THE GEOMETRY OF STRICT MAXIMALITY*

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Abstract. The notion of a strictly maximal point is a concept of proper maximality that plays an important role in the study of the stability of vector optimization problems. The aim of this paper is to study some properties of this notion with particular attention to geometrical aspects. More precisely, we individuate some relationships between strict maximality and the properties of the bases of the ordering cone. In order to prove this result, a new characterization of the existence of a bounded base for a closed convex cone is given. Moreover, we link strict maximality to the geometrical notion of strongly exposed points of a given set. Finally, we deal with the linear scalarization for the strictly maximal points.

Key words. vector optimization, proper maximality, strict maximality, linear scalarization, base for a cone

AMS subject classifications. 90C29, 46N10

DOI. 10.1137/090748858

1. Introduction. The study of the refinements of the notion of maximal point is an important issue in the theory of vector optimization. Indeed, starting from the seminal paper [15] by Kuhn and Tucker, where for the first time a concept of proper maximality (positive proper maximality) was introduced, various notions of proper maximality appeared in the literature (see, e.g., [4], [10]). The various approaches to proper maximality emphasize different aspects of the considered vector optimization problem, but all of them can be seen as a way to avoid anomalous features of the maximal points. Most of these notions require a sort of stability with respect to perturbations of the ordering structure, i.e., with respect to perturbations of the ordering cone. Among these notions, we focus on the recent notions of strict maximality, that was introduced for the first time by Bednarczuk and Song in [3]. This concept plays a key role in some results concerning the stability of the maximal frontiers of a vector optimization problem (see [3], [1], [18], and [2]) and the well-posedness of vector optimization problems (see [17] and [19]). Moreover, it is also involved in the study of the scalarizations for a vector optimization problem [21].

The first aim of this paper is to study how the notion of strict maximality is linked to the structure of the ordering cone K . Indeed, we prove that a maximal point x_0 belonging to a subset C of a normed space X is a strictly maximal point of C whenever there exists a supporting functional of C at x_0 that generates a bounded base for the ordering cone K . The essential tool to prove this result is a characterization of the existence of a bounded base for a closed convex cone K , developed in section 3. Moreover, we show that strict maximality is deeply linked to a valuable geometrical property of the set C . More precisely, we prove that a (weakly) maximal point of a set C with respect to the cone K is a strictly maximal point if and only if there exists a supporting functional of C at x_0 that strongly exposes the point x_0 .

*Received by the editors February 6, 2009; accepted for publication (in revised form) July 22, 2010; published electronically October 21, 2010.

<http://www.siam.org/journals/siopt/20-6/74885.html>

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The second aim of our work is to provide some results about the linear scalarization of strictly maximal point. With the term linear scalarization we mean the approach that individuates the (properly) maximal points of a set as the solutions of a scalar maximization problem whose objective function is built by means of a linear functional. The linear scalarization for a vector optimization problem is a widely studied topic in the theory of vector optimization (see, e.g., [9], [12]). Our geometrical viewpoint on the notion of strict maximality allow us to prove a sufficient condition for strict maximality based on linear scalarization. Moreover, given a solution x_0 of a linear maximization problem over a set $C \subset X$, we individuate the order relations in X , induced by a cone K , such that x_0 is a strict maximal point of C with respect to K . We also deal with another topic in the theory of vector optimization. As a matter of fact, our results clarify the relationships between strict maximality and positive proper maximality. Indeed, we show that a strictly maximal point is also a positive maximal point when the strictly positive linear functional involved in the definition of positive proper maximality generates a bounded base for the ordering cone. Some examples show that the situation is completely different when we consider an unbounded base of the ordering cone.

This paper is organized as follows. In section 2, we introduce some notations and we recall some known facts about closed convex cones. Section 3 is devoted to a characterization of the existence of bounded base for a closed convex cone K in terms of functionals that strongly expose the origin in K . In section 4, we introduce the notions of maximality, weak maximality, positive proper maximality, and strict maximality. Moreover, we give some examples that illustrate the main features of the strictly maximal points and their relationships with the notion of positive maximality. Section 5 contains the main results concerning the geometrical aspects of the notion of strict maximality. In this section we also prove a result concerning the relationships between strictly and positive proper maximality. Finally, in section 6 we apply our result to study the linear scalarization for strictly maximal points.

2. Notations and preliminaries. Let X be a normed space, and let X^* be the norm dual of X . Let

$$B_X = \{x \in X : \|x\| \leq 1\} \text{ and } S_X = \{x \in X : \|x\| = 1\}.$$

Let A a subset of X . We denote by $\text{cl}A$ the closure of A , by $\text{int}A$ the interior of A , and by $\text{bd}A$ the boundary of A (with respect to the norm topology). Moreover, $\text{conv}(A)$ denotes the convex hull of A and, given $x \in X$, we set $d(x, A) = \inf_{a \in A} \|x - a\|$. A nonempty subset K of X is called a *cone* if $\lambda K \subseteq K$ for every real number $\lambda \geq 0$. If, in addition, $K \cap (-K) = \{0\}$, we say that K is a *pointed* cone. A pointed and convex cone K generates a partial order $\leq$, which is compatible with the linear structure of the space X , in the following way:

$$x \leq y \iff y - x \in K.$$

Let A be a subset of X . We denote by $\text{cone}(A)$ the smallest convex cone containing A . If A is a convex set, we have

$$\text{cone}(A) = \{\lambda a : a \in A, \lambda \geq 0\}.$$

The *polar* (or dual) *cone* of K is the set

$$K^* = \{x^* \in X^* : x^*(k) \geq 0 \ \forall k \in K\}.$$

The elements of K^* are called *positive* (on K) functionals. We also introduce the subset of K^* whose elements are the *strictly positive* functionals:

$$K^{*s} = \{x^* \in X^* : x^*(k) > 0 \ \forall k \in K \setminus \{0\}\}.$$

This set is known in the literature as the quasi-interior of the polar cone K^* (see, e.g., [9]). It is worth pointing out that $\text{int}K^* = \text{int}K^{*s}$, but K^{*s} is not necessarily the interior of K^* . Indeed, for example, if K is the nonnegative orthant of ℓ^p , $1 < p < \infty$, then $\text{int}K^* = \emptyset$ but K^{*s} is nonempty. Moreover, the set K^{*s} may be empty. For example, if we consider the space $B([a, b])$ of all bounded functions on the real interval $[a, b]$ endowed with the usual “sup” norm and the standard positive cone

$$K = \{f \in B([a, b]) : f(t) \geq 0 \ \forall t \in [a, b]\},$$

then K^{*s} is empty [20, p. 27].

To fulfill our aims, it is interesting to relate the nonemptiness of K^{*s} to the notion of a based cone. We say that the nontrivial convex cone K is *based* if there exists a convex set B such that $0 \notin \text{cl } B$ and $\text{cone}(B) = K$ (see, e.g., Definition 2.2.14 in [9]). A based cone is necessarily pointed. The following remark characterizes cones with K^{*s} nonempty.

Remark 2.1 (see [6, Remark 2.2]). Let $K \subset X$ be a convex cone. K is based if and only if K^{*s} is nonempty.

In what follows, we always suppose that *the cone K is convex and K^{*s} is nonempty*.

Now, we recall that a set B is a *base* for the convex cone K when B is a nonempty, convex set and each nonzero element of K has a unique representation αb where $\alpha > 0$ and $b \in B$ (see, e.g., Definition 2.1.14 in [9]).

It is easy to see that, if $x^* \in K^{*s}$, the set

$$B_{x^*} = \{x \in K : x^*(x) = 1\}$$

is a base for the cone K . Moreover, if K is a based cone, from a standard separation argument between 0 and $\text{cl } B$ it follows that a base B_{x^*} for K , defined by a functional $x^* \in K^{*s}$, always exists. Moreover, when the cone K is based and the set B is bounded, a base B_{x^*} is bounded whenever $x^* \in K^{*s}$ is a functional that separates B from 0 .

Finally, we recall some other notions that will be used in what follows. Let $C \subset X$ be a nonempty set and let $x \in C$. A functional $x^* \in X^*$ is called a *supporting functional* of C at x if $\|x^*\| = 1$ and $x^*(x) = \sup_{c \in C} x^*(c)$. Let δ be a positive real number, the set

$$S(x_0^*, \delta, C) = \left\{x \in C : x_0^*(x) + \delta \geq \sup_{c \in C} x_0^*(c)\right\}$$

is called *slice* of the set $C \subset X$ given by the functional x^* .

DEFINITION 2.2 (see, e.g., [7]). *Let $C \subset X$ be a nonempty set. We say that a functional $x^* \in X^*$ strongly exposes a point $x_0 \in C$ when x^* is a supporting functional of C at x_0 and*

$$\text{diam } S(x_0^*, \delta, C) \rightarrow 0 \quad \text{whenever} \quad \delta \rightarrow 0.$$

where, given a nonempty set $A \subset X$,

$$\text{diam } A = \sup \{\|a_1 - a_2\| : a_1, a_2 \in A\}.$$

Moreover, we say that $x_0 \in C$ is a strongly exposed point when there exists a linear functional in X^* that strongly exposes x_0 in C .

3. Cones with bounded base. In this section, we give a characterization of the boundedness of a base B_{x^*} of a closed and convex cone K generated by a strictly positive functional $x^* \in K^{*s}$. This result will play a key role in studying the geometrical properties of strictly maximal points.

First, we recall a well-known result that characterizes the existence of a bounded base B_{x^*} for a given cone K .

THEOREM 3.1 (Theorem 3.8.4 in [13]). B_{x^*} is bounded if and only if $x^* \in \text{int} K^{*s}$.

The existence of a bounded base for a closed and convex cone is a deeply studied property (see, e.g., [11], [14], and the references therein). Now, we introduce a new characterization of the boundedness of the base B_{x^*} for a cone K based on the notion of a functional that strongly exposes the origin in the cone K .

In what follows, we will use the following well-known lemma (see, e.g., Proposition 2.8 in [8]).

LEMMA 3.2. Let $x^* \in S_{X^*}$; then, for every $x \in X$,

$$d(x, \text{Ker } x^*) = |x^*(x)|.$$

We denote the set of continuous linear functionals that strongly expose 0 in K by $\text{Se}(K)$. We observe that every functional that strongly exposes 0 in K is a functional strictly negative on K , i.e., $\text{Se}(K) \subset -K^{*s}$. In the following, we find a property equivalent to the fact that a functional strongly exposes 0 in K . This lemma will be useful to obtain this result.

LEMMA 3.3. Let $x^* \in X^*$. It holds that $\text{diam } S(x^*, \delta_n, K) \rightarrow 0$ whenever $\delta \rightarrow 0$ if and only if for every sequence $\{x_n\} \subset \text{Ker } x^*$ such that $d(x_n, K) \rightarrow 0$ we have that $x_n \rightarrow 0$.

Proof. By contradiction, let us suppose that there exists a sequence $\{\delta_n\}$ of positive real numbers and a positive real number ε such that $\delta_n \rightarrow 0$ and

$$\text{diam } S(x^*, \delta_n, K) \geq \varepsilon > 0 \quad \text{for every } n.$$

Hence, we can find a sequence $\{k_n\} \subset K$ such that $\|k_n\| \geq \frac{\varepsilon}{2}$ and $x^*(k_n) \rightarrow 0$. Now, let $z \in X$ be such that $x^*(z) = 1$. Let us consider the sequence $\{y_n\}$ defined by

$$y_n = k_n - x^*(k_n)z.$$

It is easy to see that $\{y_n\} \subset \text{Ker } x^*$. Moreover, we have

$$\|k_n - y_n\| = |x^*(k_n)| \|z\| \rightarrow 0.$$

Therefore, $\{y_n\} \subset \text{Ker } x^*$ is such that $d(y_n, K) \rightarrow 0$ and $y_n \not\rightarrow 0$, a contradiction.

Now, again by contradiction, let $\{x_n\} \subset \text{Ker } x^*$ be such that $d(x_n, K) \rightarrow 0$ and $x_n \not\rightarrow 0$. It follows that there exists a sequence $\{p_n\} \subset K$ such that

$$\|p_n - x_n\| \rightarrow 0.$$

Hence it holds that $p_n \not\rightarrow 0$

$$|x^*(p_n)| = |x^*(p_n) - x^*(x_n)| \rightarrow 0.$$

Then we conclude that $\text{diam } S(x^*, \delta_n, K) \rightarrow 0$ as $\delta \rightarrow 0$, a contradiction. $\square$

The following result completely characterizes the elements of the interior of the cone $-K^{*s}$, showing that the set $\text{Se}(K)$ coincides with the interior of $-K^{*s}$.

THEOREM 3.4. *Let $x^* \in X^*$, then $x^* \in \text{Se}(K)$ if and only if $x^* \in -\text{int}K^{*s}$.*

Proof. We begin to prove that if $x^* \in \text{Se}(K)$, then $x^* \in (-\text{int}K^{*s})$.

Now, by contradiction, let us suppose that $y^* \in \text{Se}(K) \setminus (-\text{int}K^{*s})$. Then there exists a sequence $\{y_n^*\}$ such that $\|y_n^*\| \leq \frac{1}{n}$ and $y^* + y_n^* \notin -K^{*s}$ for every n . Hence there exists a sequence $\{k_n\} \subset K \setminus \{0\}$ such that $(y^* + y_n^*)(k_n) \geq 0$. Let $z_n = \frac{k_n}{\|k_n\|}$, hence $y^*(z_n) + y_n^*(z_n) \geq 0$ for every n . Since $y^* \in -K^{*s}$, we obtain that $y_n^*(z_n) \geq -y^*(z_n) > 0$ for every n , since $z_n \in K$ and $y^* \in \text{Se}(K) \subset -K^{*s}$. Using Lemma 3.2 we have that

$$d(z_n, \text{Ker } y^*) = -(y^*(z_n) + y_n^*(z_n)) + y_n^*(z_n) \leq y_n^*(z_n) \leq \frac{1}{n}.$$

The relation above implies that there exists a sequence $\{y_n\} \subset \text{Ker } y^*$ such that $\|y_n - z_n\| \rightarrow 0$. Therefore $d(y_n, K) \rightarrow 0$, and $\|y_n\| \rightarrow 1$ (since $\|z_n\| = 1$). Hence, by Lemma 3.3, $y^* \notin \text{Se}(K)$, a contradiction.

Now we prove that if $x^* \in -\text{int}K^{*s}$, then $x^* \in \text{Se}(K)$. Preliminarily we observe that if $-\text{int}K^{*s}$ is empty, then also $\text{Se}(K)$ is empty.

Let $x^* \in -\text{int}K^{*s}$. There exists $\varepsilon > 0$ such that $x^* + \varepsilon B_{X^*} \subset -\text{int}K^{*s}$. Let us consider a sequence $\{x_n\} \subset \text{Ker } x^*$ such that $d(x_n, K) \rightarrow 0$. Therefore, we can find a sequence $\{h_n\} \subset K$ such that $x_n - h_n \rightarrow 0$. Let $\{y_n^*\} \subset S_{X^*}$ be a sequence such that $y_n^*(x_n) = \|x_n\|$ for every n . Since $|(x^* + \varepsilon y_n^*)(x_n - h_n)| \leq (\|x^*\| + \varepsilon) \|x_n - h_n\|$ we have that

$$(3.1) \quad (x^* + \varepsilon y_n^*)(x_n - h_n) \rightarrow 0.$$

Moreover, there exists a sequence $\{\alpha_n\}$ of positive real numbers such that

$$\begin{aligned} (x^* + \varepsilon y_n^*)(x_n - h_n) &= \varepsilon y_n^*(x_n) - (x^* + \varepsilon y_n^*)(h_n) \\ &= \varepsilon \|x_n\| + \alpha_n. \end{aligned}$$

Hence, by (3.1) we obtain $\|x_n\| \rightarrow 0$. The thesis follows from Lemma 3.3. $\square$

From the previous result and Theorem 3.1, it is straightforward to prove a characterization of the boundedness of a base B_{x^*} for the cone K .

COROLLARY 3.5. *Let $x^* \in X^*$. The base B_{x^*} for the cone K is bounded if and only if $-x^* \in \text{Se}(K)$.*

It is worth noting that we can obtain some information about the geometrical structure of the cones with a bounded base combining Proposition 3 in [14] and our previous corollary.

Remark 3.6. Let $C \subset X$ be a convex set. We recall that a point $x_0 \in C$ is a denting point of C if for each positive real number ε , x_0 does not belong to the closure of the convex hull of the set $\{x \in C : \|x - x_0\| \geq \varepsilon\}$. Moreover, $x_0 \in C$ is a point of continuity of C if the identity map $(C, \text{weak}) \rightarrow (C, \|\cdot\|)$ is continuous at x_0 , where (C, weak) and $(C, \|\cdot\|)$ denote the set C equipped with the induced weak topology and the induced norm topology, respectively.

Now, let $K \subset X$ be a closed, convex cone. Putting together the results of Proposition 3 in [14] and of Corollary 3.5, we have that the following statements are equivalent:

- (i) K has a bounded base B such that $0 \notin \text{cl}(B)$.
- (ii) 0 is a strongly exposed point of K .
- (iii) 0 is a denting point of K .

Moreover, if X is a Banach space, all the statements above are equivalent to:

- (iv) 0 is a point of continuity of K .

4. Strictly maximal points. First, we recall some notions concerning the maximal points of a given set with respect to the order induced by the closed, convex, and pointed cone K .

DEFINITION 4.1. We say that $x_0 \in C$ is a maximal point of C with respect to K when

$$(x_0 + K) \cap C = \{x_0\}.$$

We denote the set of maximal points of C by $\text{Max}(C, K)$. We say that $x_0 \in C$ is a positive properly maximal point of C with respect to K when there exists $x^* \in K^{*s}$ such that $x^*(x_0) \geq x^*(c)$ for every $c \in C$. We denote the set of proper maximal points of C by $\text{Pos}(C, K)$. If $\text{int}K$ is nonempty, we say that $x_0 \in C$ is a weakly maximal point of C with respect to K when

$$(x_0 + \text{int}K) \cap C = \emptyset.$$

We denote the set of weakly maximal points of C by $\text{WMax}(C, K)$.

Since the notion of weak maximality has no meaning when $\text{int}K = \emptyset$, we adopt the following convention: $\text{WMax}(C, K) = \emptyset$ whenever $\text{int}K = \emptyset$.

Remark 4.2. It is easy to see that if there exists a supporting functional x^* of C at x_0 such that $x^* \in K^{*s}$, then $x_0 \in \text{Pos}(C, K)$.

Here, we mainly focus on a refinement of the notion of maximality introduced in [3]: the strict maximality.

DEFINITION 4.3. We say that $x_0 \in C$ is a strictly maximal point of C with respect to K when for every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that

$$(C - x_0) \cap (\delta B_X + K) \subseteq \varepsilon B_X.$$

We denote the set of strictly maximal points of C by $\text{StMax}(C, K)$.

This notion plays an important role in the stability theory of vector optimization (see [3], [1], and [18]) and in the study of well-posed vector optimization problem [17]. Moreover, strict maximality is involved in other aspects of the theory of vector optimization like nonlinear scalarization of vector optimization problems [21]. The following proposition gives a characterization of strictly maximal points useful in what follows.

PROPOSITION 4.4 (Proposition 3.3 in [1]). Let $C \subset X$ and $x_0 \in \text{Max}(C, K)$. Then $x_0 \in \text{StMax}(C, K)$ if and only if for every sequences $\{x_n\}$, $\{y_n\}$ such that $\{x_n\} \subset C$, $x_n \in y_n - K$, and $y_n \rightarrow x_0$ it must be $x_n \rightarrow x_0$.

It is well known that

$$\text{Pos}(C, K) \subseteq \text{Max}(C, K) \subseteq \text{WMax}(C, K).$$

for every closed, convex, and pointed cone K (the last relation is meaningful if and only if $\text{int}K \neq \emptyset$).

The relationships of strict maximality with the other notions of maximality are much more involved. Indeed, even if it is straightforward to see that $\text{StMax}(C, K) \subseteq \text{Max}(C, K)$ (see, e.g., [1]), the relationship with positive proper maximality depends on the structure of the cone K , as shown in the following examples.

Example 4.5. Let $X = \ell^2$, $K = \ell_+^2$. Hence

$$(\ell_+^2)^{*s} = \{x^* = (x^{i*}) \in \ell^2 : x^{i*} > 0 \text{ for every } i = 1, 2, \dots\}$$

Now, let $x^* = (x^{i*}) \in (\ell_+^2)^{*s}$ and let

$$C = \ker x^*.$$

It holds that $0 \in \text{Pos}(C, K)$ but $0 \notin \text{StMax}(C, K)$. Indeed let $x_n = x^{1*}e_n - x^{n*}e_1$ for every n , where $\{e_n\}$ is the standard unit-vector basis of ℓ^2 . Hence $\{x_n\} \subset C$, $x_n \in x^{n*}B_X + K$, and $\|x_n\| \geq x^{1*} > 0$ for every n .

Example 4.6. Let $X = \ell^1$, $K = \ell_+^1$. Hence,

$$(\ell_+^1)^{*s} = \{x^* = (x^{i*}) \in \ell^\infty : x^{i*} > 0 \text{ for every } i = 1, 2, \dots\}$$

Now, let $x^* = (\frac{1}{i}) \in (\ell_+^1)^{*s}$ and let

$$C = \ker x^*.$$

We have $0 \in \text{Pos}(C, K)$ but $0 \notin \text{StMax}(C, K)$. Indeed let $x_n = e_n - \frac{1}{n}e_1$ for every n , where $\{e_n\}$ is the standard unit-vector basis of ℓ^1 . Hence $\{x_n\} \subset C$, $x_n \in \frac{1}{n}B_X + K$, and $\|x_n\| \geq 1$ for every n . It is easily seen that the functional x^* reproduces in ℓ^1 the same situation of Example 4.5. Nevertheless, not all of the functionals of $(\ell_+^1)^{*s}$ exhibit the same behavior in this framework. Indeed, let $y^* = (1, \dots, 1, \dots) \in (\ell_+^1)^{*s}$ and let again

$$C = \ker y^*.$$

We have that $0 \in \text{Pos}(C, K)$ and $0 \in \text{StMax}(C, K)$. Indeed, let us consider a sequence $\{c_n = (\gamma_n^i)\} \subset C$ such that there exists a sequence $\{a_n = (\alpha_n^i)\}$ satisfying $a_n \rightarrow 0$ and

$$(4.1) \quad c_n \in a_n - K \quad \text{for each } n.$$

Now, we split the sequence $\{c_n = (\gamma_n^i)\}$ into two sequences:

$$\{+c_n = (+\gamma_n^i)\} \quad \text{and} \quad \{-c_n = (-\gamma_n^i)\},$$

where

$$+\gamma_n^i = \begin{cases} \gamma_n^i & \text{if } \gamma_n^i \geq 0, \\ 0 & \text{if } \gamma_n^i < 0 \end{cases}$$

and

$$-\gamma_n^i = \begin{cases} -\gamma_n^i & \text{if } \gamma_n^i \leq 0, \\ 0 & \text{if } \gamma_n^i > 0. \end{cases}$$

It is easy to check that $+c_n, -c_n \in K$ for every n . Moreover, it holds that $c_n = +c_n - -c_n$ for every n . Since $\{c_n\} \subset C$, we have

$$y^*(c_n) = \sum_{i=1}^{+\infty} (+\gamma_n^i - -\gamma_n^i) = 0$$

for every n . It follows that $\|+c_n\| = \|-c_n\| = \frac{1}{2}\|c_n\|$ for every n . From (4.1) we obtain that

$$+c_n \in +a_n - K$$

or, equivalently,

$$0 \leq +\gamma_n^i \leq +\alpha_n^i \quad \text{for every } i, n.$$

Since $a_n \rightarrow 0$, we conclude that $-a_n \rightarrow 0$, hence that $+\alpha_n^i \rightarrow 0$ for every i , and finally that $+\gamma_n^i \rightarrow 0$. Therefore, $+c_n \rightarrow 0$ since the sequence $\{e_n^*\}$ of the biorthogonal functionals are norming (see, e.g., [8]). Finally, since the Schur property holds in ℓ^1 , we obtain that $+c_n \rightarrow 0$ and hence $c_n \rightarrow 0$.

The considered examples show that there exist strictly maximal points that are not positive properly maximal points. This is an interesting fact since, under the assumption of weak compactness of the base of the ordering cone K , positive proper maximality is the most restrictive notion of proper maximality (see, e.g., [10]), while the situation is completely different in a general framework. Moreover, these examples show that the structure of the cone K is deeply involved in the relationship between strict maximality and positive proper maximality. Indeed, the cone ℓ_+^2 is such that $\text{int}(\ell_+^2)^{**} \neq \emptyset$ whereas $(\ell_+^1)^{**} \neq \text{int}(\ell_+^1)^{**} \neq \emptyset$. A cone K that exhibits the behavior of ℓ_+^1 , i.e. $K^{**} \neq \text{int } K^{**} \neq \emptyset$, is named mixed based cone. We recall that, if X is a Banach space, the existence of a mixed based cone characterizes the nonreflexivity of the space X . A study of the properties of this class of cones has been developed in [5].

In what follows we will clarify the relationships between the structure of the ordering cone and strict maximality. To reach our aim, we will use the characterization of bounded bases of a closed and convex cone developed in section 3.

5. Strongly exposed points and strict maximality. The aim of this section is to study how the structure of the ordering cone K affects the existence of strictly maximal points of a nonempty set $C \subset X$. Moreover, we will show that the notion of strict maximality is deeply linked with the geometrical notion of strongly exposed points of C .

We begin to consider the particular case where the set C is the kernel of a linear functional. The examples at the end of the previous section suggest a generalization. Indeed, by Theorem 3.4 we immediately obtain the following simple result, that also provides a characterization of the existence of a bounded base for a cone.

PROPOSITION 5.1. *Let $x^* \in X^*$, $x_0 \in \text{Ker } x^*$, and $k \in K$ be such that $x^*(k) > 0$. Then $x_0 \in \text{StMax}(\text{Ker } x^*, K)$ if and only if the set*

$$B_{x^*} = \{k \in K : x^*(k) = 1\}$$

is a bounded base of the cone K .

Proof. Let us suppose that $x_0 \in \text{StMax}(\text{Ker } x^*, K)$. Since, by assumption, there exists $k \in K$ such that $x^*(k) > 0$, we have that $x^* \in K^{**}$. Without loss of generality, we can take $x_0 = 0$. Now, let $\{x_n\} \subset \text{Ker } x^* = \text{Ker}(-x^*)$ be such that $d(x_n, K) \rightarrow 0$. Since 0 is a strictly maximal point of $\text{Ker}(-x^*)$, then $x_n \rightarrow 0$. By Lemma 3.3, $-x^* \in \text{Se}(K)$. Hence, by Corollary 3.5, B_{x^*} is a bounded base of K .

Now let $x^* \in X^*$ be such that B_{x^*} is a bounded base of K . Corollary 3.5 implies that $-x^* \in \text{Se}(K)$. Therefore, it is easy to see that $0 \in \text{Max}(\text{Ker}(-x^*), K)$. Moreover, let $\{x_n\}$ and $\{y_n\}$ be two sequences such that $\{x_n\} \subset \text{Ker}(-x^*)$, $x_n \in y_n - K$, and $y_n \rightarrow 0$. Hence we have $d(x_n, K) \rightarrow 0$. Then, by Lemma 3.3, $x_n \rightarrow 0$. Therefore, we conclude that $0 \in \text{StMax}(\text{Ker}(-x^*), K) = \text{StMax}(\text{Ker } x^*, K)$ by Proposition 4.4. $\square$

From the previous proposition it follows that a point $x_0 \in \text{Pos}(\text{Ker } x^*, K)$ is such that $x_0 \in \text{StMax}(\text{Ker } x^*, K)$ if and only if the set B_{x^*} is a bounded base of the cone K .

This fact allows us to explain the different behaviors which appeared in Examples 4.5 and 4.6. Indeed, in Example 4.5 every strictly positive functional on ℓ_+^2 generates an unbounded base for ℓ_+^2 . Moreover, in Example 4.6, the base for ℓ_+^1 associated to the functional x^* is unbounded since $x^* \in (\ell_+^1)^{*s} \setminus \text{int}(\ell_+^1)^{*s}$, whereas the base associated to $y^* \in \text{int}(\ell_+^1)^{*s}$ is bounded.

Now, let us consider a nonempty subset C of the space X . When is a point $x_0 \in \text{Max}(C, K)$ also a strictly maximal point of C with respect to the cone K ? The previous proposition, together with Corollary 3.5, implies that x_0 is a strictly maximal point of C if the cone K does not flatten on the hyperplane generated by the supporting functional of C at x_0 . More precisely the following result holds.

PROPOSITION 5.2. *Let $C \subset X$ be a nonempty set and $x_0 \in \text{Max}(C, K)$. If there exists a supporting functional $x^* \in X^*$ of C at x_0 such that the set*

$$B_{x^*} = \{k \in K : x^*(k) = 1\}$$

is a bounded base of the cone K , then $x_0 \in \text{StMax}(C, K)$.

Proof. Let x^* be a supporting functional of C at x_0 that satisfies the assumption of the proposition. By Proposition 5.1, $x_0 \in \text{StMax}(\text{Ker } x^*, K)$. Since x^* is a supporting functional of C at x_0 , it is easy to see that $x_0 \in \text{StMax}(C, K)$. $\square$

We remark that the converse of Proposition 5.2 does not hold, as shown in the following easy example.

Example 5.3. Let $X = \mathbb{R}^2$ be endowed with the usual Euclidean norm, $C = B_X$ and $K = \mathbb{R}_+^2 = \{(x_1, x_2) : x_i \geq 0, i = 1, 2\}$. Then the point $x_0 = (1, 0) \in \text{StMax}(C, K)$ and the unique supporting functional of C at x_0 is $x^* = (1, 0)$. It is easy to see that $x^* \notin K^{*s} = \{(x^1, x^2) : x^i > 0, i = 1, 2\}$. Hence, the set B_{x^*} is not a base for the cone K .

As recalled at the end of section 4, the relationships between positive properly maximal points and strictly maximal points are not completely clear. Now we can clarify the situation concerning this matter. Indeed, the following corollary of Proposition 5.2 holds.

COROLLARY 5.4. *Let $C \subset X$ be a nonempty set, and let $K \subseteq X$ be a closed, convex, and pointed cone. A point $x_0 \in \text{Pos}(C, K)$ is such that $x_0 \in \text{StMax}(C, K)$ whenever the functional $x^* \in K^{*s}$ involved in the definition of positive proper maximality of x_0 is such that the base $B_{x^*} = \{k \in K : x^*(k) = 1\}$ of the cone K is bounded.*

By Corollary 3.5, Proposition 5.2 links strict maximality to the fact that 0 is a strongly exposed point of the cone K . Dually, we now show that strict maximality is linked also to the notion of strongly exposed point of C . We recall that we adhere to the convention that $\text{WMax}(C, K) = \emptyset$ whenever $\text{int}K = \emptyset$.

THEOREM 5.5. *Let $C \subseteq X$ be a nonempty and closed set and let $x_0 \in \text{bd } C$. If there exists a supporting functional x_0^* of C at x_0 such that x_0^* does not strongly expose x_0 , then there exists a closed, convex, and pointed cone $K_0 \subset X$ such that*

$$x_0 \in (\text{Max}(C, K_0) \cup \text{WMax}(C, K_0)) \setminus \text{StMax}(C, K_0).$$

Proof. Without loss of generality, let $x_0 = 0 \in \text{bd}C$, hence $\sup_{c \in C} x_0^*(c) = 0$. Since x_0^* is a supporting functional for C at $x_0 = 0$ such that

$$\text{diam } S(x_0^*, \delta, C) \geq 2\eta > 0 \quad \text{for every } \delta > 0,$$

there exists a sequence $\{x_n\} \subset C$ such that $x_n \in S(x_0^*, \frac{1}{n}, C)$. We can always suppose that $\|x_n\| = \eta$ for every n .

We divide the proof into two distinct parts.

Let us suppose that the sequence $\{x_n\}$ does not weakly converge to 0. Hence, there exists a positive real number $\varepsilon < 2$ and a functional y^* such that $\|y^*\| = 1$ and $y^*(x_n) \geq \varepsilon\eta$ for every n (up to subsequence).

Now, let

$$K_0 = \{x \in X : x_0^*(x) \geq 0\} \cap \left\{x \in X : y^*(x) \geq \frac{\varepsilon}{2} \|x\|\right\}.$$

It is easy to see that K_0 is a nonempty, closed, convex, and pointed cone with nonempty interior. Moreover, $0 \in \text{WMax}(C, K_0)$. Let $x' \in X$ be such that $\|x'\| = 1$ and $x_0^*(x') = \frac{1}{2}$. Now, there exists n_0 such that for every $n \geq n_0$

$$\begin{aligned} \frac{\varepsilon}{2} \left\| x_n + 2 \frac{x'}{n} \right\| &\leq \frac{\varepsilon}{2} \|x_n\| + \frac{\varepsilon}{n} \|x'\| \leq \frac{\varepsilon}{2} \eta + \frac{\varepsilon}{n} \\ &= \varepsilon\eta + \varepsilon \left(\frac{1}{n} - \frac{\eta}{2} \right) \leq \varepsilon\eta - \frac{2}{n} \\ &\leq y^* \left(x_n + 2 \frac{x'}{n} \right). \end{aligned}$$

Moreover,

$$x_0^* \left(x_n + 2 \frac{x'}{n} \right) \geq 0$$

for every n . Therefore,

$$\left(x_n + 2 \frac{x'}{n} \right) \in K_0 \quad \text{for every } n \geq n_0$$

and $0 \notin \text{StMax}(C, K_0)$.

Now, let us suppose that the sequence $\{x_n\}$ weakly converges to 0. We can always suppose that $\{x_n\}$ is a set of linearly independent elements. Let $\tilde{x} \in X$ be such that $\|\tilde{x}\| = 1$ and $x_0^*(\tilde{x}) = \frac{1}{2}$. Let us consider the sequence $\{y_n\}$ where

$$y_n = 4 \frac{\tilde{x}}{n} + x_n.$$

Let

$$K_0 = (\text{cone}(\text{cl}(\text{conv}\{y_n\}))),$$

where $\text{conv}\{y_n\}$ denotes the convex hull of the sequence $\{y_n\}$. It is easy to see that K_0 is a nonempty, closed, and convex cone.

In order to prove that the cone K_0 is pointed, it is sufficient to show that there exists a linear functional z^* such that $z^*(k) > 0$ for every $k \in K_0 \setminus \{0\}$ (i.e., $z^* \in K_0^{*s}$). For every $k \in K_0$, there exists a sequence $\{\alpha_n\}$ such that $\alpha_n \geq 0$ for every n and $k = \sum_{n=1}^{+\infty} \alpha_n y_n$. Therefore,

$$x_0^*(k) = \sum_{n=1}^{+\infty} \alpha_n x_0^*(y_n) = \sum_{n=1}^{+\infty} \alpha_n x_0^* \left(4 \frac{\tilde{x}}{n} + x_n \right) \geq \sum_{n=1}^{+\infty} \frac{\alpha_n}{n} \geq 0$$

for every $k \in K_0$. Moreover, $x_0^*(k) = 0$ if and only if $k = 0$. Indeed,

$$x_0^*(k) = \sum_{n=1}^{+\infty} \alpha_n x_0^*(y_n) = \sum_{n=1}^{+\infty} \alpha_n x_0^* \left(4 \frac{\tilde{x}}{n} + x_n \right) = 0$$

implies that

$$(5.1) \quad 2 \sum_{n=1}^{+\infty} \frac{\alpha_n}{n} = - \sum_{n=1}^{+\infty} \alpha_n x_0^*(x_n).$$

Since $-\frac{1}{n} \leq x_0^*(x_n) \leq 0$,

$$0 \leq - \sum_{n=1}^{+\infty} \alpha_n x_0^*(x_n) \leq \sum_{n=1}^{+\infty} \frac{\alpha_n}{n};$$

therefore, $\sum_{n=1}^{+\infty} \frac{\alpha_n}{n} = 0$. The last relation implies $\alpha_n = 0$ for every n and $k = 0$. Now, it holds that

$$0 \in \text{Pos}(C, K_0) \subseteq \text{Max}(C, K_0)$$

since $x_0^* \in K_0^{*s}$ and $x_0^*(C) \leq 0$ but

$$0 \notin \text{StMax}(C, K_0).$$

Indeed, we have that

$$\|x_n\| \geq \eta > 0 \quad \text{and} \quad x_n \in \left(\frac{1}{n} B_X + K_0 \right) \cap C$$

for every n . $\square$

Remark 5.6. The last result can be reformulated in the following way. Let $C \subseteq X$ be a nonempty, closed subset and $x_0 \in C$. If for every closed, convex, and pointed cone $K \subset X$ such that $x_0 \in \text{Max}(C, K) \cup \text{WMax}(C, K)$, we have that $x_0 \in \text{StMax}(C, K)$, then every supporting functional of C at x_0 strongly exposes the point x_0 .

It holds a sort of converse of Theorem 5.5.

PROPOSITION 5.7. *Let $C \subseteq X$ be a nonempty, closed set, and let $K \subseteq X$ be a closed, convex, and pointed cone. If $x_0 \in \text{Max}(C, K) \cup \text{WMax}(C, K)$ and there exists a supporting functional x_0^* of C at x_0 that strongly exposes x_0 , then $x_0 \in \text{StMax}(C, K)$.*

Proof. Without loss of generality we can take $x_0 = 0$. By contradiction, we suppose $0 \notin \text{StMax}(C, K)$. Hence, there exists a positive real number η and two sequences $\{q_n\} \subset X$ and $\{k_n\} \subset K$ such that

- $\lim_{n \rightarrow +\infty} q_n = 0$,
- $q_n + k_n \in C$ for every n ,
- $\|q_n + k_n\| > \eta$ for every n .

Now $x_0^*(c) \leq 0$ for every $c \in C$ since x_0^* is a supporting functional. Moreover, $x_0^*(k) \geq 0$ for every $k \in K$ since $0 \in \text{Max}(C, K) \cup \text{WMax}(C, K)$. Hence $x_0^*(q_n + k_n) \leq 0$ for every n . Since $\lim_{n \rightarrow +\infty} q_n = 0$, we obtain that $\lim_{n \rightarrow +\infty} x_0^*(k_n) = 0$.

Therefore, we have that $q_n + k_n \in S(x_0^*, \delta_n, C)$, where

$$\delta_n = \frac{1}{n} - x_0^*(q_n + k_n) \xrightarrow{n \rightarrow +\infty} 0.$$

Hence, we have that $\text{diam}S(x_0^*, \delta_n, C) > \eta > 0$ for every n , a contradiction. $\square$

We have shown that the existence of a supporting functional that strongly exposes a point of a nonempty set C is deeply linked to strict maximality. Hence, it is interesting to find a condition that ensures the existence of such a functional. Therefore, we conclude this section by showing a condition that implies the existence of a supporting functional that strongly exposes a point of a set C . The condition that we prove is a geometrical condition and is based on the enforcement of convexity related to uniform convexity: a convex set $C \subset X$ is *locally uniformly convex* at $x_0 \in C$ when for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $\frac{x+x_0}{2} + \delta_\varepsilon B_X \subset C$ whenever $x \in C$ and $\|x_0 - x\| = \varepsilon$.

PROPOSITION 5.8. *Let $C \subset X$ be a nonempty set and let $x_0 \in \text{bd } C$. If C is locally uniformly convex at x_0 , then we have that every supporting functional x^* for C at x_0 strongly exposes x_0 .*

Proof. By contradiction, let us suppose that there exist a support functional x_0^* for C at x_0 and $\bar{\varepsilon} > 0$ such that

$$\text{diam } S(x_0^*, \delta, C) > 2\bar{\varepsilon} \quad \text{for every } \delta > 0.$$

Hence, there exists a sequence $\{x_n\} \subset C$ such that

$$x_n \in S\left(x_0^*, \frac{1}{n}, C\right) \quad \text{and} \quad \|x_n - x_0\| = \bar{\varepsilon}.$$

Let $M = \sup x_0^*(C)$; then $x_0^*(x_n) \geq M - \frac{1}{n}$ for every n .

Let us suppose that (i) holds. Hence, there exists $\delta_{\bar{\varepsilon}} > 0$ such that

$$\frac{x_n + x_0}{2} + \delta_{\bar{\varepsilon}} B_X \subset C \quad \text{for every } n.$$

Let v be such that $\|v\| = 1$ and $x_0^*(v) \geq \frac{1}{2}$. Hence, we have

$$x_0^*\left(\frac{x_n + x_0}{2} + \delta_{\bar{\varepsilon}} v\right) \geq M - \frac{1}{n} + \frac{\delta_{\bar{\varepsilon}}}{2} \quad \text{for every } n.$$

Since $\frac{x_n + x_0}{2} + \delta_{\bar{\varepsilon}} v \in C$, we have

$$M - \frac{1}{n} + \frac{\delta_{\bar{\varepsilon}}}{2} \leq M \quad \text{for every } n.$$

Hence, $\delta_{\bar{\varepsilon}} \leq \frac{2}{n}$ for every n , a contradiction. $\square$

Finally we remark that, combining the last proposition with Proposition 5.7, we obtain a known sufficient condition for a maximal point $x_0 \in C$ to be a strictly maximal point.

PROPOSITION 5.9 (Proposition 3.9 in [18]). *Let $K \subset X$ be a closed, convex, and pointed cone, let $C \subset X$ be a nonempty set, and let $x_0 \in \text{Max}(C, K)$. If C is locally uniformly convex at x_0 , then $x_0 \in \text{StMax}(C, K)$.*

6. Linear scalarization for strictly maximal points. The scalarization procedures, i.e., to find a solution of a vector optimization problem by solving a scalar optimization problem, play an important role in the theory and practice of vector optimization. Among the various approaches proposed in the literature, the linear scalarization has great importance and is widely used (see, e.g., the monographs [9],

[12], [16], and the references therein). This method is based on the idea of finding the solutions of a given vector optimization problem using scalar optimization problems formulated by means of positive (on the ordering cone) functionals.

Our study of the geometrical aspects of the notion of strictly maximal points, developed in the previous sections, allows us to give a complete view of linear scalarization for strictly maximal points.

First, we give a sufficient condition for strictly maximal points.

THEOREM 6.1. *Let $K \subset X$ be a closed, convex, and pointed cone, and let $C \subset X$ be a nonempty set. If there exists a linear functional $x_0^* \in K^*$ and an element $x_0 \in C$ such that*

$$(6.1) \quad x_0^*(x) \leq x_0^*(x_0) \quad \forall x \in C$$

and one of the following statements holds:

- (i) $x_0^* \in \text{int } K^{*s}$,
- (ii) x_0^* strongly exposes x_0 in C ,

then $x_0 \in \text{StMax}(C, K)$.

Proof. Let us suppose that $x_0^* \in K^*$ and $x_0 \in C$ satisfy (6.1).

We begin by considering the case where assumption (i) holds. We recall that, by Theorem 3.1, the set $B_{x_0^*} = \{k \in K : x_0^*(k) = 1\}$ is a bounded base for the cone K . Moreover, from a well-known scalarization result (see Theorem 5.18 (b) in [12]), we have that $x_0 \in \text{Max}(C, K)$. Hence, the thesis follows immediately from Proposition 5.2.

Now, let us consider $x_0^* \in K^*$ satisfying (6.1) which holds assumption (ii). From Theorem 5.18 (a) in [12], we have that $x_0 \in \text{Max}(C, K)$. Therefore, $x_0 \in \text{StMax}(C, K)$, by Proposition 5.7. $\square$

Some simple examples show that conditions (i) and (ii) are mutually independent. Moreover, we remark that the sufficient condition stated above does not become necessary even if we add a convexity assumption, as shown in the following simple example.

Example 6.2. Let $X = \mathbb{R}^2$ be endowed with the usual Euclidean norm,

$$C = B_X \cup \{(x_1, x_2) : -1 \leq x_1 \leq 1; x_2 \leq 0\}$$

and $K = \mathbb{R}_+^2 = \{(x_1, x_2) : x_i \geq 0, i = 1, 2\}$. Then the point $x_0 = (1, 0) \in \text{StMax}(C, K)$ and the unique functional $x_0^* \in K^*$ such that

$$x_0^*(x) \leq x_0^*(x_0) \quad \forall x \in C$$

is $x_0^* = (1, 0)$. It is easy to see that $x_0^* \notin \text{int } K^*$ and x_0^* do not strongly expose x_0 in C .

In Theorem 6.1 we fix a set C and an ordering cone K and we prove a sufficient condition for strict maximality of a given point. This is the usual point of view in the vector optimization theory. Nevertheless, we deem that also to be a sort of dual perspective is worth being considered in the vector optimization theory.

Let us consider a given nonempty set $C \subset X$ and let $x_0 \in C$. Now, let $C_{x_0}^+$ be the set of the linear functionals $x^* \in X^*$ such that x_0 is a solution of the scalar optimization problem

$$(6.2) \quad \max_{x \in C} x^*(x).$$

Now, we may ask what ordering cone $K \subset X$ is such that $x_0 \in \text{Max}(C, K)$. The well-known sufficient condition for the maximality obtained by linear scalarization (see, e.g., [9] or [12]) immediately gives an answer: when $C_{x_0}^+ \cap K^{*s} \neq \emptyset$, then the cone K is such that $x_0 \in \text{Max}(C, K)$.

If we formulate the same question for strict maximality, then the answer is quite involved but our study of geometrical properties of strict maximality allows us to give a complete answer.

PROPOSITION 6.3. *Let $C \subset X$ be a nonempty set, let $x_0 \in C$, and let $x_0^* \in C_{x_0}^+$. We have two distinct cases:*

(i) *If x_0^* strongly exposes x_0 in C , then $x_0 \in \text{StMax}(C, K)$ for every closed, convex, and pointed cone $K \subset X$ such that $x_0^* \in K^*$.*

(ii) *If x_0^* does not strongly expose x_0 in C , then $x_0 \in \text{StMax}(C, K)$ for every closed, convex, and pointed cone $K \subset X$ such that $x_0^* \in \text{int} K^*$. Moreover, there exists a closed, convex, and pointed cone $K_0 \subset X$ such that $x_0^* \in K^*$ and $x_0 \notin \text{StMax}(C, K_0)$.*

Proof. Let $x_0^* \in C_0^+$ be such that x_0^* strongly exposes x_0 in C . The statement of point (i) follows immediately from Theorem 6.1.

Now, let us consider $x_0^* \in C_0^+$ that does not strongly expose x_0 in C . By Theorem 6.1 we have that $x_0 \in \text{StMax}(C, K)$ for every closed, convex, and pointed cone $K \subset X$ such that $x_0^* \in \text{int} K^*$. The second part of point (ii) can be proven considering a cone K_0 obtained following the same construction developed in the proof of Theorem 5.5. $\square$

Finally, we remark that, in point (ii) of the previous proposition, the cone K_0 such that $x_0^* \in K^*$ and $x_0 \notin \text{StMax}(C, K_0)$ can be built in two different ways, as apparent from the proof of Theorem 5.5. Indeed, either K_0 is a subcone of a cone of the form

$$\{x \in X : y^*(x) \geq \alpha \|x\|\},$$

where $0 < \alpha < 1$ and y^* is a fixed element of X^* (hence K_0 has a bounded base) or K_0 is such that $x_0^* \notin \text{int} K_0^*$ (i.e., equivalently, x_0^* does not generate a bounded base for K_0).

Acknowledgments. The authors wish to thank E. Molho for some useful comments that improved the presentation of this paper. Moreover, the authors thank the anonymous referees, whose remarks helped them to substantially improve this work.

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