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Group invariance and Pohozaev identity in Moser type inequalities

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We study the so-called limiting Sobolev cases for embeddings of the spaces $W_0^{1,n}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Differently from J. Moser, we consider optimal embeddings into Zygmund spaces: we derive related Euler-Lagrange equations, and show that Moser's concentrating sequences are the solutions of these equations and thus realize the best constants of the corresponding embedding inequalities. Furthermore, we exhibit a group invariance, and show that Moser's sequence is generated by this group invariance and that the solutions of the limiting equation are unique up to this invariance. Finally, we derive a Pohozaev-type identity, and use it to prove that equations related to perturbed optimal embeddings do not have solutions.

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1. Introduction

In the last decades an extensive literature has been devoted to the study of Sobolev spaces and related extensions and generalizations. In particular, a prominent role is played by embedding theorems and the associated inequalities; for an overview on these topics, see e.g. [1,22,8,6]. As a reference example consider the classical Sobolev space $W_0^{1,p}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ a bounded domain, which is the completion of compactly supported smooth functions with respect to the norm $\|\nabla \cdot\|_p$, $p \geq 1$. Then, for $1 \leq p < n$, the following classical Sobolev embedding holds

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \ , \quad \text{namely} \quad \|u\|_{p^*} \leq \frac{1}{S} \, \|\nabla u\|_p \ , \ \forall \ u \in \mathcal{C}_0^\infty(\Omega) \ , \tag{1.1}$$

where $p^* := \frac{np}{n-p}$ is the so-called *critical Sobolev exponent*, and the positive constant S is the *Sobolev constant* which is best possible, explicitly known, and depending just on n and p ; see [21].

Critical growth manifests itself through various well-known phenomena, namely:

i) For the subcritical case $2 \leq q < p^*$, the embeddings $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ are compact; hence the best embedding constants are achieved, and the corresponding elliptic equations $-\Delta_p u = |u|^{q-2}u$ have a solution in $W_0^{1,p}(\Omega)$, where $\Delta_p = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator.

ii) For $q = p^*$ compactness is lost because of concentration phenomena which appear due to a group invariance which both norms $\|\nabla \cdot \|_p$ and $\|\cdot\|_{p^*}$ share: this is the so-called *bubbling of spheres* phenomenon, see [16].

iii) Extremals for (1.1) are solutions of the critical equation $-\Delta_p u = |u|^{p^*-2}u$ with Dirichlet boundary conditions; via Pohožaev type variational identities this equation has no solutions on starshaped domains, and as a consequence the best Sobolev constant S is not achieved (this can be proved also directly by exploiting that S is domain independent), see [20]. Existence of solutions can be recovered by lower order perturbations of Brezis-Nirenberg type, see [7].

The case $p = n$ is the so-called *limiting Sobolev case* for the embedding (1.1): one has the embedding $W_0^{1,n}(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \leq p < \infty$, but $W_0^{1,n}(\Omega) \not\subset L^\infty(\Omega)$ if $n > 1$. The maximal degree of summability for functions in $W^{1,n}(\Omega)$ was established independently by Pohožaev [18] and Trudinger [23] and is of exponential type. More precisely, $W_0^{1,n}(\Omega)$ embeds into an Orličz class of functions, namely:

$$u \in W_0^{1,n}(\Omega) \implies \int_\Omega e^{|u|^{\frac{n}{n-1}}}\, dx < \infty$$

While the Sobolev case $p < n$ is by now well understood, criticality in the limiting case is a very delicate issue and still presents open questions. Indeed, J. Moser proposed in the seminal paper [17] a notion of criticality by means of the following uniform bound:

$$\sup_{u \in \mathcal{C}_0^\infty(\Omega), \|\nabla u\|_n \leq 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}}\, dx \;\leq\; c_0(\alpha)\,|\Omega| \quad \begin{cases} < \infty\ , & \text{if} \quad \alpha \leq \alpha_n \\ = \infty\ , & \text{if} \quad \alpha > \alpha_n \end{cases} \tag{1.2}$$

where α_n is explicitly known: $\alpha_n = n^n {\omega_n}^{1/(n-1)}$, where ω_n denotes the measure of the unit ball in $\mathbb{R}^n$. As further developed in [16,12], the functional

$$J : W_0^{1,n}(\Omega) \longrightarrow \mathbb{R} \quad , \quad J(u) = \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}}\, dx \tag{1.3}$$

is compact for $\alpha < \alpha_n$. Therefore, concerning aspects i) and ii) above, α_n plays the role of the critical Sobolev exponent, and again compactness is lost in the critical case $\alpha = \alpha_n$, cf. [16]. However, in striking contrast with the Sobolev case (point iii) above), it was shown by Carleson-Chang [9] that the supremum in (1.2) for the best constant α_2 is achieved if Ω is a ball. The result was extended to any domain in $\mathbb{R}^2$ by Flucher [14] and to any dimension n by Lin [15]. Moreover, corresponding semilinear elliptic equations with nonlinearities in the critical growth range do have variational solutions [2,10,12]. From the analysis carried out in [12], this is related to the fact that the functional (1.3) retains enough energy to avoid the critical non-compactness level. The difficulties to gain further insight seem to lie in the fact that for the Trudinger-Moser case (1.2) no Pohozaev type identity nor a group invariance are known. As we are going to see, exploiting finer norms than the Orlicz norm will yield new insights into these phenomena.

In [11] we proposed, in the case $n = 2$, a different approach by moving the attention from the uniform bound (1.2) to an embedding inequality for the space $W_0^{1,2}(\Omega)$ into a suitable Zygmund space. Here we further develop this line by extending the results of [11] to the general case.

In particular, we derive the Euler-Lagrange equation associated to the corresponding Moser-Zygmund inequality; for this we rely on the notion of *subdifferential*, since the Zygmund norm is not differentiable.

Then, we show that this equation allows a group invariance, and that its solutions, which are unique up to this group invariance, are given by the well-known *Moser-sequence* (which in fact is generated by this group invariance, as already mentioned by Moser, see also Adimurthi-Tintarev [3]). Thus, we have an analogous situation as in ii) above, with the "bubbles" given by the Moser sequence.

Furthermore, we find a Pohozaev type identity which is obtained, as usual, by applying a multiplier which is derived from the infinitesimal generator of the group invariance. We recall that for the original Trudinger-Moser embedding no group invariance nor Pohozaev identity are known.

Finally, we will consider slight variations of the Zygmund norm, and we will determine, as a consequence of the Pohozaev identity, the threshold between attainability and non-attainability of the best constants in the corresponding Moser-Zygmund embeddings and hence between the existence and non existence of solutions of the associated Euler-Lagrange equations.

Before stating our main results we briefly recall the definitions and basic properties of Zygmund spaces; for more details we refer to Section 2.

The *Zygmund space* $Z^\alpha(\Omega)$, $\alpha > 0$, consists of all measurable functions $u(x)$ on a bounded set $\Omega \subset \mathbb{R}^n$ such that

$$\int_\Omega e^{\lambda|u|^{\frac{1}{\alpha}}}\,dx < \infty, \qquad \forall\ \lambda > 0 \tag{1.4}$$

The quantity

$$\|u\|_{Z^\alpha_\varepsilon} = \sup_{t\in(0,|\Omega|)} \frac{u^*(t)}{\left[\varepsilon + \log\left(\frac{|\Omega|}{t}\right)\right]^\alpha}\,, \quad \varepsilon > 0 \tag{1.5}$$

where u^* denotes the monotone decreasing rearrangement of u, defines an ε-varying family of quasinorms on $Z^\alpha(\Omega)$ which are equivalent to a real norm (by replacing

$u^*(t)$ with the maximal function $u^{**}(t) := 1/t \int_0^t u^*(s)\,ds)$ which turns $Z^\alpha(\Omega)$ into a Banach space). As $\varepsilon \to 0$ the quasinorm (1.5) may degenerate; in the literature the quantity

$$[\,u\,]_\alpha := \|u\|_{Z_0^\alpha} = \sup_{t\in(0,|\Omega|)} \frac{u^*(t)}{\left[\log\left(\frac{|\Omega|}{t}\right)\right]^\alpha} \tag{1.6}$$

seems to appear first in [4], where the following result was proved:

Theorem [Alvino, 1977] *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Then, the following inequality holds*

$$[\,u\,]_{\frac{n-1}{n}} \leq \frac{1}{n\omega_n^{1/n}}\,\|\nabla u\|_n \tag{1.7}$$

for any $u \in W_0^{1,n}(\Omega)$, where ω_n is the measure of the unit ball. Moreover, the constant appearing in (1.7) is the best possible.

Our main results are the following:

Theorem 1. *The inequality (1.7) is attained and when $\Omega \subset \mathbb{R}^n$ is the unit ball, the (normalized) radial extremals of inequality (1.7) satisfy the following Euler-Lagrange equation*

$$\begin{cases} -\left(|y'|^{n-2}y'\,r^{n-1}\right)' = (n\omega_n)^{\frac{1-n}{n}}\,\mu^{\frac{1-n}{n}}\,\delta_{r_\mu}\ ,\ \ r\in(0,1) \\[2ex] y(r_\mu) = \dfrac{\mu^{\frac{n-1}{n}}}{(n\omega_n)^{1/n}} \\[2ex] y'(0) = y(1) = 0\ , \end{cases} \tag{1.8}$$

where $r_\mu := e^{-\mu}$ and $\mu \in (0,+\infty)$ is an arbitrary parameter; furthermore, δ_{r_μ} denotes the Dirac delta function in the point $r_\mu \in (0,1)$.

An explicit solution of (1.8), for $\mu = 1$ and $r_1 = e^{-1}$, is given by

$$u_1(r) = \frac{1}{(n\omega_n)^{1/n}} \begin{cases} -\log r\ ,\ \frac{1}{e}\leq r\leq 1 \\[2ex] 1\ ,\quad 0\leq r\leq \frac{1}{e} \end{cases} \tag{1.9}$$

For fixed $\mu = 1$, the solution $u_1(r)$ is unique. On other hand, the group action

$$T_\mu\ :\ u(r)\ \mapsto\ \mu^{-\frac{n-1}{n}}\,u(r^\mu)\ ,\ r\geq 0,\ \mu\in(0,+\infty) \tag{1.10}$$

generates the entire family of Moser functions

$$u_\mu(r) = \frac{1}{(n\omega_n)^{1/n}} \begin{cases} -\mu^{-1/n}\log r\ ,\ e^{-\mu}\leq r\leq 1 \\[2ex] \mu^{\frac{n-1}{n}}\ ,\qquad 0\leq r\leq e^{-\mu} \end{cases}, \tag{1.11}$$

which constitute all solutions of equation (1.8).

We recall that the *Moser sequence* was used by J. Moser in [17] to prove that the number α_n in inequality (1.2) is optimal.

We have formally an analogue situation to the classical Sobolev case: loss of compactness in the critical case due to the appearance of a group action. Differently from the Sobolev case, the full group is present in bounded domains, and thus does not obstruct the attainability of the best constant.

The group action (1.10) leaves invariant both $[\,u\,]_{\frac{n-1}{n}}$ and the Sobolev norm $\|\nabla u\|_n$, while this invariance fails to hold for the quasi-norm $\|u\|_{Z_\varepsilon^{\frac{n-1}{n}}}$ as long as $\varepsilon > 0$. As a consequence, we complement Alvino's theorem and Theorem 1 in the following way:

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Then, the following inequality holds*

$$\|u\|_{Z_\varepsilon^{\frac{n-1}{n}}} \leq \frac{1}{n\omega_n^{1/n}} \|\nabla u\|_n, \quad \varepsilon \geq 0 \tag{1.12}$$

for any $u \in W_0^{1,n}(\Omega)$, where ω_n is the measure of the unit ball. Moreover, the constant appearing in (1.12) *is sharp for any domain and, when Ω is a ball, it is never achieved by radial functions as long as $\varepsilon > 0$.*

(As a byproduct, for $\varepsilon = 0$ we give a new direct proof of Alvino's result which was obtained by interpolation arguments.)

The proof of Theorem 2 is based on the following "Pohozaev-type" identity:

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be the unit ball, and let $y \in W_{0,rad}^{1,n}(\Omega)$ be a solution of the radial equation*

$$-(|y'|^{n-2}y'r^{n-1})' = c\delta_{r_0} \tag{1.13}$$

where c is constant, and δ_{r_0} is the distribution with support concentrated in r_0, with $r_0 \in (0,1)$. Then y is $\mathcal{C}^1$-piecewise and there holds

$$\int_0^1 |y'|^n r^{n-1} dr \;=\; \left[|y'_+(r_0)|^n - |y'_-(r_0)|^n\right] r_0^n |\log r_0| \tag{1.14}$$

where $|y'_\pm|(r_0) = \lim_{r\to r_0^\pm} |y'|(r)$.

The connection between the group action (1.10) and the identity (1.14) is given by the fact that the function $r \log r\, y'$, used as a multiplier in the proof of Proposition 1, is the generator of the family of scaled maps $\{y_\mu(r) = y(r^\mu),\ \mu \in (0,\infty)\}$.

2. Preliminaries

Let us first recall some basic definitions and properties of the Zygmund space (see also [5,6]).

Let $u : \Omega \to \mathbb{R}$ a measurable function; denoting by $|S|$ the Lebesgue measure of a measurable set $S \subset \mathbb{R}^n$, let

$$\mu_u(s) = \big|\{x \in \Omega : |u(x)| > s\}\big|, \quad s \geq 0$$

be the distribution function of u. The *monotone decreasing rearrangement* $u^* : [0, +\infty) \to [0, +\infty]$ of u is defined as the distribution function of μ_u, that is

$$u^*(t) := \big|\{s \in [0, \infty) : \mu_u(s) > t\}\big| = \sup\Big\{s > 0 \,:\, \big|\,\{x \in \mathbb{R}^n \,:\, |u(x)| > s\}\,\big| > t\Big\},$$

$0 \leq t \leq |\Omega|$, whereas the *spherically symmetric rearrangement* $u^\sharp$ of u is defined as

$$u^\sharp(x) = u^*(\omega_n|x|^n), \quad x \in \Omega^\sharp \ ;$$

here $\Omega^\sharp$ is the open ball with center in the origin which satisfies $|\Omega^\sharp| = |\Omega|$. Clearly, u^* is a nonnegative, non-increasing and right-continuous function on $[0, \infty)$; moreover, the (nonlinear) rearrangement operator enjoys the following properties:

i) Positively homogeneous: $(\lambda u)^* = |\lambda| u^*, \quad \lambda \in \mathbb{R}$
ii) Sub-additive: $(u + v)^*(t + s) \leq u^*(t) + v^*(s), \quad t, s \geq 0$
iii) Monotone: $0 \leq u(x) \leq v(x)$ a.e. in $\Omega \Rightarrow u^*(t) \leq v^*(t), \quad t \in (0, |\Omega|)$
iv) u and u^* are equidistributed and in particular (a version of the Cavalieri Principle):

$$\int_\Omega A(|u(x)|)\, dx = \int_0^{|\Omega|} A(u^*(s))\, ds$$

for any continuous funtion $A : [0, \infty] \to [0, \infty]$, nondecreasing and such that $A(0) = 0$

v) The following inequality holds (Hardy-Littlewood):

$$\int_\Omega u(x)v(x)\, dx \leq \int_0^{|\Omega|} u^*(s)v^*(s)\, ds$$

provided the integrals are defined.

vi) The map $u \mapsto u^*$ preserves Lipschitz regularity, namely $^* : Lip(\Omega) \longrightarrow Lip(0, |\Omega|)$.

The *Zygmund space* $Z^\alpha(\Omega)$, $\alpha > 0$, consists of all measurable functions $u(x)$ on a bounded set $\Omega \subset \mathbb{R}^n$ such that

$$\int_\Omega e^{\lambda |u|^{\frac{1}{\alpha}}}\, dx < \infty \ , \qquad \forall\, \lambda > 0$$

The Zygmund space $Z^\alpha(\Omega)$ can be equivalently defined as measurable $u : \Omega \to \mathbb{R}$ such that:

$$\sup_{0<t<|\Omega|} \frac{u^*(t)}{[1 + \log(\frac{|\Omega|}{t})]^\alpha} < \infty \ \text{ and } \ \lim_{t \to 0} \frac{u^*(t)}{[1 + \log(\frac{|\Omega|}{t})]^\alpha} = 0$$

The proof of this equivalence can be found in [6] (Theorem D, p.15); here we prove only one inclusion. Let u be a measurable function such that

$$\sup_{0<t<|\Omega|} \frac{u^*(t)}{[1+\log(\frac{|\Omega|}{t})]^\alpha} = C_u \qquad \text{and} \qquad \lim_{t\to 0} \frac{u^*(t)}{[1+\log(\frac{|\Omega|}{t})]^\alpha} = 0 \ ;$$

define

$$w(s) = u^*(|\Omega| e^{-s})$$

Then

$$\sup_{0<s<\infty} \frac{w(s)}{[1+s]^\alpha} = C_u < \infty \qquad \text{and} \qquad \lim_{s\to+\infty} \frac{w(s)}{[1+s]^\alpha} = 0$$

and

$$\int_\Omega e^{\lambda |u|^{\frac{1}{\alpha}}}\, dx = \int_0^{|\Omega|} e^{\lambda |u^*|^{\frac{1}{\alpha}}}\, dt = |\Omega| \int_0^\infty e^{\lambda |w|^{\frac{1}{\alpha}} - s} ds$$

From

$$w(s) < C_u (1+s)^\alpha \ , \quad \forall\ s > 0$$

follows immediately that the integral is bounded for any $\lambda < C_u^{-1/\alpha}$. The existence of the integral for any positive λ, instead, is a consequence of the second condition: indeed, given any $\varepsilon > 0$, there exists a $\tau = \tau(\varepsilon)$ such that $w(s) < \varepsilon(1+s)^\alpha$ for any $s > \tau$; hence, for any fixed λ it suffices to chose $\varepsilon = (2\lambda)^{-\alpha}$ so that

$$\begin{aligned}\int_0^\infty e^{\lambda |w|^{\frac{1}{\alpha}} - s} ds &= \int_0^\tau e^{\lambda |w|^{\frac{1}{\alpha}} - s} ds + \int_\tau^\infty e^{\lambda |w|^{\frac{1}{\alpha}} - s} ds \\ &< \int_0^\tau e^{\lambda C_u^{\frac{1}{\alpha}} (1+s) - s} ds + \int_\tau^\infty e^{\frac{1-s}{2}}\, ds < \infty\end{aligned}$$

The quantity

$$\|u\|_{Z^\alpha} = \sup_{t\in(0,|\Omega|)} \frac{u^*(t)}{\left[1+\log\left(\frac{|\Omega|}{t}\right)\right]^\alpha} \tag{2.1}$$

defines a quasinorm on Z^α which turns out to be equivalent to a real norm. The space $Z^\alpha(\Omega)$ can also be obtained as the limiting case of *Lorentz–Zygmund spaces* $L^{p,q}(\log L)^{-\alpha}(\Omega)$, as $p, q \to \infty$, see [6,5], and can be also realized as Orličz classes $L_{e^{|u|^{1/\alpha}}}(\Omega)$, see [5], equipped with the quasinorm (2.1). Slightly varying the definition (2.1), we obtain the family of equivalent quasinorms

$$\|u\|_{Z^\alpha_\varepsilon} = \sup_{t\in(0,|\Omega|)} \frac{u^*(t)}{\left[\varepsilon+\log\left(\frac{|\Omega|}{t}\right)\right]^\alpha} \quad , \quad \varepsilon > 0$$

introduced in (1.5). Recalling that (in order to emphasize the dependence of the quasinorm (1.5) on ε) we have denoted the space $Z^\alpha(\Omega)$ endowed with $\|\cdot\|_{Z^\alpha_\varepsilon}$ by $Z^\alpha_\varepsilon(\Omega)$, the quasinorm (2.1) is a particular member of the family $\|\cdot\|_{Z^\alpha_\varepsilon}$, realized by $\varepsilon = 1$.

In order to deal with radial functions, it is more convenient to reformulate the definition of (1.5), (1.6) in terms of the spherically symmetric rearrangement $u^\sharp$ of u; recalling that $u^\sharp(r) = u^*(\omega_n r^n), r = |x|$, we have

$$\begin{aligned}\|u\|_{Z^\alpha_\varepsilon} &= \sup_{t\in(0,|\Omega|)} \frac{u^*(t)}{\left[\varepsilon + \log\left(\frac{|\Omega|}{t}\right)\right]^\alpha} = \sup_{r\in(0,(|\Omega|/\omega_n)^{1/n})} \frac{u^*(\omega_n r^n)}{\left[\varepsilon + \log\left(\frac{|\Omega|}{\omega_n r^n}\right)\right]^\alpha} \\ &= \sup_{r\in(0,R)} \frac{u^\sharp(r)}{\left[\varepsilon + n\log\left(\frac{R}{r}\right)\right]^\alpha}\ , \qquad R = (|\Omega|/\omega_n)^{1/n}\end{aligned}$$

and

$$\begin{aligned}[\,u\,]_\alpha &= \sup_{t\in(0,|\Omega|)} \frac{u^*(t)}{\left[\log\left(\frac{|\Omega|}{t}\right)\right]^\alpha} = \sup_{r\in(0,(|\Omega|/\omega_n)^{1/n})} \frac{u^*(\omega_n r^n)}{\left[\log\left(\frac{|\Omega|}{\omega_n r^n}\right)\right]^\alpha} \\ &= \sup_{r\in(0,R)} \frac{u^\sharp(r)}{\left[n\log\left(\frac{R}{r}\right)\right]^\alpha}\ , \qquad R = (|\Omega|/\omega_n)^{1/n}\end{aligned}$$

3. Proof of Theorem 1

Let $\Omega = B_1 \subset \mathbb{R}^n$ be the unit ball. We first show that the transformation (1.10), which acts on $W^{1,n}_{0,rad}$, the subspace of radial functions of $W^{1,n}_0$, leaves invariant both $[\,u\,]_{\frac{n-1}{n}}$ and the Sobolev norm $\|\nabla u\|_n$.

Proposition 2. *The Sobolev norm $\|\nabla u\|_n$ and $[\,u\,]_{\frac{n-1}{n}}$ are invariant under the group action*

$$T_\mu \ :\ u(r) \ \mapsto\ \mu^{-\frac{n-1}{n}} u(r^\mu)\ ,\ \mu \in (0,+\infty)$$

Proof. Let $u \in W^{1,n}_{0,rad}(B_1)$; then $(T_\mu u)^\sharp = T_\mu u^\sharp = \mu^{-\frac{n-1}{n}} u^\sharp(r^\mu)$. By (2.2), we have

$$\begin{aligned}[\,T_\mu u\,]_{\frac{n-1}{n}} &= \sup_{r\in(0,1)} \mu^{-\frac{n-1}{n}} \frac{u^\sharp(r^\mu)}{[n|\log r|]^{\frac{n-1}{n}}} \\ = \sup_{\rho\in(0,1)} \mu^{-\frac{n-1}{n}} &\frac{u^\sharp(\rho)}{\left[n|\log(\rho^{1/\mu})|\right]^{\frac{n-1}{n}}} = \sup_{\rho\in(0,1)} \frac{u^\sharp(\rho)}{[n|\log(\rho)|]^{\frac{n-1}{n}}} = [\,u\,]_{\frac{n-1}{n}}\end{aligned}$$

Furthermore

$$\begin{aligned}\|\nabla(T_\mu u)\|^n_n &= n\omega_n \int_0^1 \left|\frac{d}{dr}\mu^{-\frac{n-1}{n}} u(r^\mu)\right|^n r^{n-1} dr = n\omega_n\mu \int_0^1 \left|u'(r^\mu)\right|^n r^{\mu n-1} dr \\ &= n\omega_n \int_0^1 \left|u'(\rho)\right|^n \rho^{n-1} d\rho = \|\nabla u\|^n_n\end{aligned}$$

□

We know that, by testing with (1.9),

$$\frac{1}{n\omega_n^{1/n}} \ = \ \sup_{u\in W^{1,n}_0(B_1),\ \|\nabla u\|_n=1} [\,u\,]_{\frac{n-1}{n}} \tag{3.1}$$

is attained. Thanks to the Polya–Szëgo inequality [19], we may assume that extremal functions are positive, radially symmetric and decreasing; hence, we have

$$\frac{1}{n^n\omega_n} = \sup_{u\in W^{1,n}_{0,\sharp},\ \phi(u)=1} \psi_\sharp(u) \tag{3.2}$$

where we denote by $W^{1,n}_{0,\sharp}$ the subset of $W^{1,n}_0$ consisting of radially symmetric decreasing functions, namely

$$W^{1,n}_{0,\sharp} = \left\{u : [0,1] \to \mathbb{R}^+ \ :\ u(r) \text{ non increasing and } u(|x|) \in W^{1,n}_0(\Omega)\right\}$$

and

$$\psi_\sharp(u) := \sup_{r\in(0,1)} \frac{|u^\sharp(r)|^n}{|n\log r|^{n-1}} = [\,u\,]^n_{\frac{n-1}{n}}$$
$$\phi(u) := n\omega_n \int_0^1 |u'|^n r^{n-1} dr = \|\nabla u\|^n_n$$

where we have used (2.2). Note that the difference of two decreasing functions is in general not decreasing, so that the set $W^{1,n}_{0,\sharp}$ is not a subspace of W^1_0. However, the functional $\psi_\sharp(u)$ is well defined on the whole space $W^{1,n}_{0,rad}$, the subspace of $W^{1,n}_0$ consisting of radially symmetric functions; furthermore,

$$\psi_\sharp(u) = \psi(u), \qquad \forall\ u \in W^{1,n}_{0,\sharp}$$

where

$$\begin{aligned} \psi : W^{1,n}_{0,rad} &\longrightarrow [0,+\infty) \\ u \ &\mapsto\ \psi(u) = \sup_{r\in(0,1)} \frac{|u(r)|^n}{|n\log r|^{n-1}} \end{aligned}$$

which is well defined since, for any $u \in W^{1,n}_{0,rad}$:

$$\begin{aligned} |u(r)|^n &= \left|\int_r^1 -u'(\rho)d\rho\right|^n \le \left[\int_r^1 |u'(\rho)|\rho^{\frac{n-1}{n}}\rho^{-\frac{n-1}{n}}d\rho\right]^n \\ &\le \int_r^1 |u'(\rho)|^n\rho^{n-1}d\rho \cdot \left\{\int_r^1 \rho^{-1}d\rho\right\}^{n-1} \\ &\le \frac{1}{n\omega_n}\cdot\phi(u)\cdot|\log r|^{n-1} = \frac{1}{n^n\omega_n}\cdot\phi(u)\cdot|n\log r|^{n-1} \end{aligned}$$

This inequality proves also that

$$\frac{1}{n^n\omega_n} = \sup_{u\in W^{1,n}_{0,\sharp},\ \phi(u)=1} \psi_\sharp(u) = \sup_{u\in W^{1,n}_{0,\sharp},\ \phi(u)=1} \psi(u) = \sup_{u\in W^{1,n}_{0,rad},\ \phi(u)=1} \psi(u) \tag{3.3}$$

Indeed, clearly

$$\sup_{u\in W^{1,n}_{0,\sharp},\ \phi(u)=1} \psi(u) \le \sup_{u\in W^{1,n}_{0,rad},\ \phi(u)=1} \psi(u)$$

and the reverse inequality is a consequence of the previous estimate

$$|u(r)|^n \leq \frac{1}{n^n \omega_n} \cdot \phi(u) \cdot |n \log r|^{n-1}$$

Therefore, the extremals for (3.1) satisfy the Euler-Lagrange equation associated to (3.3).

Note that the functional $u \mapsto \psi(u)$ is continuous, thanks to (1.7), but not differentiable; however it is convex, as we will prove below, and its subdifferential $\partial\psi(u)$ turns out to be well defined. For the convenience of the reader, let us briefly recall definitions and some properties of the subdifferential and the subgradient of a convex function.

Definition 1. Let E be a Banach space, and $\psi : E \to \mathbb{R}$ continuous and convex. Then the subdifferential $\partial\psi(u)$ of ψ at $u \in E$ is the subset of the dual space E' characterized by

$$\eta_u \in \partial\psi(u) \iff \psi(u+v) - \psi(u) \geq \langle \eta_u, v \rangle, \quad \forall\, v \in E$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E'. An element $\eta_u \in \partial\psi(u)$ is called a subgradient of ψ at u.

Slight modifications in the proofs of Lemma 2.2, Lemma 2.3 and Corollary 2.4 in [13] yield the following two lemmas:

Lemma 1. *Let $\psi : E \to \mathbb{R}$ be convex and continuous. Assume that $\psi(x) \geq 0$ for all $x \in E$ and that for $q \geq 1$ the following holds*

$$\psi(tx) = t^q \psi(x), \quad \forall t \geq 0$$

Then

$$\eta \in \partial\psi(u) \iff \begin{cases} \langle \eta, u \rangle = q\psi(u) \\ \\ \langle \eta, v \rangle \leq \langle \eta, u \rangle, \forall v \in \psi^u := \{v \in E : \psi(v) \leq \psi(u)\} \end{cases}$$

Lemma 2. *Assume that $\phi \in \mathcal{C}^1(E; \mathbb{R})$ satisfies*

$$\langle \phi'(v), v \rangle = q\phi(v) \neq 0, \quad \forall v \in E \setminus \{0\}$$

and that $\psi : E \to \mathbb{R}$ satisfies the hypotheses of Lemma 1. If $y \in E$ is such that, for some $A > 0$,

$$\psi(y) = \sup_{u \in E,\ \phi(u)=1} \psi(u) = \frac{1}{A}$$

then

$$\phi'(y) \in \frac{1}{\psi(y)}\ \partial\psi(y) \equiv A\, \partial\psi(y).$$

In order to apply the previous two lemmas to our situation let us now verify that the functional ψ, defined by (3.3), is convex.

Lemma 3. *The functional $\psi : W^{1,n}_{0,rad} \to \mathbb{R}$ defined by* (3.3) *is convex.*

Proof. Let $u, v \in W^1_{0,rad}$ and $\lambda \in (0,1)$; clearly the function $\lambda u + (1-\lambda)v \in W^1_{0,rad}$ and

$$\begin{aligned}\psi\left(\lambda u + (1-\lambda)v\right) &= \sup_{r\in(0,1)} \frac{|(\lambda u + (1-\lambda)v)(r)|^n}{|n\log r|^{n-1}} \\ &\le \sup_{r\in(0,1)} \frac{\left[\lambda|u(r)| + (1-\lambda)|v(r)|\right]^n}{|n\log r|^{n-1}} \\ &\le \sup_{r\in(0,1)} \frac{\lambda|u(r)|^n + (1-\lambda)|v(r)|^n}{|n\log r|^{n-1}} \\ &\le \lambda \sup_{r\in(0,1)} \frac{|u(r)|^n}{|n\log r|^{n-1}} + (1-\lambda) \sup_{r\in(0,1)} \frac{|v(r)|^n}{|n\log r|^{n-1}} = \psi(u) + \psi(v)\end{aligned}$$

by the convexity of the function $t \mapsto t^n$. □

Specializing Lemma 2 to our situation in which $E = W^{1,n}_{0,rad}$, and $\psi(v)$ defined as in (3.3), we obtain

Proposition 3. *Suppose that $y \in W^{1,n}_{0,rad}$ satisfies*

$$\psi(y) = \sup_{u\in W^{1,n}_{0,rad},\ \phi(u)=1} \psi(u) = \sup_{u\in W^{1,n}_{0,\sharp},\ \phi(u)=1} \psi_\sharp(u) = \frac{1}{n^n\omega_n} \tag{3.4}$$

Then y is a weak solution of the following equation

$$\begin{cases} -\left(|y'|^{n-2}y'\,r^{n-1}\right)' = n^{n-2}\,\eta_y \\ \\ y'(0) = y(1) = 0\ , \end{cases} \tag{3.5}$$

for a suitable subgradient $\eta_y \in \partial\psi(y)$.

Proof. By Lemma 2 with $A = n^n\omega_n$ we have that y satisfies

$$n\omega_n \int_0^1 n|y'|^{n-2}y'v'r^{n-1}dr = n^n\omega_n\,\langle \eta_y, v\rangle, \quad \forall\ v \in W^{1,n}_{0,rad} \tag{3.6}$$

and the thesis is straightforward. □

It remains to determine the subgradient η_y in equation (3.5). Following [13] and in particular Lemmas 2.6, 2.7 and 2.8 of [13] we have

Proposition 4. *Let y be a maximizer for problem* (3.4) *and let*

$$K_y = \left\{ r \in (0,1) : \frac{y^n(r)}{|n \log r|^{n-1}} = \psi(y) = \frac{1}{n^n \omega_n} \right\}$$

Then

i) $\mathrm{supp}(\eta_y) \subset K_y, \quad \forall \eta_y \in \partial\psi(y)$

ii) $\langle \eta_y, y \rangle = n\,\psi(y) = \dfrac{1}{n^{n-1}\omega_n}$

iii) $K_y = \{r_0\}$

iv) $\partial\psi(y) = [\psi(y)]^{\frac{n-1}{n}} \dfrac{n}{|n \log r_0|^{\frac{n-1}{n}}}\ \delta_{r_0} = n^{-(n-2)} \omega_n^{-\frac{n-1}{n}} |n \log r_0|^{-\frac{n-1}{n}}\ \delta_{r_0}$

Proof.

i) Let $v \in \mathcal{C}_0^\infty((0,1) \setminus K_y)$. Then, for $\lambda \in \mathbb{R}$ sufficiently small we get

$$\psi(y) = \psi(y + \lambda v) \geq \psi(y) + \langle \eta_y, \lambda v \rangle$$

and hence $\langle \eta_y, v \rangle = 0$.

ii) By choosing $v = ty$ in the definition of the subgradient we have

$$\psi(y + ty) \geq \psi(y) + \langle \eta_y, ty \rangle$$

for all $t \in \mathbb{R}$, so that

$$(1+t)^n \psi(y) \geq \psi(y) + t\langle \eta_y, y \rangle$$

The thesis follows letting $t \to 0^\pm$.

iii) Let us first prove that $\mathrm{supp}(\eta_y)$ does not contain an interval. Suppose by contradiction that $\mathrm{supp}(\eta_y) \supset (r_1, r_2) =: I$, $0 \leq r_1 < r_2 \leq 1$ with $y(r) = |n \log r|^{(n-1)/n} / n\omega_n^{1/n}$, $r \in I$. From (3.6) and ii) we obtain

$$n\omega_n \int_0^1 n|y'|^{n-2} y' y' r^{n-1} dr = n^n \omega_n \langle \eta_y, y \rangle = n$$

that is

$$n\omega_n \int_0^1 |y'|^n r^{n-1} dr = 1$$

i.e. the constraint in problem (3.2). Performing the change of variable

$$w(t) = n\omega_n^{1/n} y(e^{-t/n})$$

we obtain

$$w(t) = t^{\frac{n-1}{n}}, \quad \forall\ t \in (t_1, t_2), \quad 0 \leq t_1 < t_2 \leq \infty, \qquad \int_0^\infty |w'|^n dt = 1$$

where $t_1 = -n \log r_2$ and $t_2 = -n \log r_1$. Note that

$$\int_{t_1}^{t_2} |w'|^n dt = \left(\tfrac{n-1}{n}\right)^n \int_{t_1}^{t_2} \frac{dt}{t}$$

so that $0 < t_1 < t_2 < \infty$. Hence, for some functions φ_1 and φ_2

$$w(t) = \begin{cases} \varphi_1(t) \ , \ 0 \le t \le t_1 \\ t^{\frac{n-1}{n}} \ , \ t_1 \le t \le t_2 \\ \varphi_2(t) \ , \ t_2 \le t < \infty \end{cases}$$

and letting $\delta = \int_{t_1}^{t_2} |w'|^n dt \in (0,1)$ we have

$$0 < \int_0^{t_1} |w'|^n dt = \int_0^{t_1} |\varphi_1'|^n dt \le 1 - \delta$$

On the other hand, it is easy to verify that

$$\inf \left\{ \int_0^{t_1} |\varphi_1'|^n dt \ : \ \ \varphi_1(t) \text{ is } \mathcal{C}^1\text{-piecewise} \ , \ \varphi_1(0) = 0 \, , \ \varphi_1(t_1) = t_1^{\frac{n-1}{n}} \right\}$$

is achieved by the straight line $\overline{\varphi}_1(t) = t/t_1^{1/n}$ with integral $\int_0^{t_1} |\overline{\varphi}_1(t)'|^n dt = 1$, a contradiction. Actually in the proof of point ii) we have also proved that if $r_0 \in K_y$, then y is a Moser truncated function, so that $K_y = \{r_0\}$.

iv) From point iii) we have

$$\begin{aligned} \psi(y+v) &= \sup_{r\in(0,1)} \frac{|y(r)+v(r)|^n}{|n\log r|^{n-1}} \ge \frac{|y(r_0)+v(r_0)|^n}{|n\log r_0|^{n-1}} \\ &\ge \frac{|y(r_0)|^n}{|n\log r_0|^{n-1}} + \frac{n[y(r_0)]^{n-1}v(r_0)}{|n\log r_0|^{n-1}} \\ &= \psi(y) + [\psi(y)]^{\frac{n-1}{n}} \ \frac{nv(r_0)}{|n\log r_0|^{\frac{n-1}{n}}} \end{aligned}$$

Hence $\eta_y = n\,[\psi(y)]^{\frac{n-1}{n}} \ \delta_{r_0}/|n\log r_0|^{\frac{n-1}{n}}$. Conversely, let η_y be an arbitrary element of $\partial\Psi(y)$. Since $\text{supp}(\eta_y) \subset K_y = \{r_0\}$, then $\text{supp}(\eta_y) = \{r_0\}$; hence the distribution η_y has finite order N and can be represented by $\eta_y = \sum_{i=0}^N a_1 D^i \delta_{r_0}$. Since $\eta_y \in (W^{1,n}_{0,rad})'$ one has $N = 0$ and $\eta_y = a\delta_{r_0}$; by ii) the thesis follows. □

By Propositions 3 and 4 we have that if y is a maximizer for (3.2), then it satisfies (weakly) the equation

$$\begin{cases} -\left(|y'|^{n-2}y'r^{n-1}\right)' = \omega_n^{-\frac{n-1}{n}} \ \dfrac{\delta_{r_0}}{|n\log r_0|^{\frac{n-1}{n}}} \\ \\ y'(0) = y(1) = 0 \ . \end{cases}$$

In order to simplify the notation and to emphasize the role of r_0 as parameter determining the family of maximizers, we set

$$\mu = -\log r_0 \quad \in (0, +\infty)$$

and we rename r_0 as r_μ, so that $r_\mu = e^{-\mu} \in (0,1)$. Then, the equation satisfied by the maximizer for (3.2) becomes

$$\begin{cases} -\left(|y'|^{n-2}y'r^{n-1}\right)' = (n\omega_n)^{\frac{1-n}{n}}\mu^{\frac{1-n}{n}}\delta_{r_\mu} \\ \\ y'(0) = y(1) = 0\ . \end{cases}$$

By integration against y we get $\dfrac{1}{n\omega_n} = (n\omega_n)^{(1-n)/n}\mu^{(1-n)/n}\ y(r_\mu)$, i.e. r_μ is such that

$$y(r_\mu) = \mu^{\frac{n-1}{n}}(n\omega_n)^{-\frac{1}{n}}$$

that is (1.8).

We now determine explicitly the solution of this problem: testing the equation with any smooth function φ such that $r_\mu \notin \overline{\text{supp}}\,\varphi$ we obtain

$$\int_0^1 |y'|^{n-2}y'\varphi' r^{n-1}\,dr = 0$$

that is, y is a classical solution of the two problems

$$\begin{cases} -\left(|y'|^{n-2}y'r^{n-1}\right)' = 0,\ r \in (0, r_\mu) \\ \\ y'(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} -\left(|y'|^{n-2}y'r^{n-1}\right)' = 0,\ r \in (r_\mu, 1) \\ \\ y(1) = 0 \end{cases}$$

On the other hand, for any smooth, radial function v compactly supported in the unit ball we have

$$\int_0^1 |y'|^{n-2}y'v'r^{n-1}\,dr = (n\omega_n)^{\frac{1-n}{n}}\mu^{\frac{1-n}{n}}v(r_\mu)$$

so that

$$\begin{aligned}
&(n\omega_n)^{\frac{1-n}{n}}\mu^{\frac{1-n}{n}}v(r_\mu) = \lim_{\eta\to 0^+}\left[\int_0^{r_\mu-\eta}|y'|^{n-2}y'v'r^{n-1}dr + \int_{r_\mu+\eta}^1 |y'|^{n-2}y'v'r^{n-1}dr\right] \\
&= \lim_{\eta\to 0^+}\Bigg[|y'|^{n-2}y'(r_\mu-\eta)v(r_\mu-\eta)(r_\mu-\eta)^{n-1} - \int_0^{r_\mu-\eta}\frac{d}{dr}\left(|y'|^{n-2}y'r^{n-1}\right)v\ dr \\
&\qquad -|y'|^{n-2}y'(r_\mu+\eta)v(r_\mu+\eta)(r_\mu+\eta)^{n-1} - \int_{r_\mu+\eta}^1\frac{d}{dr}\left(|y'|^{n-2}y'r^{n-1}\right)v\ dr\Bigg] \\
&= \lim_{\eta\to 0^+}\big[|y'|^{n-2}y'(r_\mu-\eta)v(r_\mu-\eta)(r_\mu-\eta)^{n-1} \\
&\qquad -|y'|^{n-2}y'(r_\mu+\eta)v(r_\mu+\eta)(r_\mu+\eta)^{n-1}\big]
\end{aligned}$$

Combining the previous information, we deduce that y satisfies the problem

$$\begin{cases} -\left(|y'|^{n-2}y'r^{n-1}\right)' = 0\ , & r \in (0,r_\mu)\cup(r_\mu,1) \\ y'(0) = y(1) = 0 & \\ r_\mu^{n-1}\left[(y'_-(r_\mu))^{n-1} - (y'_+(r_\mu))^{n-1}\right] = (n\omega_n)^{\frac{1-n}{n}}\mu^{\frac{1-n}{n}} & \end{cases}$$

and

$$y(r_\mu) = \frac{\mu^{\frac{n-1}{n}}}{(n\omega_n)^{1/n}} \tag{3.7}$$

Since $y'(0) = 0$ and $y(r_\mu)$ is given by (3.7), we get that

$$y(r)\ \text{ is constant on }\ (0,r_\mu)\ ,\quad \text{with}\ \ y(r) = \frac{\mu^{\frac{n-1}{n}}}{(n\omega_n)^{1/n}}$$

On $(r_\mu, 1)$ the solution $y(r)$ must satisfy the boundary conditions $y(1) = 0$ and (3.7), $y(r_\mu) = \mu^{\frac{n-1}{n}}(n\omega_n)^{-1/n}$; hence the solution is given by

$$y(r) = \frac{-\log r}{(n\omega_n)^{1/n}\mu^{1/n}}\quad ,\quad r_\mu < r \le 1$$

One easily checks that $y(r)$ also satisfies the "jump-condition" for y' in r_μ.

Thus, we have found the solutions (1.11). In particular, for $\mu = 1$ this is the solution $u_1(r)$ given by (1.9).

Finally, let us prove that the group action T_μ generates the entire family of Moser functions, which constitute all solutions of equation (1.8). Indeed,

$$T_\nu u_1(r) = \nu^{-\frac{n-1}{n}}u_1(r^\nu) = \frac{\nu^{-\frac{n-1}{n}}}{(n\omega_n)^{1/n}}\begin{cases} -\log(r^\nu)\ , & \text{if}\ \ r_1 = e^{-1} \le r^\nu \le 1 \\ 1\ , & \text{if}\ \ 0 < r^\nu < r_1 = e^{-1} \end{cases}$$

We get

$$T_\nu u_1(r) = \frac{1}{(n\omega_n)^{1/n}}\begin{cases} -\nu^{1/n}\log r\ , & \text{if}\ \ r_{1/\nu} = e^{-1/\nu} \le r \le 1 \\ \left(\frac{1}{\nu}\right)^{\frac{n-1}{n}}\ , & \text{if}\ \ 0 < r < r_{1/\nu} = e^{-1/\nu} \end{cases}$$

$$= \frac{1}{(n\omega_n)^{1/n}}\begin{cases} -\left(\frac{1}{\nu}\right)^{-1/n}\log r\ , & \text{if}\ \ r_{1/\nu} = e^{-1/\nu} \le r \le 1 \\ \left(\frac{1}{\nu}\right)^{\frac{n-1}{n}}\ , & \text{if}\ \ 0 < r < r_{1/\nu} = e^{-1/\nu} \end{cases}$$

that is (1.11) with $\mu = 1/\nu$; this concludes the proof of Theorem 1.

4. Proof of Theorem 2

Due to the rearrangement invariance of inequality (1.12), see e.g. [19], it is clear that the best possible constant in (1.12) is obtained when Ω is a ball and we may assume that extremals can be taken in the class of radially symmetric functions. Then, we will show that actually the best constant does not depend on the domain.

Let $u \in W^{1,n}_{0,rad}(\Omega)$ and as in [17], let us make a change of variables

$$w(t) = n\omega_n^{1/n} u^\sharp(Re^{-t/n}) \tag{4.1}$$

where $\omega_n R^n = |\Omega|$. Then w is monotone increasing on $(0, \infty)$ and by the Polya–Szëgo inequality we get

$$\int_0^\infty (w')^n dt = \int_{\Omega^\sharp} |\nabla u^\sharp|^n dx \le \int_\Omega |\nabla u|^n dx$$

By a standard density argument we may assume that $w(t) \in \mathcal{C}^1_0(0, +\infty)$, hence

$$w(t) = \int_0^t w'(s)ds \le t^{\frac{n-1}{n}} \Big(\int_0^\infty (w')^n dt \Big)^{1/n} \le t^{\frac{n-1}{n}} \|\nabla u\|_n \tag{4.2}$$

and in turn by (4.1) and (4.2)

$$u^\sharp(x) \le \frac{\|\nabla u\|_n}{n\omega_n^{1/n}} \left[\varepsilon + n \log\left(\frac{R}{|x|}\right)\right]^{\frac{n-1}{n}}, \quad \varepsilon \ge 0$$

Then we obtain

$$\begin{aligned} u^*(r) = u^\sharp\left(\left(\frac{r}{\omega_n}\right)^{1/n}\right) &\le \frac{\|\nabla u\|_n}{n\omega_n^{1/n}} \left[\varepsilon + \log\left(\frac{R^n\omega_n}{r}\right)\right]^{\frac{n-1}{n}} \\ &= \frac{\|\nabla u\|_n}{n\omega_n^{1/n}} \left[\varepsilon + \log\left(\frac{|\Omega|}{r}\right)\right]^{\frac{n-1}{n}} \end{aligned}$$

so that

$$\|u\|_{Z_\varepsilon^{\frac{n-1}{n}}} = \sup_{t \in (0,|\Omega|)} \frac{u^*(t)}{\left[\varepsilon + \log\left(\frac{|\Omega|}{r}\right)\right]^{\frac{n-1}{n}}} \le \frac{\|\nabla u\|_n}{n\omega_n^{1/n}}$$

Note that

$$\lim_{t\to\infty} \frac{w(t)}{t^{\frac{n-1}{n}}} = 0 \quad \Longrightarrow \quad \lim_{r\to 0} \frac{u^*(r)}{\left[\varepsilon + \log\left(\frac{|\Omega|}{r}\right)\right]^{\frac{n-1}{n}}} = 0, \quad \varepsilon \ge 0$$

as one can easily prove; see [17]. The inequality (1.7) is sharp, and this can be shown by using the Moser truncated functions:

$$w_k(t) = \begin{cases} \frac{t}{k^{1/n}}, & t \le k \\ k^{\frac{n-1}{n}}, & t \ge k \end{cases} \tag{4.3}$$

for which one has $\|\nabla u_k\|_n = 1$ and

$$\begin{aligned} n\omega_n^{1/n}\|u_k\|_{Z_\varepsilon^{\frac{n-1}{n}}} &= n\omega_n^{1/n} \sup_{r\in(0,|\Omega|)} \frac{u_k^*(r)}{\left[\varepsilon + \log\left(\frac{|\Omega|}{r}\right)\right]^{\frac{n-1}{n}}} \\ &= \sup_{t\in(0,\infty)} \frac{w_k(t)}{(\varepsilon+t)^{\frac{n-1}{n}}} \to 1\ , \quad \text{as} \quad k\to+\infty \end{aligned} \tag{4.4}$$

and this proves that the constant in (1.12) is optimal for any ball. Notice that the Zygmund quasinorm $\|\cdot\|_{Z_\varepsilon^\alpha(\Omega)}$ does not satisfy the zero extension property as it depends explicitly on the measure of the domain Ω. However, by exploiting scale invariance and then extending by zero does not affect the class of extremal functions (4.3). In particular, denoting the best constant by $C(\Omega)$ we get $C(\Omega^\sharp) \le C(\Omega) \le C(B_R)$, for $\Omega \subset B_R$, and this concludes the proof of the first part of Theorem 2.

We point out that by setting $\varepsilon = 0$, the proof we have just carried out yields an alternative proof of the original result of Alvino's theorem obtained by interpolation.

It remains to prove that for $\varepsilon > 0$ the constant $1/n\omega_n^{1/n}$ is not achieved by radial functions: this is a consequence of the absence of invariance under the action of the group (1.10) for the quasinorm $\|u\|_{Z_\varepsilon^{\frac{n-1}{n}}}$ as long as $\varepsilon > 0$. Indeed, let $y(|x|)$ be a radial extremal function for inequality (1.12); then, following line by line the proof of Theorem 1, one can verify that y satisfies the problem

$$\begin{cases} -\left(|y'|^{n-2}y'r^{n-1}\right)' = \omega_n^{\frac{1-n}{n}}(\varepsilon+n\mu)^{\frac{1-n}{n}}\delta_{r_\mu}, \\ \\ y'(0) = y(1) = 0, \end{cases} \tag{4.5}$$

so that y is a classical solution of

$$\begin{cases} -\left(|y'|^{n-2}y'r^{n-1}\right)' = 0 \qquad r\in(0,r_\mu)\cup(r_\mu,1) \\ y'(0) = y(1) = 0, \end{cases} \tag{4.6}$$

for some $\mu \in (0,+\infty)$ (with $r_\mu = e^{-\mu}$), with the normalization condition

$$y(r_\mu) = \frac{(\varepsilon+n\mu)^{\frac{n-1}{n}}}{n\omega_n^{1/n}} \tag{4.7}$$

Testing (4.5) with smooth, radial functions v compactly supported in the unit ball, one obtains

$$\int_0^1 |y'|^{n-2}y'v'r^{n-1}\,dr = \omega_n^{\frac{1-n}{n}}(\varepsilon+n\mu)^{\frac{1-n}{n}}v(r_\mu)$$

so that

$$
\begin{aligned}
&\omega_n^{\frac{1-n}{n}}(\varepsilon+n\mu)^{\frac{1-n}{n}}v(r_\mu)\\
&=\lim_{\eta\to 0^+}\left[\int_0^{r_\mu-\eta}|y'|^{n-2}y'v'r^{n-1}dr+\int_{r_\mu+\eta}^1|y'|^{n-2}y'v'r^{n-1}dr\right]\\
&=\lim_{\eta\to 0^+}\Bigg[|y'|^{n-2}y'(r_\mu-\eta)v(r_\mu-\eta)(r_\mu-\eta)^{n-1}-\int_0^{r_\mu-\eta}\frac{d}{dr}\left(|y'|^{n-2}y'r^{n-1}\right)v\ dr\\
&\qquad -|y'|^{n-2}y'(r_\mu+\eta)v(r_\mu+\eta)(r_\mu+\eta)^{n-1}-\int_{r_\mu+\eta}^1\frac{d}{dr}\left(|y'|^{n-2}y'r^{n-1}\right)v\ dr\Bigg]\\
&=\lim_{\eta\to 0^+}|y'|^{n-2}y'(r_\mu-\eta)v(r_\mu-\eta)(r_\mu-\eta)^{n-1}\\
&\qquad -|y'|^{n-2}y'(r_\mu+\eta)v(r_\mu+\eta)(r_\mu+\eta)^{n-1}
\end{aligned}
$$

that is,

$$
r_\mu^{n-1}\left[|y'_-(r_\mu)|^{n-2}y'_-(r_\mu)-|y'_+(r_\mu)|^{n-2}y'_+(r_\mu)\right]=\omega_n^{\frac{1-n}{n}}(\varepsilon+n\mu)^{\frac{1-n}{n}} \tag{4.8}
$$

On the other hand, applying Proposition 1,

$$
\left[|y'_+(r_\mu)|^n-|y'_-(r_\mu)|^n\right]r_0^n|\log r_\mu|=\int_0^1|y'|^nr^{n-1}\ dr=\frac{1}{n\omega_n}
$$

since y is a normalized extremal function. Now, since y is a classical solution of the problem (4.6), we easily obtain that $y(r)$ is constant in $(0,r_\mu)$, so that $y'_-(r_\mu)=0$. Then, the previous identity reduces to

$$
|y'_+(r_\mu)|^nr_\mu^n|\log r_\mu|=\frac{1}{n\omega_n} \tag{4.9}
$$

and, by (4.8),

$$
-r_\mu^{n-1}|y'_+(r_\mu)|^{n-2}y'_+(r_\mu)=\omega_n^{\frac{1-n}{n}}(\varepsilon+n\mu)^{\frac{1-n}{n}}
$$

which implies directly $y'_+(r_\mu)<0$ and the previous equation can be reformulated as follows

$$
r_\mu^{n-1}|y'_+(r_\mu)|^{n-1}=\omega_n^{\frac{1-n}{n}}(\varepsilon+n\mu)^{\frac{1-n}{n}} \tag{4.10}
$$

Combining (4.9) with (4.10) yields

$$
|\log r_\mu|=\mu+\frac{\varepsilon}{n}
$$

which contradicts the definition of $r_\mu=e^{-\mu}$.

Let us now conclude the section with the proof of Proposition 1

Proof. [Proof of Proposition 1] Let us first observe that if y is a weak solution of (1.13), then it satisfies pointwise the equation

$$
-(|y'|^{n-2}y'r^{n-1})'=0 \qquad\qquad \text{on } (0,r_0)\cup(r_0,1)
$$

Multiplying this equation by $r\log r\; y'$ and integrating on $(0,r_0-\eta)\cup(r_0+\eta,1)$ yields

$$
\begin{aligned}
0 &= \int_0^{r_0-\eta} -(|y'|^{n-2}y'r^{n-1})'r\log r y'\ dr + \int_{r_0+\eta}^{1} -(|y'|^{n-2}y'r^{n-1})'r\log r y'\ dr\\
&= -|y'(r_0-\eta)|^n(r_0-\eta)^n\log(r_0-\eta) + |y'(r_0+\eta)|^n(r_0+\eta)^n\log(r_0+\eta)\\
&\quad + \int_0^{r_0-\eta} |y'|^{n-2}y'r^{n-1}[\log r y' + y' + r\log r y'']\ dr\\
&\quad + \int_{r_0+\eta}^{1} |y'|^{n-2}y'r^{n-1}[\log r y' + y' + r\log r y'']\ dr\\
&= -|y'(r_0-\eta)|^n(r_0-\eta)^n\log(r_0-\eta) + |y'(r_0+\eta)|^n(r_0+\eta)^n\log(r_0+\eta)\\
&\quad + \int_0^{r_0-\eta} |y'|^{n}r^{n-1}[\log r + 1]\ dr + \int_{r_0+\eta}^{1} |y'|^{n}r^{n-1}[\log r + 1]\ dr\\
&\quad + \frac{1}{n}\int_0^{r_0-\eta} \left(|y'|^n\right)' r^n\log r\ dr + \frac{1}{n}\int_{r_0+\eta}^{1} \left(|y'|^n\right)' r^n\log r\ dr\\
&= -\frac{n-1}{n}\ |y'(r_0-\eta)|^n(r_0-\eta)^n\log(r_0-\eta)\\
&\quad + \frac{n-1}{n}\ |y'(r_0+\eta)|^n(r_0+\eta)^n\log(r_0+\eta)\\
&\quad + \frac{n-1}{n}\int_0^{r_0-\eta} |y'|^n r^{n-1}\ dr + \frac{n-1}{n}\int_{r_0+\eta}^{1} |y'|^n r^{n-1}\ dr
\end{aligned}
$$

Now, since $y(|x|)\in W^{1,n}_{0,rad}(\Omega)$,

$$
\int_0^1 |y'|^n r^{n-1}\ dr = \lim_{\eta\to 0}\int_0^{r_0-\eta} |y'|^n r^{n-1}\ dr + \int_{r_0+\eta}^{1} |y'|^n r^{n-1}\ dr
$$

so that, passing to the limit in the previous equation, we obtain

$$
\int_0^1 |y'|^n r^{n-1}\ dr = \left[|y'_-(r_0)|^n - |y'_+(r_0)|^n\right] r_0^n\log r_0
$$

that is our thesis (recalling that $0<r_0<1$ implies $|\log r_0| = -\log r_0$). $\square$

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