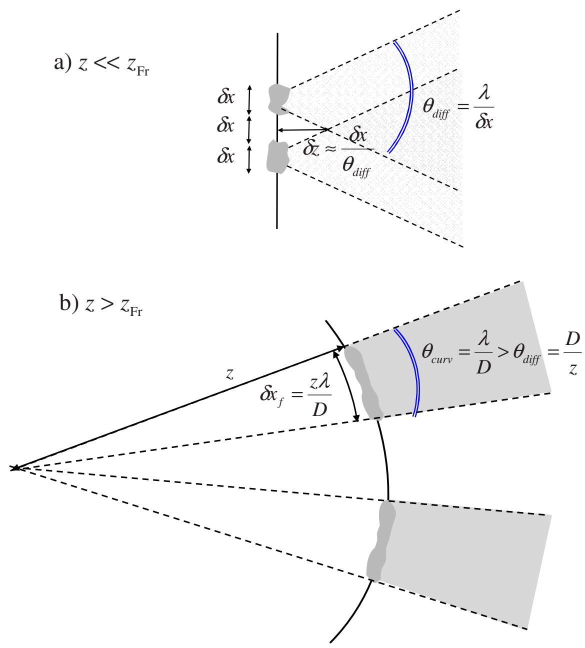

FIG. 6. (Color online) (a) In the deep Fresnel zone the beam wave front is flat, and the longitudinal coherence of speckles is ruled only by pure diffraction. $\delta z = \delta x^2/\lambda$ can be seen as the distance at which light originating from two neighbor speckles starts to interfere. (b) In the Fraunhofer zone, the beam wave front is curved. The diffraction aperture of speckles is smaller than the angle at which they spread apart because of the beam curvature, so that light originating from different speckles does not intefere any more and $\delta z \to \infty$. z is the distance from the source; $\theta_{\mathrm{curv}} = \lambda/D$ is the rate of divergence of speckles of size $\delta x_f = z\lambda/D$ due to the wave front curvature; $\theta_{\mathrm{diff}} \approx \lambda/\delta x_f = D/z$ is the pure diffraction aperture of speckles (see text for details).

negligible with respect to the rate at which they broaden and spread apart because of the wave front curvature; as a consequence, interference between speckles does not take place because, once a speckle is created, it does not superimpose any more with its neighbors. The speckles spread out with the beam without mixing, so that the light distribution broadens with distance but does not change its shape. Therefore, we can conclude that in the Fraunhofer zone, the speckles can be described as light pencils that have an infinite length, and the light distribution changes with propagation distance only by an overall scale factor: the light distributions at various distances are omothetical.

In the intermediate region, we can assume that speckles have a mixed behavior. As the distance from the source grows, the beam wave front acquires a curvature and the speckles start to spread apart from each other; because of this, light diffracted out from two neighbor speckles takes longer to interfere, and the coherence length becomes longer than the pure diffractive length $\delta x^2/\lambda$. In this intermediate region we can thus picture 3D speckles as jelly beans elongated in the direction outward from the source.

Our findings in the Fraunhofer zone are not completely unexpected (see, for example, [13] for a recent analysis that reports similar results). However, they are in partial contradiction with the picture, very common in the existing literature [10–12], of speckles like jelly beans, with the longitudinal length ruled by the diffractive law $\delta z \propto \delta x^2/\lambda$. As already remarked, this depends on the fact that several authors [10–12] adopted the approximation of a longitudinal separation $dz \ll z_1$, which loses validity on approach to the Fraunhofer zone. It should be noted that in many experimental settings the source size is on the order of millimeters and the Fraunhofer distance is on the order of several meters, so that the jelly-bean-like behavior corresponds to common laboratory observations.

The "pencil-like" behavior here predicted should also be easy to observe for smaller sources, and is surely relevant for free propagation of speckles in the atmosphere. Moreover, it should be noted that the emergence of this behavior depends on the wave front curvature: by properly controlling the curvature of the light wave front one should be able either to observe pencil-like speckles in the laboratory at shorter distances from the source, or to observe the jelly-bean-like speckles at longer distances. This issue, as well as other important issues, will be elucidated in a future presentation [22].

VII. CONCLUSIONS

In this work we have provided a detailed analysis of the 3D coherence properties of speckles, encompassing the so-called deep Fresnel zone of the source, up to the Fraunhofer zone. The analysis, in the limit of validity of the paraxial approximation, is very general because it is based on only the assumption of a quasihomegeneous planar source, without making any other specific assumption about the source intensity profile or the characterististic of the scattering medium that produces the speckles.

Our main findings are summarized in Fig. 7. As already reported in the literature [7,8], with respect to the tranverse coherence properties, the right half space of the source can be divided into two main regions: the deep Fresnel region, where the transverse coherence does not vary with propagation distance, so that the speckle transverse size remain equal to the size at the source plane, and the VCZ zone, where the tranverse coherence is ruled by the Van Cittert–Zernike theorem. However, when the full 3D coherence properties of speckles are considered, three different regions emerge. With respect to longitudinal coherence, a diffractinglike behavior dominates in the deep Fresnel region (($z < z_{\mathrm{VCZ}}$) close to the source, where speckles can be picured as jelly beans with a longitudinal length $\delta z \propto \delta x^2/\lambda$, with δx being their transverse width. Far away from the source, in the Fraunhofer region $z > z_{\mathrm{Fr}}$, speckles become like pencils, and tend to have an infinite longitudinal depth. In the intermediate region ($z_{\mathrm{VCZ}} < z < z_{\mathrm{Fr}}$) speckles have a mixed behavior, and pass from purely diffractive jelly beans to jelly beans elongated in the direction pointing outward from the source as the Fraunhofer zone is approached.

We have provided an intuitive explanation of the emergence of the pencil-like behavior, based on the fact that in the Fraunhofer zone speckles spread apart from each other because of the wave front curvature.

We believe that this work offers a complete and general view of the problem that was absent in previous literature.

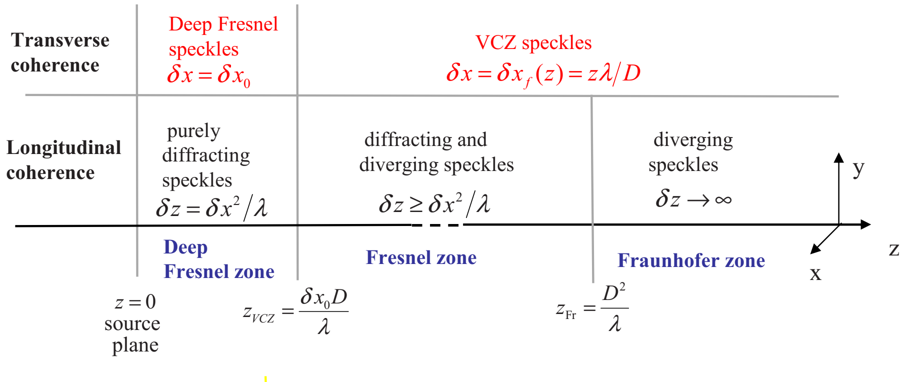

FIG. 7. (Color online) Schematic summary of transverse and longitudinal coherence properties of speckles, divided by zones with respect to the propagation distanxe z from the source. The naming of the regions (deep Fresnel zone, Fresnel zone, Fraunhofer zone) refers to the source size.

The emergence of the pencil-like behavior is unusual with respect to well-known literature, and we have provided a convincing explanation of partial discrepacy with previous works [10–12].

ACKNOWLEDGMENTS

We are grateful to B. E. A. Saleh, M. Giglio, M. A. C. Potenza, and R. Cerbino for useful discussions. D.M. and F.F. acknowledge the work of M. D. Alaimo on the longitudinal coherence properties of speckle light in the deep Fresnel region [23].

APPENDIX

This appendix illustrates the mathematical steps that bring us from Eq. (32) to Eq. (33a) and (33b). We start from Eq. (32), and in the integral on the RHS we perform the usual change of variables:

$$(\mathbf{x}_1',\mathbf{x}_2') \rightarrow \left(\mathbf{X} = \frac{\mathbf{x}_1'+\mathbf{x}_2'}{2}, \mathbf{r} = \mathbf{x}_2' - \mathbf{x}_1'\right).$$

Let us concentrate on the phase factors inside the integral in Eq. (32). Their expansion gives rise to three terms:

$$\text{(a)} \quad \exp\left[i\frac{k}{2}\left(\frac{x_2^2}{z_2} - \frac{x_1^2}{z_1}\right)\right],$$

$$\text{(b)} \quad \exp\left[i\frac{k}{2}\left(\frac{x_2'^2}{z_2} - \frac{x_1'^2}{z_1}\right)\right]$$

$$= \exp\left[i\frac{k}{2}\left(\frac{X^2}{z_2} + \frac{r^2}{4z_2} + \frac{\mathbf{X}\cdot\mathbf{r}}{z_2} - \frac{X^2}{z_1} - \frac{r^2}{4z_1} + \frac{\mathbf{X}\cdot\mathbf{r}}{z_1}\right)\right],$$

$$\text{(c)} \quad \exp\left[-ik\left(\frac{\mathbf{x}_2\cdot\mathbf{x}_2'}{z_2} - \frac{\mathbf{x}_1\cdot\mathbf{x}_1'}{z_1}\right)\right]$$

$$= \exp\left\{-i\frac{k}{z_1}\left[\mathbf{X}\cdot\left(\mathbf{x}_2\frac{z_1}{z_2} - \mathbf{x}_1\right) + \frac{\mathbf{r}}{2}\cdot\left(\mathbf{x}_2\frac{z_1}{z_2} + \mathbf{x}_1\right)\right]\right\}.$$

In the limit $z_1, z_2 \gg z_{\rm VCZ} = D\delta x_0/\lambda$, the factors in term (b) of the form

$$\frac{k\mathbf{X}\cdot\mathbf{r}}{2z_{1,2}} \approx \frac{\pi D\delta x_0}{\lambda z_{1,2}} \ll \pi$$

are negligible. Even more negligible are those factors of the form

$$\frac{kr^2}{8z_{1,2}} \approx \frac{\pi\delta x_0^2}{4\lambda z_{1,2}} \ll \pi.$$

Thus the phase factor (b) can be approximated as

$$\exp\left[i\frac{k}{2}\left(\frac{x_2'^2}{z_2} - \frac{x_1'^2}{z_1}\right)\right] \simeq \exp\left[i\frac{k}{2}X^2\left(\frac{1}{z_2} - \frac{1}{z_1}\right)\right].$$

In the VCZ limit we can thus write

$$G^{(1)}(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2) \underset{z\gg z_{\rm VCZ}}{=} \frac{1}{\lambda^2 z_1 z_2} e^{i(k/2)[(x_2^2/z_2)-(x_1^2/z_1)]} \times \int d^2\mathbf{r}\, e^{(k/2z_1)\mathbf{r}\cdot[\mathbf{x}_2(z_1/z_2)+\mathbf{x}_1]} C_0(\mathbf{r}) \times \int d^2\mathbf{X}\, e^{i(k/2)X^2[(1/z_2)-(1/z_1)]} \times e^{-i(k/z_1)\mathbf{X}\cdot[\mathbf{x}_2(z_1/z_2)-\mathbf{x}_1]} I_0(\mathbf{X}). \quad \text{(A1)}$$

In the second line of Eq. (A1) we can readily recognize the Fourier tranform of C_0. In this way, we arrive at the final result

$$G^{(1)}(\mathbf{x}_1,z_1;\mathbf{x}_2,z_2) \underset{z_1\gg z_{\rm VCZ}}{=} \frac{k^2}{z_1 z_2} e^{i(k/2)[(x_2^2/z_2)-(x_1^2/z_1)]} \times \tilde{C}_0\left[\frac{k}{2z_1}\left(\mathbf{x}_2\frac{z_1}{z_2} + \mathbf{x}_1\right)\right] \times \phi_{dz}\left[\frac{k}{z_1}\left(\mathbf{x}_2\frac{z_1}{z_2} - \mathbf{x}_1\right)\right], \quad \text{(A2a)}$$

where

$$\phi_{dz}(\mathbf{q}) = \int \frac{d^2\mathbf{X}}{2\pi} e^{i(k/2)X^2\{[1/(z_1+dz)]-(1/z_1)\}} e^{-i\mathbf{X}\cdot\mathbf{q}} I_0(\mathbf{X}), \quad \text{(A2b)}$$

which are identical to Eqs. (33a) and (33b) reported in the text.

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