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The size of a Minkowski ellipse that contains the unit ball

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ABSTRACT

In this paper we study the minimum radius of *Minkowski ellipses* (with antipodal foci on the unit sphere) necessary to contain the unit ball of a (normed or) Minkowski plane. We obtain a general upper bound depending on the modulus of convexity, and in the special case of a so-called symmetric Minkowski plane (a notion that we will recall in the paper) we prove a lower bound, and also we obtain that 3 is the exact upper bound.

1. Introduction

A 2-dimensional real normed space X will be called a *Minkowski plane* (as it is usual in the literature). We shall say that X is a *symmetric Minkowski plane* when the norm in X satisfies the following symmetry conditions

$$\big\|(a,b)\big\| = \big\|(b,a)\big\| = \big\|\big(|a|,|b|\big)\big\|. \tag{1.1}$$

Note that a symmetric Minkowski plane is just a (very) special case among rearrangement invariant spaces.

In this paper X will always be a Minkowski plane. With B and S we denote its *unit ball* and *unit sphere*, respectively. For $k > 0$ and $x \in S$ let $El(x,k) = \{y \in X;\ \|y-x\| + \|y+x\| \leqslant k\}$ denote a *Minkowski ellipse*. Some properties of ellipses and conics in Minkowski planes can be found in the papers [2–4,6,7]. In this paper we investigate the dimension of planar Minkowski ellipses needed to contain the unit ball B. More precisely, we study the minimum number k_0 such that there exists a vector x implying $B \subset El(x,k_0)$. This problem is equivalent to the evaluation of the following constant:

$$A(X) = \inf_{x\in S}\sup_{y\in S}\big(\|x+y\| + \|x-y\|\big) = \inf_{x\in S}\sup_{y\in \mathrm{ext}(S)}\big(\|x+y\| + \|x-y\|\big).$$

The study of this constant has started in [1]. More precisely, in that paper the constant $A_1(X) = \frac{A(X)}{2}$ was considered, where it is proved that in Minkowski planes we have $A(X) \leqslant \frac{1+\sqrt{33}}{2}$ ($\simeq 3.372$) and $A(X) \geqslant \frac{3+\sqrt{21}}{3}$ ($\simeq 2.528$) and, in particular, if X is a normed space in which James orthogonality is symmetric we have $A(X) \geqslant \frac{1+\sqrt{17}}{2}$ ($\simeq 2.561$). In this paper a better estimate is obtained. In the next section we will prove that $A(X) < 3.042$. Then we will prove that $A(X) \leqslant 3$ when X is a symmetric Minkowski plane, and for this case we will prove the same lower bound already obtained for spaces with symmetric James orthogonality, that is $A(X) \geqslant \frac{1+\sqrt{17}}{2}$. Finally we present an example of a Minkowski plane for which the constant $A(X)$ is "small".

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2. A general upper bound

In this section we will obtain an upper bound depending on $\delta_X(1)$ where $\delta_X(\epsilon)$ is the classical modulus of convexity of X; that is, for $0<\epsilon<2$: $\delta_X(\epsilon)=\inf\{1-\frac{\|x+y\|}{2};\ \|x\|=\|y\|=1,\ \|x-y\|\geqslant\epsilon\}$.

Lemma 1. *Let $\delta=2\delta_X(1)$. Then $A(X)\leqslant 3+\frac{\delta^2(1-2\delta)}{1-\delta+\delta^2}$.*

Proof. By definition and the fact that X is finite dimensional we can choose x and y such that $\|x\|=\|y\|=\|x-y\|=1$ and $\|x+y\|=2-\delta$. Notice that if $z_1,z_2\in S$ and $\|z_1-z_2\|=1$, then $\|z_1+z_2\|\leqslant 2-\delta$ and, in particular, $\|2x-y\|,\|x-2y\|\leqslant 2-\delta$. We have:

$$A(X)\leqslant\sup_{z\in S}\bigl(\|z+(y-x)\|+\|z-(y-x)\|\bigr).$$

When z belongs to the arc joining $y-x$ and y we have $A(X)\leqslant 3$ since $\|z-(y-x)\|\leqslant\|y-(y-x)\|=1$, and the same is true for z belonging to the arc joining $x-y$ and x. So we suppose that z lies on the arc $\widehat{xy}$. Let $\alpha\geqslant 0$ and $\beta\geqslant 0$ be such that $\|\alpha x+\beta y\|=1$. Notice that, since $\|x-y\|=1$, this implies $\alpha\leqslant 1$, $\beta\leqslant 1$ and $\alpha+\beta\geqslant 1$. Then we have

$$\begin{aligned}&\bigl\|\alpha x+\beta y-(y-x)\bigr\|+\bigl\|\alpha x+\beta y+(y-x)\bigr\|\\&\quad=\bigl\|(\alpha+2\beta-1)x-(1-\beta)(y-2x)\bigr\|+\bigl\|(2\alpha+\beta-1)y-(1-\alpha)(x-2y)\bigr\|\\&\quad\leqslant\alpha+2\beta-1+(1-\beta)(2-\delta)+2\alpha+\beta-1+(1-\alpha)(2-\delta)\\&\quad=3+\bigl\{(1+\delta)(\alpha+\beta)-1-2\delta\bigr\}.\end{aligned}\tag{2.1}$$

Now for $1/2\leqslant\lambda\leqslant 1$ we have

$$\begin{aligned}\bigl\|\lambda y+(1-\lambda)x\bigr\|&=\bigl\|\lambda(x+y)+(1-2\lambda)x\bigr\|\\&\geqslant\lambda\|x+y\|-(2\lambda-1)\|x\|=\lambda(2-\delta)-2\lambda+1=1-\delta\lambda,\end{aligned}$$

and also

$$\begin{aligned}\bigl\|\lambda y+(1-\lambda)x\bigr\|&=\bigl\|(2-\lambda)y+(\lambda-1)(2y-x)\bigr\|\\&\geqslant(2-\lambda)\|y\|-(1-\lambda)\bigl\|(2y-x)\bigr\|\\&\geqslant(2-\lambda)-(1-\lambda)(2-\delta)=\lambda(1-\delta)+\delta.\end{aligned}$$

So $\|\lambda y+(1-\lambda)x\|\geqslant\max(1-\delta\lambda,\lambda(1-\delta)+\delta)$, and this implies $\|\lambda y+(1-\lambda)x\|\geqslant 1-\delta+\delta^2$. The same result is also true for $0\leqslant\lambda\leqslant 1/2$.

Now

$$1=\|\alpha x+\beta y\|=(\alpha+\beta)\left\|\frac{\alpha}{\alpha+\beta}x+\frac{\beta}{\alpha+\beta}y\right\|\geqslant(\alpha+\beta)\bigl(1-\delta+\delta^2\bigr).$$

Using this estimate in (2.1) we obtain:

$$\begin{aligned}A(X)&\leqslant\sup_{z\in S}\bigl(\|z+(y-x)\|+\|z-(y-x)\|\bigr)\\&\leqslant\max\Bigl\{3,\sup_{\|\alpha x+\beta y\|=1,\ \alpha\geqslant 0,\ \beta\geqslant 0}\bigl\|\alpha x+\beta y-(y-x)\bigr\|+\bigl\|\alpha x+\beta y+(y-x)\bigr\|\Bigr\}\\&\leqslant 3+\left\{\frac{1+\delta}{1-\delta+\delta^2}-1-2\delta\right\}=3+\frac{\delta^2(1-2\delta)}{1-\delta+\delta^2}.\qquad\square\end{aligned}$$

Theorem 2. $A(X)\leqslant\frac{7\sqrt{3}-3}{3}$ $(\simeq 3.042)$.

Proof. This is a consequence of the fact that the function $f(\delta)=\frac{\delta^2(1-2\delta)}{1-\delta+\delta^2}$ is increasing (at least in $[0,0.3]$) and that, by a result of Nordlander (see [5]), $\delta=2\delta_X(1)\leqslant 2\delta_H(1)=2-\sqrt{3}<0.3$ where $\delta_H(\epsilon)$ is the modulus of convexity of an inner product space. $\square$

Corollary 3. *If $\delta_X(1)=0$, then we have $A(X)\leqslant 3$.*

Remark 4. Notice that if $\alpha = \beta = \frac{1}{2-\delta}$ (the "middle" point of the arc $\widehat{xy}$) using (2.1) we have:

$$\left\| \frac{x+y}{2-\delta} - (y-x) \right\| + \left\| \frac{x+y}{2-\delta} + (y-x) \right\| \leqslant 3 + \left\{ \frac{2(1+\delta)}{2-\delta} - 1 - 2\delta \right\} = 3 - \frac{\delta(1-2\delta)}{2-\delta} < 3.$$

3. An upper bound in the symmetric plane

By using a convenient Auerbach basis we see that the unit sphere of every Minkowski plane X can be represented by using a continuous concave function $\gamma : [-1, 1] \to [0, 1]$ such that $\gamma(-1) = \gamma(1) = 0$; $\gamma(0) = 1$. There will be no restriction in assuming that γ is differentiable, then $\gamma'(t) \geqslant 0$ in $[-1, 0]$ and $\gamma'(t) \leqslant 0$ in $[0, 1]$; also γ' is decreasing and $\lim_{t\to 1^-} \gamma'(t) = -\infty$, $\lim_{t\to -1^+} \gamma'(t) = +\infty$, $\gamma'(0) = 0$. The Minkowski plane X is fully described by the function γ.

Any $P \in X$ will be a pair (a, b) and, since obviously $\|(a, b)\| = \|(-a, -b)\|$, we assume that $b \geqslant 0$. Then, if $|a| \leqslant 1$, we have $\|P\| = 1 \Leftrightarrow b = \gamma(a)$; that is, S is the set of points $(t, \gamma(t))$, $t \in [-1, 1]$ together with the opposite vectors.

3.1. Parametrization and symmetry assumptions

Let the space X defined by γ be given, denote by α the positive abscissa such that $\alpha = \gamma(\alpha)$. It is easy to see that $1/2 \leqslant \alpha \leqslant 1$ (note that in the limit cases the space X is a parallelogram). We shall call this α the *parameter* of X (indeed, α parametrizes a family of spaces). We assume now that X is a symmetric Minkowski plane, i.e. that the norm in X satisfies the conditions (1.1). Consequently, for our function γ the following properties hold:

1. $\gamma(-t) = \gamma(t)$, $t \in [0, 1]$, since $\|(a, b)\| = \|(-a, b)\|$;
2. $\gamma(t) = \gamma^{-1}(t)$ (the inverse function), $t \in [0, 1]$, since $\|(a, b)\| = \|(b, a)\|$;
3. γ is determined by its values in the interval $[0, \alpha]$;
4. $\gamma'(\alpha) = -1$, since $\gamma'(\gamma(t))\gamma'(t) = 1$, and for $t = \alpha$ we have $\gamma'(\alpha)^2 = 1$;
5. $\gamma(t) \leqslant 2\alpha - t$, because γ is concave and $\gamma'(\alpha) = -1$;
6. $\gamma(t) \geqslant 1 - \frac{1-\alpha}{\alpha} t$ for $t \in [0, \alpha]$.

Let us note that the infimum which defines $A(X)$ is attained, i.e., there exists $\tau \in [-1, 1]$ such that

$$A(X) = \sup_{t\in[-1,1]} \left\{ \left\| \left(\tau + t, \gamma(\tau) + \gamma(t)\right) \right\| + \left\| \left(\tau - t, \gamma(\tau) - \gamma(t)\right) \right\| \right\}. \tag{3.1}$$

We remark that the properties of γ imply that there is no restriction assuming that $\tau \in [\alpha, 1]$, $\alpha = \gamma(\alpha)$ being the parameter of the space.

A point $(\tau, \gamma(\tau))$ such that (3.1) holds will be called a *Center* for X.

We shall use the notion of *center* for any point $c \in X$ which we choose as a candidate (surrogate) for a true *Center*; we pick such a c in order to compute the quantity

$$\sup_{\|y\|=1} \left(\|c - y\| + \|c + y\| \right).$$

Theorem 5. *If X is a symmetric Minkowski plane, then $A(X) \leqslant 3$.*

Proof. Consider a 2-dimensional symmetric space X with describing function γ and parameter α: we pick a *center* $c \in X$, and we look for an upper bound for the quantity

$$\sup_{\|y\|=1} \left(\|c - y\| + \|c + y\| \right).$$

The first observation is that using the $\pi/4$-rotation $(a, b) \to (\frac{a+b}{2\alpha}, \frac{-a+b}{2\alpha})$ we can suppose that $\alpha \in (\frac{1}{\sqrt{2}}, 1)$. We will consider only the *center* $c = (1, 0)$. Our goal is to find a (good) upper bound for the function

$$G(t) = \left\| \left(1 - t, -\gamma(t)\right) \right\| + \left\| \left(1 + t, \gamma(t)\right) \right\| = \left\| \left(1 - t, \gamma(t)\right) \right\| + \left\| \left(1 + t, \gamma(t)\right) \right\|.$$

It is enough to consider $t \in [0, 1]$ and to note that $\|(1 - t, \gamma(t))\| \leqslant 1$ if $\gamma(t) \leqslant \gamma(1 - t)$, and this is true if $t \geqslant 1/2$; since obviously $\|(1 + t, \gamma(t))\| \leqslant 2$, we conclude that $G(t) \leqslant 3$ if $t \geqslant 1/2$. All this means that it is enough to consider $G(t)$ only for $t \in [0, 1/2)$. Assume that $0 \leqslant t < 1/2$; we have

$$\left(1 + t, \gamma(t)\right) = c_1(\alpha, \alpha) + c_2\left(\gamma(1 - \alpha), 1 - \alpha\right)$$

with

$$c_1 = \frac{\gamma(t)}{\alpha} - \frac{(1-\alpha)(1+t-\gamma(t))}{\alpha(\gamma(1-\alpha)-1+\alpha)}, \qquad c_2 = \frac{1+t-\gamma(t)}{\gamma(1-\alpha)-1+\alpha}.$$

Notice that c_2 is trivially positive and c_1 is positive if and only if $\frac{1+t}{\gamma(t)} \leqslant \frac{\gamma(1-\alpha)}{1-\alpha}$. This condition is satisfied noting that the function $\frac{1+t}{\gamma(t)}$ is increasing and using the inequality $\gamma(t) \geqslant 1 - \frac{1-\alpha}{\alpha}t$:

$$\frac{1+t}{\gamma(t)} \leqslant \frac{3}{2\gamma(1/2)} \leqslant \frac{6\alpha}{6\alpha-2} \leqslant \frac{3\alpha-1-\alpha^2}{\alpha(1-\alpha)} \leqslant \frac{\gamma(1-\alpha)}{1-\alpha}.$$

The third inequality is satisfied for $1/\sqrt{2} < \alpha < 1$. Since $\|(\alpha,\alpha)\| = \|(\gamma(1-\alpha), 1-\alpha)\| = 1$, we obtain

$$\big\|\big(1+t,\gamma(t)\big)\big\| \leqslant \frac{\gamma(t)}{\alpha} - \frac{(1-\alpha)(1+t-\gamma(t))}{\alpha(\gamma(1-\alpha)-1+\alpha)} + \frac{1+t-\gamma(t)}{\gamma(1-\alpha)-1+\alpha}. \tag{3.2}$$

Similarly we have

$$\big(1-t,\gamma(t)\big) = d_1(\alpha,\alpha) + d_2\big(1-\alpha,\gamma(1-\alpha)\big)$$

with

$$d_1 = \frac{\gamma(t)}{\alpha} - \frac{\gamma(1-\alpha)(t+\gamma(t)-1)}{\alpha(\gamma(1-\alpha)-1+\alpha)}, \qquad d_2 = \frac{t+\gamma(t)-1}{\gamma(1-\alpha)-1+\alpha}.$$

Again d_2 is trivially positive, and d_1 is positive, if and only if $\frac{\gamma(t)}{1-t} \leqslant \frac{\gamma(1-\alpha)}{1-\alpha}$ and, using $t < 1/2$, and $1/\sqrt{2} < \alpha < 1$, this is true since

$$\frac{\gamma(t)}{1-t} \leqslant 2 \leqslant \frac{\alpha}{1-\alpha} = \frac{\gamma(\alpha)}{1-\alpha} \leqslant \frac{\gamma(1-\alpha)}{1-\alpha}.$$

So we obtain

$$\big\|\big(1-t,\gamma(t)\big)\big\| \leqslant \frac{\gamma(t)}{\alpha} - \frac{\gamma(1-\alpha)(t+\gamma(t)-1)}{\alpha(\gamma(1-\alpha)-1+\alpha)} + \frac{t+\gamma(t)-1}{\gamma(1-\alpha)-1+\alpha}. \tag{3.3}$$

Adding (3.2) and (3.3) we have:

$$G(t) \leqslant \frac{\gamma(t)}{\alpha} + \frac{1}{\alpha} + \frac{3\alpha-1-\gamma(1-\alpha)}{\alpha(\gamma(1-\alpha)-1+\alpha)}t.$$

Note that the coefficient of t is positive (since $1-\alpha \leqslant \gamma(1-\alpha) \leqslant 3\alpha-1$). We now can prove that $G(t) \leqslant 3$ in the interval for $t \in (1-\alpha, 1/2)$. We have $\gamma(t) \leqslant \gamma(1-\alpha)$ and $t < 1/2$. Thus

$$G(t) \leqslant \frac{\gamma(1-\alpha)+1}{\alpha} + \frac{3\alpha-1-\gamma(1-\alpha)}{2\alpha(\gamma(1-\alpha)-1+\alpha)}.$$

We have $G(t) \leqslant 3$ if

$$2\gamma^2(1-\alpha) - (4\alpha+1)\gamma(1-\alpha) + \big(11\alpha - 6\alpha^2 - 3\big) \leqslant 0;$$

and this last is true if $\gamma(1-\alpha)$ belongs to the roots interval of the second degree equation

$$2z^2 - (4\alpha+1)z + \big(11\alpha - 6\alpha^2 - 3\big) = 0.$$

The roots being $\frac{3-2\alpha}{2}$ and $(3\alpha-1)$ we have the condition

$$\frac{3-2\alpha}{2} \leqslant \gamma(1-\alpha) \leqslant 3\alpha - 1,$$

which is fulfilled. Indeed, $\gamma(1-\alpha) \geqslant (3-\alpha-1/\alpha)$ and $(3-\alpha-1/\alpha) \geqslant \frac{3-2\alpha}{2}$ if $\alpha > 1/\sqrt{2}$, which we assume, and trivially $\gamma(1-\alpha) \leqslant 1 \leqslant (3\alpha-1)$ is true.

Finally we have to consider the interval $[0, 1-\alpha)$ and, using the inequalities $\gamma(t) \leqslant 1$ and $t < (1-\alpha)$, we obtain

$$G(t) \leqslant \frac{2}{\alpha} + \frac{(1-\alpha)(3\alpha-1-\gamma(1-\alpha))}{\alpha(\gamma(1-\alpha)-1+\alpha)}.$$

Note that the function $s \to \frac{3\alpha-1-s}{s-1-\alpha}$ is decreasing, and therefore we can replace $\gamma(1-\alpha)$ with $\frac{3\alpha-1-\alpha^2}{\alpha}$ obtaining

$$G(t) \leqslant \frac{1+3\alpha-2\alpha^2}{\alpha},$$

and we have $G(t) \leqslant 3$ if $\alpha > 1/\sqrt{2}$. $\square$

4. A lower bound in the symmetric plane

We will give a general lower bound for $A(X)$ under the assumption that X is a symmetric Minkowski plane.

Theorem 6. *For any symmetric Minkowski plane X one has*

$$A(X) \geqslant \frac{1+\sqrt{17}}{2} \ (\simeq 2.5615). \tag{4.1}$$

Proof. Using the parameter α of the space X we first introduce a new norm: let us denote by $\|(\cdot,\cdot)\|_{E_\alpha}$ the norm defined by

$$\|(u,v)\|_{E_\alpha} = \begin{cases} \max(|u|,|v|) & \text{if } \min(|u|,|v|) \leqslant (2\alpha-1)\max(|u|,|v|), \\ \frac{\max(|u|,|v|)+\min(|u|,|v|)}{2\alpha} & \text{if } \min(|u|,|v|) > (2\alpha-1)\max(|u|,|v|). \end{cases} \tag{4.2}$$

It easy to verify that this is an octagonal (not regular) norm, and for any u, v one has

$$\|(u,v)\|_{E_\alpha} \leqslant \|(u,v)\|_X. \tag{4.3}$$

Define

$$V(z) = \sup_{t\in[-1,1]} \{ \|(z+t, \gamma(z)+\gamma(t))\| + \|(z-t, \gamma(z)-\gamma(t))\| \}.$$

Then we have because of symmetry

$$A(X) = \min_{z\in[0,\alpha]} V(z),$$

and also, using the observation on the $\pi/4$-rotation, we can suppose that $1/\sqrt{2} \leqslant \alpha \leqslant 1$. We get lower bounds with special choice of t, namely $t = -\alpha$:

$$V(z) \geqslant \|(z-\alpha, \gamma(z)+\alpha)\| + \|(z+\alpha, \gamma(z)-\alpha)\| = \|(\alpha-z, \alpha+\gamma(z))\| + \|(\gamma(z)-\alpha, \alpha+z)\|,$$

and by (4.3)

$$V(z) \geqslant \|(\alpha-z, \alpha+\gamma(z))\|_{E_\alpha} + \|(\gamma(z)-\alpha, \alpha+z)\|_{E_\alpha}.$$

In order to compute these norms we use (4.2). Clearly,

$$\alpha - z \leqslant \alpha + \gamma(z), \qquad \gamma(z) - \alpha \leqslant \alpha + z,$$

and setting $f(z) = \frac{\alpha-z}{\alpha+\gamma(z)}$ and $g(z) = \frac{\gamma(z)-\alpha}{\alpha+z}$ we see that f, g are decreasing ($z \in [0,\alpha]$) and both will be less or equal to $(2\alpha-1)$ if $\alpha^2 \geqslant 1/2$. By (4.2) we have

$$\|(\alpha-z, \alpha+\gamma(z))\|_{E_\alpha} = \alpha + \gamma(z), \qquad \|(\gamma(z)-\alpha, \alpha+z)\|_{E_\alpha} = \alpha + z,$$

and therefore

$$V(z) \geqslant 2\alpha + \gamma(z) + z.$$

Using $t = 1$ we obtain

$$V(z) \geqslant \|(z+1, \gamma(z))\| + \|(1-z, \gamma(z))\| \geqslant 2\|(1, \gamma(z))\| \geqslant 2\|(1, \gamma(z))\|_{E_\alpha} = \frac{1+\gamma(z)}{\alpha}.$$

So we get

$$V(z) \geqslant \max\left(2\alpha + \gamma(z) + z, \frac{1+\gamma(z)}{\alpha}\right),$$

and using again the inequality $\gamma(z) \geqslant 1 - \frac{1-\alpha}{\alpha} z$ we obtain

$$V(z) \geqslant \max\left(\frac{2}{\alpha} - \frac{1-\alpha}{\alpha^2} z, 2\alpha + 1 + \frac{2\alpha-1}{\alpha} z\right),$$

where the first term, call it f_1, is decreasing and the second, call it f_2, is increasing. Thus

$$A(X) \geqslant \min_z \max(f_1(z), f_2(z)) = S(\alpha). \tag{4.4}$$

If $\alpha \geqslant \frac{\sqrt{17}-1}{4} \sim 0.78$, then $S(\alpha) = 2\alpha + 1$ and $A(X) \geqslant \frac{1+\sqrt{17}}{2}$; if $\alpha \leqslant \frac{\sqrt{17}-1}{4}$ in (4.4), the minimum is attained when the two terms are equal, giving the value $S(\alpha) = \frac{5\alpha-2\alpha^2-1}{1+2\alpha^2-2\alpha}$. It is not hard to prove that for $\frac{1}{\sqrt{2}} \leqslant \alpha \leqslant \frac{\sqrt{17}-1}{4}$ we have $\frac{5\alpha-2\alpha^2-1}{1+2\alpha^2-2\alpha} \geqslant \frac{1+\sqrt{17}}{2}$. This proves that also for $\alpha \in [\frac{1}{\sqrt{2}}, 1]$ we have $A(X) \geqslant \frac{1+\sqrt{17}}{2}$. $\square$

This lower bound for $A(X)$ improves, in the case of symmetric norm, the general lower bound $\frac{3+\sqrt{21}}{3} \sim 2.52752$ for $A(X)$ given in [1]. We remark that in [1] our lower bound $\frac{1+\sqrt{17}}{2} \sim 2.56155$ is proved in the special case when James orthogonality is symmetric in X. This is curious since neither symmetry of the norm implies that James orthogonality is symmetric nor the symmetry of James orthogonality implies that the norm is symmetric.

5. An example and special results

5.1. A space with small $A(X)$

We now present a Minkowski plane with $A(X)$ "small". Let X be a 16-gonal space such that $||(x, y)|| = ||(|x|, |y|)||$ for any $(x, y) \in X$. Our symmetry assumption allows to consider only the following vertices in the first quadrant:

$$(0,1); \quad \left(\frac{u}{1+v}, 1\right); \quad (u, v); \quad \left(1, \frac{u-v}{u}\right).$$

The norm is defined by

$$\|(x,y)\| = \begin{cases} x & 0 \leqslant y \leqslant \frac{u-v}{u}x, \\ \frac{(uv-u+v)x+u(1-u)y}{u(2v-u)} & \frac{u-v}{u}x \leqslant y \leqslant \frac{v}{u}x, \\ \frac{(1-v^2)x+uvy}{u} & \frac{v}{u}x \leqslant y \leqslant \frac{1+v}{u}x, \\ y & y \geqslant \frac{1+v}{u}x. \end{cases}$$

We pick $(0, 1)$ as *center*, and so we compute the quantity

$$M := \sup_{\|(x,y)\|=1} \left(\|(0,1)-(x,y)\| + \|(0,1)+(x,y)\|\right).$$

Because of symmetry and convexity we have

$$\begin{aligned} A(X) \leqslant M = \max\Bigg\{ & \left\|(0,1) - \left(\frac{u}{1+v}, 1\right)\right\| + \left\|(0,1) + \left(\frac{u}{1+v}, 1\right)\right\|; \|(0,1)-(u,v)\| + \|(0,1)+(u,v)\|; \\ & \left\|(0,1) - \left(1, \frac{u-v}{u}\right)\right\| + \left\|(0,1) + \left(1, \frac{u-v}{u}\right)\right\|\Bigg\} \\ = \max\left\{2 + \frac{u}{1+v}; 1 + v + \frac{2uv-2u+1}{2v-u}; \frac{2-2v^2+2uv}{u}\right\}. \end{aligned}$$

Numerical optimization gives: $u = 0.924263$ and $v = 0.626018$. Hence we have $A(X) < 2.56811$.

We recall that for any 2-dimensional space X we have: $A(X) \geqslant \frac{3+\sqrt{21}}{3} \cong 2.5275$.

5.2. Octagons

Let O_α denote the (symmetric) octagon whose vertices in the first quadrant are $(0, 1)$, $(\alpha, \alpha = \gamma(\alpha))$, $(1, 0)$. With a simple, but somewhat lengthy computation one can calculate exactly $A(O_\alpha)$. We quote here the result:

Proposition 7. *For $1/2 \leqslant \alpha \leqslant 1$ one has*

$$A(O_\alpha) = \frac{1}{\alpha} + 2\alpha.$$

Moreover, for $\alpha^2 < 1/2$ the Centers are $(\pm\alpha, \pm\alpha)$, for $\alpha^2 > 1/2$ the Centers are $(0, \pm1)$, $(\pm1, 0)$, and for $\alpha^2 = 1/2$ every point of the unit sphere is a Center. Note that $A(O_\alpha) \geqslant 2\sqrt{2}$.

Problem 8. We finish our paper mentioning two problems which arise naturally:

1) Is it true that for every Minkowski plane $A(X) \leqslant 3$?

2) Find the exact value of $\min\{A(X)\colon X \text{ is a Minkowski plane}\}$.

References

[1] M. Baronti, E. Casini, P.L. Papini, Triangles inscribed in a semicircle, in Minkowski planes and in normed spaces, J. Math. Anal. Appl. 252 (2000) 124–146.
[2] M.A. Ghandehari, Heron's problem in the Minkowski plane, Tech. report 306, Math. Dept., University of Texas at Arlington, 76019, USA, 1997.
[3] C. Gros, T.K. Strempel, On generalizations of conics and on the generalization of the Fermat–Torricelli problem, Amer. Math. Monthly 105 (1998) 732–743.
[4] A.G. Horváth, H. Martini, Conics in normed planes, Extracta Math. 26 (1) (2011), 11 pp., in press, arXiv:1102.3008v1 [math.MG].
[5] G. Nordlander, The modulus of convexity in normed spaces, Ark. Mat. 4 (1960) 15–17.
[6] W. Senlin, J. Donghai, J. Alonso, Metric ellipses in Minkowski planes, Extracta Math. 20 (2005) 273–280.
[7] L. Tamássy, K. Bélteky, On the coincidence of two kinds of ellipses in Minkowskian spaces and in Finsler planes, Publ. Math. Debrecen 31 (1984) 157–161.