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A foundation of relative topos theory via fibrations and comorphisms of sites

Ph.D. thesis of

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Contents

Introduction

1 What is relative topos theory?

The present PhD thesis deals with the topic of relative topos theory over a base Grothendieck topos, and in particular with its treatment using the notions of comorphism of sites and of fibration: it is therefore important to understand the meaning of the word 'relative', and how relative topos theory has been studied in the past.

To the author's eye, the word 'relativity' can be interpreted in a logical and a categorical way. For the *logical interpretation*, consider the example of modules over a ring. The theory of modules over a ring is classically axiomatized with two sorts R (the ring) and M (the module), and function symbols and axioms so that we have a ring $(R, +, 0, \cdot, 1)$, an abelian group $(M, +, 0)$, and the scalar product operation $\mu : R \times M \to M$ making M into an R-module. We can, however, fix the base ring R and focus just on R-modules. In this case, we have a different axiomatization: we have one sort M plus function symbols making it into an abelian group $(M, +, 0)$, but the scalar product operation $\mu : R \times M \to M$ is substituted by a family of one-sort operations $\mu_x : M \to M$, one for each $x \in R$, representing the operation 'scalar product by x'. In this sense modules over a fixed ring are an example of a *relative mathematical theory*: there is a *base object*, which provides *parameters* for the language of our theory.

A *categorical interpretation* of relativity is what we may call *Yoneda's paradigm*, or *Grothendieck's relative point of view*: the study of an object X in a category $\mathcal{C}$ is essentially the study of all the possible points of view that the universe $\mathcal{C}$ grants us from/on the object X - more explicitly, of the representable functors $\mathcal{C}(X, -)$ and $\mathcal{C}(-, X)$, and of the coslice $X/\mathcal{C}$ and the slice $\mathcal{C}/X$. In this sense, relativity is the study of the interaction between the environment $\mathcal{C}$ and a fixed object X, with a particular emphasis on those properties that are stable when a change of base object is performed (for more on this we suggest Section 3.2.2 of [29]).

The two notions of relativity are not unrelated: for instance, an algebra A over a ring R can be thought of as a ring homomorphism $R \to A$, i.e. as a 'point of view' that R has on A. One very general way of understanding the relation between the two points of view on relativity is the duality between 'indexed objects' and 'fibred objects'. For instance, for every set X there is an equivalence

$$[X, \mathbf{Set}] \simeq \mathbf{Set}/X :$$

a covariant functor $P : X \to \mathbf{Set}$ (X being considered as a discrete category) is associated with the coproduct $\coprod_{x \in X} P(x)$ with the obvious arrow to X; on the other hand, a set $h : H \to X$ is associated with the functor $X \to \mathbf{Set}$, $x \mapsto h^{-1}(x)$ (see Chapter 2 for more details on this equivalence). The logical interpretation of relativity - the left-hand term - deals with structures *indexed by* the base structure; the categorical interpretation - the right-hand term - with structures *over* a base object.

Relative topos theory is quite simply the study of the 2-category $\mathbf{Topos}/\mathscr{E}$ of toposes over a fixed base topos $\mathscr{E}$, a point of view which is (at first) closer to the second meaning of relativity. For instance, Grothendieck toposes are the main examples of **Set**-based toposes, since every Grothendieck topos is endowed with its global sections functor to **Set**. Nonetheless, Grothendieck toposes distinguish themselves among other toposes for one fundamental property: they are equivalent to categories of sheaves over a site. The site allows for an interpretation of relative topos theory, at least over the base **Set**, closer to the first meaning of relativity, since it provides a *presentation in terms of a set of parameters* for the whole topos.

Relative topos theory is by no means a new subject. Relative techniques were developed already by the end of the fifties in Grothendieck's *entourage*, and they were in fact the foundation of their whole approach to algebraic geometry: indeed, every construction or property, be it in the realm of schemes or in that of Grothendieck toposes, was inevitably tested for base-change stability. One very good example of this is Hakim's work [15]. In the same environment, the duality between indexed structures and fibred structures was introduced and explored [14], along with the notion of stack. Grothendieck's student Jean Giraud devoted a great part of his work to the study of fibrations and stacks, in particular in relation with cohomology (see for instance [11]). We focus in particular on one of his papers, which has been a fundamental inspiration for this thesis: the article *Classifying topos*, published in 1972 [12]. In this work, Giraud shows that any (lex) stack over a base Grothendieck topos admits a canonical topology making it into a comorphism of sites, and that this tool allows for an explicit site-theoretic description of the pullback of Grothendieck toposes.

We could say that Giraud's article has it all: the relative topos theory, the change of base techniques, and the possibility of going from the categorical interpretation of relativity to its logical interpretation by using stacks as canonical presentations. Even so, his approach has been seemingly forgotten in topos theory literature. We propose a possible explanation of how this happened, by studying the different paths that stacks and topos theory have taken over the course of the years.

On the one side, the second half of the sixties and the beginning of the seventies pushed vigorously towards an elementary theory of toposes, first hinted by Lawvere's categorical axiomatization of the topos of sets [25]. The quest for a foundational approach to toposes implied that the notion of Grothendieck topos, which relies heavily on a universe of sets, was no longer adept: this led to the introduction of *elementary toposes* by Lawvere and Tierney at the end of the decade. Let us focus in particular on the article *Change of base for toposes with generators*, published by Radu Diaconescu in 1975 [9]. Though this article is much more known for the introduction of Diaconescu's theorem on the classification of geometric morphisms via flat functors, it is really a rewriting of Giraud's paper [12] in the elementary setting. Diaconescu himself states explicitly in the introduction that

> *Similar results for Grothendieck toposes were known to Giraud [12], and they served as a guide. However his proofs using descent techniques were not elementary.*

The two articles have a very similar structure and end by providing the explicit description of the pullback of toposes. In particular, Diaconescu introduces the notion (in nowaday's terminology) of *bound* for a geometric morphism $\mathscr{F} \to \mathscr{E}$, and shows that a geometric morphism $\mathscr{F} \to \mathscr{E}$ is bounded if and only if $\mathscr{F}$ is a subtopos of a topos of presheaves for an internal category of $\mathscr{E}$: in other words, $\mathscr{F}$ is a topos of *internal sheaves for an internal site of* $\mathscr{E}$. When restricting to bounded toposes, relative topos theory over an elementary topos $\mathscr{E}$ also encompasses both meanings of the words 'relativity', similarly to what happens with Grothendieck toposes[1]. The literature on the topic of internal sites is somehow sparse, Chapter C2 of [21] being to some extent the only consistent reference on the subject.

On the other hand, stacks became increasingly object of study of geometry. That is after all the reason why they were introduced in the first place (along with toposes): as powerful tools to formalize algebraic geometry. Thus we suppose that Giraud's approach was forgotten because stack theorists focused on geometry and lost interest in the development of pure topos theory, while category theorists preferred an elementary approach to relative topos theory, obtained mainly by exploiting internal categories. A thorough treatment of the problem of relative toposes in the formalism of internal categories, started by Diaconescu, can be found in Section 4.3 of [20], or in Section B3.2 of [21] (where some concepts, such as the concept of *torsor*, are introduced in an indexed formalism instead of an internal formalism).

In our perspective, the limit of the elementary approach, which was prevalent after Diaconescu's publication, is precisely the assumption that

[1] As a side note, every geometric morphism between Grothendieck toposes is bounded, and every topos bounded over a Grothendieck topos is itself a Grothendieck topos: therefore, the theory of $\mathbf{Sh}(\mathcal{C}, J)$-based Grothendieck toposes coincides precisely with the theory of toposes bounded over $\mathbf{Sh}(\mathcal{C}, J)$.

the base topos is *not* a Grothendieck topos: indexing over a base topos is *not* computationally manageable, because toposes are huge entities. Instead, working with Grothendieck toposes allows to present constructions and properties at the level of sites, making the results also useful in applications. Moreover, the passage to internal categories entails a 'rigidification' of the constructions - think about a pseudofunctor becoming a strict functor - which is cumbersome for many practical uses. It is our hope that providing relative topos theory with a more site-theoretic formalism will result in more applications to other areas of mathematics, especially those which benefit from relative techniques.

2 Structure of the thesis

This thesis is split into eight chapters (introduction excluded) and four appendices, plus a bibliography, an index of symbols and definitions and the acknowledgements (for which we felt Italian would be a more fitting linguistic choice). Each chapter and appendix begins with a short introduction describing its topics. We strove to be as consistent as possible in notation: in particular, the next section will establish some very basic notations to be used all throughout the work. The content of the thesis, aside from some results belonging to the literature on the subject, comes from the two articles [7] and [8], written by the author in collaboration with his PhD supervisor.

Chapter 1 begins by recalling some basic notions about toposes and sites: we do not dwell on details, and just provide a bird's-eye view on the subject. After that we recall the necessary notions of morphism and comorphism of sites, which provide site-theoretic presentations of geometric morphisms. In particular, Section 1.3 concludes the chapter by exploiting (co)morphisms of sites to present relative geometric morphisms over a base Grothendieck topos.

Chapter 2 starts by recalling the basic notions in the theory of indexed and fibred categories. A $\mathcal{C}$-indexed category (Definition 2.1.1) is simply a pseudofunctor

$$\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}.$$

It must be thought as a class of categories $\mathbb{D}(X)$, each indexed by an object X in $\mathcal{C}$, and of functors $\mathbb{D}(y) : \mathbb{D}(X) \to \mathbb{D}(Y)$ indexed by the arrows $y : Y \to X$ of $\mathcal{C}$. Strict functoriality of $\mathbb{D}$ is replaced by pseudofunctoriality, meaning that given two composable arrows $y : Y \to X$ and $z : Z \to Y$ in $\mathcal{C}$ the functor $\mathbb{D}(yz)$ is not strictly equal, but equivalent to $\mathbb{D}(z)\mathbb{D}(y)$. Applying the well-known Grothendieck construction to $\mathbb{D}$ yields a functor

$$p_{\mathbb{D}} : \mathcal{G}(\mathbb{D}) \to \mathcal{C},$$

which is a Grothendieck fibration (Definition 2.1.4). Finally, $\mathbb{D}$ is a stack (Section 2.3) with respect to a topology J over $\mathcal{C}$ if it satisfies a suitable generalization of the J-sheaf condition for a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$. In particular, we discuss the notion of a *Street fibration*, which is an equivalence-stable generalization of Grothendieck fibration. It is well known that many results holding for Grothendieck fibrations are also true for Street fibrations; however, it is almost impossible to find their proofs in the literature, which is why we have devoted some pages to sketching them. For instance, we prove explicitly the equivalence between the category of cloven Street fibrations over a category $\mathcal{C}$ and that of $\mathcal{C}$-indexed categories (Corollary 2.1.7), the fibred formulation of Yoneda's lemma (Proposition 2.1.10) and the characterization of (Street) prestacks in terms of Hom-functors (Proposition 2.3.7). Chapter 2 also deals with localizations of fibrations, which are instrumental in computing inverse images of fibrations along morphisms of sites, and with the truncation-inclusion adjunction

$$\mathbf{Sh}(\mathcal{C}, J) \underset{j_J}{\overset{t_J}{\underset{\hookrightarrow}{\overset{\leftarrow}{\perp}}}} \mathbf{St}^s(\mathcal{C}, J),$$

describing the connection between sheaves and (small) stacks over a site.

Chapter 3 is devoted to the various notions of colimit in a weak category. Given a diagram $D : \mathcal{C} \to \mathcal{K}$ into a 2-category, depending on the nature of the 2-cells in the cocones under R we can speak of *lax*, *oplax* or *pseudo-colimits*; moreover, these colimits can be conical or *weighted* by a functor $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$. The interest in colimits is motivated by the fact that we shall meet many colimits over the course of the tractation: for instance, we shall study a plethora of adjunctions (in particular, what we will call the *fundamental adjunction* of Chapter 6) that provide a contravariant hom-functor with a left adjoint, which thus acts as a colimit functor; moreover, the inverse image of fibrations can also be described using colimits. The first section of the chapter recalls the general notion of colimit in a 2-category, making the definition completely explicit in terms of colimit cocones, and shows that a result of commutativity of weight and diagram in a colimit holds (Proposition 3.1.2). Section 3.2 tackles instead the problem of describing weighted colimits as conical colimits, a process which we called 'conification': we show that it is possible to conify lax and oplax colimits and pseudocolimits with a discrete weight. Finally, the last section presents various possible ways of computing colimits in $\mathbf{Cat}$, based on conification, the commutativity of weights and diagrams and localizations of fibrations. Similar presentations for colimits indexed by bicategories can be found in [24].

Chapter 4 deals with the topic of base change, which we mentioned is one of the aspects that Grothendieck put at the centre of his work. After recalling what change of base means in the context of sheaves with respect to a topological space, we start by showing that a functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor

$$F_* : \mathbf{Fib}_{\mathcal{D}} \to \mathbf{Fib}_{\mathcal{C}}$$

between categories of fibrations, called the *direct image*: it is usually described as a strict pullback along F in the case of Grothendieck fibrations, and we show that for Street fibrations it is substituted by a pseudopullback; in the indexed formalism, it corresponds to mere precomposition with F (Proposition 4.1.1). The direct image admits two adjoints: we focus in particular on its left adjoint, called the *inverse image* of fibrations, and present it both from in the fibred and in the indexed formalism in Section 4.2. Section 4.3 considers instead the change of base for stacks: the two notions extend those at the level of fibrations, but in this case one must require that the functor F satisfy a topological property with respect to the base sites, namely the continuity of Definition 1.2.1. Finally, the last section revisits the base change functors for sheaves in light of the previous results, describing them from a fibrational standpoint.

Chapter 5 introduces the fundamental notion of *Giraud topos*. As we mentioned earlier, Jean Giraud showed in [12] that by endowing the domain of a stack $\mathcal{G}(\mathbb{D}) \to \mathcal{C}$ with the smallest topology $J_{\mathbb{D}}$ making it into a comorphism of sites, the resulting sheaf topos $\mathbf{Sh}(\mathcal{G}(\mathbb{D}), J_{\mathbb{D}})$ satisfies the universal property of *classifying topos of the stack*. The results reported in Chapter 5 show that the whole theory of fibrations fits naturally into the theory of *continuous* comorphisms of sites: for any fibration $p : \mathcal{D} \to \mathcal{C}$, we define its *Giraud topos* as the topos of sheaves $\mathbf{Sh}(\mathcal{D}, J_{\mathcal{D}})$, where $J_{\mathcal{D}}$ is the smallest topology on $\mathcal{D}$ making p into a comorphism of sites. In particular, in Section 5.2 we give a first example of a context in which relative topos theory benefits from exploiting fibrations and comorphisms of sites: namely, the site-theoretic description of dependent product functors.

Chapter 6 introduces the *fundamental adjunction* for a site $(\mathcal{C}, J)$, i.e. an adjunction between (a suitable 2-category of) fibrations over $\mathcal{C}$ and the 2-category of toposes over the base topos $\mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J)$. This adjunction comes as a generalization of the well-known presheaf-bundle adjunction for topological spaces, which is recalled in Section 6.1; it also generalizes the adjoints to the Grothendieck construction, which are presented in Section 6.2 and generalized to include the interaction between fibrations and comorphisms. After presenting the fundamental adjunction in Section 6.3, we show in the following section that it admits a dualizing object, the canonical stack of the base site. Finally, in Section 6.5, we hint at the new point of view on relative topos theory afforded by the fundamental adjunction and the hybrid comorphism-fibration formalism. This last section is rather short, for its content is to be expanded with future publications. Moreover, another on-going part of this project on relative topos theory is the introduction of suitable syntax and semantics for stacks, generalizing those of [37] and [35].

Having reached from the purely category-theoretic notion of relative topos a purely logical description of the same concept, we will thus have closed the gap between the two interpretations of relativity mentioned at the beginning of this introduction.

The last two chapters specialize the previous results to the discrete setting of sheaves on the base site. Chapter 7 specializes the fundamental adjunction to sheaves, obtaining a presheaf-bundle adjunction for sites that generalizes the topological one of Section 6.1; the same adjunction is specialized further to preorder categories in Section 7.2, recovering a point-free version of the topological presheaf-bundle adjunction (probably known to many authors, but never explicitly stated: see item (ii) of Remark 7.2.3). A feature of the fundamental adjunction for preorder sites is that it can be described directly at the level of sites without mentioning toposes, as explained in Section 7.3.

For a topological space X, the presheaf-bundle adjunction

$$\mathbf{Psh}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Top}/X \quad (\Lambda \dashv \Gamma)$$

allows in particular to describe the sheafification $\mathbf{Psh}(X) \to \mathbf{Sh}(X)$ as the composite $\Gamma \circ \Lambda$: the same happens for the discrete-bundle adjunction for sites and for preorder categories. Following these ideas, in Chapter 8 we apply our tools in the theory of morphisms and comorphisms of sites to provide site-theoretic description of the sheafification functor $\mathrm{a}_J : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$. The first section provides a description in terms of morphisms of sites, and the second in terms of equivalence classes of comorphisms of sites. The third section introduces a novel point of view on sheafification, showing that an element of the sheafification can be thought of as a 'locally matching family' of morphisms of fibrations: this is obtained considering our previous results on colimits of étale toposes. Finally, Section 8.4 collects all the presentations of the sheafification functor we acquired in the case of presheaves over a topological space, and shows their mutual connections.

The four appendices contain some peripheral results. Appendix A comes from [7] and is a *pendant* of Section 5.2. It is devoted to describing dependent products in elementary toposes in a fashion alternative to those found in the literature, obtained by reducing the use of higher-order constructions to the bare minimum. Appendix B is devoted to universes, and contains a couple of original results analyzing properties of geometric morphisms, described at the level of sites by morphisms or comorphisms, that are 'stable under change of universe'. Appendix C studies general conditions for a map $\pi : E \to X$ from a set to a topological space to be étale: it results that the topology to be put on E making π étale is determined by the family of open sections we choose for π. We also show that two well-known topologies for spectra of algebraic theories (the structural sheaf of a ring and of a MV-algebra) can be understood in this way. Finally, Appendix D collects some technical lemmas in category theory which we exploit throughout the text.

3 Notation

Most of the symbols will be defined along the way, and they can be retrieved using the index of symbols at the end of this work. We only set here some very basic notations.

Two adjoints forming a geometric morphism f will as always be denoted by $f^* \dashv f_*$; we will denote by $f_!$ a further left adjoint of f^*: when it exists f is said to be essential.

Natural transformations, and more in general 2-cells in 2-categories, will be denoted with a double arrows $\Rightarrow$. We set as convention that a natural transformation between geometric morphisms $\alpha : F \Rightarrow G$ is a natural transformation $\alpha : F^* \Rightarrow G^*$ between their inverse images.

In the context of a fixed model of set theory, we denote by **CAT** the 2-category of locally small categories and by **Cat** the 2-category of small categories.

Generic categories are denoted with the cursive font ($\mathcal{C}$, $\mathcal{D}$...); in particular we will use the calligraphic font $\mathscr{E}$, $\mathscr{F}$... for toposes. Pseudofunctors will usually be denoted by blackboard letters, such as $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{Cat}$.

We adopt the standard notations for opposite categories: the opposite of a 1-category $\mathcal{C}$ is denoted by $\mathcal{C}^{op}$, and the same notation will be used for the category obtained by reversing the 1-cells of a 2-category; on the other hand, if $\mathcal{C}$ is a 2-category, we shall denote by $\mathcal{C}^{co}$ the 2-category obtained by reversing its 2-cells.

Given a category $\mathcal{C}$, the hom-set of arrows $A \to B$ is denoted by $\mathcal{C}(A, B)$; similarly, we will denote by $\mathcal{K}(A, B)$ the hom-category for two objects in a 2-category $\mathcal{K}$. The only exception to this are categories of functors or of pseudofunctors, which we denote with square brackets: for instance $[\mathcal{A}, \mathcal{B}]$ is the category of functors $\mathcal{A} \to \mathcal{B}$.

When speaking about an arrow $y : Y \to X$ of $\mathcal{C}$ as an object of the slice category $\mathcal{C}/X$ (or of a comma category), we will refer to it as $[y]$ in square brackets; arrows of $\mathcal{C}/X$, on the other hand, will not have a specific notation: so for instance we write $z : [yz] \to [y]$.

Chapter 1

Toposes and comorphisms of sites

This introductory chapter is dedicated to recalling some basic facts about the theory of sites and toposes, with a special focus on comorphisms of sites. The main point of view to keep in mind is the following: sites must be thought as small, computationally manageable presentations of toposes; on the other hand, toposes must be thought as categorical completions of sites (and categories). A Grothendieck topos is almost always a huge abstract entity, but having a site of presentation allows us to reduce the weight of computations by reasoning in terms of generators: the connection is equivalent to that between a vector space and a basis for it. On the other hand, a basis of vectors is just a set, while a vector space is rich of all kinds of interesting properties: in the same way toposes enjoy many properties and features that other categories lack, therefore building a topos of sheaves from a category allows us more freedom of movement.

We will not recall any notions of basic category theory, which can be found in texts such as [27] and [34]; moreover, we will not provide any reference for results which belong to standard topos theory, since they can be found in almost every manual: our principal sources were [28] and [21]. The content of the latter sections, dedicated to comorphism of sites, comes mainly from [6] and [8]: for those results we have always provided the source.

1.1 Sites and Grothendieck toposes

Consider a category $\mathcal{C}$: a *presheaf over* $\mathcal{C}$ is any functor $P : \mathcal{C}^{op} \to \mathbf{Set}$. Presheaves over $\mathcal{C}$, along with their natural transformations, form the *category of presheaves*

$$[\mathcal{C}^{op}, \mathbf{Set}].$$

Historically, presheaves (and sheaves) over a category were born as generalizations of the notion of presheaf (and sheaf) over a topological space. Given a topological space X, a presheaf of sets P over X is given for any open subset $U \subseteq X$ by a set $P(U)$, and for every inclusion of open subsets $V \subseteq U$ by a map $\rho_{V \subseteq U} : P(U) \to P(V)$, with the extra requirements that the identities $\rho_{U=U} = \mathrm{id}_{P(U)}$ and $\rho_{W \subseteq V} \circ \rho_{V \subseteq U} = \rho_{W \subseteq U}$ always hold. One classical example of a presheaf over a topological space is that of continuous functions with values in $\mathbb{R}$, where $P(U) := \mathbf{Top}(U, \mathbb{R})$ and each map ρ acts by restriction of the domain. If we denote by $\mathcal{O}(X)$ the topology of X, seen as a posetal category, then presheaves over P are precisely contravariant functors $\mathcal{O}(X)^{op} \to \mathbf{Set}$, i.e. presheaves over the posetal category $\mathcal{O}(X)$.

The presheaf P is a sheaf if whenever we are given a collection of open subsets $U_i \subseteq X$ and data $x_i \in P(U_i)$ compatible when restricted to the intersections of the opens, the data themselves are restrictions of a unique datum $x \in P(\bigcup_i U_i)$. This is true for instance in the case of the presheaf of continuous functions: given a family of opens $U_i \subseteq X$, a family of functions $f_i : U_i \to \mathbb{R}$ which match on the intersections of the opens U_i glue to a unique function $\cup U_i \to \mathbb{R}$. The general notion of sheaf over a category $\mathcal{C}$ requires first to provide a concept of 'covering family' for objects of $\mathcal{C}$, generalizing that of open cover, and this yields the notion of Grothendieck topology; once that is done, a sheaf will be precisely a presheaf whose matching 'local' data, in the sense given by the Grothendieck topology, glue to a global datum.

Given a category $\mathcal{C}$ and an object X in $\mathcal{C}$, a *sieve* S over X is a collection of arrows with codomain X that is closed under precomposition: more explicitly, given an arrow y in S and another arrow z such that $\mathrm{cod}(z) = \mathrm{dom}(y)$, the composite $y \circ z$ also belongs to S. In particular, every object X admits a maximal sieve $\max_X$, which is the collection of all arrows with codomain X, and a minimal sieve which is the empty one. The collection of sieves over an object is closed under arbitrary unions and intersections. Finally, given an arrow $y : Y \to X$, we can define the pullback sieve of S along y,

$$y^*S := \{z : \mathrm{dom}(z) \to Y \mid y \circ z \in S\} :$$

A *Grothendieck topology* J on $\mathcal{C}$ is a collection of sieves $J(X)$ for every object $X \in \mathcal{C}$ subject to the following axioms:

(i) *maximality*: for every X in $\mathcal{C}$ the maximal sieve $\max_X$ belongs to $J(X)$;

(ii) *pullback-stability*: if $R \in J(X)$ and $y : Y \to X$ then $f^*R \in J(Y)$;

(iii) *transitivity*: if $S \in J(X)$ and R is a sieve on X such that for every $y \in S$ it holds that $y^*R \in J(\mathrm{dom} y)$, then $R \in J(X)$.

We will occasionally mention covering families: a *J-covering family* for an object X of $\mathcal{C}$ is a family of arrows $F = \{y_i : Y_i \to X \mid i \in I\}$ such that the smallest sieve containing it, usually called the *sieve generated by* F and denoted by $\langle F \rangle$, is J-covering. Grothendieck topologies admit an alternative formulation entirely in terms of covering families.

Sieves and covering families are to be thought as the generalization of the notion of open covering for an open subset of a topological space; the notion of sieve allows us to talk about sheaves as follows. Given a sieve S over an object X and a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$, a *matching family for P and S* is a collection $\underline{a} := \{a_y \mid y \in S\}$ indexed by the arrows of S, such that every a_y belongs to $P(\mathrm{dom} y)$ and such that the following matching condition holds: whenever an arrow z is precomposable to y, then $a_{yz} = P(z)(a_y)$. An *amalgamation* for $\underline{a}$ is an element $a \in P(X)$ such that for every y in S it holds that $a_y = P(y)(a)$.

The couple of a category $\mathcal{C}$ and a topology J over it is called a *site* $(\mathcal{C}, J)$. Given a site $(\mathcal{C}, J)$, a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ is said to be *J-separated* when, given $a, b \in P(X)$ and $S \in J(X)$, if for every $y \in S$ it holds that $P(y)(a) = P(y)(b)$ then $a = b$: this means that if two elements of P are locally equal, they must be equal. A presheaf is called a *J-sheaf* if every matching family for P and any J-sieve S admits a unique amalgamation: in particular, every J-sheaf is J-separate. The full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$ of J-sheaves is denoted by $\mathbf{Sh}(\mathcal{C}, J)$. Topologies can be ordered with respect to inclusion: then, whenever $J \subseteq K$, it holds that $\mathbf{Sh}(\mathcal{C}, K) \subseteq \mathbf{Sh}(\mathcal{C}, J)$. In some cases we will use the French school notations $\widehat{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ as a shorthand for respectively $[\mathcal{C}^{op}, \mathbf{Set}]$ and $\mathbf{Sh}(\mathcal{C}, J)$: this is handy whenever there is no risk of confusion on the topology considered.

A category is a *Grothendieck topos* if it is equivalent to $\mathbf{Sh}(\mathcal{C}, J)$ for a small site $(\mathcal{C}, J)$. Every Grothendieck topos

(i) is complete and cocomplete,

(ii) is a geometric category (implying in particular that it is a regular and coherent category),

(iii) is extensive,

(iv) admits a generating set of objects,

(v) admits a subobject classifier, power objects and exponentials,

(vi) is a model for constructive set theory.

Moreover, Grothendieck toposes appear in many flavours:

1. **Set**, the category of sets, is the archetypal Grothendieck topos: it can be seen for instance as the topos of presheaves over the one-object category $\mathbb{1} = \{*\}$;

2. the category with exactly one object is also a topos, the topos of sheaves over any category with respect to its maximal topology (the one containing all sieves): when shall call it the *trivial topos* and denote it by $\{0 = 1\}$;

3. every localization of a Grothendieck topos is a Grothendieck topos;
4. every slice of a Grothendieck topos is a topos;
5. the category of coalgebras for a lex accessible comonad acting over a Grothendieck topos is a Grothendieck topos;
6. the category of discrete actions of a topological group is a Grothendieck topos.

In the definition of Grothendieck topos, it is fundamental to require the smallness of the presentation site $(\mathcal{C}, J)$: indeed, if the underlying category $\mathcal{C}$ is not small, the category of sheaves $\mathbf{Sh}(\mathcal{C}, J)$ loses most of the properties listed above. There is however a way to relax this request: a locally small site $(\mathcal{C}, J)$ is said to be *small-generated* if $\mathcal{C}$ admits a small subcategory $\mathcal{A}$ which is J-dense [36], a condition which implies that there is an equivalence $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{A}, J_{|\mathcal{A}})$. Therefore, in the following we will say 'site' to mean 'small-generated site'. In particular, with this point of view every Grothendieck topos $\mathscr{E}$ becomes a site when endowed with its canonical topology $J^{can}_{\mathscr{E}}$, and moreover it is equivalent to its topos of $J^{can}_{\mathscr{E}}$-sheaves:

$$\mathscr{E} \simeq \mathbf{Sh}(\mathscr{E}, J^{can}_{\mathscr{E}}).$$

We conclude by recalling that sieves in $\mathcal{C}$ can be thought as monomorphisms in $[\mathcal{C}^{op}, \mathbf{Set}]$, a point of view which we shall exploit from time to time later on. If we denote by

$$よ_{\mathcal{C}} : \mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \mathbf{Set}]$$

the Yoneda embedding, then a sieve S over X corresponds to a subobject $m_S : S \rightarrowtail よ(X)$: in particular, the maximal sieve over X is $よ(X)$ itself, while the minimal sieve corresponds to the monic arrow from the initial object of the topos $0_{[\mathcal{C}^{op}, \mathbf{Set}]} \rightarrowtail よ(X)$, and the operations of intersection and union of sieves are simply the operations in the lattice of subobjects of $よ(X)$. In particular, given an arrow $y : Y \to X$ in $\mathcal{C}$ the pullback sieve y^*S over Y is the actual pullback of m_S along $よ(y)$:

$$\begin{array}{ccc} y^*S & \longrightarrow & S \\ \downarrow & \lrcorner & \downarrow{\scriptstyle m_S} \\ よ(Y) & \xrightarrow[よ(y)]{} & よ(X) \end{array} .$$

A matching family $\{a_y \mid y \in S\}$ for P and S is simply an arrow $\underline{a} : S \to P$ in the topos, while an amalgamation for $\underline{a}$ is an arrow $a : よ(X) \to P$ satisfying the identity $a \circ m_S = \underline{a}$. The presheaf P is J-separated if for every X in $\mathcal{C}$ and every $S \in J(X)$ the map

$$[\mathcal{C}^{op}, \mathbf{Set}](よ(X), P) \xrightarrow{-\circ m_S} [\mathcal{C}^{op}, \mathbf{Set}](S, P)$$

is injective, while P is a J-sheaf if it is an isomorphism.

As it is well known, Grothendieck toposes belong to a wider class of categories, called *elementary toposes*: an elementary topos is a first-order description of the main features of Grothendieck toposes that can be formulated without any mention of set theory. However, since the main objective of this work is to interpret relative topos theory from a site-theoretic point of view, there will be no need for elementary toposes in the following: thus we establish the convention that whenever we say 'topos' from now on we will invariably mean 'Grothendieck topos' (with the only exception of the content of Appendix A).

1.2 Arrows of toposes, arrows of sites

It is well known that a continuous map of topological spaces $f : Y \to X$ induces a pair of adjoint functors

$$\mathbf{Sh}(X) \underset{\mathbf{Sh}(f)^*}{\overset{\mathbf{Sh}(f)_*}{\rightleftarrows}} \mathbf{Sh}(Y),$$

where moreover the functor $\mathbf{Sh}(f)^*$ preserves finite limits; for more details on this, see the introduction to Chapter 4. Generalizing to arbitrary toposes we obtain the notion of geometric morphism. Given two toposes $\mathscr{E}$ and $\mathscr{F}$, a *geometric morphism* $f : \mathscr{E} \to \mathscr{F}$ is given by two adjoints

$$\mathscr{E} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathscr{F},$$

where moreover the left adjoint f^* preserves finite limits. The functor f_* is called *direct image* of the geometric morphism, while f^* is called *inverse image*. A *transformation of geometric morphisms* $\alpha : f \Rightarrow g$ is given by a natural transformation $\alpha : f^* \Rightarrow g^*$ (by adjunction, this corresponds to a natural transformation $\bar{\alpha} : g_* \Rightarrow f_*$). Toposes, geometric morphisms and transformations form a 2-category, which we shall denote by **Topos**. In the following we will also often use the notion of *essential geometric morphism*, which is a geometric morphism f such that the inverse image f^* has a further left adjoint, denoted by $f_!$ and called the *essential image* of f. Toposes, essential geometric morphisms and their transformations form a sub-2-category of **Topos**, which we will denote by **EssTopos**.

The notion of geometric morphism between toposes proves to be the right notion of arrow between toposes, because it encompasses most of the canonical functors arising between toposes. Here are some examples:

1. every inclusion
$$\iota_J : \mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{op}, \mathbf{Set}]$$
is the direct image of a geometric morphism, called an *embedding* or *inclusion* of toposes: the left adjoint is called the *sheafification functor*
$$\mathrm{a}_J : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J).$$
Composing the Yoneda embedding $よ : \mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$ with the sheafification functor a_J we obtain in particular the canonical functor
$$\ell_J : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J),$$
mapping each object of X to its corresponding representable sheaf $\ell_J(X)$.

2. every topos $\mathscr{E}$ admits exactly one (up to equivalence) geometric morphism to **Set**: the right adjoint
$$\Gamma_{\mathscr{E}} : \mathscr{E} \to \mathbf{Set}$$
maps any E in $\mathscr{E}$ to the set $\mathscr{E}(1, E)$, and is called the *global sections functor*, while the left adjoint
$$\Delta_{\mathscr{E}} : \mathbf{Set} \to \mathscr{E}$$
maps a set S to the coproduct $\coprod_{s \in S} 1_{\mathscr{E}}$. This makes **Set** into the terminal object in the 2-category of toposes.

3. for every topos $\mathscr{E}$ there exists exactly one geometric morphism (up to equivalence) from $\{0 = 1\}$ to it, making the trivial topos the initial object in the 2-category of toposes: the left adjoint $\mathscr{E} \to \{0 = 1\}$ maps everything to the unique object of $\{0 = 1\}$, while its right adjoint maps the unique object to a terminal of $\mathscr{E}$.

4. Given an accessible lex comonad T over a topos $\mathscr{E}$, the functor to its category of coalgebras
$$\mathscr{E} \to \mathscr{E}_T$$
is the direct image of a geometric morphism. Such geometric morphisms are called *surjections*. In particular, surjections and inclusions form an orthogonal factorization system (one of the many) for geometric morphisms.

5. For every arrow $e : E \to E'$ in a topos $\mathscr{E}$, the pullback functor
$$e^* : \mathscr{E}/E' \to \mathscr{E}/E$$

is part of an adjoint triple

$$\mathscr{E}/E \xleftarrow{e^*} \mathscr{E}/E', \quad \Sigma_e \dashv e^* \dashv \Pi_e \quad (\Sigma_e, \Pi_e : \mathscr{E}/E \to \mathscr{E}/E')$$

i.e. an essential geometric morphism $\mathscr{E}/E \to \mathscr{E}/E'$. We will come back to these geometric morphisms in Section 5.2 and in Appendix A.

Many geometric morphisms can be presented using the sites of definition for the toposes involved, and this can be done essentially in two ways: using morphisms of sites or using comorphisms of sites. We begin by recalling that every functor induces an essential geometric morphism at the level of presheaf toposes:

Proposition 1.2.1. *A functor $p : \mathcal{D} \to \mathcal{C}$ between small categories induces an adjoint triple*

$$[\mathcal{D}^{op}, \mathbf{Set}] \xleftarrow{p^*} [\mathcal{C}^{op}, \mathbf{Set}], \quad \mathrm{lan}_{p^{op}} \dashv p^* \dashv \mathrm{ran}_{p^{op}} \quad (\mathrm{lan}_{p^{op}}, \mathrm{ran}_{p^{op}} : [\mathcal{D}^{op}, \mathbf{Set}] \to [\mathcal{C}^{op}, \mathbf{Set}])$$

where $p^* := (- \circ p^{op})$ and its left and right adjoints are the left and right Kan extension functors along p^{op}.

Now, depending on which of the two functors $\mathrm{ran}_{p^{op}}$ or p^* we wish to restrict to sheaves, we obtain the two notions of morphism and of comorphism of sites:

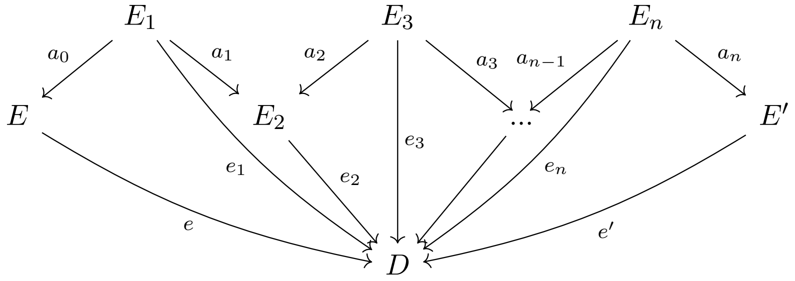
Definition 1.2.1 [6, Definition 4.7]. Consider two small sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$: a functor $p : \mathcal{D} \to \mathcal{C}$ is (K, J)-*continuous* if any of the following two equivalent conditions holds:

(i) the functor $p^* = (- \circ p^{op}) : [\mathcal{C}^{op}, \mathbf{Set}] \to [\mathcal{D}^{op}, \mathbf{Set}]$ maps J-sheaves to K-sheaves;

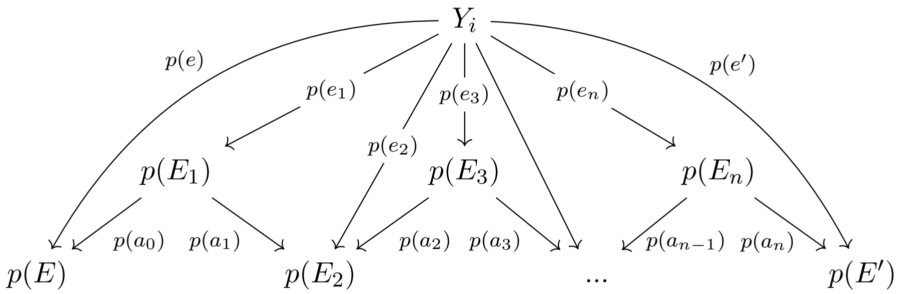
(ii) p is *cover-preserving*, i.e. if S is a K-covering sieve then $p(S)$ is a J-covering family, and it satisfies the following *cofinality condition*: for any K-covering sieve S on an object D and any commutative square

$$\begin{array}{ccc} X & \xrightarrow{g} & p(E') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p(e')} \\ p(E) & \xrightarrow[p(e)]{} & p(D) \end{array}$$

with e and e' belonging to S, there exists a J-covering family $\{y_i : Y_i \to X \mid i \in I\}$ such that for each i there exists a finite zigzag in S



and there exist arrows $\alpha_j : Y_i \to p(E_j)$ such that the diagram below is commutative:



When p is continuous, it induces an adjunction

$$\mathbf{Sh}(\mathcal{C}, J) \underset{\mathbf{Sh}(p)^*}{\overset{\mathbf{Sh}(p)_*}{\rightleftarrows}} \mathbf{Sh}(\mathcal{D}, K), \quad \top$$

where $\mathbf{Sh}(p)_*$ is the restriction of p^* and $\mathbf{Sh}(p)^*$ is defined as the composite

$$\mathbf{Sh}(\mathcal{D}, K) \xhookrightarrow{\iota_K} [\mathcal{D}^{op}, \mathbf{Set}] \xrightarrow{\mathrm{lan}_{p^{op}}} [\mathcal{C}^{op}, \mathbf{Set}] \xrightarrow{\mathrm{a}_J} \mathbf{Sh}(\mathcal{C}, J).$$

The functor p is a *morphism of sites* if moreover $\mathbf{Sh}(p)^*$ preserves finite limits, i.e. if the adjunction $\mathbf{Sh}(p)^* \dashv \mathbf{Sh}(p)_*$ provides a geometric morphism

$$\mathbf{Sh}(p) : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K).$$

We denote the 2-categories of small-generated sites, morphisms of sites and their natural transformations by **Site**.

Remarks 1.2.1. 1. Morphisms of sites can be defined purely in terms of objects and arrows of the base sites: see Definition 3.2 of [6].

2. Given a topos $\mathscr{E}$ and a small category $\mathcal{C}$, every functor $A : \mathcal{C} \to \mathscr{E}$ induces a pair of adjoints

$$\mathscr{E} \underset{L_A}{\overset{R_A}{\rightleftarrows}} [\mathcal{C}^{op}, \mathbf{Set}],\quad L_A \dashv R_A$$

where R_A maps an object E of $\mathscr{E}$ to the presehaf $\mathscr{E}(A(-), E)$, while L_A is defined on representables as $よ(X) \mapsto A(X)$. The functor A is *flat* if the functor L_A preserves finite limits, i.e. if the adjunction $L_A \dashv R_A$ is a geometric morphism. Given moreover a topology J on $\mathcal{C}$, the functor A is *J-continuous* if R_A takes image in $\mathbf{Sh}(\mathcal{C}, J)$ (implying that the adjunction $L_A \dashv R_A$ induces a geometric morphism $\mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$ by Lemma D.5). We will from time to time refer to flat J-continuous functors: however, this is a particular case of the notion of morphism of sites, since functor $A : \mathcal{C} \to \mathscr{E}$ is flat and J-continuous if and only if it is a morphism of sites $A : (\mathcal{C}, J) \to (\mathscr{E}, J^{can}_{\mathscr{E}})$.

We also state an alternative characterization of continuous functors which we will exploit later:

Lemma 1.2.2. *Consider two small-generated sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$: a functor $p : \mathcal{D} \to \mathcal{C}$ is (K, J)-continuous if and only if for every K-covering sieve $m : R \rightarrowtail よ(D)$ the arrow $\mathrm{a}_J(\mathrm{lan}_{p^{op}}(m))$ of $\mathbf{Sh}(\mathcal{C}, J)$ is an isomorphism.*

Proof. This can be proven considering, for every K-covering sieve $m : R \rightarrowtail よ(D)$ and every J-sheaf H, the commutative square of Hom-sets

$$\begin{array}{ccc} \mathbf{Sh}(\mathcal{C}, J)(\mathrm{a}_J\mathrm{lan}_{p^{op}}(よ(D)), H) & \xrightarrow{\sim} & [\mathcal{D}^{op}, \mathbf{Set}](よ(D), H \circ p^{op}) \\ {\scriptstyle -\circ \mathrm{a}_J\mathrm{lan}_{p^{op}}(m)}\downarrow & & \downarrow{\scriptstyle -\circ m} \\ \mathbf{Sh}(\mathcal{C}, J)(\mathrm{a}_J\mathrm{lan}_{p^{op}}(R), H) & \xrightarrow{\sim} & [\mathcal{D}^{op}, \mathbf{Set}](R, H \circ p^{op}) \end{array} .$$

The horizontal isomorphisms are given by the adjunction $\mathrm{a}_J \circ \mathrm{lan}_{p^{op}} \dashv p^* \circ \iota_J$. The left-hand vertical map is an isomorphism if and only if $\mathrm{a}_J\mathrm{lan}_{p^{op}}(m)$ is an isomorphism, while the right-hand one is if and only if $H \circ p^{op}$ is a K-sheaf, i.e. if and only if p is (K, J)-continuous. □

Remark 1.2.2. This result can also be obtained from Exposé II, Proposition 5.3 and Exposé III, Proposition 1.2 of [1].

We now introduce the dual notion of comorphism of sites:

Definition 1.2.2 [21, Proposition C2.3.18]. Consider two small sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$: a functor $p : \mathcal{D} \to \mathcal{C}$ is a *comorphism of sites* $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$ if any of the following equivalent conditions is satisfied:

(i) The functor $\mathrm{ran}_{p^{op}} : [\mathcal{D}^{op}, \mathbf{Set}] \to [\mathcal{C}^{op}, \mathbf{Set}]$ sends K-sheaves to J-sheaves, i.e. there exists a geometric morphism $C_p : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ satisfying $\iota_J C_{p_*} \simeq \mathrm{ran}_{p^{op}} \iota_K$; its inverse image C_p^* is defined as the composite

$$\mathbf{Sh}(\mathcal{C}, J) \xhookrightarrow{\iota_J} [\mathcal{C}^{op}, \mathbf{Set}] \xrightarrow{p^*} [\mathcal{D}^{op}, \mathbf{Set}] \xrightarrow{\mathrm{a}_K} \mathbf{Sh}(\mathcal{D}, K).$$

(ii) $p^* : [\mathcal{C}^{op}, \mathbf{Set}] \to [\mathcal{D}^{op}, \mathbf{Set}]$ maps J-dense monomorphisms to K-dense monomorphisms (we recall that a monomorphism m is J-dense if $\mathrm{a}_J(m)$ is an isomorphism).

(iii) The functor p satisfies the *covering-lifting property*, i.e. for every D in $\mathcal{D}$ and every J-covering sieve S over $p(D)$ there is a K-covering sieve R over D such that $p(R) \subseteq S$.

We denote the 2-category of small-generated sites, their comorphisms and natural transformations by **Com**.

In the following, we will not be particularly interested in morphisms of sites, but we will use continuous functors extensively, and especially continuous comorphisms of sites.

To conclude this section, we recall that every functor whose codomain is a site can be made into a comorphism of sites in a minimal way:

Proposition 1.2.3 [21, Proposition C2.3.19(i)]. *Consider a site $(\mathcal{C}, J)$ and a functor $p : \mathcal{D} \to \mathcal{C}$: there exists a smallest topology M_J^p on $\mathcal{D}$ such that $p : (\mathcal{D}, M_J^p) \to (\mathcal{C}, J)$ is a comorphism of sites. In particular, it is the smallest topology containing the the family of sieves of the form $S_D := \{f : \mathrm{dom}(f) \to D \mid p(f) \in S\}$ for any D in $\mathcal{D}$ and $S \in J(p(D))$.*

Example 1.2.1. One particular instance of this minimal topology is when $p : \mathcal{C}/X \to \mathcal{C}$ is the canonical projection functor: we will denote it simply by J_X, and one immediately sees that a family of arrows $\mathcal{F} = \{z_i : [yz_i] \to [y] \mid i \in I\}$ over $[y]$ in $\mathcal{C}/X$ is J_X-covering if and only if $\{z_i : \mathrm{dom}(z_i) \to Y \mid i \in I\}$ is J-covering in $\mathcal{C}$.

The topologies of the form M_J^p satisfy another fundamental property:

Lemma 1.2.4 [6, Corollary 3.6]. *Take a comorphism $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$ and two functors $A : \mathcal{D} \to \mathcal{E}$, $q : \mathcal{E} \to \mathcal{C}$: if $qA \cong p$ then A is a comorphism of sites $(\mathcal{D}, K) \to (\mathcal{E}, M_J^q)$.*

This allows in particular to interpret the association $p \mapsto M_J^p$ as part of an adjunction. Let us denote by $\mathbf{Com}^s$ the 2-category of small sites and their comorphisms, and by $\mathbf{Com}^s/(\mathcal{C}, J)$ its 2-slice over $(\mathcal{C}, J)$ in the sense of Definition 1.4.1. We introduce a 2-functor

$$\mathfrak{G} : \mathbf{Cat}/\mathcal{C} \to \mathbf{Com}^s/(\mathcal{C}, J)$$

by mapping a 0-cell $[p : \mathcal{D} \to \mathcal{C}]$ to $p : (\mathcal{D}, M_J^p) \to (\mathcal{C}, J)$: Lemma 1.2.4 implies that every 1-cell in $\mathbf{Cat}/\mathcal{C}$ is sent to a comorphism of sites. In fact, for *any* comorphism $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$, a 1-cell $(F, \varphi) : [p : \mathcal{D} \to \mathcal{C}] \to [q : \mathcal{E} \to \mathcal{C}]$ in $\mathbf{Cat}/\mathcal{C}$ corresponds to a 1-cell $(F, \varphi) : [p : (\mathcal{D}, K) \to (\mathcal{C}, J)] \to [q : (\mathcal{E}, M_J^q) \to (\mathcal{C}, J)]$ of $\mathbf{Com}^s/(\mathcal{C}, J)$, and so we can conclude the following:

Corollary 1.2.5 [8, Corollary 2.10.4]. *Consider a small site $(\mathcal{C}, J)$: there is a 2-adjunction (see Definition 3.1.2)*

$$\mathbf{Com}^s/(\mathcal{C}, J) \underset{\mathfrak{G}}{\overset{\mathrm{For}}{\rightleftarrows}} \mathbf{Cat}/\mathcal{C} \quad (\perp)$$

where For *is the usual forgetful functor and $\mathfrak{G}$ is defined on 0-cells by mapping $[p : \mathcal{D} \to \mathcal{C}]$ to $[p : (\mathcal{D}, M_J^p) \to (\mathcal{C}, J)]$. The 2-functor $\mathfrak{G}$ is also locally fully faithful, i.e. every functor $\mathbf{Cat}/\mathcal{C}([p], [q]) \to \mathbf{Com}^s/(\mathcal{C}, J)(\mathfrak{G}([p]), \mathfrak{G}([q]))$ is full and faithful.*

1.3 Presentation of essential geometric morphisms via continuous comorphisms of sites

We can functorialize the connection between sites and toposes, in order to describe an adjunction between the 2-category of sites and continuous comorphisms and the 2-category of toposes and essential geometric morphisms. First of all, the passage $p \mapsto \mathrm{ran}_{p^{op}}$ of Proposition 1.2.1 provides a functor from $\mathbf{Cat}$ to $\mathbf{EssTopos}$, by mapping a small category $\mathcal{C}$ to its topos of presheaves $[\mathcal{C}^{op}, \mathbf{Set}]$, and a functor $p : \mathcal{D} \to \mathcal{C}$ to the essential geometric morphism $\mathrm{ran}_{p^{op}}$. If we wish to extend this to natural transformations we have to take into account that the involution $(-)^{op}$ reverses their direction, i.e. if $\alpha : p \Rightarrow q : \mathcal{D} \to \mathcal{C}$ then $\alpha^{op} : q^{op} \Rightarrow p^{op}$. This means that there is an induced natural transformation $\alpha^* : q^* \Rightarrow p^* : [\mathcal{C}^{op}, \mathbf{Set}] \to [\mathcal{D}^{op}, \mathbf{Set}]$ acting as precomposition with α^{op}, and thus the functor $\mathbf{Cat} \to \mathbf{EssTopos}$ extends to a 2-functor

$$\mathbf{Cat} \to \mathbf{EssTopos}^{co}.$$

Similarly, if we call $\mathbf{Com}$ the category of small-generated sites and their comorphisms, the association $(\mathcal{C}, J) \mapsto \mathbf{Sh}(\mathcal{C}, J)$ and $p \mapsto C_p$ describes a functor $\mathbf{Com} \to \mathbf{Topos}$; this functor also extends to natural transformations reversing their direction, and thus we have in fact a 2-functor

$$C_{(-)} : \mathbf{Com} \to \mathbf{Topos}^{co}.$$

Finally, notice that if p is a (K, J)-continuous comorphism the inverse image $C_p^* = \mathbf{Sh}(p)_*$ admits a further left adjoint $(C_p)_! := \mathbf{Sh}(p)^*$, i.e. the induced geometric morphism C_p is *essential*. If we denote by $\mathbf{Com}_{cont}$ the 2-category of sites and their continuous comorphisms, the 2-functor $C_{(-)}$ restricts to a 2-functor

$$C_{(-)} : \mathbf{Com}_{cont} \to \mathbf{EssTopos}^{co}.$$

Proposition 1.3.1 [6, Theorem 4.20]. *Consider a small-generated site $(\mathcal{C}, J)$ and a Grothendieck topos $\mathscr{E}$: there is an equivalence of categories*

$$\mathbf{EssTopos}^{co}(\mathbf{Sh}(\mathcal{C}, J), \mathscr{E}) \simeq \mathbf{Com}_{cont}((\mathcal{C}, J), (\mathscr{E}, J_{\mathscr{E}}^{can}))$$

acting as follows:

- *An essential geometric morphism $F : \mathbf{Sh}(\mathcal{C}, J) \to \mathscr{E}$ is sent to the functor $F_! \ell_J : \mathcal{C} \to \mathscr{E}$, and a natural transformation $\Omega : F \Rightarrow G : \mathbf{Sh}(\mathcal{C}, J) \to \mathscr{E}$ to $\Omega_! \circ \ell_J$, where $\Omega_! : G_! \Rightarrow F_!$ is the 2-cell induced by $\Omega : F^* \Rightarrow G^*$ on the essential images;*
- *a $(J, J_{\mathscr{E}}^{can})$-continuous comorphism of sites $A : \mathcal{C} \to \mathscr{E}$ is sent to $C_A : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathscr{E}, J_{\mathscr{E}}^{can})$ and then composed with the canonical equivalence $\mathbf{Sh}(\mathscr{E}, J_{\mathscr{E}}^{can}) \simeq \mathscr{E}$; the same holds for a natural transformation $\omega : A \Rightarrow B : \mathcal{C} \to \mathscr{E}$.*

The previous result is evidently describing a 2-adjunction: the left adjoint is the functor $C_{(-)}$ that we already know, while its right adjoint, which we will denote by $(-)_!$, maps a topos $\mathscr{E}$ to the site $(\mathscr{E}, J_{\mathscr{E}}^{can})$, an essential geometric morphism $H : \mathscr{E} \to \mathscr{F}$ to the functors $H_!$ and a 2-cell $\omega : H \Rightarrow K$ of geometric morphisms to the 2-cell $\omega_! : K_! \Rightarrow H_!$ induced on the essential images. The only thing we need to prove is that $H_!$ is in fact a continuous $(J_{\mathscr{E}}^{can}, J_{\mathscr{F}}^{can})$-comorphism of sites, so that $(-)_!$ is well-defined; in fact, we will show that at the level of sheaf toposes it induces precisely the geometric morphism H, i.e. that there is a commutative diagram

$$\begin{array}{ccc} [\mathscr{E}^{op}, \mathbf{Set}] & \xrightarrow{\mathrm{ran}_{H_!^{op}}} & [\mathscr{F}^{op}, \mathbf{Set}] \\ \big\uparrow & & \big\uparrow \\ \mathscr{E} & \xrightarrow{\quad H \quad} & \mathscr{F} \end{array} .$$

First of all, the adjunction $H_! \dashv H^* \dashv H_*$ implies the two equivalences

$$(- \circ (H^*)^{op}) \simeq \mathrm{lan}_{H_*^{op}}, \ (- \circ H_!^{op}) \simeq \mathrm{lan}_{(H^*)^{op}},$$

by Lemma D.1. It is well known that for any functor $A : \mathcal{A} \to \mathcal{B}$ the identity $\mathrm{lan}_{A^{op}} \text{よ}_{\mathcal{A}} \simeq \text{よ}_{\mathcal{B}} A$ holds, and so in particular we have that, for any X in $\mathscr{E}$ and Y in $\mathscr{F}$, there are natural isomorphisms

$$\text{よ}(Y) \circ H_!^{op} \simeq \text{よ}(H^*(Y)), \ \mathrm{ran}_{H_!^{op}}(\text{よ}(X)) \simeq \text{よ}(H_*(X))$$

which express precisely the fact that the adjunction $\mathrm{lan}_{H_!^{op}} \dashv (- \circ H_!^{op}) \dashv \mathrm{ran}_{H_!^{op}}$ restricts to the essential geometric morphism $H : \mathscr{E} \to \mathscr{F}$, i.e. that $H_!$ is a $(J_{\mathscr{E}}^{can}, J_{\mathscr{F}}^{can})$-continuous comorphism of sites. One can then verify that the correspondence is pseudonatural in $(\mathcal{D}, K)$ and $\mathscr{E}$, and hence we obtain the following:

Corollary 1.3.2 [8, Corollary 4.1.1]. *There is a 2-adjunction*

$$\mathbf{Com}_{cont} \underset{(-)_!}{\overset{C_{(-)}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{EssTopos}^{co}$$

acting as follows:

- $C_{(-)}$ *maps a site* $(\mathcal{C}, J)$ *to the sheaf topos* $\mathbf{Sh}(\mathcal{C}, J)$*; a continuous comorphism of sites* $A : (\mathcal{C}, J) \to (\mathcal{D}, K)$ *is sent to the essential geometric morphism* $C_A : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$*, and a natural transformation* $\alpha : A \Rightarrow B$ *of comorphisms of sites to* $C_\alpha : C_B \Rightarrow C_A$*.*

- $(-)_!$ *maps a Grothendieck topos* $\mathscr{E}$ *to the site* $(\mathscr{E}, J^{can}_{\mathscr{E}})$*, an essential geometric morphism* $H : \mathscr{E} \to \mathscr{F}$ *to the continuous comorphism of sites* $H_! : (\mathscr{E}, J^{can}_{\mathscr{E}}) \to (\mathscr{F}, J^{can}_{\mathscr{F}})$ *and a natural transformation* $\omega : H \Rightarrow K$ *to the natural transformation* $\omega_! : K_! \Rightarrow H_!$*.*

Moreover, the functor $(-)_!$ *is 2-full and faithful.*

In Section 4.4 of [6] a result similar to Proposition 1.3.1 is sketched, where essential geometric morphisms $F : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ are presented using liftings of topologies to presheaf toposes. More precisely, it is shown that given a site $(\mathcal{C}, J)$ there is a topology $\widehat{J}$ on $[\mathcal{C}^{op}, \mathbf{Set}]$, called the *presheaf lifting of* J, which is the topology coinduced by J along $よ_\mathcal{C} : \mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \mathbf{Set}]$ (in the sense of [6, Proposition 6.11]): the topology $\widehat{J}$ makes $よ_\mathcal{C}$ into a comorphism of sites such that

$$C_{よ_\mathcal{C}} : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}([\mathcal{C}^{op}, \mathbf{Set}], \widehat{J})$$

is an equivalence of toposes. One can also check that the inclusion

$$\mathbf{Sh}([\mathcal{C}^{op}, \mathbf{Set}], \widehat{J}) \hookrightarrow [[\mathcal{C}^{op}, \mathbf{Set}]^{op}, \mathbf{Set}]$$

can be seen as the inclusion

$$\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{op}, \mathbf{Set}] \xhookrightarrow{よ_{[\mathcal{C}^{op}, \mathbf{Set}]}} [[\mathcal{C}^{op}, \mathbf{Set}]^{op}, \mathbf{Set}].$$

Using presheaf liftings of topologies one can prove that essential geometric morphisms $\mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ correspond to J-equivalence classes of continuous comorphisms of sites $(\mathcal{D}, K) \to ([\mathcal{C}^{op}, \mathbf{Set}], \widehat{J})$, where two comorphisms of sites to $[\mathcal{C}^{op}, \mathbf{Set}]$ are J-*equivalent* if and only if they induce the same geometric morphism (up to equivalence). The correspondence extends to natural transformations, and thus we end up with the following:

Proposition 1.3.3 [8, Proposition 4.1.2]. *There is an equivalence of categories*

$$\mathbf{EssTopos}^{co}(\mathbf{Sh}(\mathcal{D},K),\mathbf{Sh}(\mathcal{C},J)) \simeq \mathbf{Com}^{J}_{cont}((\mathcal{D},K),([\mathcal{C}^{op},\mathbf{Set}],\widehat{J})),$$

where the category on the right is the category of continuous comorphisms of sites $(\mathcal{D},K) \to ([\mathcal{C}^{op},\mathbf{Set}],\widehat{J})$ up to J-equivalence, acting as follows:

- *an essential geometric morphism $F : \mathbf{Sh}(\mathcal{D},K) \to \mathbf{Sh}(\mathcal{C},J)$ is sent to the functor $\iota_J F_! \ell_K : \mathcal{D} \to [\mathcal{C}^{op},\mathbf{Set}]$, which is a $(K,\widehat{J})$-continuous comorphism of sites; a natural transformation $\omega : F \Rightarrow G$ induces $\omega_! : G_! \Rightarrow F_!$, and we map it to the composite $\iota_J \circ \omega_! \circ \ell_K$.*

- *a continuous comorphism of sites $A : (\mathcal{D},K) \to ([\mathcal{C}^{op},\mathbf{Set}],\widehat{J})$ produces an essential geometric morphism $\mathbf{Sh}(\mathcal{D},K) \to \mathbf{Sh}([\mathcal{C}^{op},\mathbf{Set}],\widehat{J}) \simeq \mathbf{Sh}(\mathcal{C},J)$, and the same holds for natural transformations.*

Moreover, the equivalence is pseudonatural in $(\mathcal{D},K)$ and $(\mathcal{C},J)$.

Remark 1.3.1. Combining this result with Proposition 1.3.1 we get in particular an equivalence of categories

$$\begin{array}{c}\mathbf{Com}_{cont}((\mathcal{D},K),(\mathbf{Sh}(\mathcal{C},J),J^{can}_{\mathbf{Sh}(\mathcal{C},J)}))\\ \iota_J\circ- \Big\downarrow \Big\uparrow \mathrm{a}_J\circ- \\ \mathbf{Com}^{J}_{cont}((\mathcal{D},K),([\mathcal{C}^{op},\mathbf{Set}],\widehat{J})).\end{array}$$

1.4 Site-theoretic classification of geometric morphisms over a base

Morphisms and comorphisms of sites can also be exploited to obtain site-theoretic descriptions of geometric morphisms *over* a base topos: one first instance of this is Proposition 2.4 of [12], which we shall generalize in various ways in the present section.

First of all, let us set once and for all the notation for slice 2-categories, which we will exploit all throughout the work:

Definition 1.4.1. Consider a 1-category $\mathcal{C}$ and an object X: by $\mathcal{C}/X$ we shall denote the usual *slice category*, such that

- objects are arrows $y : Y \to X$ of $\mathcal{C}$, and
- arrows $z : [w : W \to X] \to [y : Y \to X]$ correspond to arrows $z : W \to Y$ of $\mathcal{C}$ such that $yz = w$.

If $\mathcal{A}$ is a strict 2-category we can perform its 1-categorical slice over an object A, which we will denote by $\mathcal{A}/_1 A$. The right 2-categorical notion for an arbitrary lax 2-category $\mathcal{A}$ is that of *slice 2-category* $\mathcal{A}/A$ defined as follows:

0-cells are arrows $b : B \to A$;

1-cells from $[p : C \to A]$ to $[b : B \to A]$ correspond to pairs (c, γ) where $c : C \to B$ and $\gamma : bc \overset{\sim}{\Longrightarrow} p$;

2-cells from (c, γ) to $(d, \delta) : [p] \to [b]$ are 2-cells $\omega : c \Rightarrow d$ satisfying the identity $\delta = \gamma(b \circ \omega)$.

If we do not require that γ in the definition of 1-cell is invertible, we obtain instead the notion of *lax-slice-2-category* $\mathcal{A} /\!\!/ A$(see [32]).

We start by classifying essential geometric morphisms whose domain is induced by a continuous comorphism of sites:

Proposition 1.4.1. *Consider a* (K, J)*-continuous comorphism of sites* $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$ *and an essential geometric morphism* $E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$*: there a pseudonatural equivalence of categories*

$$\mathbf{EssTopos}^{co} /\!\!/ \mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{D}, K), \mathscr{E}) \simeq \mathbf{Com}_{cont}((\mathcal{D}, K), (\mathscr{E}, J^{can}_{\mathscr{E}}))/E^*\ell_J p$$

Moveover, this restricts to an equivalence between

$$\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{D}, K), \mathscr{E})$$

and the full subcategory of $\mathbf{Com}_{cont}((\mathcal{D}, K), (\mathscr{E}, J^{can}_{\mathscr{E}}))/E^*\ell_J p$ *whose 1-cells are the natural transformations* $\xi : A \Rightarrow E^*\ell_J p$ *such that the composite*

$$E_! A \xrightarrow{E_! \circ \xi} E_! E^* \ell_J p \xrightarrow{\varepsilon \circ \ell_J p} \ell_J p$$

is an isomorphism, where ε *is the counit of* $E_! \dashv E^*$.

Proof. First of all, let us specify how $\mathbf{EssTopos}^{co} /\!\!/ \mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{D}, K), \mathscr{E})$ is made. Notice that the presence of ^{co} forces all the 2-cells appearing in the definition of the lax slice to change direction: so for instance an object would be a pair $(F, \varphi) : [C_p] \to [E]$, where F is an essential geometric morphism $\mathbf{Sh}(\mathcal{D}, K) \to \mathscr{E}$ and φ a natural transformation $C_p^* \Rightarrow F^*E^*$. However, we can exploit the presence of the essential images to reverse the direction of 2-cells: objects of $\mathbf{EssTopos}^{co} /\!\!/ \mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{D}, K), \mathscr{E})$ are pairs (F, φ) with $F : \mathbf{Sh}(\mathcal{D}, K) \to \mathscr{E}$ essential and $\varphi : E_! F_! \Rightarrow (C_p)_!$. Similarly, given two such 1-cells (F, φ) and (G, γ), an arrow $\omega : (F, \varphi) \Rightarrow (G, \gamma)$ is given by a natural transformation $\omega : F_! \Rightarrow G_!$ satisfying the identity $\gamma(E_! \circ \omega) = \varphi$. Notice that we can exploit the adjunction $E_! \dashv E^*$ to obtain from φ a natural transformation $\bar{\varphi} : F_! \Rightarrow E^*(C_p)_!$: then the identity $\gamma(E_! \circ \omega) = \varphi$ translates into $\bar{\varphi} = \bar{\gamma}\omega$. Finally, notice that $F_!$, $\bar{\varphi}$ and ω are defined (up to isomorphism) by their values on the generators of $\mathbf{Sh}(\mathcal{D}, K)$: i.e., they are uniquely defined by the composites $F_!\ell_K$, $\bar{\varphi} \circ \ell_K$ and $\omega \circ \ell_K$. But $F_!\ell_K$ is a continuous comorphism of sites presenting F, while $E^*(C_p)_!\ell_K \simeq E^*\ell_J p$: thence we end up with the equivalence

$$\mathbf{EssTopos}^{co} /\!\!/ \mathbf{Sh}(\mathcal{C},J)(\mathbf{Sh}(\mathcal{D},K),\mathscr{E}) \simeq \mathbf{Com}_{cont}((\mathcal{D},K),(\mathscr{E},J^{can}_{\mathscr{E}}))/E^*\ell_J p$$

Now, notice that $\varphi : E_! F_! \Rightarrow (C_p)_!$ can be recovered from $\bar{\varphi}$ as $\varphi = (\varepsilon \circ (C_p)_!)(E_! \circ \bar{\varphi})$. Restricting again to the generators of $\mathbf{Sh}(\mathcal{D},K)$ and setting $\xi := \bar{\varphi} \circ \ell_K$, φ is an isomorphism if and only if $(\varepsilon \circ \ell_J p)(E_! \circ \xi)$ is. □

Remark 1.4.1. There is a slight, but innocuous, abuse of notation in the previous result: we wrote the right-hand term as a slice category of $\mathbf{Com}_{cont}$, but the functor $E^*\ell_J p$ in general is *not* a continuous comorphism of sites.

We can also exploit the point of view of J-equivalent comorphisms expressed in Proposition 1.3.3, in order to obtain a classification of relative geometric morphisms whose domain and codomain are both induced by continuous comorphisms of sites:

Proposition 1.4.2. *Consider two continuous comorphisms $p : (\mathcal{D},K) \to (\mathcal{C},J)$ and $q : (\mathcal{E},T) \to (\mathcal{C},J)$: then there a pseudonatural equivalence of categories between*

$$\mathbf{EssTopos}^{co} /\!\!/ \mathbf{Sh}(\mathcal{C},J)(\mathbf{Sh}(\mathcal{D},K),\mathbf{Sh}(\mathcal{E},T))$$

and

$$\mathbf{Com}^{T}_{cont}((\mathcal{D},K),([\mathcal{E}^{op},\mathbf{Set}],\widehat{T}))/q^* \text{よ}_{\mathcal{C}} p.$$

Moveover, this restricts to an equivalence between

$$\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)(\mathbf{Sh}(\mathcal{D},K),\mathscr{E})$$

and the full subcategory of $\mathbf{Com}^{T}_{cont}((\mathcal{D},K),([\mathcal{E}^{op},\mathbf{Set}],\widehat{T})/q^* \text{よ}_{\mathcal{C}} p$ whose 1-cells are the natural transformations $\tau : B \Rightarrow q^* \text{よ}_{\mathcal{C}} p$ such that the composite

$$\mathrm{lan}_{q^{op}} B \xRightarrow{\mathrm{lan}_{q^{op}} \circ \tau} \mathrm{lan}_{q^{op}} q^* \text{よ}_{\mathcal{C}} p \xRightarrow{\varepsilon' \circ \text{よ}_{\mathcal{C}} p} \text{よ}_{\mathcal{C}} p.$$

is sent by a_J to an isomorphism, where ε' is the counit of $\mathrm{lan}_{q^{op}} \dashv q^*$.

Proof. Let us again use the shorthands $\mathbf{Sh}(\mathcal{C},J) := \widetilde{\mathcal{C}}$ and $[\mathcal{C}^{op},\mathbf{Set}] := \widehat{\mathcal{C}}$. The previous proposition tells us that

$$\mathbf{EssTopos}^{co} /\!\!/ \widetilde{\mathcal{C}}(\widetilde{\mathcal{D}},\widetilde{\mathcal{E}}) \simeq \mathbf{Com}_{cont}((\mathcal{D},K),(\widetilde{\mathcal{E}},J^{can}_{\widetilde{\mathcal{E}}}))/C^*_q \ell_J p$$

which starting from (F,φ) provides a natural transformation $\bar{\varphi} : F_!\ell_K \Rightarrow C^*_q \ell_J p$. First of all, notice that $C^*_q \ell_J p = C^*_q \mathrm{a}_J \text{よ}_{\mathcal{C}} p \simeq \mathrm{a}_T q^* \text{よ}_{\mathcal{C}} p$. Second, set $A := \iota_T F_! \ell_K$ and consider the composite $\iota_T \circ \bar{\varphi} : A \Rightarrow \iota_T \mathrm{a}_T q^* \text{よ}_{\mathcal{C}} p$: we are now in $\widehat{\mathcal{E}}$, and we can perform the pullback of $\iota_T \circ \bar{\varphi}$ componentwise along $\eta \circ q^* \text{よ}_{\mathcal{C}} p : q^* \text{よ}_{\mathcal{C}} p \Rightarrow \iota_T \mathrm{a}_T q^* \text{よ}_{\mathcal{C}} p$, where η is the unit of $\mathrm{a}_T \dashv \iota_T$. We obtain a natural transformation $\tau : B \Rightarrow q^* \text{よ}_{\mathcal{C}} p$, where B is the functor $B : \mathcal{D} \to \widehat{\mathcal{E}}$ mapping every object D to the pullback of $\bar{\varphi}(D)$ and $\eta_{q^* \text{よ}_{\mathcal{C}}(p(D))}$ in $[\mathcal{E}^{op}, \mathbf{Set}]$. Notice that the natural transformation $\omega : B \Rightarrow A$ satisfies the condition that $\mathrm{a}_T \circ \omega$ is an isomorphism, since it is the pullback of η: this forces B to be a $(K, \widehat{T})$-continuous comorphism of sites which induces the same geometric morphism as A, namely F. To see this, we recall that $\mathbf{Sh}(\widehat{\mathcal{E}}, \widehat{T}) \simeq \widetilde{\mathcal{E}}$ with the inclusion $\iota_{\widehat{T}} : \mathbf{Sh}(\widehat{\mathcal{E}}, \widehat{T}) \hookrightarrow [\widehat{\mathcal{E}}^{op}, \mathbf{Set}]$ being the composite functor $\text{よ}_{\widehat{\mathcal{E}}} \iota_T$. Let us first see that $(- \circ B^{op}) : [\widehat{\mathcal{E}}^{op}, \mathbf{Set}] \to \widehat{\mathcal{D}}$ restricts to sheaves. For any T-sheaf W

$$\begin{aligned}(- \circ B^{op}) \text{よ}_{\widehat{\mathcal{E}}} \iota_T(W) := \widehat{\mathcal{E}}(B(-), \iota_T(W)) &\simeq \widetilde{\mathcal{E}}(\mathrm{a}_T B(-), W) \\ &\simeq \widetilde{\mathcal{E}}(\mathrm{a}_T A(-), W) \\ &\simeq \widehat{\mathcal{E}}(A(-), \iota_T(W)),\end{aligned}$$

meaning that $(- \circ B^{op}) \circ \iota_{\widehat{T}} \simeq (- \circ A^{op}) \circ \iota_{\widehat{T}}$. But the latter functor factors through $\widetilde{\mathcal{D}}$, since A is $(K, \widehat{T})$-continuous, and hence so does the first: therefore B is $(K, \widehat{T})$-continuous. This immediately implies that $\mathrm{a}_K(- \circ B^{op})\iota_{\widehat{T}} \simeq C_A^*$, which has a right adjoint, and thus B is a comorphism of sites. Therefore, for any (F, φ) in

$$\mathbf{EssTopos}^{co} /\!\!/ \mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{D}, K), \mathbf{Sh}(\mathcal{E}, T)),$$

we may map it to $\tau : B \Rightarrow p^* \text{よ}_{\mathcal{C}} p$ in

$$\mathbf{Com}^{T}_{cont}((\mathcal{D}, K), ([\mathcal{E}^{op}, \mathbf{Set}], \widehat{T}))/q^* \text{よ}_{\mathcal{C}} p,$$

and this provides our equivalence.

Finally, we know that φ is an isomorphism if and only if

$$(C_q)_! F_! \ell_K \xrightarrow{(C_q)_! \circ \bar{\varphi}} (C_q)_! C_q^* \ell_J p \xrightarrow{\varepsilon \circ \ell_J p} \ell_J p$$

is an isomorphism, where ε is the counit of $(C_q)_! \dashv C_q^*$, but a routine computation shows that $(\varepsilon \circ \ell_J p)((C_q)_! \circ \bar{\varphi})$ is the image through a_J of

$$\mathrm{lan}_{q^{op}} B \xrightarrow{\mathrm{lan}_{q^{op}} \circ \tau} \mathrm{lan}_{q^{op}} q^* \text{よ}_{\mathcal{C}} p \xrightarrow{\varepsilon' \circ \text{よ}_{\mathcal{C}} p} \text{よ}_{\mathcal{C}} p.$$

□

The next result will instead classify *all* geometric morphisms whose codomain is of the form $[C_p]$, but this time using morphisms of sites:

Theorem 1.4.3. *Consider a comorphism of sites $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$ and a geometric morphism $E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$: then there is a pseudonatural equivalence of categories*

$$\mathbf{Topos} /\!\!/ \mathbf{Sh}(\mathcal{C}, J)([E], [C_p]) \simeq \mathbf{Site}((\mathcal{D}, K), (\mathscr{E}, J^{can}_{\mathscr{E}}))/E^* \ell_J p$$

which is pseudonatural in both $[E]$ and p.

Proof. We start by defining the equivalence at the level of objects. An object of **Topos** $/\!\!/$ $\mathbf{Sh}(\mathcal{C}, J)([E], [C_p])$ is a pair (F, φ) where $F : \mathscr{E} \to \mathbf{Sh}(\mathcal{D}, K)$ is a geometric morphism and $\varphi : F^*C_p^* \Rightarrow E^*$. By Corollary D.2 this is the same as a natural transformation $\varphi \mathrm{a}_J : F^*C_p^*\mathrm{a}_J \Rightarrow E^*\mathrm{a}_J$; but since $C_p^*\mathrm{a}_J \simeq \mathrm{a}_K p^*$, we have for now a natural transformation $\varphi \circ \mathrm{a}_J : F^*\mathrm{a}_K p^* \Rightarrow E^*\mathrm{a}_J$. Now, since $\mathrm{lan}_{p^{op}} \dashv p^*$ we have that $- \circ p^* \dashv - \circ \mathrm{lan}_{p^{op}}$ by Lemma D.1, and thus $\varphi \circ \mathrm{a}_J$ corresponds to a natural transformation $\varphi' : F^*\mathrm{a}_K \Rightarrow E^*\mathrm{a}_J\mathrm{lan}_{p^{op}}$. Finally, such a natural transformation is uniquely determined (up to isomorphism) by its values on the generators of $\mathbf{Sh}(\mathcal{D}, K)$, i.e. by the composite $\bar{\varphi} := \varphi' \circ \text{よ}_{\mathcal{D}} : F^*\ell_K \Rightarrow E^*\mathrm{a}_J\mathrm{lan}_{p^{op}}\text{よ}_{\mathcal{D}} \simeq E^*\ell_J p$. Notice that $F^*\ell_K$ is the flat K-continuous functor $\mathcal{D} \to \mathscr{E}$ that generates the geometric morphism F, and thus it is indeed a morphism of sites $(\mathcal{D}, K) \to (\mathscr{E}, J_{\mathscr{E}}^{can})$.

To extend the equivalence to arrows, consider two 1-cells (F, φ) and $(G, \gamma) : [E] \to [C_p]$ and 2-cell $\omega : (F, \varphi) \Rightarrow (G, \gamma)$, i.e. a natural transformation $\omega : F^* \Rightarrow G^*$ such that $\gamma(\omega \circ C_p^*) = \varphi$. Notice that ω is uniquely defined by its restriction $\bar{\omega} = \omega \circ \ell_K$ to the generators of $\mathbf{Sh}(\mathcal{D}, K)$: then a rapid computation shows that $\gamma(\omega \circ C_p^*) = \varphi$ holds if and only if $\bar{\gamma}\bar{\omega} = \bar{\varphi}$. Thus the association $\omega \mapsto \bar{\omega}$ defines the equivalence

$$\mathbf{Topos} /\!\!/ \mathbf{Sh}(\mathcal{C}, J)([F], [C_p]) \simeq \mathbf{Site}((\mathcal{D}, K), (\mathscr{E}, J_{\mathscr{E}}^{can}))/E^*\ell_J p$$

on arrows. The pseudonaturality is lengthy but straightforward to check. □

Remark 1.4.2. Let us make explicit the relationship between the two natural transformations $\varphi : F^*C_p^* \Rightarrow E^*$ and $\bar{\varphi} : F^*\ell_K \Rightarrow E^*\ell_L p$, for it will come in handy later. Denote by η and ε the unit and counit of $\mathrm{lan}_{p^{op}} \dashv p^*$. Starting from φ, we obtain $\bar{\varphi}$ as the composite

$$\begin{array}{ccc}
F^*\ell_K & = & F^*\mathrm{a}_K\text{よ}_{\mathcal{D}} \\
 & & \Downarrow {\scriptstyle F^*\mathrm{a}_K \circ \eta \circ \text{よ}_{\mathcal{D}}} \\
 & & F^*\mathrm{a}_K p^*\mathrm{lan}_{p^{op}}\text{よ}_{\mathcal{D}} \\
\Downarrow {\scriptstyle \bar{\varphi}} & & \Downarrow {\scriptstyle \wr} \\
 & & F^*C_p^*\mathrm{a}_J\mathrm{lan}_{p^{op}}\text{よ}_{\mathcal{D}} \\
 & & \Downarrow {\scriptstyle \varphi \circ \mathrm{a}_J\mathrm{lan}_{p^{op}}\text{よ}_{\mathcal{D}}} \\
E^*\ell_J p & \overset{\sim}{\Longleftarrow} & E^*\mathrm{a}_J\mathrm{lan}_{p^{op}}\text{よ}_{\mathcal{D}}
\end{array}$$

Conversely, start from $\bar{\varphi} : F^*\ell_K \Rightarrow E^*\ell_J p$, which we can see as a 2-cell $F^*\mathrm{a}_K\text{よ}_{\mathcal{D}} \Rightarrow E^*\mathrm{a}_J\mathrm{lan}_{p^{op}}\text{よ}_{\mathcal{D}}$: then $\bar{\varphi}$ induces a natural transformation $\tilde{\varphi} : F^*\mathrm{a}_K \Rightarrow E^*\mathrm{a}_J\mathrm{lan}_{p^{op}}$. We can then consider the composite

$$F^*C_p^*\mathrm{a}_J \cong F^*\mathrm{a}_K p^* \xRightarrow{\tilde{\varphi} \circ p^*} E^*\mathrm{a}_J\mathrm{lan}_{p^{op}}p^* \xRightarrow{E^*\mathrm{a}_J \circ \varepsilon} E^*\mathrm{a}_J :$$

its restriction to sheaves (which coincides with its composition with the functor $\iota_K : \mathbf{Sh}(\mathcal{D}, J) \to [\mathcal{D}^{op}, \mathbf{Set}]$) provides the natural transformation $\varphi : F^* C_p^* \Rightarrow E^*$.

The components of $(E^* \mathrm{a}_J \circ \varepsilon)(\tilde{\varphi} \circ p^*)$, and thus those of φ, can be stated directly in terms of the components of $\bar{\varphi}$ using colimits. Let us start by considering a representable presheaf $\text{よ}(X) : \mathcal{C}^{op} \to \mathbf{Set}$: the composite

$$F^* \mathrm{a}_K p^* \text{よ}(X) \xrightarrow{\tilde{\varphi}(p^*(\text{よ}(X)))} E^* \mathrm{a}_J \mathrm{lan}_{p^{op}} p^*(\text{よ}(X)) \xrightarrow{E^* \mathrm{a}_J(\varepsilon_{\text{よ}(X)})} E^* \mathrm{a}_J \text{よ}(X)$$

can be described using the fact in $[\mathcal{D}^{op}, \mathbf{Set}]$ the presheaf $p^* \text{よ}(X)$ can be presented as the colimit of representables $p^*(\text{よ}(X)) \simeq \mathrm{colim}_{y:p(D) \to X} \text{よ}(D)$. First of all, we recall that $\varepsilon_{\text{よ}(X)}$ is the map

$$\varepsilon_{\text{よ}(X)} : \mathrm{colim}_{y:P(D) \to X} \text{よ}(p(D)) \to \text{よ}(X)$$

induced by the cocone whose y-indexed leg is the map

$$\text{よ}(y) : \text{よ}(p(D)) \to \text{よ}(X).$$

The composite $E^* \mathrm{a}_J(\varepsilon_{\text{よ}(X)})$ is computed thus as the arrow

$$\mathrm{colim}_{y:P(D) \to X} E^* \ell_J(p(D)) \to E^* \ell_J(X),$$

induced by the cocone whose y-indexed leg is the arrow

$$E^* \ell_J(y) : E^* \ell_J(p(D)) \to E^* \ell_J(X).$$

On the other hand, the arrow

$$\tilde{\varphi}(p^* \text{よ}(X)) : F^* \mathrm{a}_K p^* \text{よ}(X) \to E^* \mathrm{a}_J \mathrm{lan}_{p^{op}} p^*(\text{よ}(X))$$

is the morphism

$$\mathrm{colim}_{y:p(D) \to X} F^* \ell_K(D) \to \mathrm{colim}_{y:p(D) \to X} E^* \ell_J(p(D))$$

induced by colimit property by the maps

$$\bar{\varphi}(D) : F^* \ell_K(D) \to E^* \ell_J(p(D)) :$$

therefore, globally we have that

$$(E^* \mathrm{a}_J \circ \varepsilon)(\tilde{\varphi} \circ p^*)(\text{よ}(X)) : F^* \mathrm{a}_K p^* \text{よ}(X) \to E^* \mathrm{a}_J \text{よ}(X)$$

is an arrow

$$\mathrm{colim}_{y:p(D) \to X} F^* \ell_K(D) \to E^* \ell_J(X)$$

induced by the cocone whose y-indexed leg is the arrow

$$F^* \ell_K(D) \xrightarrow{\bar{\varphi}(D)} E^* \ell_J(p(D)) \xrightarrow{E^* \ell_J(y)} E^* \ell_J(X).$$

If now we take any presheaf $H : \mathcal{C}^{op} \to \mathbf{Set}$, we can exploit the colimit $H \simeq \mathrm{colim}_{x\in H(X)} \text{よ}(X)$: then the arrow

$$(E^*\mathrm{a}_J \circ \varepsilon)(\tilde{\varphi} \circ p^*)(H) : F^*C_p^*\mathrm{a}_J(H) \to E^*\mathrm{a}_J(H)$$

is a morphism

$$\mathrm{colim}_{x\in H(X)} F^*\mathrm{a}_K p^* \text{よ}(X) \to \mathrm{colim}_{x\in H(X)} E^*\ell_J(X)$$

induced componentwise by the arrows $(E^*\mathrm{a}_J \circ \varepsilon)(\tilde{\varphi} \circ p^*)(\text{よ}(X))$ we described above: thus we can conclude that $(E^*\mathrm{a}_J \circ \varepsilon)(\tilde{\varphi} \circ p^*)(H)$ is induced by colimit property by the arrows

$$\alpha_{x,y} : F^*\ell_K(D) \xrightarrow{\bar{\varphi}(D)} E^*\ell_J(p(D)) \xrightarrow{E^*\ell_J(y)} E^*\ell_J(X)$$

indexed by $x \in H(X)$ and $y : p(D) \to X$.

The previous results admit an alternative formulation using comma categories:

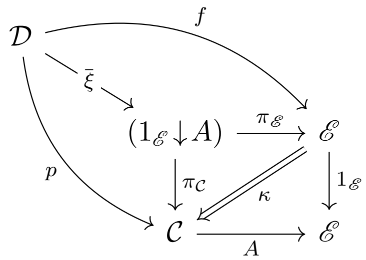
Definition 1.4.2. Consider two functors $A : \mathcal{A} \to \mathcal{C}$ and $B : \mathcal{B} \to \mathcal{C}$: the *comma category* $(A{\downarrow}B)$ is the category whose objects are triples $(X \in \mathcal{A},\ Y \in \mathcal{B},\ f : A(X) \to B(Y))$ and whose morphism $(\alpha, \beta) : (X, Y, f) \to (X', Y', f')$ are pairs of arrows $\alpha : X \to X'$ and $\beta : Y \to Y'$ such that $B(\beta) \circ f = f' \circ A(\alpha)$.

The comma category has two obvious canonical projections to $\mathcal{A}$ and $\mathcal{B}$ and a natural transformation $\varphi : A \circ p_{\mathcal{A}} \to B \circ p_{\mathcal{B}}$ such that $\varphi(X, Y, f)$ is the arrow $f : A(X) \to B(Y)$ (for any object (X, Y, f) of $(A{\downarrow}B)$):

$$\begin{array}{ccc} (A{\downarrow}B) & \xrightarrow{p_B} & \mathcal{B} \\ {\scriptstyle p_A}\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle B} \\ \mathcal{A} & \xrightarrow[A]{} & \mathcal{C} \end{array} .$$

The comma category $(A{\downarrow}B)$ satisfies a strict 2-limit universal property in **CAT**: for every other pair of functors $F_A : \mathcal{D} \to \mathcal{A}$, $F_B : \mathcal{D} \to \mathcal{B}$ and natural transformation $\psi : AF_A \Rightarrow BF_B$ there is a unique functor $F : \mathcal{D} \to (A{\downarrow}B)$ such that $F_A = p_A F$, $F_B = p_B F$ and $\psi = \varphi \circ F$. This universal property extends immediately to 2-cells.

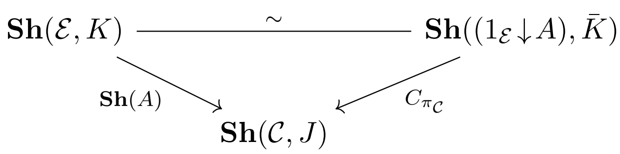
Using comma categories we can provide an alternative description of the category $\mathbf{Site}((\mathcal{D}, K), (\mathscr{E}, J_{\mathscr{E}}^{can}))/E^*\ell_J p$. Let us set $E^*\ell_J := A$: then a 1-cell $[\xi : f \Rightarrow Ap]$ corresponds to a unique $\bar{\xi} : \mathcal{D} \to (1_{\mathscr{E}}{\downarrow}A)$ as in the following diagram:



and a similar correspondence holds for the 2-cells. We only need to take into account that we want the composite $\pi_{\mathscr{E}}\bar{\xi}$ to be a morphism of sites $(\mathcal{D}, K) \to (\mathscr{E}, J^{can}_{\mathscr{E}})$. To do so, we shall exploit the following result:

Theorem 1.4.4 [6, Theorem 3.16]. *Consider a morphism of sites $A : (\mathcal{C}, J) \to (\mathcal{E}, K)$. Consider the topology $\bar{K}$ on the comma category $(1_{\mathcal{E}} \downarrow A)$, whose covering sieves are exactly those whose image in $\mathcal{E}$ is K-covering: then*

- *the projection $\pi_{\mathcal{C}} : (1_{\mathcal{E}} \downarrow A) \to \mathcal{C}$ is a comorphism of sites,*
- *the projection $\pi_{\mathcal{E}} : (1_{\mathcal{E}} \downarrow A) \to \mathcal{E}$ is a morphism and a comorphism of sites inducing an equivalence of toposes,*
- *the diagram of geometric morphisms*



  is commutative.

Seeing A as a morphism of sites $(\mathcal{C}, J) \to (\mathscr{E}, J^{can}_{\mathscr{E}})$, we can apply the theorem above to obtain an equivalence $\mathbf{Sh}((1_{\mathscr{E}} \downarrow A), \overline{J^{can}_{\mathscr{E}}}) \simeq \mathscr{E}$ that identifies the geometric morphisms E and $C_{\pi_{\mathcal{C}}}$. One verifies immediately that $\pi_{\mathscr{E}} \circ \bar{\xi}$ is a morphism of sites if and only if $\bar{\xi}$ is, since at the level of toposes the two functors induce essentially the same geometric morphism, and so we end up with the following result:

Proposition 1.4.5. *Consider a geometric morphism $E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$ with corresponding flat J-continuous functor $A : \mathcal{C} \to \mathscr{E}$ and a comorphism of sites $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$: then there is an equivalence of categories between*

$$\mathbf{Topos} /\!\!/ \mathbf{Sh}(\mathcal{C}, J)([E], [C_p])$$

and the full subcategory of $\mathbf{Site}((\mathcal{D}, K), ((1_{\mathscr{E}} \downarrow A), \overline{J^{can}_{\mathscr{E}}}))$ whose objects are the morphisms of sites $\xi : (\mathcal{D}, K) \to ((1_{\mathscr{E}} \downarrow A), \overline{J^{can}_{\mathscr{E}}})$ such that $\pi_{\mathcal{C}} \circ \xi = p$.

With the same argument we can also derive the following corollary of Proposition 1.3.1:

Proposition 1.4.6. *Consider a geometric morphism* $E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$ *with corresponding flat* J*-continuous functor* $A : \mathcal{C} \to \mathscr{E}$ *and a continuous comorphism of sites* $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$*: then there is an equivalence of categories between*

$$\mathbf{EssTopos}^{co} /\!\!/ \mathbf{Sh}(\mathcal{C}, J)([C_p], [E])$$

and the full subcategory of $\mathbf{Com}_{cont}((\mathcal{D}, K), ((1_{\mathscr{E}}{\downarrow}A), \overline{J^{can}_{\mathscr{E}}}))$ *whose objects are the continuous comorphisms of sites* $\xi : (\mathcal{D}, K) \to ((1_{\mathscr{E}}{\downarrow}A), \overline{J^{can}_{\mathscr{E}}})$ *such that* $\pi_{\mathcal{C}} \circ \xi = p$.

Chapter 2

Fibrations and stacks

As Jean Giraud already made evident in [12], fibrations and stacks are a fundamental tool to develop relative topos theory: we have collected in this chapter all the technical tools about them that will be needed in this thesis. Unless stated otherwise, all the results in this chapter can be found in Chapter 2 of [8].

The first section deals with the notions of $\mathcal{C}$-indexed category and Street fibration. Street fibrations are the equivalence-stable generalization of the more notorious Grothendieck fibrations (see [33] and its references), therefore all classical results for Grothendieck fibrations that are equivalence-stable hold also for Street fibrations: nonetheless, we have provided sketches of proofs for these results, both to set the notation and as a reference. This means that in the following, unless stated otherwise, 'fibration' will always mean in the sense of Street. The second section deals with localizations of fibrations, which will be useful when computing base change functors in Chapter 4. We will then introduce stacks, both in fibrational and indexed terms, and in particular the canonical stack of a site; finally, we will analyse the connection between stacks and sheaves, called the truncation functor, which can be understood both in terms of an orthogonal factorization system for geometric morphisms and in terms of one for functors between sites.

2.1 Indexed categories and Street fibrations

Fibrations are best understood as the gluing of information coming from a $\mathcal{C}$-indexed category, so we start from there. Intuitively, a $\mathcal{C}$-indexed category is nothing but a presheaf taking values in **CAT** instead of **Set**; however, it proves fruitful to weaken the rigid functoriality of presheaves as follows:

Definition 2.1.1. Consider a category $\mathcal{C}$. A *pseudofunctor* $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ is the datum of

- a category $\mathbb{D}(X)$ for each object X in $\mathcal{C}$, called the *fibre of* $\mathbb{D}$ *over* X,

- a functor $\mathbb{D}(y) : \mathbb{D}(X) \to \mathbb{D}(Y)$ for each arrow $y : Y \to X$ in $\mathcal{C}$, called *transition morphism*,
- a family of natural isomorphisms

$$\varphi_X^{\mathbb{D}} : 1_{\mathbb{D}(X)} \xRightarrow{\sim} \mathbb{D}(1_X),$$

one for each X in $\mathcal{C}$, and a family of natural isomorphisms

$$\varphi_{y,z}^{\mathbb{D}} : \mathbb{D}(z)\mathbb{D}(y) \xRightarrow{\sim} \mathbb{D}(yz),$$

one for each composable pair y, z of arrows in $\mathcal{C}$, which are the *structural isomorphisms* of $\mathbb{D}$,

satisfying the following compatibility conditions: for any $y : Y \to X$

$$\varphi_{1_X,y}^{\mathbb{D}}(\mathbb{D}(y) \circ \varphi_X^{\mathbb{D}}) = \varphi_{y,1_Y}^{\mathbb{D}}(\varphi_Y^{\mathbb{D}} \circ \mathbb{D}(y)) = \mathrm{id}_{\mathbb{D}(y)} : \mathbb{D}(y) \Rightarrow \mathbb{D}(y),$$

and for any $w : W \to Z$, $z : Z \to Y$, $y : Y \to X$,

$$\varphi_{y,zw}^{\mathbb{D}}(\varphi_{z,w}^{\mathbb{D}} \circ \mathbb{D}(y)) = \varphi_{yz,w}^{\mathbb{D}}(\mathbb{D}(w) \circ \varphi_{y,z}^{\mathbb{D}}) : \mathbb{D}(w)\mathbb{D}(z)\mathbb{D}(y) \Rightarrow \mathbb{D}(yzw).$$

In short, a pseudofunctor $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ is functorial up to canonical 2-isomorphisms; a *strict pseudofunctor* is a functor $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ in the usual sense.

A *pseudonatural transformation* $F : \mathbb{D} \Rightarrow \mathbb{E}$ consists of the following data. For every X in $\mathcal{C}$ we have a functor $F_X : \mathbb{D}(X) \to \mathbb{E}(X)$ and for every $y : Y \to X$ of $\mathcal{C}$ we have a natural isomorphism $F_y : \mathbb{E}(y)F_X \xRightarrow{\sim} F_Y\mathbb{D}(y)$ satisfying the following compatibility conditions: for every X in $\mathcal{C}$

$$F_X \circ \varphi_X^{\mathbb{D}} = F_{1_X}(\varphi_X^{\mathbb{E}} \circ F_X) : F_X \Rightarrow F_X\mathbb{D}(1_X)$$

and for every $z : Z \to Y$, $y : Y \to X$,

$$F_{yz}(\varphi_{y,z}^{\mathbb{E}} \circ F_X) = (F_Z \circ \varphi_{y,z}^{\mathbb{D}})(F_z \circ \mathbb{D}(y))(\mathbb{E}(z) \circ F_y) : \mathbb{E}(z)\mathbb{E}(y)F_X \Rightarrow F_Z\mathbb{D}(yz).$$

If we do not ask the components F_y to be invertible, we have what is called a *lax natural transformation*; if moreover we ask that the F_y's go in the opposite direction, we can suitably adapt the two axioms above to obtain the definition of *oplax natural transformation*.

A *modification of pseudo-/oplax/lax natural transformations*, denoted by $\xi : F \Rrightarrow G$, consists for every X in $\mathcal{C}$ of a natural transformation $\xi_X : F_X \Rightarrow G_X$ such that for every $y : Y \to X$ the identity

$$G_y(\mathbb{E}(y) \circ \xi_X) = (\xi_Y \circ \mathbb{D}(y))F_y : \mathbb{E}(y)F_X \Rightarrow G_Y\mathbb{D}(y)$$

is satisfied.

Pseudofunctors $\mathcal{C}^{op} \to \mathbf{CAT}$, their pseudonatural transformations and modifications are respectively the 0-cells, 1-cells and 2-cells of a 2-category which we shall denote by

$$[\mathcal{C}^{op}, \mathbf{CAT}]_{ps}.$$

If instead of pseudonatural transformations we consider lax natural transformations as 1-cells we still have a 2-category, denoted by $[\mathcal{C}^{op}, \mathbf{CAT}]_{lax}$; similarly, if the 1-cells are oplax natural transformations we shall use the notation $[\mathcal{C}^{op}, \mathbf{CAT}]_{oplax}$.

Pseudofunctors $\mathcal{C}^{op} \to \mathbf{CAT}$ can also be thought as *$\mathcal{C}$-indexed categories*; their pseudonatural transformations are thus called *$\mathcal{C}$-indexed functors* and the modifications *$\mathcal{C}$-indexed natural transformations*. This justifies the more compact notation $\mathbf{Ind}_{\mathcal{C}}$ for the 2-category $[\mathcal{C}^{op}, \mathbf{CAT}]_{ps}$, which we will adopt whenever the focus is not really on the kind of transformation considered.

If we consider a set I and a I-indexed family of sets $\{X_i \mid i \in I\}$, there is an obvious way of gluing them all together: one considers their disjoint union $\coprod_{i \in I} X_i$. The original sets can now be retrieved as fibres of the canonical projection map $\coprod_i X_i \to I$. In a similar fashion, the whole information contained in a $\mathcal{C}$-indexed category $\mathbb{D}$ can be glued together in one single category over $\mathcal{C}$, exploiting the well-known *Grothendieck construction*:

Definition 2.1.2. Any $\mathcal{C}$-indexed category $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ is canonically associated to a functor $p_{\mathbb{D}} : \mathcal{G}(\mathbb{D}) \to \mathcal{C}$, defined as follows: the objects of $\mathcal{G}(\mathbb{D})$ are pairs (X, U) with X in $\mathcal{C}$ and U in $\mathbb{D}(X)$, and arrows of $\mathcal{G}(\mathbb{D})$ are pairs $(y, a) : (X, V) \to (X, U)$ where $y : Y \to X$ and $a : V \to \mathbb{D}(y)(U)$ in $\mathbb{D}(Y)$. The identity arrow of (X, U) is the pair $(1_X, \varphi_X^{\mathbb{D}}(U))$, while composition of arrows is defined by the equation $(y, a) \circ (z, b) = (yz, \varphi_{y,z}(U)\mathbb{D}(z)(a)b)$. The functor $p_{\mathbb{D}}$ acts by forgetting the second component.

From a $\mathcal{C}$-indexed functor $F : \mathbb{D} \to \mathbb{E}$ we can obtain a functor $\mathcal{G}(F) : \mathcal{G}(\mathbb{D}) \to \mathcal{G}(\mathbb{E})$ such that $p_{\mathbb{E}}\mathcal{G}(F) = p_{\mathbb{D}}$. We set $\mathcal{G}(F)(X, U) := (X, F_X(U))$, and for any arrow $(y, a) : (Y, V) \to (X, U)$ in $\mathcal{G}(\mathbb{D})$ we set $\mathcal{G}(F)(y, a)$ to be the arrow $(t, F_y(U)^{-1}F_Y(a)) : (Y, F_Y(V)) \to (X, F_Y(U))$. Moreover, a $\mathcal{C}$-indexed natural transformation $\xi : F \Rightarrow G$ is sent to a natural transformation $\mathcal{G}(\xi) : \mathcal{G}(F) \Rightarrow \mathcal{G}(G)$ satisfying the identity $p_{\mathbb{E}} \circ \mathcal{G}(\xi) = \mathrm{id}_{p^{\mathbb{D}}}$. This means that $\mathcal{G}$ provides a strict 2-functor

$$\mathcal{G} : \mathbf{Ind}_{\mathcal{C}} \to \mathbf{CAT}/\mathcal{C}$$

which moreover factors through the sub 2-category of strictly commutative triangles over $\mathcal{C}$.

Remark 2.1.1. The Grothendieck construction can be performed on a much wider class of functors, namely that of lax functors $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ and their oplax natural transformations, and one can again obtain a 2-functor $\mathbf{Lax}(\mathcal{C}^{op}, \mathbf{CAT})_{oplax} \to \mathbf{CAT}/\mathcal{C}$: this is implied, though not explicitly stated, in [19]. For the following results though (in particular the equivalence in Corollary 2.1.7), it is still necessary to restrict to $\mathcal{C}$-indexed categories.

As it happened in the example with indexed families of sets, the $\mathcal{C}$-indexed category $\mathbb{D}$ can be recovered from $p : \mathcal{G}(\mathbb{D}) \to \mathcal{C}$: indeed, once fixed X in $\mathcal{C}$, the fibre $\mathbb{D}(X)$ is isomorphic to the collection of objects (X, U) of $\mathcal{G}(\mathbb{D})$ and of morphisms of the form $(1, a)$ between them. We can similarly recover each functor $\mathbb{D}(y)$.

Example 2.1.1. Let us see some examples of $\mathcal{C}$-indexed categories:

(i) Any presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ can be seen as a particular kind of $\mathcal{C}$-indexed category which is strict and *discrete*: that is, it is a strict functor and its fibres are all discrete categories. In this case one usually writes $\int P$ for $\mathcal{G}(P)$ and calls it the *category of elements* of P: objects are still pairs (X, U) with $U \in P(X)$, and arrows are of the form $y : (Y, V) \to (X, U)$ where $y : Y \to X$ and $P(y)(U) = V$. The functor $p_P : \int P \to \mathcal{C}$ simply forgets the second component. A morphism of presheaves $f : P \Rightarrow Q$ is mapped to a functor $\int f : \int P \to \int Q$ operating by sending $y : (Y, Py(U)) \to (X, U)$ to $y : (Y, f_Y Py(U)) \to (X, f_X(U))$. In particular, we remark that $\int よ(X) \simeq \mathcal{C}/X$, and that for $y : Y \to X$, $\int よ(y)$ (which we shall denote sumply by $\int y$) acts as the postcomposition functor $y \circ - : \mathcal{C}/Y \to \mathcal{C}/X$.

(ii) Given a geometric morphism $F : \mathscr{F} \to \mathscr{E}$, we can define a $\mathscr{E}$-indexed category $\mathbb{I}_F$ by sending an object E of $\mathscr{E}$ to the category $\mathscr{F}/F^*(E)$ and any arrow $g : E \to E'$ in $\mathscr{E}$ to the pullback functor $\mathscr{F}/F^*(E') \to \mathscr{F}/F^*(E)$ along the arrow $F^*(g) : F^*(E) \to F^*(E')$. Applying the Grothendieck construction to $\mathbb{I}_F$ we obtain that $\mathcal{G}(\mathbb{I}_F)$ is the comma category $(\mathscr{F} \downarrow F^*)$, with the canonical projection to $\mathscr{E}$.

(iii) If $\mathcal{C}$ is a category with finite limits and a canonical choice of pullbacks, it admits a $\mathcal{C}$-indexed category $\mathbb{P} : \mathcal{C}^{op} \to \mathbf{CAT}$ which maps every object X to the slice category $\mathcal{C}/X$, and every morphism $y : Y \to X$ to the pullback functor $y^* : \mathcal{C}/X \to \mathcal{C}/Y$. Applying the Grothendieck construction one sees that $\mathcal{G}(\mathbb{P}) \cong Mor(\mathcal{C})$, the category of morphisms of $\mathcal{C}$ and their commutative squares, with its structural functor being the codomain functor $\mathrm{cod} : Mor(\mathcal{C}) \to \mathcal{C}$.

(iv) If we consider a Grothendieck topos $\mathscr{E}$ and its identity geometric morphism $1_{\mathscr{E}}$, the $\mathscr{E}$-indexed category $\mathbb{I}_{1_{\mathscr{E}}}$ of item (ii) coincides with the one in item (iii): it is what we shall call the *canonical stack over* $\mathscr{E}$ (see Section 2.4).

(v) We are always able to build from a $\mathcal{C}$-indexed category $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$, another pseudofunctor $\mathbb{D}^V : \mathcal{C}^{op} \to \mathbf{CAT}$ by setting $\mathbb{D}^V(X) := \mathbb{D}(X)^{op}$ for every X in $\mathcal{C}$, and $\mathbb{D}^V(y) := \mathbb{D}(y)^{op} : \mathbb{D}(X)^{op} \to \mathbb{D}(Y)^{op}$. for every $y : Y \to X$: we will make extensive use of these pseudofunctors in the following. The notation adopted is consistent with that of [12].

Fibrations over $\mathcal{C}$ are introduced by providing an axiomatic description of functors of the kind $\mathcal{G}(\mathbb{D}) \to \mathcal{C}$:

Definition 2.1.3. Consider a functor $p : \mathcal{D} \to \mathcal{C}$: an arrow $f : A \to B$ in $\mathcal{D}$ is said to be *cartesian* if for every other $h : C \to B$ and any $g : p(C) \to p(A)$ such that $p(h) = p(f)g$ there is a unique $k : C \to A$ such that $p(k) = g$ and $h = fk$.

$$\begin{array}{ccccccc} C & & & & p(C) & & \\ {\scriptstyle h}\downarrow & \searrow^{!k} & & \overset{p}{\mapsto} & {\scriptstyle p(h)}\downarrow & \searrow^{g} & \\ B & \xleftarrow[f]{} & A & & p(B) & \xleftarrow[p(f)]{} & p(A) \end{array}$$

Remarks 2.1.2. Let us list here some useful properties of cartesian arrows:

(i) Given a cartesian arrow $f : A \to B$, any other arrow $k : C \to A$ is determined uniquely by the composite fk and the projection $p(k)$ in $\mathcal{C}$. We will often use this in the following to show that two arrows in the domain of a fibration are equal.

(ii) If f is cartesian, then fg is cartesian if and only if g is cartesian.

(iii) Consider an arrow $x : X \to p(U)$ and two distinct cartesian arrows $a : A \to U$ and $b : B \to U$ with isomorphisms $\alpha : X \xrightarrow{\sim} p(A)$ and $\beta : X \xrightarrow{\sim} p(B)$ such that $x = p(a)\alpha = p(b)\beta$. Exploiting the cartesianity of both a and b it follows that there is a unique $\gamma : A \xrightarrow{\sim} B$ such that $b\gamma = a$ and $p(\gamma) = \beta\alpha^{-1}$. This has the immediate consequence that if $x : X \to p(U)$ is an isomorphism, then all its *cartesian lifts* (see the following definition) are isomorphisms: this happens because x admits among its lifts the identity arrow 1_U, and all other lifts of are isomorphic to it.

Definition 2.1.4. A functor $p : \mathcal{D} \to \mathcal{C}$ is called a *(Street) fibration* if for every A in $\mathcal{D}$ and any $x : X \to p(A)$ in $\mathcal{C}$ there are a cartesian arrow $f : B \to A$ and an isomorphism $\theta : X \xrightarrow{\sim} p(B)$ such that $p(f)\theta = x$:

$$\begin{array}{ccc} p(B) & & \\ \wr\uparrow{\scriptstyle \theta} & \searrow^{p(f)} & \\ X & \xrightarrow[x]{} & p(A) \end{array}$$

The arrow f is called a *cartesian lift* for x. The fibration p is *cloven*, or *with cleavage*, if for every x and A we have a choice of a cartesian lift $\hat{x}_A : \mathrm{dom}(\hat{x}_A) \to A$ and an isomorphism $\theta_{f,A} : X \to \mathrm{dom}(p(\hat{x}_A))$ as above. In particular, p is a *Grothendieck fibration* if every x has a cartesian lift f such that $p(f) = x$; a *Grothendieck cleavage* is a cleavage where all the isomorphisms $\theta_{f,A}$ are identities.

Proposition 2.1.1. *Consider a $\mathcal{C}$-indexed category $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$: then $p : \mathcal{G}(\mathbb{D}) \to \mathcal{C}$ is a cloven Grothendieck fibration.*

Proof. Consider an arrow $y : Y \to X$ of $\mathcal{C}$ and an object (X, U) in the fibre of X. Then the arrow $(y, 1_{\mathbb{D}(y)(U)}) : (Y, \mathbb{D}(y)(U)) \to (X, U)$ is a cartesian lift of y: it obviously projects to y, and for any other $(h, b) : (Z, W) \to (X, U)$ of $\mathcal{G}(\mathbb{D})$ such that $h = yz$ for some z, then $(h, b) = (y, 1)(z, \varphi_{y,z}^{-1} b)$, and $(z, \varphi_{y,z}^{-1} b)$ is a unique lift for z. □

Remarks 2.1.3. (i) The 'evilness' of the definition of Grothendieck fibration, which explains resorting to Street fibrations, stems from the fact that the condition $p(f) = x$ on arrows forces the equality $p(B) = X$ on objects. We will provide in Corollary 2.1.8 an explicit proof later that Street fibrations are precisely Grothendieck fibrations up to equivalence.

(ii) Given a Grothendieck fibration $p : \mathcal{D} \to \mathcal{C}$, an arrow is said to be *vertical* if its image via p is an identity arrow. In the case of Street fibrations, we define the 'non-evil' version of this as follows: an arrow is *vertical* if its image via p is invertible.

(iii) The Grothendieck construction, and the symbol $\mathcal{G}$, will always mean for us the construction just defined, *even when applied to covariant pseudofunctors*. Indeed, it will happen later that we will consider covariant pseudofunctors $R : \mathcal{C} \to \mathbf{CAT}$: by seeing them as $\mathcal{C}^{op}$-indexed categories, we will perform the Grothendieck construction to obtain a category over $\mathcal{C}^{op}$,

$$p_R : \mathcal{G}(R) \to \mathcal{C}^{op}.$$

We stress this, because there is also a notion of *covariant* Grothendieck construction for covariant pseudofunctors (cf. Paragraph 2 of [31]), but it is *not* the same as applying the contravariant Grothendieck construction to R, seen as a $\mathcal{C}^{op}$-indexed category: instead, the covariant Grothendieck construction associates R with the category over $\mathcal{C}$

$$p_{R^V}^{op} : \mathcal{G}(R^V)^{op} \to \mathcal{C}.$$

In general, an *opfibration* is any functor p such that its opposite p^{op} is a fibration: therefore, the functor $p_{R^V}^{op}$ is usually called the Grothendieck opfibration associated to R.

We can now define a 2-category of fibred categories over $\mathcal{C}$:

Definition 2.1.5. We will define the *2-category of fibrations* over $\mathcal{C}$, denoted by $\mathbf{Fib}_{\mathcal{C}}$, as the sub-2-category of $\mathbf{CAT}/\mathcal{C}$ defined thus:

0-cells: they are Street fibrations $p : \mathcal{D} \to \mathcal{C}$;

1-cells: given two fibrations $p : \mathcal{D} \to \mathcal{C}$ and $q : \mathcal{E} \to \mathcal{C}$, a *morphism of fibrations* is a pair (F, φ), where $F : \mathcal{D} \to \mathcal{E}$ is a functor mapping cartesian arrows to cartesian arrows and φ is a natural isomorphism $q \circ F \overset{\sim}{\Longrightarrow} p$;

2-cells: given two fibrations $[p]$ and $[q]$ over $\mathcal{C}$ and two morphisms of fibrations (F, φ), $(G, \gamma) : [p] \to [q]$, a 2-cell $\alpha : (F, \varphi) \Rightarrow (G, \gamma)$ is given by a natural transformation $\alpha : F \Rightarrow G$ such that $\varphi = \gamma(q \circ \alpha)$.

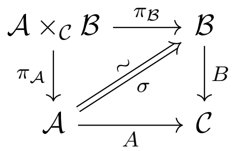
In particular, we will denote by $\mathbf{cFib}_{\mathcal{C}}$ the full sub-2-category of cloven fibrations. We will denote the (non full) sub-2-category of Grothendieck fibrations by $\mathbf{Fib}_{\mathcal{C}}^{Gr}$, and by $\mathcal{U} : \mathbf{Fib}_{\mathcal{C}}^{Gr} \to \mathbf{Fib}_{\mathcal{C}}$ the inclusion functor; analogously, $\mathbf{cFib}_{\mathcal{C}}^{Gr}$ will denote the full sub-2-category of $\mathbf{Fib}_{\mathcal{C}}^{Gr}$ of Grothendieck fibrations endowed with a Grothendieck cleavage.

Therefore, the Grothendieck construction provides a 2-functor

$$\mathcal{G} : \mathbf{Ind}_{\mathcal{C}} \to \mathbf{cFib}_{\mathcal{C}}^{Gr}.$$

Let us finally provide the notion of fibre of a Street fibration: to do so we shall exploit a 2-categorical notion of pullback.

Definition 2.1.6. Given two functors $A : \mathcal{A} \to \mathcal{C}$ and $B : \mathcal{B} \to \mathcal{C}$, their *strict pseudopullback* $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ (using the naming convention of [30]) is the category whose objects are triples (X, U, f), where X is an object of $\mathcal{A}$, U is an object of $\mathcal{B}$ and $f : A(X) \overset{\sim}{\to} B(U)$ in $\mathcal{C}$, while morphisms are pairs $(r, s) : (X, U, f) \to (Y, V, g)$ where $r : X \to Y$, $s : U \to V$ and $g \circ A(r) = B(s) \circ f$. There are two forgetful functors from $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ to $\mathcal{A}$ and $\mathcal{B}$ and a natural isomorphism σ as in the diagram



The strict pseudopullback $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ satisfies the following universal property: for every other functors $P : \mathcal{D} \to \mathcal{A}$ and $Q : \mathcal{D} \to \mathcal{B}$ and natural isomorphism $\tau : AP \overset{\sim}{\Longrightarrow} BQ$ there is a unique functor $H : \mathcal{D} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ such that $\pi_{\mathcal{A}} \circ H = P$, $\pi_{\mathcal{B}} \circ H = Q$ and $\sigma \circ H = \tau$. Moreover, for any pair of two such cones (P, Q, τ) and (P', Q', τ'), if there are natural transformations $\alpha : P \Rightarrow P'$ and $\beta : Q \Rightarrow Q'$ such that $\tau'(A \circ \alpha) = (B \circ \beta)\tau$ then there exists a unique natural transformation $\eta : H \Rightarrow H'$ such that $\pi_{\mathcal{A}} \circ \eta = \alpha$ and $\pi_{\mathcal{B}} \circ \eta = \beta$.

Definition 2.1.7. Consider a functor $p : \mathcal{D} \to \mathcal{C}$: the *essential fibre of p at X*, which we will simply call *fibre* and denote by $\mathbb{D}(X)$, is the strict pseudopullback

$$\begin{array}{ccc} \mathbb{D}(X) & \longrightarrow & \mathcal{D} \\ \big\downarrow & \overset{\sim}{\nearrow\!\!\!\nearrow} & \big\downarrow p \\ \mathbb{1} & \xrightarrow[e_X]{} & \mathcal{C} \end{array},$$

where the bottom functor is the constant functor with value X. In other words, $\mathbb{D}(X)$ is the category whose objects are pairs $(A, \alpha : X \xrightarrow{\sim} p(A))$, while given two objects (A, α) and (B, β) in $\mathbb{D}(X)$, an arrow $\gamma : A \to B$ in $\mathcal{D}$ satisfying the identity $p(\gamma) \circ \alpha = \beta$ yields an arrow $(A, \alpha) \to (B, \beta)$ in $\mathbb{D}(X)$ (which we still label $\gamma : (A, \alpha) \to (B, \beta)$).

Remark 2.1.4. Strict pseudopullbacks are stable under isomorphism of categories but not under equivalence, so one might be concerned with their compatibility with the theory of Street fibrations; however, they act as canonical (and manageable) representatives of pseudopullbacks: indeed, any category which is equivalent to the strict pseudopullback $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ will be a *pseudopullback* of A and B, i.e. the induced functor H above will be unique up to a unique 2-isomorphism, and pseudopullbacks are indeed stable under equivalence. For more details on this we refer again to [30].

Our first purpose is to show that cloven Street fibrations are equivalent to pseudofunctors, generalizing the well known result about Grothendieck fibrations: to do so, we will exploit the definition of fibre just provided to build from a fibration $p : \mathcal{D} \to \mathcal{C}$ a pseudofunctor $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$.

First of all, let us introduce two families of canonical arrows that exist for every cloven fibration, relating lifts of compositions of morphisms, which we will exploit in a moment. Consider $y : Y \to X$ and $x : X \to p(U)$ in $\mathcal{C}$: we want to compare the two lifts $\widehat{x}_U$ and $\widehat{xy}_U$. To do so, consider the arrow $\theta_{x,U} \circ y : Y \to p(\mathrm{dom}(\widehat{x}_U))$ and its cartesian lift $\widehat{\theta_{x,U} \circ y}_{\mathrm{dom}(\widehat{x}_U)}$: then the composite $\widehat{x}_U \widehat{\theta_{x,U} \circ y}_{\mathrm{dom}(\widehat{x}_U)}$ is still cartesian, and moreover it lifts xy, as the following commutative diagram shows:

$$\begin{array}{ccccc} Y & \xrightarrow{\qquad\qquad y \qquad\qquad} & X & & \\ {\scriptstyle \theta_{\theta_{x,U} \circ y, \mathrm{dom}(\widehat{x}_U)}} \big\downarrow \wr & & \wr \big\downarrow {\scriptstyle \theta_{x,U}} & \overset{x}{\searrow} & \\ p(\mathrm{dom}(\widehat{\theta_{x,U} \circ y}_{\mathrm{dom}(\widehat{x}_U)})) & \xrightarrow[p(\widehat{\theta_{x,U} \circ y}_{\mathrm{dom}(\widehat{x}_U)})]{} & p(\mathrm{dom}(\widehat{x}_U)) & \xrightarrow[p(\widehat{x}_U)]{} & p(U) \end{array}$$

Hence, there is a canonical isomorphism that compares $\widehat{x}_U \widehat{\theta_{x,U} \circ y}_{\mathrm{dom}(\widehat{x}_U)}$ with $\widehat{xy}_U$, which is a unique $\chi_{x,y,U} : \mathrm{dom}(\widehat{xy}_U) \xrightarrow{\sim} \mathrm{dom}(\widehat{\theta_{x,U} \circ y}_{\mathrm{dom}(\widehat{x}_U)})$ such that $\widehat{xy}_U = \widehat{x}_U \widehat{\theta_{x,U} \circ y}_{\mathrm{dom}(\widehat{x}_U)} \chi_{x,y,U}$ and $p(\chi_{x,y,U}) = \theta_{\theta_{x,U} \circ y, \mathrm{dom}(\widehat{x}_U)} \theta^{-1}_{xy,U}$.

Let us moreover introduce the notation

$$\lambda_{x,y,U} := \widehat{\theta_{x,U} \circ y}_{\mathrm{dom}(\widehat{x}_U)} \chi_{x,y,U} : \mathrm{dom}(\widehat{xy}_U) \to \mathrm{dom}(\widehat{x}_U).$$

The arrow $\lambda_{x,y,A}$ can be defined alternatively as the unique arrow satisfying $\widehat{x}_U \lambda_{x,y,U} = \widehat{xy}_U$ and $p(\lambda_{x,y,U}) = \theta_{x,U} \circ y \circ \theta^{-1}_{xy,U}$. Since both $\widehat{x}_U$ and $\widehat{xy}_U$ are cartesian, $\lambda_{x,y,U}$ is also cartesian. In particular, it is easy to verify that the following identities also hold: $\lambda_{x,1_X,A} = 1_{\mathrm{dom}(\widehat{x}_A)}$, and $\lambda_{x,y,A}\lambda_{xy,z,A} = \lambda_{x,yz,A}$.

Proposition 2.1.2. *There is a strict 2-functor* $\mathfrak{I} : \mathbf{cFib}_{\mathcal{C}} \to \mathbf{Ind}_{\mathcal{C}}$ *operating as follows:*

0-cells: consider a cloven fibration $p : \mathcal{D} \to \mathcal{C}$. *The* $\mathcal{C}$*-indexed category*

$$\mathfrak{I}(p) = \mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$$

is defined on objects by taking every X *in* $\mathcal{C}$ *to the fibre* $\mathbb{D}(X)$, *while for* $y : Y \to X$ *in* $\mathcal{C}$ *the functor* $\mathbb{D}(y) : \mathbb{D}(X) \to \mathbb{D}(Y)$ *is defined as*

$$\Big[(A,\alpha) \xrightarrow{\omega} (B,\beta)\Big] \mapsto \Big[(\mathrm{dom}(\widehat{\alpha y}_A), \theta_{\alpha y,A}) \xrightarrow{\mathbb{D}(y)(w)} (\mathrm{dom}(\widehat{\beta y}_B), \theta_{\beta y,B})\Big]$$

where $\mathbb{D}(y)(\omega) : \mathrm{dom}(\widehat{\alpha y}_A) \to \mathrm{dom}(\widehat{\beta y}_B)$ *is the unique arrow satisfying the identities* $\widehat{\beta y}_B \mathbb{D}(y)(\omega) = \omega \circ \widehat{\alpha y}_A$ *and* $p(\mathbb{D}(y)(\omega)) = \theta_{\beta y,B}\theta^{-1}_{\alpha y,A}$.

1-cells: A morphism of fibrations $(F,\varphi) : [p : \mathcal{D} \to \mathcal{C}] \to [q : \mathcal{E} \to \mathcal{C}]$ *produces a* $\mathcal{C}$*-indexed functor*

$$\mathfrak{I}^{(F,\varphi)} : \mathbb{D} \Rightarrow \mathbb{E}$$

built as follows. For every X *in* $\mathcal{C}$, $\mathfrak{I}^{(F,\varphi)}_X : \mathbb{D}(X) \to \mathbb{E}(X)$ *is defined as*

$$\mathfrak{I}^{(F,\varphi)}_X : \Big[(A,\alpha) \xrightarrow{\gamma} (B,\beta)\Big] \mapsto \Big[(F(A), \varphi_A^{-1}\alpha) \xrightarrow{F(\gamma)} (F(B), \varphi_B^{-1}\beta)\Big]$$

while for every arrow $y : Y \to X$ *in* $\mathcal{C}$ *the canonical isomorphism* $\mathfrak{I}^{(F,\varphi)}_y : \mathbb{E}(y)\mathfrak{I}^{(F,\varphi)}_X \overset{\sim}{\Longrightarrow} \mathfrak{I}^{(F,\varphi)}_Y \mathbb{D}(y)$ *is defined componentwise thus: the arrow*

$$\mathfrak{I}^{(F,\varphi)}_y(A,\alpha) : \mathrm{dom}(\widehat{\varphi_A^{-1}\alpha y}_{F(A)}) \to F(\mathrm{dom}(\widehat{\alpha y}_A))$$

is the unique satisfying the identities $F(\widehat{\alpha y}_A)\mathfrak{I}^{(F,\varphi)}_y(A,\alpha) = \widehat{\varphi_A^{-1}\alpha y}_{F(A)}$ *and* $q(\mathfrak{I}^{(F,\varphi)}_y(A,\alpha)) = \varphi^{-1}_{\mathrm{dom}(\widehat{\alpha y}_A)}\theta_{\alpha y,A}\theta^{-1}_{\varphi_A^{-1}\alpha y,F(A)}$.

2-cells: Given a 2-cell of fibrations $\xi : (F,\varphi) \Rightarrow (G,\gamma) : [p] \to [q]$, *the corresponding* $\mathcal{C}$*-indexed natural transformation*

$$\mathfrak{I}^{\xi} : \mathfrak{I}^{(F,\varphi)} \Rrightarrow \mathfrak{I}^{(G,\gamma)}$$

is defined componentwise as follows: for X *in* $\mathcal{C}$, *the natural transformation* $\mathfrak{I}^{\xi}_X : \mathfrak{I}^{(F,\varphi)}_X \Rightarrow \mathfrak{I}^{(G,\gamma)}_X$ *is defined componentwise, for every* (A,α) *in* $\mathbb{D}(X)$, *as* $\mathfrak{I}^{\xi}_X(A,\alpha) = \xi_A : (FA, \varphi_A^{-1}\alpha) \to (GA, \gamma_A^{-1}\alpha)$.

Proof. We only provide the definitions of the relevant structures, but omit verifications, for they are routine calculations: we stress that all equalities of arrows in the fibrations are verified by exploiting what we said in Remark 2.1.2(i).

The arrow $\mathbb{D}(y)(\gamma)$ is well defined, and it is easy to see that its uniqueness implies that $\mathbb{D}(y)$ is a functor. To see that $\mathbb{D}$ is a pseudofunctor, the following canonical natural isomorphisms must be considered: for every X in $\mathcal{C}$, we define $\varphi^{\mathbb{D}}_X : Id_{\mathbb{D}(X)} \overset{\sim}{\Longrightarrow} \mathbb{D}(1_X)$ componentwise by setting $\varphi^{\mathbb{D}}_X(A,\alpha)$ equal to

$$(A,\alpha) \xrightarrow{\widehat{\alpha}_A^{-1}} (\mathrm{dom}(\widehat{\alpha}_A), \theta_{\alpha,A});$$

for every $z : Z \to Y$ and $y : Y \to X$, we define $\varphi^{\mathbb{D}}_{y,z} : \mathbb{D}(z)\mathbb{D}(y) \overset{\sim}{\Longrightarrow} \mathbb{D}(yz)$ componentwise by setting $\varphi^{\mathbb{D}}_{y,z}(A,\alpha)$ equal to

$$(\mathrm{dom}(\widehat{\theta_{\alpha y,A} \circ z}_{\widehat{\alpha y}_A}), \theta_{\theta_{\alpha y,A}\circ z, \mathrm{dom}(\widehat{\alpha y}_A)}) \xrightarrow{\chi^{-1}_{\alpha y,z,A}} (\mathrm{dom}(\widehat{\alpha y z}_A), \theta_{\alpha yz,A}).$$

We have already shown that the arrows χ are isomorphisms, while $\widehat{\alpha}_A$ is an isomorphism since it lifts the isomorphism α. Here one needs to check that they are arrows of the fibres, that their components are natural and that the identities in the definition of a $\mathcal{C}$-indexed category are satisfied.

For the $\mathfrak{I}^{(F,\varphi)}$, notice that $\mathfrak{I}^{(F,\varphi)}_y(A,\alpha)$ is well defined because F preserves the cartesianity of arrows: indeed, it is easy to see that since $\widehat{\alpha y}_A$ lifts αy via p then $F(\widehat{\alpha y}_A)$ lifts $\varphi_A^{-1}\alpha y$ via q, and hence it is canonically isomorphic to $\widehat{\varphi_A^{-1}\alpha y}_{F(A)}$. The verification that $\mathfrak{I}^{(F,\varphi)}$ is a pseudonatural transformation is a matter of calculations.

Finally, the verification that the arrows $\mathfrak{I}^{\xi}_X(A,\alpha)$ provide a natural transformation is an explicit check, as is the verification that $\mathfrak{I}^{\xi}$ is a $\mathcal{C}$-indexed natural transformation. To conclude, one can check that for every $[p]$ and $[q]$ as above, we have defined a functor $\mathbf{cFib}_{\mathcal{C}}([p],[q]) \to \mathbf{Ind}_{\mathcal{C}}(\mathbb{D},\mathbb{E})$, that $\mathfrak{I}^{Id_{\mathcal{D}}} = Id_{\mathbb{D}}$ and that for $(F,\varphi) : [p] \to [q]$ and $(G,\gamma) : [q] \to [r]$, $\mathfrak{I}^{(G,\gamma)(F,\varphi)} = \mathfrak{I}^{(G,\gamma)}\mathfrak{I}^{(F,\varphi)}$. This makes $\mathfrak{I}$ into a strict 2-functor of 2-categories. □

Though a fibration may admit different cleavages, they are all equivalent, in the sense that they produce essentially the same pseudofunctor:

Proposition 2.1.3. *Given two cleavages for a fibration $p : \mathcal{D} \to \mathcal{C}$, the corresponding pseudofunctors obtained by applying $\mathfrak{I}$ are equivalent up to a pseudonatural isomorphism.*

Proof. Consider $f : X \to p(A)$ in $\mathcal{C}$: we will denote by $(\widehat{f}_A, \theta_{f,A})$ the cartesian lift for the first cleavage, and by $\mathbb{D}$ the pseudofunctor built from it by applying $\mathfrak{I}$, and similarly by $(\widetilde{f}_A, \tilde{\theta}_{f,A})$ the cartesian lift for the second cleavage, and by $\tilde{\mathbb{D}}$ the relative pseudofunctor.

The proof is straightforward. First of all, $\mathbb{D}$ and $\tilde{\mathbb{D}}$ behave in exactly the same way on the objects of $\mathcal{C}$, because the definition of fibre is independent from the cleavage, which only affects the construction of the transition morphisms. But recall that, given $y : Y \to X$ and (A, α) in $\mathbb{D}(X)$, we have

$$\mathbb{D}(y)(A, \alpha) = (\widehat{\alpha y}_A, \theta_{\alpha y, A}),\ \tilde{\mathbb{D}}(y)(A, \alpha) = (\widetilde{\alpha y}_A, \widetilde{\theta}_{\alpha y, A}) :$$

now, since $\widehat{\alpha y}_A$ and $\widetilde{\alpha y}_A$ are both cartesian lifts of αy, there is a unique well defined isomorphism $z_y(A, \alpha) : (\widehat{\alpha y}_A, \theta_{\alpha y, A}) \xrightarrow{\sim} (\widetilde{\alpha y}_A, \widetilde{\theta}_{\alpha y, A})$ in $\mathbb{D}(Y)$, which provides the components for a pseudonatural isomorphism from $\mathbb{D}$ to $\tilde{\mathbb{D}}$. □

We denote by

$$\mathfrak{I}^{Gr} : \mathbf{cFib}_{\mathcal{C}}^{Gr} \leftrightarrows \mathbf{Ind}_{\mathcal{C}} : \mathcal{G}$$

the equivalence between Grothendieck cloven fibrations and pseudofunctors, which appears for instance in [21, Section B1.3]. When working with Grothendieck fibrations, we obtain essentially the same $\mathcal{C}$-indexed category whether we apply $\mathfrak{I}$ or $\mathfrak{I}^{Gr}$:

Proposition 2.1.4. *Consider a cloven Grothendieck fibration $p : \mathcal{D} \to \mathcal{C}$: then $\mathbb{D}^{Gr} := \mathfrak{I}^{Gr}(p)$ and $\mathbb{D} := \mathfrak{I}(p)$ are equivalent pseudofunctors. This extends to 2-natural isomorphism between the 2-functors*

$$\begin{array}{ccc} \mathbf{cFib}_{\mathcal{C}} & \xrightarrow{\mathfrak{I}} & \mathbf{Ind}_{\mathcal{C}} \\ {\scriptstyle \mathcal{U}}\uparrow & \nearrow_{\mathfrak{I}^{Gr}} & \\ \mathbf{cFib}_{\mathcal{C}}^{Gr} & & \end{array} ,$$

where $\mathcal{U}$ is the obvious inclusion.

Proof. Since the cleavage for p is a Grothendieck cleavage, all the lifting isomorphisms θ are actually identities.

There is an obvious full and faithful functor $H_X : \mathbb{D}^{Gr}(X) \hookrightarrow \mathbb{D}(X)$ mapping A to $(A, 1_{p(A)})$ and $g : A \to B$ to $g : (A, 1_{p(A)}) \to (B, 1_{p(B)})$. There is also a functor $K_X : \mathbb{D}(X) \to \mathbb{D}^{Gr}(X)$ in the opposite direction, which acts as follows:

$$\left[(A, \alpha) \xrightarrow{g} (B, \beta)\right] \mapsto \left[\mathrm{dom}(\widehat{\alpha}_A) \xrightarrow{\mathbb{D}(1_X)(g)} \mathrm{dom}(\widehat{\beta}_B)\right],$$

where by definition $\mathbb{D}(1_X)(g)$ is the unique arrow satisfying

$$\widehat{\beta}_B \mathbb{D}(1_X)(g) = g \circ \widehat{\alpha}_A,\ p(\mathbb{D}(1_X)(g)) = 1_X.$$

A computation shows that each K_X is a quasi-inverse for H_X and that they are compatible with the transition functors $\mathbb{D}(y) : \mathbb{D}(X) \to \mathbb{D}(Y)$ and $\mathbb{D}^{Gr}(y) : \mathbb{D}^{Gr}(X) \to \mathbb{D}^{Gr}(Y)$. In fact, the composites $\mathbb{D}(y)K_X$ and $K_Y \mathbb{D}^{Gr}(y)$ are exactly equal. This shows that $\mathbb{D}^{Gr} \cong \mathbb{D}$.

To prove that this extends to a pseudoequivalence of 2-functors, we build for every morphism $F : [p] \to [q]$ in $\mathbf{cFib}_{\mathcal{C}}^{Gr}$ an invertible modification

$$\begin{array}{ccc} \mathbb{D}^{Gr} & \xrightarrow{K^{[p]}} & \mathbb{D} \\ {\scriptstyle \mathfrak{I}^{Gr,F}}\big\downarrow & \overset{\sim}{\underset{\kappa^F}{\Lleftarrow}} & \big\downarrow{\scriptstyle \mathfrak{I}^{(F,1)}} \\ \mathbb{E}^{Gr} & \xrightarrow[K^{[q]}]{} & \mathbb{E} \end{array} ,$$

where $(F, 1)$ is just $\mathcal{U}(F)$. Now, a computation shows that the two pseudonatural transformations $K^{[q]}\mathfrak{I}^{Gr,F}$ and $\mathfrak{I}^{(F,1)}K^{[p]}$ are actually equal, and thus κ^F can be defined as the identity modification. This concludes the proof. $\square$

Corollary 2.1.5. *The two functors*

$$Id_{\mathbf{Ind}_{\mathcal{C}}},\ \mathfrak{I}\mathcal{U}\mathcal{G} : \mathbf{Ind}_{\mathcal{C}} \to \mathbf{Ind}_{\mathcal{C}}$$

are equivalent.

Proof. It is immediate: $Id_{\mathbf{Ind}_{\mathcal{C}}} \cong \mathfrak{I}^{Gr}\mathcal{G}$ is a standard result, and we have just shown that $\mathfrak{I}^{Gr} \cong \mathfrak{I}\mathcal{U}$, whence $Id_{\mathbf{Ind}_{\mathcal{C}}} \cong \mathfrak{I}\mathcal{U}\mathcal{G}$. $\square$

The converse is also true, as the following result shows:

Proposition 2.1.6. *A cloven fibration $p : \mathcal{D} \to \mathcal{C}$ is equivalent to the Grothendieck fibration $\pi : \mathcal{G}(\mathbb{D}) \to \mathcal{C}$. This extends to an equivalence of pseudofunctors*

$$Id_{\mathbf{cFib}_{\mathcal{C}}},\ \mathcal{U}\mathcal{G}\mathfrak{I} : \mathbf{cFib}_{\mathcal{C}} \to \mathbf{cFib}_{\mathcal{C}},$$

i.e. there is an invertible 2-natural transformation $\mathcal{U}\mathcal{G}\mathfrak{I} \overset{\sim}{\Longrightarrow} Id_{\mathbf{cFib}_{\mathcal{C}}}$.

Proof. We remark that objects of $\mathcal{G}(\mathbb{D})$ are couples $(X, (A, \alpha))$, where X is an object of $\mathcal{C}$ and (A, α) an object of $\mathbb{D}(X)$, i.e. $\alpha : X \overset{\sim}{\to} p(A)$ in $\mathcal{C}$.

We begin by defining a functor $T : \mathcal{D} \to \mathcal{G}(\mathbb{D})$ as

$$\left[A \xrightarrow{g} B\right] \mapsto \left[(p(A), (A, 1_{p(A)})) \xrightarrow{(p(g);\bar{g})} (p(B), (B, 1_{p(B)}))\right],$$

where $\bar{g} : A \to \mathrm{dom}(\widehat{p(g)}_B)$ is the unique arrow such that $\widehat{p(g)}_B\bar{g} = g$ and $p(\bar{g}) = \theta_{p(g),B}$: the verification that it is a functor is based on the uniqueness of the arrow $\bar{g}$ (notice in particular that $\overline{1_A} = \widehat{1_{p(A)}}_A^{-1} = \varphi_A(A, 1_{p(A)})$, which implies the preservation of identities). If we suppose that $g : A \to B$ is cartesian, it is also a lift for $p(g)$. This implies that $\bar{g}$ is an isomorphism and hence $(p(g), \bar{g})$ is a cartesian arrow of $\mathcal{G}(\mathbb{D})$. Moreover, it is immediate to see that $\pi T = p$, and hence we have a morphism of fibrations $(T, 1) : [p] \to [\pi]$.

We define its quasi-inverse $S : \mathcal{G}(\mathbb{D}) \to \mathcal{D}$ as

$$\left[(Y, (B, \beta)) \xrightarrow{(y,g)} (X, (A, \alpha))\right] \mapsto \left[B \xrightarrow{\widehat{\alpha y}_A g} A\right].$$

One can verify that it is functorial, and also that it maps cartesian arrows to cartesian arrows: indeed, (y, g) is cartesian if g is an isomorphism, which means that its image $\widehat{\alpha y}_A g$ is also cartesian (as $\widehat{\alpha y}_A$ is). Finally, there is a natural isomorphism $\sigma : ps \overset{\sim}{\Longrightarrow} \pi$ defined componentwise as

$$\left[pS(X,(A,\alpha)) \xrightarrow{\sigma(X,(A,\alpha))} \pi(X,(A,\alpha)) \right] := \left[p(A) \xrightarrow{\alpha^{-1}} X \right].$$

The two compositions $(S, \sigma)(T, 1)$ and $(T, 1)(S, \sigma)$ are indeed the two components of an equivalence of fibrations.

Now consider again the morphisms $(S, \sigma) : \mathcal{G}(\mathbb{D}) \to [p]$: we add a superscript $S^{[p]}$, $\sigma^{[p]}$ to specify that they stem from the fibration p. To extend these data into an invertible pseudonatural transformation we also need for each $(F, \varphi) : [p] \to [q]$ a 2-isomorphism

$$\begin{array}{ccc} \mathcal{G}(\mathbb{D}) & \xrightarrow{(S^{[p]},\sigma^{[p]})} & [p] \\ {\scriptstyle \mathcal{G}\mathfrak{I}^{(F,\varphi)}}\downarrow & \overset{\sim}{\swarrow} {\scriptstyle \zeta^{(F,\varphi)}} & \downarrow{\scriptstyle (F,\varphi)} \\ \mathcal{G}(\mathbb{E}) & \xrightarrow[(S^{[q]},\sigma^{[q]})]{} & [q] \end{array} :$$

but the two composites $(F, \varphi)(S^{[p]}, \sigma^{[p]})$ and $(S^{[q]}, \sigma^{[q]})\mathcal{G}\mathfrak{I}^{(F,\varphi)}$ are equal, so ζ can be set as the identity, and the pseudonaturality axioms are quickly verified. □

Combining the previous results, we obtain the equivalence we wanted:

Corollary 2.1.7. *The two 2-functors*

$$\mathfrak{I} : \mathbf{cFib}_{\mathcal{C}} \to \mathbf{Ind}_{\mathcal{C}},\ \mathcal{U}\mathcal{G} : \mathbf{Ind}_{\mathcal{C}} \to \mathbf{cFib}_{\mathcal{C}}$$

form an equivalence of 2-categories.

As a corollary we obtain a result mentioned multiple times, namely that Street fibrations are Grothendieck fibrations up to equivalence:

Corollary 2.1.8. *A functor $p : \mathcal{D} \to \mathcal{C}$ is a fibration if and only there are a Grothendieck fibration $q : \mathcal{E} \to \mathcal{C}$, an equivalence of categories $F : \mathcal{D} \xrightarrow{\sim} \mathcal{E}$ and a natural isomorphism φ as in the diagram:*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow[\sim]{F} & \mathcal{E} \\ & {\scriptstyle p}\searrow \ \overset{\sim}{\swarrow}{\scriptstyle \varphi} & \downarrow{\scriptstyle q} \\ & & \mathcal{C} \end{array}$$

Proof. We have shown previously that if p is a fibration then it is equivalent to $\mathcal{G}(\mathbb{D}) \to \mathcal{C}$, which is Grothendieck. Conversely, suppose given F and φ: without loss of generality we may assume that F is the left adjoint of an adjoint equivalence, so that the identities $G(\varepsilon) = \eta_G^{-1}$ and $F(\eta) = \varepsilon_F^{-1}$ hold for the unit and counit of the adjunction. Consider an arrow $x : X \to p(D)$ in $\mathcal{C}$: then the composite $\varphi_D^{-1}x : X \to qF(D)$ admits a cartesian lift $f : Y \to F(D)$ through q. A check shows that the arrow $\hat{x} := \eta_D^{-1}G(f) : G(Y) \to D$ is still cartesian, and that $x = p(\hat{x})\varphi_{GY}q(\varepsilon_Y^{-1})$, making $\hat{x}$ into a lift for x. □

To conclude, let us consider split Street fibrations. We recall that a Grothendieck cleavage is said to be a *splitting* if it is compatible with identities and compositions, or equivalently if the corresponding indexed category is a strict functor. By suitably generalizing the notion of splitting, we obtain an analogous result for Street fibrations:

Proposition 2.1.9. *Given a cloven fibration $p : \mathcal{D} \to \mathcal{C}$, the following are equivalent:*

- *The corresponding $\mathcal{C}$-indexed category $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ is a strict functor (i.e. all natural transformations $\varphi_X^{\mathbb{D}}$ and $\varphi_{y,z}^{\mathbb{D}}$ are identities);*
- *the cleavage for p is a* splitting*, meaning that the following conditions are satisfied:*
  - *(a) for every $\alpha : X \xrightarrow{\sim} p(A)$, $\widehat{\alpha}_A = 1_A$;*
  - *(b) for every $z : Z \to Y$, $y : Y \to p(A)$, $\widehat{yz}_A = \widehat{y}_A\widehat{\theta_{y,A}z}_{\mathrm{dom}(\widehat{y}_A)}$ and $\theta_{yz,A} = \theta_{\theta_{y,A}z,\mathrm{dom}(\widehat{y}_A)}$.*

This restricts the equivalence $\mathbf{cFib}_\mathcal{C} \simeq \mathbf{Ind}_\mathcal{C}$ to an equivalence $\mathbf{sFib}_\mathcal{C} \simeq [\mathcal{C}^{op}, \mathbf{CAT}]$, where $\mathbf{sFib}_\mathcal{C}$ denotes the full subcategory of $\mathbf{cFib}_\mathcal{C}$ of split fibrations.

Proof. This is immediate recalling the definition of the natural isomorphisms φ_X and $\varphi_{y,z}$ above. Since $\varphi_X^{\mathbb{D}}(A, \alpha) := \widehat{\alpha}_A^{-1}$, $\varphi^{\mathbb{D}}$ is an identity if and only if the cleavage lifts of isomorphisms are identity arrows. For the second condition, $\varphi_{y,z}^{\mathbb{D}}(A, \alpha) = \chi_{\alpha y,z,A}^{-1}$. By the definition of χ, it is the identity if and only if for all choices of α, y and z it holds that $\widehat{\alpha yz}_A = \widehat{\alpha y}_A\widehat{\theta_{\alpha y,A}z}_{\mathrm{dom}(\widehat{\alpha y}_A)}$ and $\theta_{\alpha yz,A} = \theta_{\theta_{\alpha y,A}z,\mathrm{dom}\widehat{\alpha y}_A}$, which is evidently equivalent to the condition (b) stated above. □

Finally, we present one last capital result in the theory of fibrations, the fibred form of Yoneda's lemma:

Proposition 2.1.10 (Fibred Yoneda lemma). *Given a cloven Street fibration $p : \mathcal{D} \to \mathcal{C}$, there is an equivalence of categories*

$$\mathbf{Fib}_\mathcal{C}(\mathcal{C}/X, \mathcal{D}) \simeq \mathbb{D}(X)$$

which is pseudonatural in both components, i.e. for any $y : Y \to X$ in $\mathcal{C}$ and $(F, \varphi) : \mathcal{D} \to \mathcal{E}$ in $\mathbf{cFib}_{\mathcal{C}}$ the two squares

$$\begin{array}{ccc} \mathbb{D}(X) & \xrightarrow{\sim} & \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D}) \\ {\scriptstyle \mathbb{D}(y)}\downarrow & & \downarrow{\scriptstyle -\circ \int y} \\ \mathbb{D}(Y) & \xrightarrow{\sim} & \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/Y, \mathcal{D}) \end{array} \qquad \begin{array}{ccc} \mathbb{D}(X) & \xrightarrow{\sim} & \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D}) \\ {\scriptstyle F_{|\mathbb{D}(X)}}\downarrow & & \downarrow{\scriptstyle (F,\varphi)\circ -} \\ \mathbb{E}(X) & \xrightarrow{\sim} & \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{E}) \end{array}$$

commute up to canonical natural isomorphisms.

Proof. Let us sketch the proof, omitting all computations.

One direction of the equivalence is easy: starting with a morphism of fibrations $(F, \varphi) : \mathcal{C}/X \to \mathcal{D}$, we consider the pair $(F([1_X]), \varphi^{-1}_{[1_X]} : X \xrightarrow{\sim} pF([1_X]))$, which is an object of $\mathbb{D}(X)$; for a 2-cell $\alpha : (F, \varphi) \Rightarrow (G, \gamma)$, we set $\alpha_{[1_X]} : (F([1_X]), \varphi^{-1}_{[1_x]}) \to (G([1_X]), \gamma^{-1}_{[1_X]})$ as its image. This definition is evidently functorial, hence we have a functor $\Phi : \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D}) \to \mathbb{D}(X)$.

The building of the quasi-inverse exploits the cleavage. Consider an object $(A, \alpha : X \xrightarrow{\sim} p(A))$ in $\mathbb{D}(X)$, then we define a morphism of fibrations $(F_{(A,\alpha)}, \varphi_{(A,\alpha)}) : \mathcal{C}/X \to \mathcal{D}$ as follows:

- we define $F_{(A,\alpha)}([y]) := \mathrm{dom}(\widehat{\alpha y}_A)$;
- for $z : [yz] \to [y]$ we set $F_{(A,\alpha)}(z) := \lambda_{\alpha y,z,A}$.

Since arrows of the form λ are cartesian, this functor maps cartesian arrows to cartesian arrows. The natural isomorphism $\varphi_{(A,\alpha)} : pF \xRightarrow{\sim} p_X$ is defined componentwise as $\varphi_{(A,\alpha)}([y]) = \theta^{-1}_{\alpha y,A} : pF([y]) \xrightarrow{\sim} Y$ and it is indeed natural.

To define our functor on arrows, we consider $\gamma : (A, \alpha) \to (B, \beta)$ in $\mathbb{D}(X)$, i.e. an arrow $\gamma : A \to B$ of $\mathcal{D}$ such that $p(\gamma) = \beta \circ \alpha^{-1}$. Calling (F, φ) and (F', φ') the images of (A, α) and (B, β) for brevity, we want to build a 2-cell F_γ between them. To do so, it is sufficient to see that for any $y : Y \to X$ the identity $p(\gamma \circ \widehat{\alpha y}_A) = p(\widehat{\beta y}_B)\theta_{\beta y,B}\theta^{-1}_{\alpha y,A}$ holds: then by cartesianity there is a unique $F_\gamma([y]) : \mathrm{dom}(\widehat{\alpha y}_A) \to \mathrm{dom}(\widehat{\beta y}_B)$ such that $\widehat{\beta y}_B F_\gamma([y]) = \gamma \circ \widehat{\alpha y}_A$ and $p(F_\gamma([y])) = \theta_{\beta y,B}\theta^{-1}_{\alpha y,A}$. The components $F_\gamma([y])$ define a natural transformation $F \Rightarrow F'$ (it follows from the definition of the arrows λ), and the identity $\varphi = \varphi'(p \circ F_\gamma)$ is immediately verified: thus F_γ is a 2-cell of $\mathbf{Fib}_{\mathcal{C}}$. The uniqueness in the definition of the arrows $F_\gamma([y])$ assures us that this construction is functorial, hence we have a functor $\Psi : \mathbb{D}(X) \to \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D})$.

Now to show that the two functors are quasi-inverses. Starting from (A, α) in $\mathbb{D}(X)$, consider $\Phi\Psi(A, \alpha) = (F_{(A,\alpha)}([1_X]), \varphi_{(A,\alpha)}([1_X])^{-1})$ in $\mathbb{D}(X)$. Notice that the arrow $\widehat{\alpha}_A : F([1_X]) \to A$ is an isomorphism, since it is a cartesian lift for α, which is invertible. Since moreover $\alpha = p(\hat{\alpha}_A)\theta_{\alpha,A}$, it is an isomorphism $\hat{\alpha}_A : (\mathrm{dom}(\widehat{\alpha}_A), \theta_{\alpha,A}) \to (A, \alpha)$. It is easy to check that it is also natural in (A, α), and hence we have a natural isomorphism $\Phi\Psi \overset{\sim}{\Longrightarrow} Id_{\mathbb{D}(X)}$.

Conversely, start from a morphism of fibrations $(F, \varphi) : \mathcal{C}/X \to \mathcal{D}$, consider $\Phi(F, \varphi) = (F([1_X]), \varphi^{-1}_{[1_X]})$ in $\mathbb{D}(X)$ and then $\Psi\Phi(F, \varphi) = (G, \gamma) : \mathcal{C}/X \to \mathcal{D}$. Notice that for any $y : [y] \to [1_X]$ in $\mathcal{C}/X$, the arrow $F(y) : F([y]) \to F([1_X])$ is cartesian since F is a morphism of fibrations; moreover, a computation shows that it lifts αy: thus there is a unique isomorphism $\kappa_{[y]} : F[y] \overset{\sim}{\longrightarrow} \mathrm{dom}(\widehat{\alpha y}_A)$, since both lift the same arrow. It is immediate to check that the $\kappa_{[y]}$ are the components of a natural transformation $\kappa : F \Rightarrow G$ and that it is in fact an invertible 2-cell $\kappa : (F, \varphi) \Rightarrow (G, \gamma)$. The naturality in (F, φ) is also a straightforward check, so we conclude that there is a natural isomorphism $\Psi\Phi \overset{\sim}{\Longrightarrow} Id_{\mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D})}$.

Finally, it is lengthy but straightforward to verify that there exist natural isomorphisms making the squares above commute, and this concludes the proof. □

Corollary 2.1.11. *Every Street fibration is equivalent to a split Street fibration.*

Proof. The fibred Yoneda lemma states precisely that $\mathbb{D} \simeq \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/-, \mathcal{D})$ and since $\mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/-, \mathcal{D})$ is a functor it corresponds to a split Street fibration. □

2.2 Localizations of fibrations

As we already mentioned in the introduction, the operation of base change for fibrations (see Chapter 4) can be described using localizations, so it is meaningful to collect some results about this topic. All these come from Section 2.8 of [8].

We begin by proving that the category of fibrations is closed under localization with respect to vertical arrows.

Proposition 2.2.1. *Let $p : \mathcal{D} \to \mathcal{C}$ be a fibration, W a class of arrows of $\mathcal{D}$ and j_W the canonical functor $\mathcal{D} \to \mathcal{D}[W^{-1}]$. Then the following conditions are equivalent:*

(i) There is a fibration $p_W : \mathcal{D}[W^{-1}] \to \mathcal{C}$ such that j_W is a morphism of fibrations $p \to p_W$:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{j_W} & \mathcal{D}[W^{-1}] \\ {\scriptstyle p}\downarrow & \swarrow {\scriptstyle p_W} & \\ \mathcal{C} & & \end{array}$$

(ii) Every arrow in W is vertical with respect to p and, denoting by p_W the unique functor (determined by the universal property of the localization)

$\mathcal{D}[W^{-1}] \to \mathcal{C}$ such that $p_W \circ j_W = p$, j_W sends arrows which are cartesian with respect to p to arrows which are cartesian with respect to p_W.

Proof. (i) ⇒ (ii) Let f be an arrow in W; then $j_W(f)$ is an isomorphism by definition of the localization j_W, so $p_W(j_W(f))$ is also an isomorphism by functoriality; but $p_W(j_W(f)) = p(f)$, so $p(f)$ is an isomorphism, that is, f is vertical. The fact that j_W sends cartesian arrows to cartesian arrow follows from the fact that j_W is a morphism of fibrations.

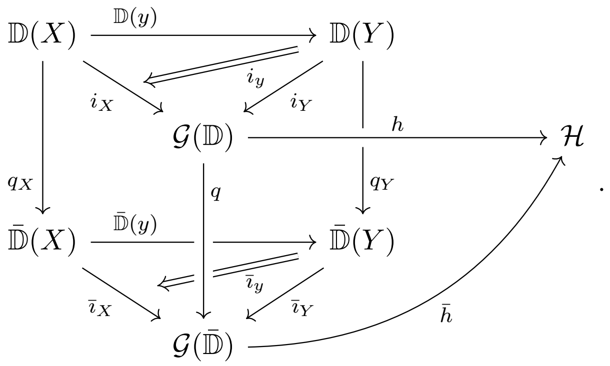
(ii) ⇒ (i) Since every arrow in W is vertical with respect to p, we have a functor $\mathcal{D}[W^{-1}] \to \mathcal{C}$ such that $p_W \circ j_W = p$. It remains to show that this functor is a fibration. But this follows immediately from the fact that p is a fibration by using the fact that j_W sends cartesian arrows to cartesian arrows. Indeed, the functor j_W is essentially surjective by the construction of $\mathcal{D}[W^{-1}]$, and given an arrow $c \to p_W(j_W(d)) = p(d)$ in $\mathcal{C}$ and a cartesian lift $g : d' \to d$ of it with respect to p, the arrow $j_W(g)$ is clearly a cartesian lift of it with respect to p_W, by the equality $p_W \circ j_W = p$. □

In particular, when working with a pseudofunctor $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{Cat}$ and its Grothendieck fibration $\mathcal{G}(\mathbb{D})$, the request that $\mathcal{G}(\mathbb{D})$ is localized with respect to a family W of vertical arrows can be understood as a localization which already takes place at the level of fibres. In this case, one can verify that computing the fibration $\mathcal{G}(\mathbb{D})$ and then localizing with respect to W is the same as performing a fibrewise localization of $\mathbb{D}$ and then moving to the corresponding fibration:

Lemma 2.2.2. *Consider a pseudofunctor $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{Cat}$ and suppose given for each object X in $\mathcal{C}$ a class of arrows S_X of $\mathbb{D}(X)$ such that each transition morphism $\mathbb{D}(y) : \mathbb{D}(X) \to \mathbb{D}(Y)$ restricts to a transition morphism $\mathbb{D}(X)[S_X^{-1}] \to \mathbb{D}(X)[S_Y^{-1}]$. If $\bar{\mathbb{D}} : \mathcal{C}^{op} \to \mathbf{Cat}$ is the pseudofunctor obtained by the pointwise localization of $\mathbb{D}$, then $\mathcal{G}(\bar{\mathbb{D}})$ is a localization of $\mathcal{G}(\mathbb{D})$ with respect to all arrows $(y, a) : (Y, V) \to (X, U)$ such that y is invertible and a belongs to S_Y.*

Proof. We set the notations as in the following diagram (where as usual

$y : Y \to X$ in $\mathcal{C}$):



.

First of all, we know by Proposition 3.3.1 that $\mathrm{colim}_{lax}(\mathbb{D}) \simeq \mathcal{G}(\mathbb{D})$, with i_X and i_y the components of its colimit cocone. Notice that by hypothesis all the squares such as that in the background of the diagram commute up to isomorphism, and thus there exists an essentially unique functor q simply by the universal property of colimits. Our aim is to show that q is in fact the localization of $\mathcal{G}(\mathbb{D})$ with respect to the class of arrows $(y, a) : (Y, V) \to (X, U)$ such that y is invertible and $a \in S_Y$. Notice that it is actually enough to show that q localizes with respect to all vertical arrows $(1, a) : (X, U) \to (X, U')$ with $a \in S_X$ for some X. To show this, consider a functor $h : \mathcal{G}(\mathbb{D}) \to \mathcal{H}$ such that every arrow $(1, a) : (X, U) \to (X, U')$ with $a \in S_X$ is inverted: this means that the composite functor hi_X factors through $\bar{\mathbb{D}}(X)$. If h inverts the vertical arrows in $i_X(S_X)$ for each X in $\mathcal{C}$, we can therefore build a lax cocone under the diagram $\bar{\mathbb{D}}$, and thus an essentially unique functor $\bar{h} : \mathcal{G}(\bar{\mathbb{D}}) \to \mathcal{H}$ which factors h. This entails that q presents $\mathcal{G}(\bar{\mathbb{D}})$ as the localization of $\mathcal{G}(\mathbb{D})$ we desired. □

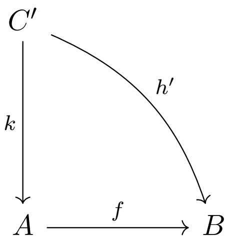
Localizations are conveniently calculated when the class W of morphisms to be inverted admits a calculus of fractions. The following proposition shows that if W admits a right calculus of fractions then p_W is automatically a fibration and j_W a morphisms of fibrations from p to p_W:

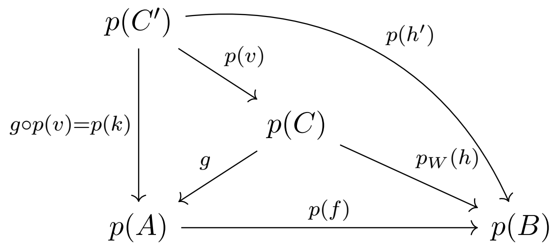
Proposition 2.2.3. *Let $p : \mathcal{D} \to \mathcal{C}$ be a fibration and W a class of vertical arrows of $\mathcal{D}$ admitting a right calculus of fractions. Then p_W is a fibration and j_W yields a morphism of fibrations from p to p_W.*

Proof. By Proposition 2.2.1, we only have to show that the canonical functor $j_W : \mathcal{D} \to \mathcal{D}[W^{-1}]$ sends p-cartesian arrows to p_W-cartesian arrows.

Let $f : A \to B$ be a p-cartesian arrow in $\mathcal{D}$. We want to show that $j_W(f) : j_W(A) \to j_W(B)$ is p_W-cartesian. For this, we suppose that $g : p_W(j_W(C)) \to p_W(j_W(A))$ is an arrow in $\mathcal{C}$ and $h : j_W(C) \to j_W(B)$ is an arrow in $\mathcal{D}[W^{-1}]$ such that $p(f) \circ g = p_W(h)$. We want to show the existence and uniqueness of an arrow $r : j_W(C) \to j_W(A)$ such that $g = p_W(r)$ and $j_W(f) \circ r = h$. Let us start with the existence proof.

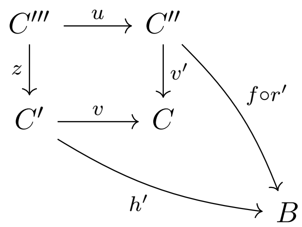
Let us represent h as $j_W(h') \circ j_W(v)^{-1}$, where h' is an arrow $C' \to B$ in $\mathcal{D}$ and v is an arrow $C' \to C$ in W. Since $p(f) \circ g = p_W(h)$, composing both sides with $p(v)$ we get $p(f) \circ (g \circ p(v)) = p(h')$, whence, since f is p-cartesian, there is an arrow (in fact, a unique one) $k : C' \to A$ in $\mathcal{D}$ such that $f \circ k = h'$ and $g \circ p(v) = p(k)$:





Therefore the arrow $j_W(k) \circ (j_W(v))^{-1}$ satisfies the desired property.

It now remains to prove uniqueness. We shall do so by showing that any arrow $r : j_W(C) \to j_W(A)$ such that $g = p_W(r)$ and $j_W(f) \circ r = h$ is necessarily equal to $j_W(k) \circ (j_W(v))^{-1}$ in $\mathcal{D}[W^{-1}]$. Let us represent r as $j_W(r') \circ j_W(v')^{-1}$, where r' is an arrow $C'' \to A$ in $\mathcal{D}$ and v is an arrow $C'' \to C$ in W. The equality $j_W(f) \circ r = h$ in $\mathcal{D}[W^{-1}]$ implies, by the construction of the localization at a class admitting a right calculus of fractions, that we can find arrows $u : C''' \to C$ in $\mathcal{D}$ and $z : C''' \to C'$ such that $v' \circ u = v \circ z \in W$ and the following diagram commutes:

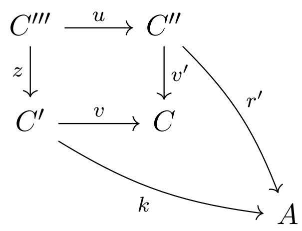


That is, $f \circ r' \circ u = h' \circ z$. Now, consider the arrows $r' \circ u$ and $k \circ z$. We have

$$\begin{aligned} p(r' \circ u) &= p_W(j_W(r') \circ j_W(u)) = p_W(r \circ j_W(v') \circ j_W(u)) \\ &= p_W(r \circ j_W(v) \circ j_W(z)) = p_W(r) \circ p_W(j_W(v)) \circ p_W(j_W(z)) \\ &= g \circ p(v) \circ p(z) = p(k) \circ p(z) = p(k \circ z). \end{aligned}$$

Also, as remarked above, $f \circ (r' \circ u) = h' \circ z = f \circ k \circ z$. Therefore, as f is p-cartesian, we can conclude that $r' \circ u = k \circ z$. This in turn implies that

$r = j_W(k) \circ (j_W(v))^{-1}$ in $\mathcal{D}[W^{-1}]$, since the following diagram commutes:



□

Remark 2.2.1. Notice that the family $W = \bigcup_{X \in \mathcal{C}} S_X$ in $\mathcal{G}(\mathbb{D})$ satisfies condition (ii) in Proposition 2.2.1.

2.3 Stacks

Consider a site $(\mathcal{C}, J)$ and a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$: in the first chapter we recalled that P is called a J-separated presheaf (resp. J-sheaf) if for every object X of $\mathcal{C}$ and every J-covering sieve $m_S : S \rightarrowtail \text{よ}(X)$, the map

$$[\mathcal{C}^{op}, \mathbf{Set}](\text{よ}(X), P) \xrightarrow{-\circ m_S} [\mathcal{C}^{op}, \mathbf{Set}](S, P)$$

is injective (resp. a bijection). The extension of this definition to the fibrational context is immediate and provides the fundamental notion of *stack*.

Definition 2.3.1. Consider a site $(\mathcal{C}, J)$ and a fibration $p : \mathcal{D} \to \mathcal{C}$: then p is a *J-prestack* (resp. *J-stack*) if for every J-sieve $m_S : S \rightarrowtail \text{よ}(X)$ the functor

$$\mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D}) \xhookrightarrow{-\circ \int m_S} \mathbf{Fib}_{\mathcal{C}}(\textstyle\int S, \mathcal{D})$$

is full and faithful (resp. an equivalence).

The notion of stack is in fact an expansion of that of sheaf, as established by the next result.

Proposition 2.3.1 [39, Proposition 4.9]. *Given a site $(\mathcal{C}, J)$, a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ is J-separated (resp. J-sheaf) if and only if the fibration $\int P \to \mathcal{C}$ is a J-prestack (resp. J-stack).*

In Section 2.5 we will study in more detail the connection between sheaves and stacks, which can be formulated in terms of a truncation-inclusion adjunction.

Similarly to sheaves, stacks can be defined in terms of matching families and amalgamations, a definition which is usually more 'operatively' useful. In order to do so, we restrict to cloven fibrations and translate the definition above into the $\mathcal{C}$-indexed language: given a site $(\mathcal{C}, J)$ and a $\mathcal{C}$-indexed category $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$, then $\mathbb{D}$ is a J-prestack (resp. J-stack) if and only if for every sieve $m_S : S \rightarrowtail よ(X)$ the functor

$$\mathbf{Ind}_{\mathcal{C}}(よ(X), \mathbb{D}) \xrightarrow{-\circ m_S} \mathbf{Ind}_{\mathcal{C}}(S, \mathbb{D})$$

is full and faithful (resp. an equivalence), where both $よ(X)$ and S are interpreted as discrete $\mathcal{C}$-indexed categories. This definition can be made explicit by specifying what a pseudonatural transformation $\alpha : S \Rightarrow \mathbb{D}$ is. It consists of an object $U_y \in \mathbb{D}(Y)$ for every $y \in S(Y)$, and of a family of isomorphisms $\alpha_{y,z} : \mathbb{D}(z)(U_y) \simeq U_{yz}$ of $\mathbb{D}(Z)$ for every $y \in S(Y)$ and every $z : Z \to Y$, such that the following identities hold for any $w : W \to Z$, $z : Z \to Y$, $y \in S(Y)$ in $\mathcal{C}$:

$$\alpha_{y,1_Y} = \varphi^{\mathbb{D}}_Y(U_y)^{-1} : \mathbb{D}(1_Y)(U_y) \to U_y$$
$$\alpha_{y,zw} \circ \varphi^{\mathbb{D}}_{z,w}(U_y) = \alpha_{yz,w} \circ \mathbb{D}(w)(\alpha_{y,z}) : \mathbb{D}(w)\mathbb{D}(z)(U_y) \to U_{yzw}$$

The collection $\alpha = (U_y, \alpha_{y,z})_{y\in S}$ is called a *descent datum* for $\mathbb{D}$ and S. It is the generalization of a matching family, where the pseudonaturality allows for some elasticity: for each $y : Y \to X$ in S we have an object U_y in $\mathbb{D}(Y)$, and these objects are mutually compatible *up to canonical isomorphisms*.

The definition of a morphism in the category of descent data, i.e. an arrow $\xi : (U_y, \alpha_{y,z})_{y\in S} \to (V_y, \beta_{y,z})_{y\in S}$, can be retrieved analogously by expanding the definition of a modification $\xi : \alpha \Rrightarrow \beta$. It is given by an arrow $\xi_y : U_y \to V_y$ in $\mathbb{D}(Y)$ for each $y \in S(Y)$, subject to the condition $\beta_{y,z} \circ \mathbb{D}(Z)(\xi_y) = \xi_{yz} \circ \alpha_{y,z}$. We can therefore consider the category of descent data for $\mathbb{D}$ and S, which we will denote by $\mathbb{D}(S)$.

Now, recall that by the fibred Yoneda lemma for Grothendieck fibrations, there is a pseudonatural equivalence $\mathbf{Ind}_{\mathcal{C}}(よ(X), \mathbb{D}) \simeq \mathbb{D}(X)$: then the functor $(- \circ m_S)$ can be expressed as a functor $L_S : \mathbb{D}(X) \to \mathbb{D}(S)$ acting on objects as follows:

$$U \in \mathbb{D}(X) \mapsto (\mathbb{D}(y)(X), \varphi^{\mathbb{D}}_{y,z}(U))_{y\in S}.$$

A descent datum $(U_y, \alpha_{y,z})_{y\in S}$ is *effective* if it lies in the essential image of L_S, i.e. there are U in $\mathbb{D}(X)$ and an isomorphism of descent data $(\mathbb{D}(y)(U), \varphi^{\mathbb{D}}_{y,z}(U))_S \simeq (U_y, \alpha_{y,z})_S$; the object U is the generalized version of an amalgamation. We end up with the following definition of (pre)stack:

Definition 2.3.2. Given a site $(\mathcal{C}, J)$, a $\mathcal{C}$-indexed category $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ is called a *J-prestack* (resp. *J-stack*) if for every J-sieve $S \rightarrowtail よ(X)$ the functor

$$L_S : \mathbb{D}(X) \to \mathbb{D}(S)$$

is full and faithful (resp. an equivalence). More explicitly, $\mathbb{D}$ is a stack if and only if for every sieve S of J all descent data for $\mathbb{D}$ and S are effective.

Stacks over a site $(\mathcal{C}, J)$ form a 2-fully faithful subcategory of $\mathbf{Ind}_{\mathcal{C}}$, denoted by $\mathbf{St}(\mathcal{C}, J)$. In particular, we shall denote by $\mathbf{St}(\mathscr{E})$ the category of stacks on a topos $\mathscr{E}$ with respect to the canonical topology on it. These 2-categories of stacks are Grothendieck 2-toposes in the sense of [38].

In short, for a stack, compatible local data along the arrows of a covering sieve S can be glued together into a global datum in the fibre over the codomain of said sieve, in a similar way to sheaves. We remark that if the category $\mathcal{C}$ has pullbacks, the notion of descent data admits a more manageable definition using J-covering families, as in [39, § 4.1.2].

A further similarity between stacks and sheaves is the fact that, similarly to the sheafification process for presheaves, there exists a stackification process for fibrations:

Theorem 2.3.2 [11, Chapter II, Section 2]. *Consider a site $(\mathcal{C}, J)$. There exists a 2-functor*

$$s_J : \mathbf{Fib}_{\mathcal{C}} \to \mathbf{St}(\mathcal{C}, J),$$

called stackification *(or the* associated stack functor*), which is left adjoint to the inclusion $i_J : \mathbf{St}(\mathcal{C}, J) \to \mathbf{Fib}_{\mathcal{C}}$.*

We will now show that Street fibrations that are prestacks can be characterized via a special class of presheaves, by generalizing the following result:

Proposition 2.3.3 [39, Proposition 4.7]. *Consider a small-generated site $(\mathcal{C}, J)$ and a Grothendieck fibration $p : \mathcal{D} \to \mathcal{C}$ with corresponding pseudofunctor $\mathbb{D}$. For any X in $\mathcal{C}$ and any pair of objects A, B in $\mathbb{D}(X)$, consider the presheaf*

$$\mathrm{Hom}(A, B) : (\mathcal{C}/X)^{op} \to \mathbf{Set}$$

defined, for an object $[y : Y \to X]$ of $\mathcal{C}/X$, as

$$\mathrm{Hom}(A, B)([y]) := \mathbb{D}(Y)\left(\mathbb{D}(y)(A), \mathbb{D}(y)(B)\right).$$

The fibration p is a prestack if and only if for every X in $\mathcal{C}$ and any A and B in $\mathbb{D}(X)$ the presheaf $\mathrm{Hom}(A, B)$ is a J_X-sheaf, where J_X is defined as in Example 1.2.1.

We must first extend the notion of Hom-presheaf to Street fibrations:

Definition 2.3.3. Consider a cloven fibration $p : \mathcal{D} \to \mathcal{C}$ and two objects $(A, \alpha : X \xrightarrow{\sim} p(A))$ and $(B, \beta : X \xrightarrow{\sim} p(B))$ of $\mathbb{D}(X)$. We define the Hom-presheaf

$$\mathrm{Hom}((A, \alpha), (B, \beta)) : (\mathcal{C}/X)^{op} \to \mathbf{Set}$$

as follows:

- for any $[y : Y \to X]$ in $\mathcal{C}/X$,

  $$\mathrm{Hom}((A, \alpha), (B, \beta))([y]) := \mathbb{D}(Y)\left(\mathbb{D}(y)(A, \alpha), \mathbb{D}(y)(B, \beta)\right).$$

  More explicitly, elements of $\mathrm{Hom}((A, \alpha), (B, \beta))([y])$ can be seen as arrows $\gamma : \mathrm{dom}(\widehat{\alpha y}_A) \to \mathrm{dom}(\widehat{\beta y}_B)$ such that $p(\gamma)\theta_{\alpha y, A} = \theta_{\beta y, B}$;

- for $z : [yz] \to [y]$ and $\gamma \in \mathrm{Hom}((A,\alpha),(B,\beta))([y])$, we define the arrow $\mathrm{Hom}((A,\alpha),(B,\beta))(z)(\gamma)$ as the composite $\chi^{-1}_{\beta y,z,B}\mathbb{D}(z)(\gamma)\chi_{\alpha y,z,A}$; explicitly, it is the unique arrow $\gamma' : \mathrm{dom}(\widehat{\alpha yz}_A) \to \mathrm{dom}(\widehat{\beta yz}_B)$ satisfying the identities
$$\widehat{\beta y}_B \gamma \lambda_{\alpha y,z,A} = \widehat{\beta yz}_B \gamma'$$
$$p(\gamma') = \theta_{\beta yz,B}\theta^{-1}_{\alpha yz,A}.$$

Remark 2.3.1. Consider $\mathrm{Hom}((A,\alpha),(B,\beta)) : (\mathcal{C}/X)^{op} \to \mathbf{Set}$ and a J_X-sieve S over $[y : Y \to X]$ in $\mathcal{C}/X$.

A *matching family* for $\mathrm{Hom}((A,\alpha),(B,\beta))$ and S consists, for every $f \in S$, of an arrow $\gamma_f : \mathrm{dom}(\widehat{\alpha yf}_A) \to \mathrm{dom}(\widehat{\beta yf}_B)$ of $\mathcal{D}$, such that $p(\gamma_f) = \theta_{\beta yf,B}\theta^{-1}_{\alpha yf,A}$, with the condition that whenever g is precomposable to f then $\gamma_{fg} = \mathrm{Hom}((A,\alpha),(B,\beta))(g)(\gamma_f)$, i.e. $\gamma_{fg} : \mathrm{dom}(\widehat{\alpha yfg}_A) \to \mathrm{dom}(\widehat{\beta yfg}_B)$ is the unique arrow such that $\widehat{\beta yf}_B \gamma_f \lambda_{\alpha y,fg,A} = \widehat{\beta yfg}_B \gamma_{fg}$.

An *amalgamation* for this matching family is an arrow $\gamma : \mathrm{dom}(\widehat{\alpha y}_A) \to \mathrm{dom}(\widehat{\beta y}_B)$ such that $p(\gamma) = \theta_{\beta y,B}\theta^{-1}_{\alpha y,A}$ and that for every f in S the arrow γ_f is the unique arrow such that $\widehat{\beta y}_B \gamma \lambda_{\alpha y,f,A} = \widehat{\beta yf}_B \gamma_f$.

We will also need some technical results about matching families for Hom-functors.

Lemma 2.3.4 [8, Lemma 2.6.4]. *Consider a site $(\mathcal{C}, J)$, an arrow $y : Y \to X$ of $\mathcal{C}$ and a sieve $S \in J(Y)$. Denoting by $S_{[1_Y]}$ (resp. $S_{[y]}$) the J_Y-sieve over $[1_Y]$ (resp. J_X-sieve over $[y]$) whose arrows are those of S, we have $\mathrm{lan}_{(\int y)^{op}}(S_{[1_Y]}) \simeq S_{[y]}$ naturally.*

Proof. Consider a presheaf $H : (\mathcal{C}/X)^{op} \to \mathbf{Set}$. An arrow $\alpha : S_{[y]} \to H$ in $[(\mathcal{C}/X)^{op}, \mathbf{Set}]$ is a matching family for H and $S_{[y]}$, i.e. the given for the arrows $z : [yz] \to [y]$ in $S_{[y]}$ of compatible elements $x_z \in H([yz])$. It is immediate to see this is the same as a matching family for $S_{[1_Y]}$ and $H \circ (\int y)^{op}$, providing a natural bijection
$$[(\mathcal{C}/X)^{op}, \mathbf{Set}](S_{[y]}, H) \simeq [(\mathcal{C}/Y)^{op}, \mathbf{Set}](S_{[1_Y]}, H \circ (\textstyle\int y)^{op})$$
which implies $S_{[y]} \simeq \mathrm{lan}_{(\int y)^{op}}(S_{[1_Y]})$. □

From this it follows that all matching families of the Hom-functors can be interpreted, if we allow a change of slice category, as matching families over the terminal object of the slice:

Corollary 2.3.5 [8, Corollary 2.6.5]. *Consider a site $(\mathcal{C}, J)$, a fibration $\mathcal{D} \to \mathcal{C}$, a J-sieve S over Y and two objects (A,α) and (B,β) of $\mathbb{D}(X)$. A matching family for $\mathrm{Hom}((A,\alpha),(B,\beta))$ and S, seen as a J_X-sieve over $[y]$ in $\mathcal{C}/X$, is the same as a matching family for $\mathrm{Hom}(\mathbb{D}(y)(A,\alpha), \mathbb{D}(y)(B,\beta))$ and the sieve S seen as a J_Y-sieve over $[1_Y]$ in $\mathcal{C}/Y$. The same holds for amalgamations of matching families.*

Proof. A rapid computation shows that $\mathrm{Hom}(\mathbb{D}(y)(A,\alpha), \mathbb{D}(y)(B,\beta))$ is isomorphic to $\mathrm{Hom}((A,\alpha),(B,\beta)) \circ (\int y)^{op}$, and the claim follows from the previous lemma. □

The following lemma relates matching families for Hom-functors with 2-cells of fibrations.

Lemma 2.3.6 [8, Lemma 2.6.6]. *Consider a site $(\mathcal{C}, J)$, a fibration $p : \mathcal{D} \to \mathcal{C}$, a J-sieve R over X and two objects (A,α) and (B,β) of $\mathbb{D}(X)$. A matching family for* $\mathrm{Hom}((A,\alpha),(B,\beta))$ *and R, seen as a J_X-sieve over $[1_X]$ in $\mathcal{C}/X$, is the same thing as a 2-cell of fibrations $\Psi(A,\alpha)\circ m_R \Rightarrow \Psi(B,\beta)\circ m_R$, where $m_R : \int R \hookrightarrow \mathcal{C}/X$ is the canonical inclusion functor. Analogously, an amalgamation for a matching family as above corresponds to a 2-cell of fibrations $\Psi(A,\alpha) \Rightarrow \Psi(B,\beta)$, i.e. to a morphism $(A,\alpha) \to (B,\beta)$.*

Proof. Remember that $\Psi(A,\alpha)\circ m_R : \int R \to \mathcal{D}$ operates as follows: every $[y]$ object of $\int R$, i.e. every arrow y in R, is sent to $\mathrm{dom}(\widehat{\alpha y}_A)$, and every morphism $z : [yz] \to [y]$ to $\lambda_{\alpha y,z,A} : \mathrm{dom}(\widehat{\alpha yz}_A) \to \mathrm{dom}(\widehat{\alpha y}_A)$. It is now immediate to see that the components of a matching family for $\mathrm{Hom}((A,\alpha),(B,\beta))$, being arrows $\gamma_y : \mathrm{dom}(\widehat{\alpha y}_A) \to \mathrm{dom}(\widehat{\beta y}_B)$, provide exactly the components for a 2-cell of fibrations $\Psi(A,\alpha) \circ m_R \Rightarrow \Psi(B,\beta) \circ m_R$, and viceversa. □

The following results allow us to prove the characterization of prestacks in terms of Hom-functors:

Proposition 2.3.7 [8, Proposition 2.6.7]. *Consider a site $(\mathcal{C}, J)$ and a cloven fibration $p : \mathcal{D} \to \mathcal{C}$. Then p is a J-prestack if and only if for every X in $\mathcal{C}$ and every (A,α), (B,β) in $\mathbb{D}(X)$ the presheaf* $\mathrm{Hom}((A,\alpha),(B,\beta)) : (\mathcal{C}/X)^{op} \to$ **Set** *is a J_X-sheaf.*

Proof. The proof is a generalization of the usual argument for Grothendieck fibrations. In the following we will use the notation $(F,\varphi) := \Psi(A,\alpha)$ and $(G,\gamma) = \Psi(B,\beta)$.

Firstly, by the previous lemma we may reduce to considering matching families over the terminal $[1_X]$ of $\mathcal{C}/X$. So suppose $\mathcal{D}$ is a prestack and consider a J-covering sieve R over X and a matching family for R over $[1_X]$ and $\mathrm{Hom}((A,\alpha),(B,\beta))$. By the previous lemma, the matching family corresponds to a 2-cell $\alpha : (F_{|R}, \varphi_{|R}) \Rightarrow (G_{|R}, \gamma_{|R})$ in $\mathbf{Fib}_{\mathcal{C}}(\int R, \mathcal{D})$. If $\mathcal{D}$ is a prestack the functor $\mathbb{D}(X) \simeq \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D}) \to \mathbf{Fib}_{\mathcal{C}}(\int R, \mathcal{D})$ is full and faithful, therefore α is the image of a unique $\bar{\alpha} : (A,\alpha) \to (B,\beta)$, which corresponds to an amalgamation for the original matching family and hence the Hom-presheaf we were considering is a J_X-sheaf.

If conversely all Hom-presheaves are sheaves, start by considering (F,φ) and (G,γ) as above and a 2-cell $\alpha : (F_{|R}, \varphi_{|R}) \Rightarrow (G_{|R}, \gamma_{|R})$. The 2-cell α corresponds to a matching family for R over $[1_X]$ and $\mathrm{Hom}((A,\alpha),(B,\beta))$, by the previous lemma, and has a unique amalgamation. Such an amalgamation corresponds to a unique 2-cell $\bar{\alpha} : (F, \varphi) \Rightarrow (G, \gamma)$ extending the original α, and hence $\mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D}) \to \mathbf{Fib}_{\mathcal{C}}(\int R, \mathcal{D})$ is fully faithful. □

2.4 The canonical stack of a site

We have seen in Example 2.1.1 that $\mathcal{C}$-indexed categories and fibrations can be built in many ways, for instance from the pullbacks of a category or from a geometric morphism. It is fundamental to know that every site $(\mathcal{C}, J)$ is canonically associated with a $\mathcal{C}$-indexed category, which is furthermore a J-stack, as follows:

Definition 2.4.1. Consider a site $(\mathcal{C}, J)$: we can define the $\mathcal{C}$-indexed category $\mathcal{S}_{(\mathcal{C},J)} : \mathcal{C}^{op} \to \mathbf{CAT}$ by setting

$$\left[Y \xrightarrow{y} X\right] \xmapsto{\mathcal{S}_{(\mathcal{C},J)}} \left[\mathbf{Sh}(\mathcal{C}, J)/\ell_J(X) \xrightarrow{\ell_J(y)^*} \mathbf{Sh}(\mathcal{C}, J)/\ell_J(Y)\right].$$

Notice that $\mathcal{S}_{(\mathcal{C},J)}$ is always well-defined, since there is a canonical choice of pullbacks in $\mathbf{Sh}(\mathcal{C}, J)$.

A standard result in topos theory is the equivalence between $\mathbf{Sh}(\mathcal{C}, J)/\ell_J(X)$ and the sheaf topos $\mathbf{Sh}(\mathcal{C}/X, J_X)$, where J_X is the topology of Example 1.2.1 (cf. Theorem 5.2.1). In this way, the pseudofunctor $\mathcal{S}_{(\mathcal{C},J)}$ admits an alternative description, by setting

$$\left[Y \xrightarrow{y} X\right] \xmapsto{\mathcal{S}_{(\mathcal{C},J)}} \left[\mathbf{Sh}(\mathcal{C}/X, J_X) \xrightarrow{C^*_{\int y}} \mathbf{Sh}(\mathcal{C}/Y, J_Y)\right],$$

where in particular $C^*_{\int y}$ acts as $- \circ (\int y)^{op}$ (by continuity of $\int y$: see Proposition 5.1.3).

Theorem 2.4.1. *The $\mathcal{C}$-indexed category $\mathcal{S}_{(\mathcal{C},J)}$ associated to a site $(\mathcal{C}, J)$ is a J-stack, called the* canonical stack of $(\mathcal{C}, J)$.

We can use both definitions of $\mathcal{S}_{(\mathcal{C},J)}$ to describe its associated fibration.

Using the first description of $\mathcal{S}_{(\mathcal{C},J)}$, it is immediate to see that the fibration $\mathcal{G}(\mathcal{S}_{(\mathcal{C},J)}) \to \mathcal{C}$ is the comma category $(1_{\mathbf{Sh}(\mathcal{C},J)} \downarrow \ell_J) \to \mathcal{C}$. Indeed, its objects are arrows $h : H \to \ell_J(X)$ of $\mathbf{Sh}(\mathcal{C}, J)$, and arrows $(k : K \to \ell_J(Y)) \to (h : H \to \ell_J(X))$ are pairs (g, y) where $g : K \to H$, $y : Y \to X$ and $\ell_J(y)k = hg$. In particular, (g, y) is cartesian if the square formed with h and k is a pullback square in $\mathbf{Sh}(\mathcal{C}, J)$.

Using the second description, we see that objects of $\mathcal{G}(\mathcal{S}_{(\mathcal{C},J)})$ are pairs $(X, P : \mathcal{C}/X^{op} \to \mathbf{Set})$, where X is an object of $\mathcal{C}$ and P is a J_X-sheaf. Arrows of $\mathcal{G}(\mathcal{S}_{(\mathcal{C},J)})$ are pairs $(y, \alpha) : (Y, Q) \to (X, P)$, where $y : Y \to X$ in $\mathcal{C}$ and $\alpha : Q \Rightarrow P \circ (\int y)^{op}$. In particular, a cartesian arrow of $\mathcal{G}(\mathcal{S}_{(\mathcal{C},J)})$ is of the form $(y, \alpha) : (Y, Q) \to (X, P)$ with α an isomorphism. In the following we will denote by $\mathcal{S}_{(\mathcal{C},J)}$ both the fibration and the $\mathcal{C}$-indexed category.

Remark 2.4.1. Using the well-known equivalence $\mathscr{E} \simeq \mathbf{Sh}(\mathscr{E}, J_{\mathscr{E}}^{can})$, one can immediately see that the canonical fibration over $(\mathscr{E}, J_{\mathscr{E}}^{can})$ coincides with the fibration $\mathrm{cod} : Mor(\mathscr{E}) \to \mathscr{E}$ in Example 2.1.1(iv).

Theorem 2.4.1 is a standard result in the theory of stacks, which can be found for instance as Proposition 3.4.4 of [11]. We will provide two proofs of it in the sequel. The first one is Corollary 4.3.3, and views $\mathcal{S}_{(\mathcal{C},J)}$ as the image along a direct image functor of the canonical stack over $\mathbf{Sh}(\mathcal{C}, J)$ (which we must assume to be a stack); the second proof relies on the fundamental adjunction, and appears as Corollary 6.4.2. The canonical stack of a site is a main protagonist in relative topos theory. It acts as a dualizing object for the fundamental adjunction, as well as being the 'stack-theoretic embodiment' of the base topos $\mathbf{Sh}(\mathcal{C}, J)$. This means that, when we will consider relative (pre)sheaf toposes we will consider classes of morphisms of fibrations with values in $\mathcal{S}_{(\mathcal{C},J)}$, in the same wat that (pre)sheaf toposes in the usual sense are functors with values in $\mathbf{Set}$. We will come back to this topic in Section 6.5.

2.5 The truncation functor

We conclude the chapter by studying the relationship between sheaves and stacks through an adjunction which can be interpreted both at the level of toposes and of comorphisms of sites. All the results mentioned here appear in Section 2.7 of [8]. The comparison requires that we focus on stacks with values in **Cat** instead of **CAT**: we shall call them *small* stacks and denote their class by $\mathbf{St}^s(\mathcal{C}, J)$.

Proposition 2.3.1 states that a presheaf over $\mathcal{C}$ is a J-sheaf if and only if, when seen as a discrete $\mathcal{C}$-indexed category, it is a J-stack. This provides us with an inclusion functor

$$j_J : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{St}^s(\mathcal{C}, J),$$

of J-sheaves into the category of small J-stacks. In fact, if we consider the (2-)adjunction

$$\mathbf{Set} \underset{Disc}{\overset{\pi_0}{\underset{\bot}{\leftrightarrows}}} \mathbf{Cat},$$

where $Disc$ maps each set to the corresponding discrete category while π_0 sends a category to its set of connected components, then j_J acts by mapping a J-sheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ to the composite $Disc \circ P$. By standard categorical considerations, the functor $j_J := (Disc \circ -)$ admits a left adjoint, and their adjunction restricts to sheaves and stacks:

Proposition 2.5.1. *Given a site $(\mathcal{C}, J)$, there is an adjunction*

$$\mathbf{Sh}(\mathcal{C}, J) \underset{j_J}{\overset{t_J}{\underset{\perp}{\leftrightarrows}}} \mathbf{St}^s(\mathcal{C}, J),$$

*where j_J includes J-sheaves into $\mathbf{St}^s(\mathcal{C}, J)$ as small discrete stacks, while the J-*truncation functor *t_J maps each small J-stack $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{Cat}$ to the J-sheaf $\mathrm{a}_J(\pi_0 \circ \mathbb{D})$. In other words, the J-truncation of a stack is computed by considering its presheaf of connected components and sheafifying it with respect to J.*

In particular, when J is the trivial topology over $\mathcal{C}$, we shall denote the truncation-inclusion adjunction simply by $t_\mathcal{C} \dashv j_\mathcal{C}$.

Proof. Since $\pi_0 \dashv Disc$, by standard considerations about adjoints and functor categories (dual to those of Lemma D.1) the functor $j_\mathcal{C} := (Disc \circ -)$ has a left adjoint $t_\mathcal{C} := (\pi_0 \circ -)$. Now consider the commutative diagram

$$\begin{array}{ccc}
\mathbf{St}^s(\mathcal{C}, J) & \underset{i_J}{\overset{s_J}{\underset{\perp}{\leftrightarrows}}} & \mathbf{Ind}_\mathcal{C} \\
{\scriptstyle t_J} \downarrow \dashv \uparrow {\scriptstyle j_J} & & {\scriptstyle t_\mathcal{C}} \downarrow \dashv \uparrow {\scriptstyle j_\mathcal{C}} \\
\mathbf{Sh}(\mathcal{C}, J) & \underset{\iota_J}{\overset{\mathrm{a}_J}{\underset{\perp}{\leftrightarrows}}} & [\mathcal{C}^{op}, \mathbf{Set}]
\end{array} :$$

The identity $j_\mathcal{C} \circ \iota_J = i_J \circ j_J$ is obvious, and by standard arguments about adjoints (see Lemma D.5) one concludes that the composite $\mathrm{a}_J \circ t_\mathcal{C} \circ i_J$ provides a left adjoint to j_J. □

Truncation functors also interact well when multiple topologies are at play:

Lemma 2.5.2. *Consider a site $(\mathcal{C}, J)$, a further topology $K \supseteq J$ and the diagram*

$$\begin{array}{ccc}
\mathbf{St}^s(\mathcal{C}, K) & \underset{i_K}{\overset{s_K}{\underset{\perp}{\leftrightarrows}}} & \mathbf{St}^s(\mathcal{C}, J) \\
{\scriptstyle t_K} \downarrow \dashv \uparrow {\scriptstyle j_K} & & {\scriptstyle t_J} \downarrow \dashv \uparrow {\scriptstyle j_J} \\
\mathbf{Sh}(\mathcal{C}, K) & \underset{\iota_K}{\overset{\mathrm{a}_J}{\underset{\perp}{\leftrightarrows}}} & \mathbf{Sh}(\mathcal{C}, J)
\end{array} .$$

Then the following hold:

$$\begin{aligned}
j_J \circ \iota_K &= i_K \circ j_K, \\
\mathrm{a}_J \circ t_J &\cong t_K \circ s_K, \\
\mathrm{a}_K &\cong t_K \circ s_K \circ j_J.
\end{aligned}$$

Proof. The first identity is obvious. Since the composites in the second isomorphism are the left adjoints of those in the first equality, the second isomorphism must also hold. Finally, the third isomorphism follows from the second one and the fact that $t_J \circ j_J \cong \mathrm{id}_{\mathbf{Sh}(\mathcal{C},J)}$. $\square$

So far we have provided an indexed perspective on the truncation functor. We conclude the section by presenting two other possible point of view of the truncation of stacks, exploiting some tools which will be developed in the sequel: a topos-theoretic interpretation, which follows from the *fundamental adjunction* of Chapter 6, describes the truncation process using the *(terminally connected, local homeomorphism)-factorization system* for essential geometric morphisms; and a site-theoretic (fibrational) interpretation of truncation, which exploits instead the *relative comprehensive factorization system* of functors, first introduced in [6, Section 4.7].

For the topos-theoretic interpretation, we shall need the adjunction of Corollary 6.3.9,

$$\mathbf{St}(\mathcal{C},J) \underset{\Gamma'}{\overset{\Lambda'}{\rightleftarrows}} \mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J), \qquad \Lambda' \dashv \Gamma'$$

where in particular Γ' maps an essential $\mathbf{Sh}(\mathcal{C},J)$-topos $\mathscr{E}$ to the J-stack

$$\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)(\mathbf{Sh}(\mathcal{C}/-,J_{(-)}),\mathscr{E}) : \mathcal{C}^{op} \to \mathbf{CAT},$$

while Λ' maps a J-stack $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ to its Giraud topos $\mathrm{Gir}_J(\mathbb{D}) \to \mathbf{Sh}(\mathcal{C},J)$ (see Definition 5.1.1). This adjunction can be restricted to small stacks and relatively small toposes (see Definition 7.1.1).

Consider now the adjunction

$$\mathbf{Sh}(\mathcal{C},J) \underset{\mathbf{Sh}(\mathcal{C},J)/-}{\overset{L}{\leftrightarrows}} \mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J), \qquad L \dashv \mathbf{Sh}(\mathcal{C},J)/-$$

where the right adjoint $\mathbf{Sh}(\mathcal{C},J)/-$ maps a J-sheaf P to the topos $\mathbf{Sh}(\mathcal{C},J)/P \to \mathbf{Sh}(\mathcal{C},J)$, and it is full and faithful by Lemma 7.1.2. On the other hand, the left adjoint L maps a topos $E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C},J)$ to the image $E_!(1_{\mathscr{E}})$ of the terminal object via the essential image. The fact that the two form an adjunction follows immediately from Lemma 4.59 of [6].

Remark 2.5.1. Alternatively, we could derive this adjunction from the discrete adjunction of Proposition 7.1.3, by specializing it to J-sheaves and essential $\mathbf{Sh}(\mathcal{C},J)$-toposes.

By composing the two adjunctions, we can recover j_J and t_J:

Proposition 2.5.3. *Consider the composite adjunction*

$$\mathbf{Sh}(\mathcal{C},J) \underset{\Gamma'\circ\mathbf{Sh}(\mathcal{C},J)/-}{\overset{L\circ\Lambda'}{\bot}} \mathbf{St}^s(\mathcal{C},J),$$

then $\Gamma' \circ \mathbf{Sh}(\mathcal{C},J)/- \cong j_J$ *and* $L \circ \Lambda' \cong t_J$*. In particular, the truncation of a stack* $\mathbb{D}$ *may be defined as either of the two* J*-sheaves*

$$\begin{aligned} t_J(\mathbb{D}) &\simeq (C_{p_\mathbb{D}})_!(1_{\mathrm{Gir}_J(\mathbb{D})}) \\ &\simeq \mathrm{colim}_{(X,U)\in\mathcal{G}(\mathbb{D})}\, \ell_J(X). \end{aligned}$$

Proof. It is sufficient to notice that for a J-sheaf P the following chain of equivalences holds:

$$\begin{aligned} (\Gamma'\circ\mathbf{Sh}(\mathcal{C},J)/-)(P) &:= \Gamma'(\mathbf{Sh}(\mathcal{C},J)/P) \\ &= \mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)(\mathrm{Gir}_J(\mathcal{C}/-),\mathbf{Sh}(\mathcal{C},J)/P) \\ &\simeq \mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)(\mathbf{Sh}(\mathcal{C},J)/\ell_J(-),\mathbf{Sh}(\mathcal{C},J)/P) \\ &\simeq \mathbf{Sh}(\mathcal{C},J)(\ell_J(-),P) \\ &\simeq P. \end{aligned}$$

The third line is justified since for each X there is a natural equivalence $\mathrm{Gir}_J(\mathcal{C}/-) \simeq \mathbf{Sh}(\mathcal{C},J)/\ell_J(-)$, by Theorem 5.2.1; the fourth line holds by Lemma 7.1.2 and the final by Yoneda's lemma. Therefore, the right adjoint $\Gamma' \circ \mathbf{Sh}(\mathcal{C},J)/-$ acts as the inclusion $j_J : \mathbf{Sh}(\mathcal{C},J) \hookrightarrow \mathbf{St}(\mathcal{C},J)$, meaning that the composite $L \circ \Lambda'$ is isomorphic to the truncation functor t_J: by spelling out explicitly the composite $L \circ \Lambda'$ we get the first description of $t_J(\mathbb{D})$ in the claim. The second expression is obtained by seeing the terminal $1_{\mathrm{Gir}_J(\mathbb{D})}$ as the colimit of all the representables in $\mathrm{Gir}_J(\mathbb{D})$, and exploiting the commutativity of $(C_{p_\mathbb{D}})_!$ with colimits and the equivalence $(C_{p_\mathbb{D}})_! \circ \ell_{J_\mathbb{D}} \cong \ell_J \circ p_\mathbb{D}$ we have that

$$\begin{aligned} t_J(\mathbb{D}) &\simeq (C_{p_\mathbb{D}})_! \left(\mathrm{colim}_{(X,U)\in\mathcal{G}(\mathbb{D})}\, \ell_{J_\mathbb{D}}(X,U)\right) \\ &\simeq \mathrm{colim}_{(X,U)\in\mathcal{G}(\mathbb{D})}\, ((C_{p_\mathbb{D}})_!\ell_{J_\mathbb{D}}(X,U)) \\ &\simeq \mathrm{colim}_{(X,U)\in\mathcal{G}(\mathbb{D})}\, \ell_J(X). \end{aligned}$$

□

Propositions 4.62 of [6] states that an essential geometric morphism $f : \mathscr{E} \to \mathscr{F}$ always admits a factorization

$$\mathscr{E} \xrightarrow{f'} \mathscr{F}/f_!(1_\mathscr{E}) \xrightarrow{\prod_{f_!(1_\mathscr{E})}} \mathscr{F}.$$

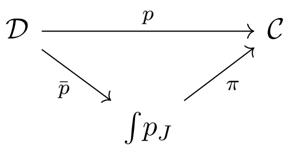
into a terminally connected geometric morphism followed by a local homeomorphism (see Definition 7.1.2). In particular, consider a J-stack $\mathbb{D} : \mathcal{C}^{op} \to$ **Cat** and its Giraud topos

$$C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), J_{\mathbb{D}}) \to \mathbf{Sh}(\mathcal{C}, J),$$

which is an essential geometric morphism (see Chapter 5): if we apply the (terminally connected, local homeomorphism)-factorization to $C_{p_{\mathbb{D}}}$, we have that the object of $\mathbf{Sh}(\mathcal{C}, J)$ providing the local homeomorphism factor is $(C_{p_{\mathbb{D}}})_!(1_{\mathrm{Gir}_J(\mathbb{D})})$, i.e. the J-truncation of $\mathbb{D}$ by our last result. Thus from a topos-theoretic point of view the process of truncating a stack is equivalent to one of the many factorization systems for geometric morphisms.

The (terminally connected, local homeomorphism)-factorization can also be presented at the level of sites (cf. [6, Proposition 4.70(ii)]) using the J-comprehensive factorization of functors, implying a description of the truncation of a stack from a fibrational standpoint.

Definition 2.5.1 [6, Definition 4.67]. Consider a site $(\mathcal{C}, J)$. Every functor $p : \mathcal{D} \to \mathcal{C}$ admits a *J-comprehensive (orthogonal) factorization*



,

where

- p_J is the J-sheaf $\mathrm{colim}(\ell_J \circ p)$ and π its associated discrete J-stack;
- $\bar{p}$ is a M_J^{π}-cofinal functor (cfr. [6, Definition 2.23]), where M_J^{π} is Giraud's topology for p_J (see Definition 5.1.1).

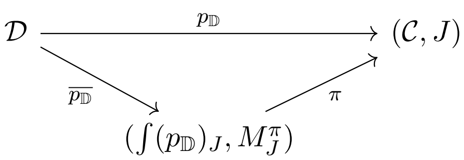
Moreover, if p is a continuous comorphism of sites, at the level of toposes its J-comprehensive factorization induces the (terminally connected, local homeomorphism)-factorization of C_p.

By applying this to a J-stack $\mathbb{D} : \mathcal{C}^{op} \to$ **Cat** and the corresponding fibration $p_{\mathbb{D}}$ we can immediately conclude the following:

Corollary 2.5.4. *From a fibrational point of view, the J-truncation functor*

$$t_J : \mathbf{St}^s(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{C}, J)$$

acts by mapping a J-stack $p : \mathcal{D} \to \mathcal{C}$ to the second component π in its J-comprehensive factorization



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Chapter 3

Colimits of categories

We will meet many instances of bicategorical colimits (cf. [19, Chapter 5]), both when working with base change for stacks and in the context of the fundamental adjunction. The present chapter, which deals with the content of Section 2.9 in [8], is devoted to some technical results about pseudocolimits, and their computation in **Cat**: in particular, the explicit description of pseudocolimits of categories in terms of localizations of Grothendieck fibrations will be useful when describing the inverse image of fibrations in chapter 4.

3.1 Bicategorical colimits

Definition 3.1.1. Consider two weak 2-categories $\mathcal{C}$ and $\mathcal{K}$, a pseudofunctor $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ and a pseudofunctor $R : \mathcal{C} \to \mathcal{K}$: the $\mathbb{D}$-*weighted lax colimit of* R is an object $\mathrm{colim}^{\mathbb{D}}_{lax} R$ of $\mathcal{K}$ such that there a pseudonatural equivalence

$$\mathcal{K}(\mathrm{colim}^{\mathbb{D}}_{lax} R, K) \simeq [\mathcal{C}^{op}, \mathbf{CAT}]_{lax}(\mathbb{D}, \mathcal{K}(R(-), K)).$$

Similarly, the $\mathbb{D}$-*weighted oplax colimit* $\mathrm{colim}^{\mathbb{D}}_{oplax} R$ will satisfy the condition

$$\mathcal{K}(\mathrm{colim}^{\mathbb{D}}_{oplax} R, K) \simeq [\mathcal{C}^{op}, \mathbf{CAT}]_{oplax}(\mathbb{D}, \mathcal{K}(R(-), K)).$$

If we further restrict to pseudonatural transformations, we obtain the notion of $\mathbb{D}$-*weighted pseudocolimit* $\mathrm{colim}^{\mathbb{D}}_{ps} R$:

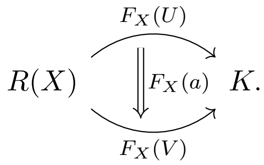
$$\mathcal{K}(\mathrm{colim}^{\mathbb{D}}_{ps} R, K) \simeq [\mathcal{C}^{op}, \mathbf{CAT}]_{ps}(\mathbb{D}, \mathcal{K}(R(-), K)).$$

Any of these colimits is said to be *conical* if the weight is the constant pseudofunctor $\Delta\mathbb{1} : \mathcal{C}^{op} \to \mathbf{CAT}$ with value the terminal category $\mathbb{1}$: in this case we will omit mentioning the weight altogether and adopt the notations $\mathrm{colim}_{lax} R$, $\mathrm{colim}_{oplax} R$ and $\mathrm{colim}_{ps} R$.

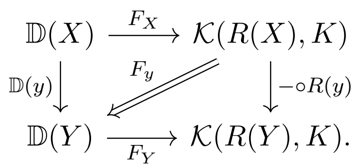
Remark 3.1.1. In the 2-categorical literature what we have just defined is usually called *bicolimit*, while a *colimit* is an object producing a natural *isomorphism* of hom-categories; since we will not have to draw the distinction between the two concepts anywhere in the sequel, we shall drop the *bi-* prefix.

The lax/oplax/pseudonatural transformations appearing in the definition of colimit are called the *lax/oplax/pseudonatural cocones with vertex X under the diagram R*. Let us describe explicitly, for instance, the data of a lax transformation $F : \mathbb{D} \Rightarrow \mathcal{K}(R-, K)$ of pseudofunctors from $\mathcal{C}^{op}$ to **CAT**. It consists of:

(i) a functor $F_X : \mathbb{D}(X) \to \mathcal{K}(R(X), K)$ for every X in $\mathcal{C}$: that is, for every X in $\mathcal{C}$ and every U in $\mathbb{D}(X)$ we have a 1-cell $F_X(U) : R(X) \to K$ in $\mathcal{K}$, and for every $a : U \to V$ in $\mathbb{D}(X)$ we have a 2-cell

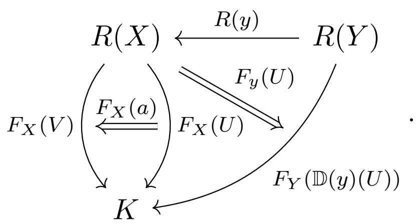


(ii) a natural transformation F_y for every arrow $y : Y \to X$ in $\mathcal{C}$ as in the following diagram:



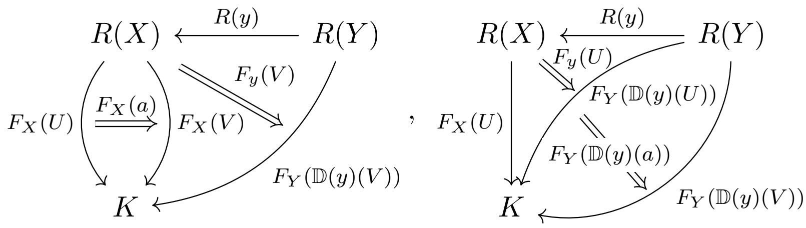
The component of F_y at every U of $\mathbb{D}(X)$ is a 2-cell $F_y(U) : F_X(X) \circ R(y) \Rightarrow F_Y(\mathbb{D}(y)(U))$ of $\mathcal{K}$. Moreover, F_y satisfies the same axioms of a pseudonatural transformation (see Definition 2.1.1).

We can visualize the cocone F in $\mathcal{K}$ as in the following figure, for $y : Y \to X$ in $\mathcal{C}$ and $a : U \to V$ in $\mathbb{D}(X)$:



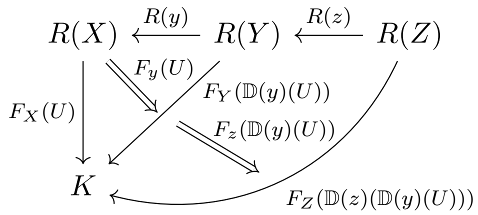
As we can see, the arrows of $\mathcal{C}$ produce the usual triangles of the cocone, while the arrows in each $\mathbb{D}(X)$ produce a 'spindle' underneath $R(X)$. The compatibility conditions that said arrows must satisfy, aside from the functoriality of F_X, are the following:

(i) naturality of F_y: for each $a : U \to V$ in $\mathbb{D}(X)$ and each $y : Y \to X$,the two diagrams

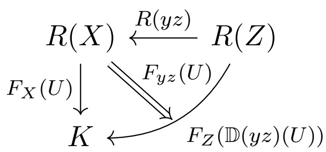


coincide in $\mathcal{K}$.

(ii) lax transformation axioms: up to canonical 2-isomorphisms, for every $y : Y \to X$ and $z : Z \to Y$ in $\mathcal{C}$ and every U in $\mathbb{D}(X)$ the two diagrams



and



coincide in $\mathcal{K}$. Moreover, up to canonical 2-isomorphisms, the 2-cell $F_{1_X}(U) : F_X(U) \circ R(1_X) \Rightarrow F_X(\mathbb{D}(1_X)(U))$ coincides with the identity of $F_X(U)$.

If we consider an *oplax cocone*, what changes is the direction of all the natural transformations of the kind $F_y(U)$. Finally, if we consider *pseudonatural transformations* all the $F_y(U)$ are natural isomorphisms. In particular if the weight were $\Delta\mathbb{1} : \mathcal{C}^{op} \to \mathbf{CAT}$ we would have no 'spindle' underneath each of the $R(X)$, only triangles: this explains why the said 2-colimits are called conical.

Colimits can be interpreted as a particular kind of 2-adjoint:

Definition 3.1.2. Consider two 2-categories $\mathcal{A}$ and $\mathcal{B}$ and two 2-functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$: then there is a *2-adjunction* $L \dashv R$ if and only if there is an equivalence

$$\mathcal{B}(L(X), Y) \simeq \mathcal{A}(X, R(Y))$$

pseudonatural in X and Y.

Thus for instance the existence of weighted lax colimits for a certain diagram $R : \mathcal{C} \to \mathcal{K}$ can be understood as the existence of a left adjoint, $\operatorname{colim}^{(-)}_{lax} R$, to the functor

$$\mathcal{K}(R(=), -) : \mathcal{K} \to [\mathcal{C}^{op}, \mathbf{CAT}]_{lax},\ K \mapsto \mathcal{K}(R(=), K),$$

As in the 1-categorical context, (limits and) colimits interact well with 2-adjoints:

Lemma 3.1.1. *Left 2-adjoints preserve any kind of bicolimit: more explicitly, given two 2-functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ such that $L \dashv R$, a category $\mathcal{C}$, two pseudofunctors $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{Cat}$ and $I : \mathcal{C} \to \mathcal{A}$, then*

$$L(\operatorname{colim}^{\mathbb{D}}_{\bullet} I) \simeq \operatorname{colim}^{\mathbb{D}}_{\bullet}(L \circ I),$$

where $\bullet$ is either lax, oplax or ps.

Proof. This is a consequence for the following chain of natural equivalences, which hold for every B in $\mathcal{B}$:

$$\begin{aligned}
\mathcal{B}\left(\operatorname{colim}^{\mathbb{D}}_{\bullet}(L \circ I), B\right) &\simeq [\mathcal{C}^{op}, \mathbf{Cat}]_{\bullet}\left(\mathbb{D}, \mathcal{B}((L \circ I)(-), B)\right)\\
&\simeq [\mathcal{C}^{op}, \mathbf{Cat}]_{\bullet}\left(\mathbb{D}, \mathcal{A}(I(-), R(B))\right)\\
&\simeq \mathcal{A}\left(\operatorname{colim}^{\mathbb{D}}_{\bullet} I, R(B)\right)\\
&\simeq \mathcal{A}\left(L(\operatorname{colim}^{\mathbb{D}}_{\bullet} I), B\right).
\end{aligned}$$

□

Finally, weights and colimits play a symmetric role in colimits:

Proposition 3.1.2. *Consider a category $\mathcal{C}$ and two pseudofunctors $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ and $\mathbb{E} : \mathcal{C} \to \mathbf{CAT}$. There are isomorphisms of categories*

$$\begin{aligned}
[\mathcal{C}^{op}, \mathbf{CAT}]_{lax}(\mathbb{D}, \mathbf{CAT}(\mathbb{E}(-), \mathcal{K})) &\simeq [\mathcal{C}, \mathbf{CAT}]_{oplax}(\mathbb{E}, \mathbf{CAT}(\mathbb{D}(-), \mathcal{K}))\\
[\mathcal{C}^{op}, \mathbf{CAT}]_{oplax}(\mathbb{D}, \mathbf{CAT}(\mathbb{E}(-), \mathcal{K})) &\simeq [\mathcal{C}, \mathbf{CAT}]_{lax}(\mathbb{E}, \mathbf{CAT}(\mathbb{D}(-), \mathcal{K}))\\
[\mathcal{C}^{op}, \mathbf{CAT}]_{ps}(\mathbb{D}, \mathbf{CAT}(\mathbb{E}(-), \mathcal{K})) &\simeq [\mathcal{C}, \mathbf{CAT}]_{ps}(\mathbb{E}, \mathbf{CAT}(\mathbb{D}(-), \mathcal{K}))
\end{aligned}$$

that are natural in $\mathcal{K}$. This implies in particular the equivalences

$$\operatorname{colim}^{\mathbb{D}}_{lax} \mathbb{E} \simeq \operatorname{colim}^{\mathbb{E}}_{oplax} \mathbb{D},\ \operatorname{colim}^{\mathbb{D}}_{oplax} \mathbb{E} \simeq \operatorname{colim}^{\mathbb{E}}_{lax} \mathbb{D},\ \operatorname{colim}^{\mathbb{D}}_{ps} \mathbb{E} \simeq \operatorname{colim}^{\mathbb{E}}_{ps} \mathbb{D}.$$

Proof. Let us consider the first isomorphism of categories. An object in the left-hand category is a lax natural transformation $F : \mathbb{D} \Rightarrow \mathbf{CAT}(\mathbb{E}(-), \mathcal{K})$, i.e. the given, for every X in $\mathcal{C}$, of a functor $F_X : \mathbb{D}(X) \to \mathbf{CAT}(\mathbb{E}(X), \mathcal{K})$, and for every arrow $y : Y \to X$ of a natural transformation

$$\begin{array}{ccc}
\mathbb{D}(X) & \xrightarrow{F_X} & \mathbf{CAT}(\mathbb{E}(X), \mathcal{K})\\
{\scriptstyle \mathbb{D}(y)}\downarrow & \swArrow_{F_y} & \downarrow{\scriptstyle -\circ\mathbb{E}(y)}\\
\mathbb{D}(Y) & \xrightarrow[F_Y]{} & \mathbf{CAT}(\mathbb{E}(Y), \mathcal{K})
\end{array}$$

satisfying suitable compatibility conditions.

Consider now any object M in $\mathbb{E}(X)$. If we set the rule $R_X(M)(-) := F_X(-)(M)$, where the blank space stands either for an object or an arrow of $\mathbb{D}(X)$, then this defines a functor $R_X(M) : \mathbb{D}(X) \to \mathcal{K}$. By the definition, it follows that $R_X : \mathbb{E}(X) \to \mathbf{CAT}(\mathbb{D}(X), \mathcal{K})$ is a functor if and only if F_X is a functor. Then we consider for every $y : Y \to X$ in $\mathcal{C}$ the components of F_y, i.e. the natural transformations $F_y(U) : F_X(U) \circ \mathbb{E}(y) \Rightarrow F_Y(\mathbb{D}(y)(U))$: fixing M in $\mathbb{E}(Y)$, we obtain arrows $F_y(U)(M) : F_X(U)(\mathbb{E}(y)(M)) \to F_Y(\mathbb{D}(y)(U))(M)$ in the category $\mathcal{K}$. Setting $R_y(M)(U) := F_y(U)(M)$ we obtain a natural transformation $R_y(M) : R_X(\mathbb{E}(y)(M)) \Rightarrow R_Y(M) \circ \mathbb{D}(y)$, and all the natural transformations $R_y(M)$ in turn provide the components of a natural transformation

$$\begin{array}{ccc} \mathbb{E}(X) & \xrightarrow{R_X} & \mathbf{CAT}(\mathbb{D}(X), \mathcal{K}) \\ {\scriptstyle \mathbb{E}(y)}\uparrow & \overset{R_y}{\searrow\!\!\!\searrow} & \uparrow{\scriptstyle -\circ\mathbb{D}(y)} \\ \mathbb{E}(Y) & \xrightarrow[R_Y]{} & \mathbf{CAT}(\mathbb{D}(Y), \mathcal{K}) \end{array} .$$

The transformation R_y is natural if and only if F_y is a natural transformation. Finally, one can also check that F_y satisfies the axioms of a lax natural transformation if and only if R_y satisfies those of an oplax natural transformation. A similar correspondence can be established between modifications, and this proves the first isomorphism of categories. The second and third isomorphism of categories are proved in the exact same fashion.

Finally, the three equivalences between colimits are a straightforward consequence of the former three equivalences of hom-categories. □

Remarks 3.1.2. (i) Formula 3.9 of [23, Section 3.1] states the equivalence $\mathrm{colim}^R \mathbb{D} \cong \mathrm{colim}^{\mathbb{D}} R$ in the enriched setting. In particular, it is shown that both colimits can be computed as the *coend* of the functor $R \cdot \mathbb{D} : \mathcal{C} \times \mathcal{C}^{op} \to \mathbf{Cat}$ acting as $(R \cdot \mathbb{D})(X, Y) := R(X) \times \mathbb{D}(Y)$: thus the commutativity of the product in **Cat**, which allows to switch R and $\mathbb{D}$, is the abstract reason behind the commutativity of weights and diagrams in colimits.

(ii) Given a small category $\mathcal{C}$ and a functor $A : \mathcal{C} \to \mathbf{Set}$, there is a functor

$$- \otimes_{\mathcal{C}} A : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Set}$$

acting as left Kan extension of A along $よ_{\mathcal{C}}$, i.e. the left adjoint to the functor

$$R_A(\bullet) := \mathbf{Set}(A(-), \bullet) : \mathbf{Set} \to [\mathcal{C}^{op}, \mathbf{Set}]$$

defined by $R_A(S) := \mathbf{Set}(A(-), S)$. For any presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ there is an isomorphism

$$P \otimes_{\mathcal{C}} A \cong A \otimes_{\mathcal{C}^{op}} P$$

of sets (cf. Section 5.1.4 of [5]). The commutativity of weights and diagrams is thus a generalization of the commutativity of the tensor of functors. Indeed, if we introduce the notation

$$- \otimes_{\mathcal{C}} \mathbb{E} := \mathrm{colim}^{(-)}_{ps}\, \mathbb{E} : [\mathcal{C}^{op}, \mathbf{Cat}]_{ps} \to \mathbf{Cat}$$

for the left adjoint of the 2-functor

$$\mathbf{Cat}(\mathbb{E}(=), -) : \mathbf{Cat} \to [\mathcal{C}^{op}, \mathbf{Set}]_{ps},$$

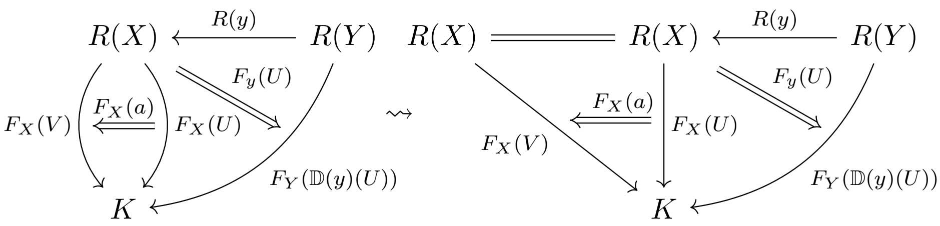
we have precisely that

$$\mathbb{D} \otimes_{\mathcal{C}} \mathbb{E} \simeq \mathbb{E} \otimes_{\mathcal{C}^{op}} \mathbb{D}.$$

3.2 Conification of colimits

Lax weighted colimits can be 'conified', i.e. they can be interpreted as conical colimits: we will exploit this to compute colimits in **Cat**, for there is an easy way of computing lax conical colimits via the Grothendieck construction.

The idea behind conification is simple: if we take a $\mathbb{D}$-weighted lax cocone under a diagram $R : \mathcal{C} \to \mathbf{CAT}$, we can open up the spindles under each node $R(X)$ into triangles, to obtain a conical cocone,



Since the nodes of the 'conified' cocone are indexed by the pairs (X, U) where U belongs to $\mathbb{D}(X)$, it is no longer under the diagram R, but under a diagram over the category $\mathcal{G}(\mathbb{D})$ instead. The formal statement of this is contained in the next results.

Proposition 3.2.1. *Consider two $\mathcal{C}$-indexed categories $\mathbb{D}$, $\mathbb{E} : \mathcal{C}^{op} \to \mathbf{CAT}$, and by $p_{\mathbb{D}} : \mathcal{G}(\mathbb{D}) \to \mathcal{C}$ the Grothendieck fibration associated to $\mathbb{D}$: then*

$$[\mathcal{C}^{op}, \mathbf{CAT}]_{oplax}(\mathbb{D}, \mathbb{E}) \cong [\mathcal{G}(\mathbb{D})^{op}, \mathbf{CAT}]_{oplax}(\Delta \mathbb{1}, \mathbb{E} \circ p^{op}_{\mathbb{D}}),$$

$$[\mathcal{C}^{op}, \mathbf{CAT}]_{lax}(\mathbb{D}, \mathbb{E}) \cong [\mathcal{G}(\mathbb{D}^{V})^{op}, \mathbf{CAT}]_{lax}(\Delta \mathbb{1}, \mathbb{E} \circ p^{op}_{\mathbb{D}^{V}}),$$

where the notation $(-)^{V}$ was introduced in Example 2.1.1(v).

If moreover $\mathbb{D}$ is discrete, i.e. it is in fact a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$, then the equivalences above reduce to

$$[\mathcal{C}^{op}, \mathbf{CAT}]_{ps}(P, \mathbb{E}) \cong [(\textstyle\int P)^{op}, \mathbf{CAT}]_{ps}(\Delta \mathbb{1}, \mathbb{E} \circ p_{P}).$$

Proof. An oplax transformation $F : \mathbb{D} \Rightarrow \mathbb{E}$ is a collection of functors $F_X : \mathbb{D}(X) \to \mathbb{E}(X)$ indexed by objects X in $\mathcal{C}$ and of natural transformations $F_y : F_Y \mathbb{D}(y) \Rightarrow \mathbb{E}(y) F_X$ indexed by arrows $y : Y \to X$; on the other side, an oplax transformation $\bar{F} : \Delta \mathbb{1} \to \mathbb{E} \circ p^{op}$ consists of an object $\bar{F}_{(X,U)} \in \mathbb{E}(X)$ for each (X, U) in $\mathcal{G}(\mathbb{D})$ and of an arrow $\bar{F}_{(y,a)} : \bar{F}_{(Y,V)} \to \mathbb{E}(y)(\bar{F}_{(X,U)})$ for each $(y, a) : (Y, V) \to (X, U)$. Starting from F, we can define $\bar{F}$ as follows:

$$\bar{F}_{(X,U)} := F_X(U), \ \bar{F}_{(y,a)} := F_y(U) F_Y(a) \text{ for } (y, a) : (Y, V) \to (X, U)$$

Conversely, starting from $\bar{F}$ we can define F by setting

$$F_X(U) := \bar{F}_{(X,U)}, \ F_X(a) := \varphi^{\mathbb{E}}_X(\bar{F}_{(X,U')})^{-1} \bar{F}_{(1_X, \varphi^{\mathbb{D}}_X(U')a)} \text{ for } a : U \to U'$$

$$F_y(U) := \bar{F}_{(y, 1_{\mathbb{D}(y)(U)})}$$

Then F satisfies the axioms of an oplax transformation if and only if $\bar{F}$ does; moreover, the associations $F \mapsto \bar{F}$ and $\bar{F} \mapsto F$ can be extended to modifications, and they provide the isomorphism of categories in the claim.

The second identity follows from the first one by applying the isomorphism

$$[\mathcal{C}^{op}, \mathbf{CAT}]_{lax}(\mathbb{D}, \mathbb{E}^V) \cong [\mathcal{C}^{op}, \mathbf{CAT}]_{oplax}(\mathbb{D}^V, \mathbb{E}),$$

which can be readily checked, and the equality $(\mathbb{E} \circ p^{op}_{\mathbb{D}^V})^V = \mathbb{E}^V \circ p^{op}_{\mathbb{D}^V}$.

The last claim is an immediate consequence of the equivalence we defined above. If $\mathbb{D}$ is a presheaf P, the only arrows in its fibres are identity morphisms, implying that $\bar{F}_{(y, 1_{\mathbb{D}(y)(U)})} = F_y(U)$ and hence that F_y is invertible if and only if every $\bar{F}_{(y, 1_{\mathbb{D}}(y)(U))}$ is. Thus F is pseudonatural if and only if $\bar{F}$ is. □

Now, if we consider in particular a pseudofunctor $R : \mathcal{C} \to \mathcal{K}$, an object K in $\mathcal{K}$ and set $\mathbb{E} := \mathcal{K}(R(-), K) : \mathcal{C}^{op} \to \mathbf{CAT}$, the previous result has the following corollary:

Corollary 3.2.2. *Consider a $\mathcal{C}$-indexed category $\mathbb{D}$, with $p_{\mathbb{D}} : \mathcal{G}(\mathbb{D}) \to \mathcal{C}$ its corresponding Grothendieck fibration, and a pseudofunctor $R : \mathcal{C} \to \mathcal{K}$. The $\mathbb{D}$-weighted lax colimit of R is isomorphic to the conical colimit of $R \circ p_{\mathbb{D}^V}$:*

$$\operatorname{colim}^{\mathbb{D}}_{lax} R \simeq \operatorname{colim}_{lax}(R \circ p_{\mathbb{D}^V}).$$

Similarly, the $\mathbb{D}$-weighted oplax colimit of R is isomorphic to the conical colimit of $R \circ p_{\mathbb{D}}$:

$$\operatorname{colim}^{\mathbb{D}}_{oplax} R \simeq \operatorname{colim}_{oplax}(R \circ p_{\mathbb{D}}).$$

*For a presheaf $P : \mathcal{C}^{op} \to$ **Set** with Grothendieck fibration $p : \int P \to \mathcal{C}$, the P-weighted pseudocolimit of R is isomorphic to the conical colimit of $R \circ p$:*

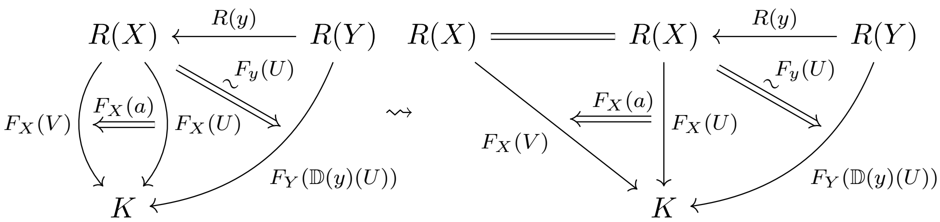
$$\operatorname{colim}^{P}_{ps} R \simeq \operatorname{colim}_{ps}(R \circ p).$$

The commutativity of weights and diagrams expressed by Proposition 3.1.2 can also be expressed in a conified version as follows:

Corollary 3.2.3. *Consider two pseudofunctors* $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ *and* $\mathbb{E} : \mathcal{C} \to \mathbf{CAT}$*: then*

$$\mathrm{colim}_{lax}(\mathbb{E} \circ p_{\mathbb{D}^V}) \simeq \mathrm{colim}_{oplax}(\mathbb{D} \circ p_{\mathbb{E}}).$$

Remark 3.2.1. Corollary 3.2.2 states that we can conify a pseudocolimit with a discrete weight. However, this does not hold for a general pseudocolimit. To understand this, consider two pseudofunctors $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$, $R : \mathcal{C} \to \mathbf{CAT}$ and a $\mathbb{D}$-weighted cocone under R as the one on the left: when we open the spindles of the cocone, we obtain the diagram on the right.



The transformations of the form $F_y(U)$ are invertible, but in general not those of the form $F_X(a)$, and hence the cone on the right cannot be the cocone of a *pseudo*-colimit. Thus in general none of the colimits

$$\mathrm{colim}^{\mathbb{D}}_{ps}\,\mathbb{E},\ \mathrm{colim}_{ps}(\mathbb{E} \circ p_{\mathbb{D}})\ \mathrm{colim}_{ps}(\mathbb{E} \circ p_{\mathbb{D}^V})$$

are equivalent: we shall provide an explicit example of this using Giraud toposes in Remark 6.3.4. This justifies the interest in *lax* colimits: the commutativity of weights and diagrams in their conified expression allows for some extra elasticity in their expression and their computation.

3.3 Weighted colimits in Cat

Using the Grothendieck construction we can provide an explicit way of computing weighted pseudocolimits in **Cat**. The starting point is the following well-known result:

Proposition 3.3.1. *Consider a pseudofunctor* $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{Cat}$*: then*

$$\mathrm{colim}_{lax}\,\mathbb{D} \simeq \mathcal{G}(\mathbb{D}),$$

$$\mathrm{colim}_{oplax}\,\mathbb{D} \simeq \mathcal{G}(\mathbb{D}^V)^{op}.$$

Proof. The first claim is proved in [19, Section 10.2], while the second is sketched for covariant pseudofunctors in the paragraph *As an oplax colimit* of [31]. The second claim can also be proved as a consequence of the first by the following chain of natural equivalences:

$$\begin{aligned}
\mathbf{Cat}(\operatorname{colim}_{oplax}\mathbb{D},\mathcal{K}) &\simeq [\mathcal{C},\mathbf{Cat}]_{oplax}(\Delta\mathbb{1},\mathbf{Cat}(\mathbb{D}(-),\mathcal{K}))\\
&\simeq [\mathcal{C},\mathbf{Cat}]_{oplax}((\Delta\mathbb{1})^{V},\mathbf{Cat}(\mathbb{D}(-),\mathcal{K}))\\
&\simeq [\mathcal{C},\mathbf{Cat}]_{lax}(\Delta\mathbb{1},\mathbf{Cat}(\mathbb{D}(-),\mathcal{K})^{V})\\
&\simeq [\mathcal{C},\mathbf{Cat}]_{lax}(\Delta\mathbb{1},\mathbf{Cat}(\mathbb{D}^{V}(-),\mathcal{K}^{op}))\\
&\simeq \mathbf{Cat}(\operatorname{colim}_{lax}(\mathbb{D}^{V}),\mathcal{K}^{op})\\
&\simeq \mathbf{Cat}((\operatorname{colim}_{lax}(\mathbb{D}^{V}))^{op},\mathcal{K}).
\end{aligned}$$

□

The colimit cocone of $\mathcal{G}(\mathbb{D})$ is made of the following triangles:

$$\begin{array}{ccc}
\mathbb{D}(X) & \xrightarrow{\mathbb{D}(y)} & \mathbb{D}(Y)\\
 & {\scriptstyle i_X}\searrow \quad \overset{i_y}{\Leftarrow} \quad \swarrow{\scriptstyle i_Y} & \\
 & \mathcal{G}(\mathbb{D}) &
\end{array}\ .$$

Each functor $i_X : \mathbb{D}(X) \to \mathcal{G}(\mathbb{D})$ is the usual inclusion of fibres, which maps an object U to the object (X, U) and acts on arrows accordingly. The natural transformation i_y is defined componentwise, for U in $\mathbb{D}(X)$, as the arrow

$$i_y(U) := (y, 1_{\mathbb{D}(y)(U)}) : (Y, \mathbb{D}(y)(U)) \to (X, U).$$

On the other hand, objects of the category $\mathcal{G}(\mathbb{D}^V)^{op}$ are still pairs (X, U) with X in $\mathcal{C}$ and U in $\mathbb{D}(X)$, but an arrow $(y, a) : (X, U) \to (Y, V)$ is indexed by an arrow $y : Y \to X$ and an arrow $a : \mathbb{D}(y)(U) \to V$. The colimit cocone

$$\begin{array}{ccc}
\mathbb{D}(X) & \xrightarrow{\mathbb{D}(y)} & \mathbb{D}(Y)\\
 & {\scriptstyle j_X}\searrow \quad \overset{j_y}{\Rightarrow} \quad \swarrow{\scriptstyle j_Y} & \\
 & \operatorname{colim}_{oplax}\mathbb{D} &
\end{array}$$

is defined as follows: the functor j_X maps an object U in $\mathbb{D}(X)$ to (X, U), and acts on arrows accordingly. The natural transformation $j_y : j_X \Rightarrow j_Y \circ \mathbb{D}(y)$ is defined componentwise, for some U in $\mathbb{D}(X)$, as

$$j_y(U) := (y, 1_{\mathbb{D}(y)(U)}) : (X, U) \to (Y, \mathbb{D}(y)(U)).$$

Weighted op-/lax colimits can be computed by applying conification (Corollary 3.2.2). Weighted *pseudo*colimits can be computed instead by localizing op-/lax colimits, as the next result shows:

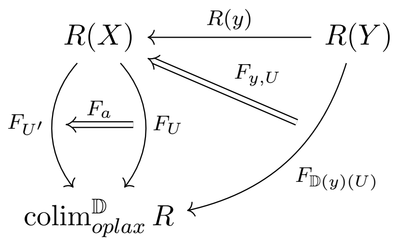
Proposition 3.3.2. *Consider a category $\mathcal{C}$ and two pseudofunctors $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ and $R : \mathcal{C} \to \mathbf{CAT}$. By Corollary 3.2.2 and Proposition 3.1.2, the following two chains of equivalences hold:*

$$\operatorname{colim}_{lax}(R \circ p_{\mathbb{D}^V}) \simeq \operatorname{colim}^{\mathbb{D}}_{lax} R \simeq \operatorname{colim}^{R}_{oplax} \mathbb{D} \simeq \operatorname{colim}_{oplax}(\mathbb{D} \circ p_R),$$

$$\operatorname{colim}_{oplax}(R \circ p_{\mathbb{D}}) \simeq \operatorname{colim}^{\mathbb{D}}_{oplax} R \simeq \operatorname{colim}^{R}_{lax} \mathbb{D} \simeq \operatorname{colim}_{lax}(\mathbb{D} \circ p_{R^V}).$$

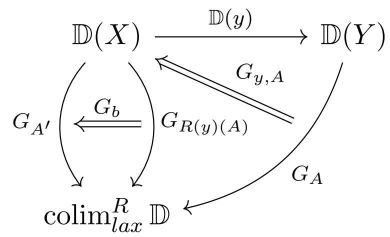
Then $\operatorname{colim}^{\mathbb{D}}_{ps} R$ can be presented as a localization of any of the eight categories above, as follows:

(i) as a localization of $\operatorname{colim}^{\mathbb{D}}_{oplax} R$ or of $\operatorname{colim}^{\mathbb{D}}_{lax} R$: if



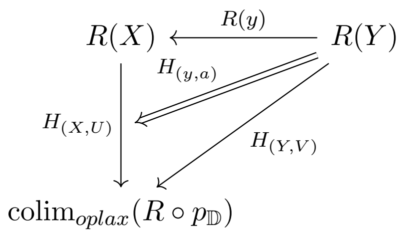
is the colimit cocone of $\operatorname{colim}^{\mathbb{D}}_{oplax} R$ (with $y : Y \to X$ in $\mathcal{C}$ and $a : U \to U'$ in $\mathbb{D}(X)$), the essentially unique functor $\operatorname{colim}^{\mathbb{D}}_{oplax} R \to \operatorname{colim}^{\mathbb{D}}_{ps} R$ is a localization with respect to the components of each natural transformation of the kind $F_{y,U}$. A similar statement holds by considering the colimit cocone of $\operatorname{colim}^{\mathbb{D}}_{lax} R$.

(ii) as a localization of $\operatorname{colim}^{R}_{lax} \mathbb{D}$ or of $\operatorname{colim}^{R}_{oplax} \mathbb{D}$: if



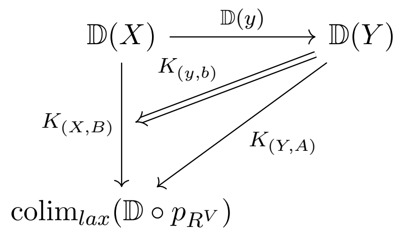
is the colimit cocone of $\operatorname{colim}^{R}_{lax} \mathbb{D}$ (with $y : Y \to X$ in $\mathcal{C}$, A in $R(Y)$ and $b : R(y)(A) \to A'$ in $R(X)$), then the essentially unique functor $\operatorname{colim}^{R}_{lax} \mathbb{D} \to \operatorname{colim}^{\mathbb{D}}_{ps} R$ is a localization with respect to the components of each natural transformation $G_{y,A}$. A similar argument holds for $\operatorname{colim}^{R}_{oplax} \mathbb{D}$.

(iii) *as a localization of* $\mathrm{colim}_{oplax}(R \circ p_{\mathbb{D}})$ *or of* $\mathrm{colim}_{lax}(R \circ p_{\mathbb{D}^V})$*: if*



is the colimit cocone of $\mathrm{colim}_{oplax}(R \circ p_{\mathbb{D}})$ *(with* $(y, a) : (Y, V) \to (X, U)$ *in* $\mathcal{G}(\mathbb{D})$*), then the induced functor* $\mathrm{colim}_{oplax}(R \circ p_{\mathbb{D}}) \to \mathrm{colim}^{\mathbb{D}}_{ps} R$ *is a localization with respect to the components of each natural transformation of the kind* $H_{(y,a)}$ *such that* $(y, a) : (Y, V) \to (X, U)$ *is a cartesian arrow of* $\mathcal{G}(\mathbb{D})$ *(i.e. the component a is invertible). The same considerations hold for* $\mathrm{colim}_{lax}(R \circ p_{\mathbb{D}^V})$*.*

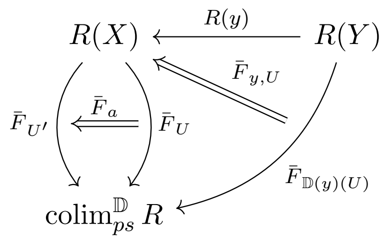
(iv) *as a localization of* $\mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V})$ *or of* $\mathrm{colim}_{oplax}(\mathbb{D} \circ p_R)$*: if*



is the colimit cocone of $\mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V})$ *(with* $(y, b) : (X, B) \to (Y, A)$ *in* $\mathcal{G}(R^V)$*), then the induced functor* $\mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V}) \to \mathrm{colim}^{\mathbb{D}}_{ps} R$ *is a localization with respect to the components of all natural transformations of the form* $K_{(y,b)}$ *where* (y, b) *is a cartesian arrow of* $\mathcal{G}(R^V)$ *(i.e. b is invertible). Similar considerations hold for* $\mathrm{colim}_{oplax}(\mathbb{D} \circ p_R)$*.*

Proof. Items (ii) to (iv) will follow from item (i) by showing how the colimit cocones relate to one another.

(i) Let



.

be the colimit cocone of $\mathrm{colim}^{\mathbb{D}}_{ps} R$. Notice that, being $\mathrm{colim}^{\mathbb{D}}_{ps} R$ a pseudocolimit, the components of each natural transformation $\bar{F}_y(U)$ are invertible. There is a (essentially) unique functor $\xi : \mathrm{colim}^{\mathbb{D}}_{oplax} R \to$

$\operatorname{colim}^{\mathbb{D}}_{ps} R$. It satisfies in particular the identity $\xi \circ F_y(U) = \bar{F}_y(U)$, and thus it maps the components of every $F_y(U)$ to an invertible map. It is now obvious that any functor $K : \operatorname{colim}^{\mathbb{D}}_{oplax} R \to \mathcal{K}$ to any category $\mathcal{K}$ factors through $\operatorname{colim}^{\mathbb{D}}_{ps} R$ if and only if K inverts all the components of the natural transformations of the form $F_y(U)$: therefore, $\operatorname{colim}^{\mathbb{D}}_{ps} R$ is obtained as a localization of $\operatorname{colim}^{\mathbb{D}}_{oplax} R$ with respect to all the components of all the natural transformations of the form $F_y(U)$.

(ii) We can apply the natural equivalences in Proposition 3.1.2 in order to translate the colimit cocone of $\operatorname{colim}^{\mathbb{D}}_{oplax} R$ into the colimit cocone of $\operatorname{colim}^{R}_{lax} \mathbb{D}$: in particular, for any U in $\mathbb{D}(X)$ and A in $R(Y)$ the identity $G_{y,A}(U) = F_{y,U}(A)$ holds. Thus, since $\operatorname{colim}^{\mathbb{D}}_{oplax} R \to \operatorname{colim}^{\mathbb{D}}_{ps} R$ localizes with respect to all the components of the natural transformations $F_{y,U}$, the functor $\operatorname{colim}^{R}_{lax} \mathbb{D} \to \operatorname{colim}^{\mathbb{D}}_{ps} R$ localizes with respect to all the components of the natural transformations $G_{y,A}$.

(iii) We can apply the natural equivalences in Proposition 3.2.1 in order to translate the colimit cocone of $\operatorname{colim}^{\mathbb{D}}_{oplax} R$ into the colimit cocone of $\operatorname{colim}_{oplax}(R \circ p_{\mathbb{D}})$. In particular, the natural transformations $H_{(y,a)}$ are defined componentwise, for A in $R(Y)$, as $H_{(y,a)}(A) := F_{y,U}(A) \circ F_a(A)$. Therefore, since $\operatorname{colim}^{\mathbb{D}}_{oplax} R \to \operatorname{colim}^{\mathbb{D}}_{ps} R$ localizes with respect to the components of each $F_{y,U}$ it follows that $\operatorname{colim}_{oplax}(R \circ p_{\mathbb{D}}) \to \operatorname{colim}^{\mathbb{D}}_{ps} R$ localizes with respect to the components of each $H_{(y,1)}$. Finally, we recall that if a is invertible then F_a is invertible, by functoriality, and thus localizing with respects to the components of the $H_{(y,1)}$ is exactly the same as localizing with respect to the components of the $H_{(y,a)}$ with $(y, a) : (Y, V) \to (X, U)$ cartesian in $\mathcal{G}(\mathbb{D})$.

(iv) We can apply the natural equivalences of Proposition 3.2.1 in order to translate the colimit cocone of $\operatorname{colim}^{R}_{lax} \mathbb{D}$ into the colimit cocone of $\operatorname{colim}_{lax}(\mathbb{D} \circ p_{R^V})$. In particular each natural transformation $K_{(y,b)}$ is defined componentwise, for U in $\mathbb{D}(X)$, as $K_{(y,b)}(U) := G_b(U) \circ G_{y,A}(U)$. With considerations similar to those of the previous item, we can conclude that $\operatorname{colim}_{lax}(\mathbb{D} \circ p_{R^V}) \to \operatorname{colim}^{\mathbb{D}}_{ps} R$ is a localization with respect to the components of all natural transformations $K_{(y,b)}$ with $(y, b) : (X, B) \to (Y, A)$ cartesian in $\mathcal{G}(R^V)$.

□

Remark 3.3.1. The previous result is a generalization of the considerations in Paragraph 6.4.0 of [2, Exposé VI], where it was first shown that since $\mathcal{G}(\mathbb{D}) \simeq \operatorname{colim}_{lax} \mathbb{D}$, the category $\operatorname{colim}_{ps} \mathbb{D}$ is the localization of $\mathcal{G}(\mathbb{D})$ with respect to its cartesian arrows.

By the previous result, to compute explicit a colimit $\operatorname{colim}^{\mathbb{D}}_{ps} R$ in **Cat** we can resort to the representation which we deem more fitting or easier to compute. The two representations in items (i) and (ii) still maintain the distinction between weight and diagram; on the other hand, the conified representations in items (iii) and (iv) are localizations of the 'mixed structures' of the form

$$\operatorname{colim}_{lax}(\mathbb{D}\circ p_{R^V}),\ \operatorname{colim}_{lax}(R\circ p_{\mathbb{D}^V}),\ \operatorname{colim}_{oplax}(R\circ p_{\mathbb{D}}),\ \operatorname{colim}_{oplax}(\mathbb{D}\circ p_R).$$

We will compute explicitly the first of these colimits in the next result, along with its localization to $\operatorname{colim}^{\mathbb{D}}_{ps} R$, to show the 'mixed structure' at play:

Proposition 3.3.3. *Consider two pseudofunctors* $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ *and* $R : \mathcal{C} \to \mathbf{CAT}$*. The colimit*

$$\operatorname{colim}_{lax}(\mathbb{D}\circ p_{R^V})$$

is equivalent to the category whose objects are triples (X, U, B)*, with* X *in* $\mathcal{C}$*,* U *in* $\mathbb{D}(X)$ *and* B *in* $R(X)$*, while an arrow*

$$(y, a, b) : (Y, V, A) \to (X, U, B)$$

is indexed by three arrows $y : Y \to X$ *in* $\mathcal{C}$*,* $a : V \to \mathbb{D}(y)(U)$ *in* $\mathbb{D}(Y)$ *and* $b : R(y)(A) \to B$ *in* $R(X)$*. There is a square*

$$\begin{array}{ccc} \operatorname{colim}_{lax}(\mathbb{D}\circ p_{R^V}) & \xrightarrow{\ q\ } & \mathcal{G}(\mathbb{D}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_{\mathbb{D}}} \\ \mathcal{G}(R^V)^{op} & \xrightarrow[p^{op}_{R^V}]{} & \mathcal{C} \end{array},$$

where p *is a Grothendieck fibration and* q *maps* p*-cartesian arrows to* $p_{\mathbb{D}}$*-cartesian arrows. The functor* p *forgets the second component, the functor* q *the third and the diagonal both components. In particular, the square is a strict pullback.*

The functor

$$\operatorname{colim}_{lax}(\mathbb{D}\circ p_{R^V}) \to \operatorname{colim}^{\mathbb{D}}_{ps} R$$

acts by localizing $\operatorname{colim}_{lax}(\mathbb{D}\circ p_{R^V})$ *with respect to all of the morphisms that are* p*-cartesian and that are mapped through* q *to* $p_{\mathbb{D}}$*-cartesian arrows: that is, it localizes with respect to all morphisms* (y, a, b) *where both* a *and* b *are invertible.*

Proof. The colimit $\operatorname{colim}_{lax}(\mathbb{D}\circ p_{R^V})$ can be computed, by Proposition 3.3.1, as the Grothendieck fibration associated to the pseudofunctor

$$\mathcal{G}(R^V) \xrightarrow{p_{R^V}} \mathcal{C}^{op} \xrightarrow{\mathbb{D}} \mathbf{CAT}.$$

The fibration $p_{R^V} : \mathcal{G}(R^V) \to \mathcal{C}^{op}$ is made of objects (X, B) with B in $R(X)$, and arrows $(y, b) : (X, B) \to (Y, A)$ such that $y : Y \to X$ and

$b : R(y)(A) \to B$. Thus the fibration $p : \mathcal{G}(\mathbb{D} \circ p_{R^V}) \to \mathcal{G}(R^V)^{op}$ is made of triples $((X, B), U)$, where (X, B) belongs to $\mathcal{G}(R^V)$ and $U \in \mathbb{D}(p_{R^V}(X, B)) = \mathbb{D}(X)$, and an arrow

$$((Y, A), V) \xrightarrow{((y,b),a)} ((X, B), U)$$

is indexed by an arrow $(y, a) : (X, B) \to (Y, A)$ in $\mathcal{G}(R^V)$ and another arrow $a : V \to \mathbb{D}(p_{R^V}(y, b))(U) = \mathbb{D}(y)(U)$ in $\mathbb{D}(Y)$: it is obvious that this description of $\mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V})$ is equivalent to the one provided in the claim.

The composite functor

$$\mathcal{G}(\mathbb{D} \circ p_{R^V}) \xrightarrow{p} \mathcal{G}(R^V)^{op} \xrightarrow{p_{R^V}^{op}} \mathcal{C}$$

acts by forgetting the second and the third components. Notice that it is not a fibration, as it originates from the composition of the fibration p with the opfibration $p_{R^V}^{op}$. This functor factors through $p_{\mathbb{D}} : \mathcal{G}(\mathbb{D}) \to \mathcal{C}$, and the factor

$$q : \mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V}) \to \mathcal{G}(\mathbb{D})$$

acts by forgetting the third components in $\mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V})$.

Finally, if we explicit the colimit cocone

$$\begin{array}{ccc} \mathbb{D}(X) & \xrightarrow{\mathbb{D}(y)} & \mathbb{D}(Y) \\ & \overset{K_{(y,b)}}{\Leftarrow} & \\ {\scriptstyle K_{(X,B)}}\downarrow & \swarrow {\scriptstyle K_{(Y,A)}} & \\ \mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V}) & & \end{array},$$

we see that the legs act thus,

$$K_{(X,B)} : \left[U \xrightarrow{a} U' \right] \mapsto \left[(X, U, B) \xrightarrow{(1,a,1)} (X, U', B) \right],$$

while for each U in $\mathbb{D}(X)$ the component of $K_{(y,b)}$ at U is the arrow

$$K_{(y,b)}(U) := \left[(Y, \mathbb{D}(y)(U), A) \xrightarrow{(y,1,b)} (X, U, B) \right].$$

We have omitted, for the sake of readability, any reference to the canonical isomorphisms of the pseudofunctors R and $\mathbb{D}$. The functor

$$\mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V}) \to \mathrm{colim}^{\mathbb{D}}_{ps} R$$

localizes with respect to the components of the natural transformations $K_{(y,b)}$ such that (y, b) is cartesian, i.e. with respect to all the arrows $(y, 1, b)$ with b invertible, but this is the same as localizing with respect to the arrows in the statement. □

Remark 3.3.2. The pullback square above, which shows the connection between $\mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V})$ and $\mathcal{G}(\mathbb{D})$, is a particular instance of the concept of direct image of a $\mathcal{C}$-indexed category along a functor, which we will analyse in Section 4.1. For further details, see Remark 4.1.1(ii).

Every category $\mathcal{C}$ is endowed in particular with the covariant pseudofunctor $\mathcal{C}/- : \mathcal{C} \to \mathbf{CAT}$, which maps each object X to the slice over it. The previous result has the consequence that weighted pseudocolimits for $\mathcal{C}/-$ can be interpreted as lax colimits of the weight:

Corollary 3.3.4. *Consider $R : \mathcal{C} \to \mathbf{CAT}/\mathcal{C}$ mapping each X to $\mathcal{C}/X$: then the $\mathbb{D}$-weighted colimit of R is equivalent to the lax colimit of $\mathbb{D}$.*

Proof. For the sake of brevity, in this proof we shall denote by $\mathcal{K}$ the category $\mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V})$, and also suppress any mention of $\mathbb{D}$'s structural isomorphisms. Applying the previous result to $R := \mathcal{C}/- : \mathcal{C} \to \mathbf{CAT}$ provides the following description of $\mathcal{K}$: objects are triples $(X, U \in \mathbb{D}(X), w : W \to X)$ with X,W and w in $\mathcal{C}$ and U in $\mathbb{D}(X)$, and an arrow

$$(Y, V, z : Z \to Y) \xrightarrow{(y,a,b)} (X, U, w : W \to X)$$

is indexed by two arrows $y : Y \to X$, $b : Z \to W$ in $\mathcal{C}$ such that $w \circ b = y \circ z$, and a further arrow $a : V \to \mathbb{D}(y)(U)$ in $\mathbb{D}(Y)$. We can then define a functor

$$L : \mathcal{K} \to \mathcal{G}(\mathbb{D})$$

acting as follows:

$$\left[(Y, V, [z]) \xrightarrow{(y,a,b)} (X, U, [w])\right] \mapsto \left[(Z, \mathbb{D}(z)(V)) \xrightarrow{(b,\mathbb{D}(z)(a))} (W, \mathbb{D}(w)(U))\right].$$

If we denote by S the class of arrows (y, a, b) in $\mathcal{K}$ with both components a and b invertible, then L inverts all arrows in S: if we prove that L satisfies the universal property of the localization with respect to S, we can conclude that

$$\mathrm{colim}_{lax}\, \mathbb{D} \simeq \mathcal{G}(\mathbb{D}) \simeq \mathcal{K}[S^{-1}] \simeq \mathrm{colim}^{\mathbb{D}}_{ps}\, R,$$

by the previous result and by Proposition 3.3.1. To show this consider any functor $H : \mathcal{K} \to \mathcal{A}$ mapping any arrow in S to an invertible arrow in $\mathcal{A}$: we want to define an essentially unique functor

$$\bar{H} : \mathcal{G}(\mathbb{D}) \to \mathcal{A}$$

such that $\bar{H} \circ L \cong H$. This condition forces the definition of $\bar{H}$, up to natural isomorphism: indeed, if we consider the invertible arrows

$$L(X, U, 1_X) = (X, \mathbb{D}(1_X)(U)) \xrightarrow{(1,1)} (X, U)$$

of $\mathcal{G}(\mathbb{D})$, they must be mapped via $\bar{H}$ to isomorphisms

$$\bar{H} \circ L(X, U, 1_X) = H(X, U, 1_X) \xrightarrow{\bar{H}(1,1)} \bar{H}(X, U).$$

Therefore we can define $\bar{H}$ as follows: for any $(y, a) : (Y, V) \to (X, U)$ in $\mathcal{G}(\mathbb{D})$, we set its image via $\bar{H}$ as the arrow

$$H(Y, V, 1_Y) \xrightarrow{H(y,a,y)} H(X, U, 1_X).$$

Finally, we prove that $\bar{H} \circ L \cong H$. For this, notice that every object $(X, U, w : W \to X)$ in $\mathcal{K}$ admits a canonical morphism

$$(W, \mathbb{D}(w)(U), 1_W) \xrightarrow{(w,1,1)} (X, U, w),$$

which belongs to S: its image via H is therefore an isomorphism

$$H(W, \mathbb{D}(w)(U), 1_W) = \bar{H}(W, \mathbb{D}(w)(U)) = \bar{H} \circ L(X, U, w) \xrightarrow{(w,1,1)} H(X, U, w),$$

and it provides the component in (X, U, w) of a natural isomorphism $\bar{H} \circ L \cong H$. $\square$

Remarks 3.3.3. (i) We will provide a further point of view on this result in Remark 4.2.1, by using inverse images of fibrations. It can also be seen as a consequence of Corollary 6.2.2, which shows that $\mathrm{colim}^{\mathbb{D}}_{ps}(\mathcal{C}/-) \simeq \mathcal{G}(\mathbb{D})$ by exploiting the right adjoint of the 2-functor $\mathcal{G}$.

(ii) Curiously, the localization $L : \mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V}) \to \mathrm{colim}^{\mathbb{D}}_{ps} R \simeq \mathcal{G}(\mathbb{D})$ acts by 'restriction' of the diagram, and this because every fibre $R(X) = \mathcal{C}/X$ of R has a terminal object $[1_X]$. First of all, notice that

$$\mathcal{G}(R^V) \xrightarrow{p_{R^V}} \mathcal{C}^{op}$$

is the opposite of the codomain functor $\mathrm{cod} : Mor(\mathcal{C}) \to \mathcal{C}$: objects in $\mathcal{G}(R^V)$ are arrows $[w : W \to X]$, and a morphism $(y, b) : [w : W \to X] \to [z : Z \to Y]$ is given by two arrows $y : Y \to X$ and $b : Z \to W$ such that $w \circ b = y \circ z$. The fact that each fibre of R has a terminal implies that we can define a functor

$$T : \mathcal{C} \to \mathcal{G}(R^V)^{op} = Mor(\mathcal{C})$$

by mapping every object X to $[1_X]$ and every $y : Y \to X$ to $(y, y) : [1_Y] \to [1_X]$. This functor is (a right adjoint and) a section of cod, i.e. $\mathrm{cod} \circ T = \mathrm{id}_{\mathcal{C}}$: therefore, since $p_{R^V} \cong \mathrm{cod}^{op}$, we can conclude that

$$\mathrm{colim}_{lax}\, \mathbb{D} \simeq \mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V} \circ T^{op}) = \mathrm{colim}_{lax}\left((\mathbb{D} \circ p_{R^V}) \circ T^{op}\right).$$

We can really think of $\mathcal{G}(\mathbb{D})$ as obtained by restricting the cocone of $\mathrm{colim}_{lax}(\mathbb{D} \circ p_{R^V})$ along the functor T. As we have formulated it, it is evident that this happens everytime the covariant pseudofunctor R we consider (not just $R = \mathcal{C}/-$) has fibres with terminal objects.

Chapter 4

Change of base for fibrations

Sheaves over topological spaces can be ‘moved along’ continuous maps, in a process known as *change of base* (see Section 6.1 for a quick summary on topological sheaves). More precisely, a continuous map of topological spaces $f : X \to Y$ induces a geometric morphism

$$\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$$

acting as follows:

- given a sheaf $P \in \mathbf{Sh}(X)$, the *direct image* of P along f, i.e. the presheaf $\mathbf{Sh}(f)_*(P)$, is defined for any $U \subseteq Y$ open as

$$\mathbf{Sh}(f)_*(P)(U) := P(f^{-1}(U));$$

- starting from a sheaf $Q \in \mathbf{Sh}(Y)$, and denoting by $p_Q : E_Q \to Y$ its étale bundle over Y, we can pull it back along $f : X \to Y$ in order to obtain an étale bundle over X: the corresponding sheaf is the *inverse image* $\mathbf{Sh}(f)^*(Q)$ of Q along f.

This process generalizes to the notion of geometric morphism induced by a morphism of sites, which we recalled in Definition 1.2.1. In the present chapter we will consider the same kind of change of base processes when working with fibrations and stacks: a classic reference for these constructions is provided by Sections I.2 and II.3 of [11]. The material presented in this chapter comes from Chapter 3 of [8].

The generalization of Kan extensions to indexed categories allows to extend to categories of fibrations and stacks the constructions of the geometric morphisms induced by morphisms or comorphisms of sites: in particular, in this chapter we shall provide an explicit description of the inverse image of fibrations as a pseudocolimit of categories, and as a localization of a particular fibration of generalized elements. We will also study the relationship between base change and the relative comprehensive factorization of functors introduced in [6], and we will show that pullbacks of Giraud toposes can be computed at the level of sites, if one adopts the point of view of comorphisms of sites.

4.1 Direct image of fibrations

The *direct image* along a functor is the simplest change of base technique for fibrations, as it is a categorical pullback. Indeed, it is well known that if $q : \mathcal{E} \to \mathcal{D}$ is a Grothendieck fibration, its pullback along a functor $F : \mathcal{C} \to \mathcal{D}$ provides a fibration over $\mathcal{C}$. Let us show here that weakening to pseudopullbacks allows us to talk about the direct image of Street fibrations:

Proposition 4.1.1. *Consider a functor $F : \mathcal{C} \to \mathcal{D}$, a fibration $q : \mathcal{E} \to \mathcal{D}$ and the strict pseudopullback (cf. Definition 2.1.6)*

$$\begin{array}{ccc} \mathcal{C} \times_{\mathcal{D}} \mathcal{E} & \xrightarrow{\pi} & \mathcal{E} \\ {\scriptstyle p}\downarrow & \overset{\sim}{\swarrow} & \downarrow{\scriptstyle q} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

the functor p is a fibration over $\mathcal{C}$, called the direct image of q along F, *and is denoted by $F^*(q)$; the functor π maps p-cartesian arrows to q-cartesian arrows. In particular, if q is cloven then its inverse image p is also cloven. Computing the direct image of fibrations over $\mathcal{D}$ along the functor F provides a 2-functor*

$$F^* : \mathbf{Fib}_{\mathcal{D}} \to \mathbf{Fib}_{\mathcal{C}},$$

defined by mapping a morphism $(A, \alpha) : [q : \mathcal{E} \to \mathcal{D}] \to [r : \mathcal{R} \to \mathcal{D}]$ to the functor

$$F^*(A, \alpha) : \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{R},$$

defined on objects as $F^*(A, \alpha)(X, E, f : q(E) \xrightarrow{\sim} F(X)) := (X, A(E), f\alpha_E)$ and on arrows as $F^*(A, \alpha)(t, s) = (t, A(s))$. Similarly, a 2-cell $\gamma : (A, \alpha) \Rightarrow (B, \beta)$ of morphisms of fibrations is sent to the 2-cell $F^*(\gamma) : F_*(A, \alpha) \Rightarrow F_*(B, \beta)$, defined componentwise by

$$F^*(\gamma)(X, E, f) := (1, \gamma_E) : (X, A(E), f\alpha_E) \to (X, B(E), f\beta_E).$$

If we consider the pseudofunctor $\mathfrak{I}([q]) = \mathbb{E}$, then $\mathfrak{I}(F^*([q])) \simeq \mathbb{E} \circ F^{op}$: that is, on $\mathcal{D}$-indexed categories the direct image along F can be described as precomposition with F^{op}.

Proof. Start with an object $(X, E, f : F(X) \xrightarrow{\sim} q(E))$ of $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ and an arrow $y : Y \to X$: we want to build a cartesian lift of y with codomain (X, E, f). To do so, consider the composite arrow $f \circ F(y) : F(Y) \to q(E)$ in $\mathcal{D}$: since q is a fibration, there are a cartesian arrow $\bar{y} : E' \to E$ in $\mathcal{E}$

and an isomorphism $g : F(Y) \xrightarrow{\sim} q(E')$ such that $q(\bar{y}) \circ g = f \circ F(y)$. This provides an arrow $(y, \bar{y}) : (Y, E', g) \to (X, E, f)$ such that $p(y, \bar{y}) = y$, which is a cartesian lift for y.

It is obvious that if q is cloven then p is also cloven, since a canonical choice of lifting for q entails a canonical choice of liftings for p. The fact that F^* is a 2-functor is straightforward: in particular, the functor $F^*(A, \alpha)$ exists by the universal property of pseudopullbacks, while the fact that it preserves cartesian arrows follows from the fact that A does.

Finally, by unwinding the definition of $\mathfrak{I}$ we have that for an object X in $\mathcal{C}$ the category $\mathfrak{I}(F^*([q]))(X)$ has as objects quadruples

$$(Y, E, f : F(Y) \xrightarrow{\sim} q(E), \gamma : X \xrightarrow{\sim} Y),$$

where Y belongs to $\mathcal{C}$ and E to $\mathcal{E}$. We can associate to each object $(E, \alpha : F(X) \xrightarrow{\sim} q(E))$ of $\mathfrak{I}([q])(F(X))$ the object $(X, E, \alpha, 1_X)$ of $\mathfrak{I}(F^*([q]))(X)$, and conversely to the object (Y, E, f, γ) above one can associate the object $(E, f \circ F(\gamma) : F(X) \xrightarrow{\sim} q(E))$ of $\mathfrak{I}([q])(F(X))$. This association extends to arrows and provides an equivalence $\mathfrak{I}(F^*([q]))(X) \simeq \mathfrak{I}([q])(F(X))$, which is also pseudonatural in X. We can thus conclude that F^* acts on $\mathcal{D}$-indexed categories as a precomposition 2-functor. □

Remarks 4.1.1. (i) By their very definition, we can now think of fibres as direct images. Given a fibration $p : \mathcal{D} \to \mathcal{C}$ and an object X in $\mathcal{C}$, the fibre of p at X is defined as the pseudopullback of p along the functor from the one-object category,

$$e_X : \mathbb{1} \to \mathcal{C},$$

which selects the object X. In short, the fibre of p at X is the domain of $e_X^*([p])$.

(ii) The fact that the pullback of a fibration is still a fibration can be interpreted as a result about the compatibility of pullbacks and colimits. Indeed, given a pseudofunctor $\mathbb{E} : \mathcal{D}^{op} \to \mathbf{CAT}$, by Proposition 3.3.1 we know that $\mathcal{G}(\mathbb{E}) \simeq \mathrm{colim}_{lax}\, \mathbb{E}$ in $\mathbf{CAT}$, and thus *a fortiori* in $\mathbf{CAT}/\mathcal{D}$. If now we consider the previous result, we have that for any functor $F : \mathcal{C} \to \mathcal{D}$ the pseudopullback of $\mathrm{colim}_{lax}(\mathbb{E})$ along F is equivalent to the colimit $\mathrm{colim}_{lax}(\mathbb{E} \circ F^{op})$.

(iii) The argument in the previous item allows us to apply arguments about direct images of fibrations also to the topic of commutativity of weights and diagrams. Given two pseudofunctors $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ and $R : \mathcal{C} \to \mathbf{CAT}$, we know by Corollary 3.2.3 that there exists an equivalence $\mathrm{colim}_{oplax}(R^V \circ p_{\mathbb{D}}) \simeq \mathrm{colim}_{lax}(\mathbb{D} \circ p_R)$. However, we can think of

$\mathrm{colim}_{lax}(\mathbb{D} \circ p_R)$ as the fibration associated to the direct image of $\mathbb{D}$ along p_R^{op}; in other words, we have a pullback diagram

$$\begin{array}{ccccc}
\mathrm{colim}_{lax}(\mathbb{D} \circ p_R) & \xrightarrow{\sim} & \mathcal{G}(\mathbb{D} \circ p_R) & \xrightarrow{(p_R^{op})^*[p_{\mathbb{D}}]} & \mathcal{G}(R)^{op} \\
\big\downarrow & & \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p_R^{op}} \\
\mathrm{colim}_{lax}(\mathbb{D}) & \xrightarrow{\sim} & \mathcal{G}(\mathbb{D}) & \xrightarrow[p_{\mathbb{D}}]{} & \mathcal{C}
\end{array}\;.$$

If we apply $(-)^{op}$ to the square on the right, it remains a pullback square, but this time we can see it as computing the pullback of p_R along the functor $p_{\mathbb{D}}^{op}$, i.e. the inverse image of R along $p_{\mathbb{D}}^{op}$:

$$\begin{array}{ccc}
\mathrm{colim}_{lax}(R \circ p_{\mathbb{D}}) & \longrightarrow & \mathrm{colim}_{lax}(R) \\
\wr\big| & & \big|\wr \\
\mathcal{G}(R \circ p_{\mathbb{D}}) & \longrightarrow & \mathcal{G}(R) \\
{\scriptstyle (p_{\mathbb{D}}^{op})^*[p_R]}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p_R} \\
\mathcal{G}(\mathbb{D})^{op} & \xrightarrow[p_{\mathbb{D}}^{op}]{} & \mathcal{C}^{op}
\end{array}\;.$$

This implies that there is an equivalence of categories

$$\mathrm{colim}_{lax}(R \circ p_{\mathbb{D}}) \simeq \mathrm{colim}_{lax}(\mathbb{D} \circ p_R)^{op},$$

which expresses an alternative form of the commutativity of diagrams and colimits. Combining this with the identity from Corollary 3.2.3 we also obtain the further equivalence

$$\mathrm{colim}_{oplax}(R^V \circ p_{\mathbb{D}}) \simeq \mathrm{colim}_{lax}(R \circ p_{\mathbb{D}})^{op}.$$

4.2 Inverse image of fibrations

We have said that the direct image of fibrations, from an indexed perspective, is a precomposition functor. Its restriction to presheaves,

$$(- \circ F^{op}) : [\mathcal{D}^{op}, \mathbf{Set}] \to [\mathcal{C}^{op}, \mathbf{Set}],$$

was the starting point of the notion of morphism and comorphism of sites in Section 1.2. In particular, Proposition 1.2.1 shows that $- \circ F^{op}$ admits both a left and a right adjoint, which are the Kan extension functors: the same happens at the level of indexed categories, providing the notion of *pseudo-Kan extensions.*

Proposition 4.2.1. *We denote by* $\mathbf{Ind}^s_{\mathcal{C}}$ *the sub-2-category of* $\mathbf{Ind}_{\mathcal{C}}$ *of pseudofunctors with values in* **Cat** *(i.e. 'small'* $\mathcal{C}$*-indexed categories). Consider any functor* $F : \mathcal{C} \to \mathcal{D}$ *and the* direct image *2-functor*

$$F^* : \mathbf{Ind}^s_{\mathcal{D}} \to \mathbf{Ind}^s_{\mathcal{C}}$$

which acts by precomposition with F^{op}. The 2-functor F^* has both a left and a right 2-adjoint, denoted respectively by $\mathrm{Lan}_{F^{op}}$ *and* $\mathrm{Ran}_{F^{op}}$*, which act as follows:*

- *for any D in $\mathcal{D}$ denote by $\pi_F^D : (D{\downarrow}F) \to \mathcal{C}$ the canonical projection functor: then for $\mathbb{E} : \mathcal{C}^{op} \to \mathbf{Cat}$, its image $\mathrm{Lan}_{F^{op}}(\mathbb{E}) : \mathcal{D}^{op} \to \mathbf{Cat}$ is defined componentwise as*

$$\mathrm{Lan}_{F^{op}}(\mathbb{E})(D) = \mathrm{colim}_{ps}\left((D{\downarrow}F)^{op} \xrightarrow{(\pi_F^D)^{op}} \mathcal{C}^{op} \xrightarrow{\mathbb{E}} \mathbf{Cat}\right).$$

  The pseudofunctor $\mathrm{Lan}_{F^{op}}(\mathbb{E})$ *is called the* inverse image of $\mathbb{E}$ along F*.*

- *for any D in $\mathcal{D}$ denote by $\pi'^D_F : (F{\downarrow}D) \to \mathcal{C}$ the canonical projection functor: then for $\mathbb{E} : \mathcal{C}^{op} \to \mathbf{Cat}$, its image $\mathrm{Ran}_{F^{op}}(\mathbb{E}) : \mathcal{D}^{op} \to \mathbf{Cat}$ is defined componentwise as*

$$\mathrm{Ran}_{F^{op}}(\mathbb{E})(D) = \mathrm{lim}_{ps}\left((F{\downarrow}D)^{op} \xrightarrow{(\pi'^D_F)^{op}} \mathcal{C}^{op} \xrightarrow{\mathbb{E}} \mathbf{Cat}\right).$$

Proof. The approach used to compute enriched Kan extensions in [23, Chapter 4] is still viable in this context. For instance, it is shown *loc. cit.* that the left Kan extension $\mathrm{Lan}_{F^{op}}\mathbb{E}$ can be computed componentwise in D as the pseudocolimit $\mathrm{colim}_{ps}^{\mathcal{D}(D,F(-))}\mathbb{E}$; the equivalence with our expression above results from applying the last part of Corollary 3.2.2 and noticing that π_F^D is the opfibration corresponding to $\mathcal{D}(D, F(-))$. □

Consider now the functor

$$\tau_F : \mathcal{C} \to \mathbf{cFib}_{\mathcal{D}},$$

which maps every object X in $\mathcal{C}$ to the discrete fibration $(1_{\mathcal{D}}{\downarrow}F(X)) \to \mathcal{D}$. We can also think of τ_F as a functor with values in $\mathbf{Ind}_{\mathcal{D}}$, where each X is mapped to the pseudofunctor $(-{\downarrow}F(X)) = \mathcal{D}(-, F(X)) : \mathcal{D}^{op} \to \mathbf{Cat}$: with this in mind, we can improve the pointwise description of inverse images in the previous result, obtaining a description in terms of weighted colimits of indexed categories.

Proposition 4.2.2. *Consider a functor $F : \mathcal{C} \to \mathcal{D}$ and a pseudofunctor $\mathbb{E} : \mathcal{C}^{op} \to \mathbf{Cat}$: the inverse image $\mathrm{Lan}_{F^{op}}(\mathbb{E})$ is computed as the weighted pseudocolimit*

$$\mathrm{colim}_{ps}^{\mathbb{E}}(\tau_F).$$

Proof. This follows from the identity $\tau_F(-) = \mathcal{D}/F(-)$ and Yoneda's lemma: for every $\mathbb{F} : \mathcal{D}^{op} \to \mathbf{Cat}$ we have the chain of natural equivalences

$$\begin{aligned}\mathbf{Ind}_{\mathcal{D}}(\mathrm{colim}_{ps}^{\mathbb{E}}\,\tau_F, \mathbb{F}) &\simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{E}, \mathbf{Ind}_{\mathcal{D}}(\tau_F(-), \mathbb{F}))\\ &\simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{E}, \mathbf{Ind}_{\mathcal{D}}(\mathcal{D}/F(-), \mathbb{F}))\\ &\simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{E}, \mathbb{F}(F(-)))\\ &\simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{E}, F^*(\mathbb{F})),\end{aligned}$$

implying that $\mathrm{Lan}_{F^{op}}(\mathbb{E}) \simeq \mathrm{colim}^{\mathbb{E}}_{ps}\, \tau_F$. $\square$

Remarks 4.2.1. (i) This result justifies in particular the equivalence

$$\mathrm{colim}^{\mathbb{D}}_{ps}\, \mathcal{C}/- \simeq \mathrm{colim}_{lax}\, \mathbb{D}$$

of Corollary 3.3.4. One can easily see that $\mathcal{C}/-$ is the functor

$$\tau_{1_{\mathcal{C}}} : \mathcal{C} \to \mathbf{cFib}_{\mathcal{C}},$$

and thus the first colimit is the inverse image of $p_{\mathbb{D}}$ along the functor $1_{\mathcal{C}}$: but obviously direct and inverse images along an identity functor are identity functors themselves, and hence we have that

$$\mathrm{colim}^{\mathbb{D}}_{ps}\, \mathcal{C}/- \simeq \mathrm{Lan}_{1_{\mathcal{C}}^{op}}[p_{\mathbb{D}}] \simeq \mathcal{G}(\mathbb{D}) \simeq \mathrm{colim}_{lax}\, \mathbb{D}$$

as fibrations over $\mathcal{C}$.

(ii) The restriction to small $\mathcal{C}$-indexed categories is needed to ensure the existence of the colimits involved in the definition of $\mathrm{Lan}_{F^{op}}$. However, inverse images of non-small indexed categories may still be computable resorting to small subdiagrams or to cofinality arguments.

The fibrewise description of $\mathrm{Lan}_{F^{op}}(\mathbb{E})$ of Proposition 4.2.1 can be recovered from Proposition 4.2.2 using evaluation functors. For a fixed object D of $\mathcal{D}$, the *evaluation 2-functor*

$$\mathrm{Ev}_D : \mathbf{cFib}_{\mathcal{D}} \to \mathbf{CAT}$$

maps each fibration $q : \mathcal{Q} \to \mathcal{D}$ to its fibre in D, or analogously, each $\mathcal{D}$-indexed category $\mathbb{F}$ to its fibre $\mathbb{F}(D)$. Notice that the 2-functor Ev_D behaves as the direct image along the functor $e_D : \mathbb{1} \to \mathcal{D}$ which selects the object D in $\mathcal{D}$. From a fibrational point of view this is trivial, because the fibre at D is computed precisely as the direct image along e_D (see Remark 4.1.1(i)). From an indexed point of view, the inverse image of a pseudofunctor $\mathbb{F} : \mathcal{D}^{op} \to \mathbf{CAT}$ along e_D is the composite

$$\mathbb{1}^{op} \xrightarrow{(e_D)^{op}} \mathcal{D}^{op} \xrightarrow{\mathbb{F}} \mathbf{CAT},$$

which via the equivalence $[\mathbb{1}^{op}, \mathbf{CAT}]_{ps} \simeq \mathbf{CAT}$ is uniquely determined by its value $\mathbb{F}(D)$. This implies that Ev_D has both a left and a right adjoint, i.e. the right and left Kan extensions of Proposition 4.2.1, and thus that it preserves all kinds of limits and colimits by Lemma 3.1.1. This means that the following equivalences hold for any diagram $I : \mathcal{C} \to \mathbf{Ind}_{\mathcal{D}}$ with weight $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{Cat}$:

$$\begin{aligned}
\mathrm{Ev}_D(\mathrm{colim}^{\mathbb{D}}_{lax}(I)) &\simeq \mathrm{colim}_{lax}(\mathrm{Ev}_D \circ I),\\
\mathrm{Ev}_D(\mathrm{colim}^{\mathbb{D}}_{oplax}(I)) &\simeq \mathrm{colim}^{\mathbb{D}}_{oplax}(\mathrm{Ev}_D \circ I),\\
\mathrm{Ev}_D(\mathrm{colim}^{\mathbb{D}}_{ps}(I)) &\simeq \mathrm{colim}^{\mathbb{D}}_{ps}(\mathrm{Ev}_D \circ I).
\end{aligned}$$

Thus colimits of any kind are computed fibrewise in $\mathbf{Ind}^s_{\mathcal{D}}$.

Now, we can apply these considerations to diagrams of the kind

$$\tau_F^D : \mathcal{C} \xrightarrow{\tau_F} \mathbf{cFib}_{\mathcal{D}} \xrightarrow{\mathrm{Ev}_D} \mathbf{Set}.$$

Each of these functors takes values in **Set**, since $\tau_F^D(X) = \mathcal{D}(D, F(X))$: indeed, it is the fibre in D of $(1_{\mathcal{D}} \downarrow F(X))$, which is a discrete fibration over $\mathcal{D}$. Since evaluation functors commute with bicolimits, we have that

$$\begin{aligned}\mathrm{Lan}_{F^{op}}(\mathbb{E})(D) &:= \mathrm{Ev}_D(\mathrm{Lan}_{F^{op}}(\mathbb{E}))\\ &= \mathrm{Ev}_D(\mathrm{colim}^{\mathbb{E}}_{ps}\, \tau_F)\\ &\simeq \mathrm{colim}^{\mathbb{E}}_{ps}(\tau_F^D)\\ &\simeq \mathrm{colim}^{\tau_F^D}_{ps}\, \mathbb{E},\end{aligned}$$

where the last step holds by the commutativity of weights and diagrams (see Proposition 3.1.2). But the last colimit is precisely the one defined as $\mathrm{Lan}_{F^{op}}(\mathbb{E})(D)$ in the proof of Proposition 4.2.1.

If we want to compute inverse images explicitly, we can apply the techniques about colimits and localizations developed in Chapters 2 and 3. First of all, by Proposition 3.3.2 the colimit $\mathrm{Lan}_{F^{op}}(\mathbb{E}) \simeq \mathrm{colim}^{\mathbb{E}}_{ps}\, \tau_F$ is a localization of either of the following:

$$\mathrm{colim}_{lax}(\tau_F \circ p_{\mathbb{E}^V}) \simeq \mathrm{colim}^{\mathbb{E}}_{lax}\, \tau_F \simeq \mathrm{colim}^{\tau_F}_{oplax}\, \mathbb{E} \simeq \mathrm{colim}_{oplax}(\mathbb{E} \circ p_{\tau_F}),$$

$$\mathrm{colim}_{oplax}(\tau_F \circ p_{\mathbb{E}}) \simeq \mathrm{colim}^{\mathbb{E}}_{oplax}\, \tau_F \simeq \mathrm{colim}^{\tau_F}_{lax}\, \mathbb{E} \simeq \mathrm{colim}_{lax}(\mathbb{E} \circ p_{\tau_F^V}).$$

Each of these localizations is performed with respect to the components of a certain class of natural transformations appearing in the colimit cocone; moreover, Remark 4.1.1(iii) implies that $\mathrm{colim}^{\mathbb{E}}_{ps}\, \tau_F$ can also be seen as a localization of $\mathrm{colim}_{lax}(\tau_F^V \circ p_{\mathbb{E}})^{op}$. In particular, we will focus on the description of the localization

$$\mathrm{colim}_{lax}(\mathbb{E} \circ p_{\tau_F^V}) \to \mathrm{colim}^{\mathbb{E}}_{ps}\, \tau_F$$

of Proposition 3.3.3, which is based on an application of Proposition 3.3.1 to explicitly compute lax colimits using Grothendieck fibrations.

Firstly, let us compute the fibrations associated with the functors τ_F and τ_F^D. If we consider the functor

$$\tau_F : \mathcal{C} \to \mathbf{cFib}_{\mathcal{D}},$$

as a functor with values in **Cat**, a quick computation shows that the opfibration corresponding to τ_F, i.e. the opposite of the functor $p_{\tau_F^V} : \mathcal{G}(\tau_F^V) \to \mathcal{C}^{op}$ (see Remark 2.1.3(i)), is precisely the comma category

$$\pi_{\mathcal{C}} : (1_{\mathcal{D}} \downarrow F) \to \mathcal{C},$$

with $\pi_{\mathcal{C}}$ forgetting the component in $\mathcal{D}$. The same category is also fibred over $\mathcal{D}$, by considering the functor

$$\pi_{\mathcal{D}} : (1_{\mathcal{D}} {\downarrow} F) \to \mathcal{D}$$

which forgets the component in $\mathcal{C}$. In a similar way, each functor

$$\tau_F^D : \mathcal{C} \to \mathbf{Set}$$

is associated with the opfibration

$$\pi_{\mathcal{C}}^D : (D {\downarrow} F) \to \mathcal{C}.$$

We are now ready to prove the following:

Proposition 4.2.3. *Consider a pseudofunctor $\mathbb{E} : \mathcal{C}^{op} \to \mathbf{Cat}$ and a functor $F : \mathcal{C} \to \mathcal{D}$. Consider the category $(D {\downarrow} F \circ p_{\mathbb{E}})$, whose objects are arrows $[d : D \to F \circ p_{\mathbb{E}}(X, U)]$ and whose morphisms $(y, a) : [d' : D \to F \circ p_{\mathbb{E}}(Y, V)] \to [d : D \to F \circ p_{\mathbb{E}}(X, U)]$ are given by arrows $(y, a) : (Y, V) \to (X, U)$ such that $F(y) \circ d' = d$. Consider the class of arrows*

$$S_D := \left\{ [d'] \xrightarrow{(y,a)} [d] \;\middle|\; (y, a) \text{ cartesian in } \mathcal{G}(\mathbb{E}) \right\} :$$

then

$$\mathrm{Lan}_{F^{op}}(\mathbb{E})(D) \simeq (D {\downarrow} F \circ p_{\mathbb{E}})[S_D^{-1}].$$

From a fibred point of view, let $p : \mathcal{P} \to \mathcal{C}$ be a cloven fibration. Consider the fibration of generalized elements

$$(1_{\mathcal{D}} {\downarrow} (F \circ p)) \xrightarrow{r} \mathcal{D}$$

of the functor $F \circ p$, whose objects are arrows $[d : D \to (F \circ p)(U)]$ of $\mathcal{D}$, and a morphism

$$(e, \alpha) : [d' : D' \to (F \circ p)(V)] \to [d : D \to (F \circ p)(U)]$$

is given by an arrow $e : D' \to D$ in $\mathcal{D}$ and an arrow $\alpha : V \to U$ in $\mathcal{P}$ such that $(F \circ p)(\alpha) \circ d' = d \circ e$. Consider the class of arrows

$$S := \{(e, \alpha) : [d'] \to [d] \mid (e, \alpha) \ r\text{-vertical}, \ \alpha \text{ cartesian in } \mathcal{P}\} :$$

then

$$\mathrm{Lan}_{F^{op}}([p]) \simeq (1_{\mathcal{D}} {\downarrow} (F \circ p))[S^{-1}].$$

Proof. Proposition 3.3.3 states that we can consider the pullback

$$\begin{array}{ccc} \mathrm{colim}_{lax}(\mathbb{E} \circ p_{(\tau_F^D)^V}) & \xrightarrow{\pi_{\mathbb{E}}^D} & \mathcal{G}(\mathbb{E}) \\ {\scriptstyle (\pi^D)'} \downarrow & & \downarrow {\scriptstyle p_{\mathbb{E}}} \\ \mathcal{G}((\tau_F^D)^V)^{op} & \xrightarrow[p^{op}_{(\tau_F^D)^V}]{} & \mathcal{C} \end{array} ,$$

and $\mathrm{colim}_{ps}^{\tau_F^D}\mathbb{E}$ is the localization of $\mathrm{colim}_{lax}(\mathbb{E}\circ p_{(\tau_F^D)^V})$ with respect to the morphisms that are $(\pi^D)'$-cartesian and are mapped via $\pi_{\mathbb{E}}^D$ to $p_{\mathbb{E}}$-cartesian arrows. Now we can perform the following simplifications: first of all $(\tau_F^D)^V \cong \tau_F^D$, since it is a discrete functor; secondly, as we mentioned earlier, the functor $p^{op}_{\tau_F^D}$ corresponds to the canonical projection

$$\pi_{\mathcal{C}}^D : (D{\downarrow}F) \to \mathcal{C};$$

finally, the explicit description of $\mathrm{colim}_{lax}(\mathbb{E}\circ p_{(\tau_F^D)^V})$ of Proposition 3.3.3 provides, in this case, the category $(D{\downarrow}F\circ p_{\mathbb{E}})$. Therefore, the pullback above can be rewritten as

$$\begin{array}{ccc} (D{\downarrow}F\circ p_{\mathbb{E}}) & \xrightarrow{\pi_{\mathbb{E}}^D} & \mathcal{G}(\mathbb{E}) \\ {\scriptstyle (\pi^D)'}\downarrow & & \downarrow{\scriptstyle p_{\mathbb{E}}} \\ (D{\downarrow}F) & \xrightarrow[\pi_F^D]{} & \mathcal{C} \end{array} .$$

Finally, notice that an arrow

$$[d' : D \to F\circ p_{\mathbb{E}}(Y,V)] \xrightarrow{(y,a)} [d : D \to F\circ p_{\mathbb{E}}(X,U)]$$

in $(D{\downarrow}F\circ p_{\mathbb{E}})$ is $(\pi^D)'$-cartesian if and only if (y,a) is $p_{\mathbb{E}}$-cartesian as an arrow in $\mathcal{G}(\mathbb{E})$, which is the same as to say that it is mapped to a $p_{\mathbb{E}}$-cartesian arrow via the forgetful functor $\pi_{\mathbb{E}}^D$: therefore, $\mathrm{Lan}_{F^{op}}(\mathbb{E})(D)$ is computed localizing $(D{\downarrow}F\circ p_{\mathbb{E}})$ with respect to all its arrows whose component in $\mathcal{G}(\mathbb{E})$ is cartesian, i.e. with respect to the class S_D.

Now, if we consider each category $(D{\downarrow}F\circ p)$ as the fibre in D of the pseudofunctor

$$(-{\downarrow}F\circ p) : \mathcal{D}^{op} \to \mathbf{Cat},$$

then $\mathrm{Lan}_{F^{op}}[p](D)$ is a localization of it; moreover, said localization is compatible with transition morphisms, i.e. given $g : D' \to D$ in $\mathcal{D}$, the transition morphism

$$-\circ e : (D{\downarrow}F\circ p) \to (D'{\downarrow}F\circ p)$$

restricts to a functor

$$(D{\downarrow}F\circ p)[S_D^{-1}] \to (D'{\downarrow}F\circ p)[S_{D'}^{-1}].$$

Therefore, we may apply Lemma 2.2.2: the fibration associated to the localized pseudofunctor

$$(-{\downarrow}F\circ p)[S_{(-)}^{-1}] : \mathcal{D}^{op} \to \mathbf{Cat}$$

is the localization of the fibration associated to $(-{\downarrow}F\circ p)$, which is the fibration of generalized elements

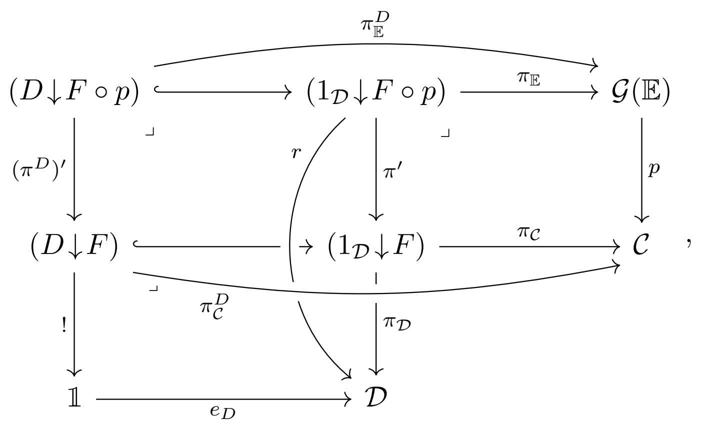
$$(1_{\mathcal{D}}{\downarrow}F\circ p) \xrightarrow{r} \mathcal{D},$$

with respect to the class of arrows

$$(e, \alpha) : [d' : D' \to Fp(V)] \to [d : D \to Fp(U)]$$

that are $\pi_{\mathcal{D}}$-vertical (i.e. e is invertible) and such that their component α in the fibre belongs to $S_{D'}$. This provides the description of $\mathrm{Lan}_{F^{op}}([p])$ in the statement. $\square$

The categories involved can be summed up in the following diagram:



where all the squares are pullbacks since they correspond to the computation of direct images of fibrations: this implies that all the vertical functors are fibrations. On the other hand, $\pi_{\mathcal{C}} = p^{op}_{\tau_F^V}$ is an opfibration, and thus its pullback $\pi_{\mathbb{E}}$ is also an opfibration. The fibration $\mathrm{Lan}_{F^{op}}[p]$ is the localization of $(1_{\mathcal{D}}{\downarrow}F \circ p)$ at the class of arrows that are r-vertical and whose image via $\pi_{\mathbb{E}}$ is p-cartesian. Similarly, the single fibre $\mathrm{Lan}_{F^{op}}(\mathbb{E})(D)$ is the localization of $(D{\downarrow}F \circ p)$ at the class of arrows whose image via $\pi_{\mathbb{E}}^{D}$ is p-cartesian: notice that in this case there is no need to consider the vertical arrows, since every arrow in $(D{\downarrow}F \circ p)$ is vertical with respect to the unique functor to $\mathbb{1}$.

Remarks 4.2.2. (i) This approach recovers the fibration $\mathrm{Lan}_{F^{op}}([p])$ first by localizing a pseudofunctor fibrewise and then by computing the associated fibration. Alternatively, we could have computed $\mathrm{Lan}_{F^{op}}([p])$ as the colimit $\mathrm{colim}^{\mathbb{D}}_{ps}(\tau_F)$ of indexed categories, going instead first to the associated fibration and then localizing it.

(ii) We can obtain the fibrational description of inverse images without resorting to the indexed formalism, but instead by showing explicitly the equivalence of categories

$$\mathbf{cFib}_{\mathcal{C}}([p], F^*[q]) \simeq \mathbf{cFib}_{\mathcal{D}}((1_{\mathcal{D}}{\downarrow}F \circ p)[S^{-1}], [q])$$

for $p : \mathcal{P} \to \mathcal{C}$ and $q : \mathcal{Q} \to \mathcal{D}$ fibrations. The proof goes as follows: by definition of pseudopullback, a pair (G, γ) as in the diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{G} & \mathcal{C} \times_{\mathcal{D}} \mathcal{Q} \\ & \underset{p}{\searrow} \;\; \overset{\sim}{\Leftarrow}_{\gamma} & \downarrow{\scriptstyle F^*(q)} \\ & & \mathcal{C} \end{array}$$

corresponds to a pair (G_1, γ_1) as in the diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{G} & \mathcal{Q} \\ {\scriptstyle p}\downarrow & \overset{\sim}{\swarrow}_{\gamma_1} & \downarrow{\scriptstyle q} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array} ;$$

since q is a cloven fibration we can apply Corollary 6.2.5, and we obtain a unique pair (G_2, γ_2) as in the diagram

$$\begin{array}{ccc} (1_{\mathcal{D}}{\downarrow}F \circ p) & \xrightarrow{G_2} & \mathcal{Q} \\ & \underset{\pi_{\mathcal{D}}}{\searrow} \;\; \overset{\sim}{\Leftarrow}_{\gamma_2} & \downarrow{\scriptstyle q} \\ & & \mathcal{D} \end{array} ,$$

and moreover the functor G_2 is a morphism of fibrations. What we said so far is true for any *functor* G: one can then verify explicitly that G is a morphism of fibrations if and only if G_2 inverts all arrows in S, and thus factors through the localization as a functor $\bar{G} : (1_{\mathcal{D}}{\downarrow}F \circ p) \to \mathcal{Q}$. The correspondence between G and $\bar{G}$ describes (on objects) the equivalence between hom-categories given by the adjunction.

4.3 Change of base for stacks

Starting from two sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, we now study direct and inverse images of stacks along a functor $F : \mathcal{C} \to \mathcal{D}$. In analogy with the case of presheaves, the main ingredient is the (J, K)-*continuity* of F, necessary to restrict the adjunction $\mathrm{lan}_{F^{op}} \dashv F^* : [\mathcal{D}^{op}, \mathbf{Set}] \to [\mathcal{C}^{op}, \mathbf{Set}]$ to an adjunction $\mathbf{Sh}(F)^* \dashv \mathbf{Sh}(F)_* : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$. Indeed, continuous functors are precisely those whose direct image preserves the property of being a stack, extending the content of Definition 1.2.1:

Proposition 4.3.1. *Consider two sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ and a functor $F : \mathcal{C} \to \mathcal{D}$: then F is (J, K)-continuous functor if and only if $F^* : \mathbf{Ind}_{\mathcal{D}} \to \mathbf{Ind}_{\mathcal{C}}$ restricts to a 2-functor $\mathbf{St}(\mathcal{D}, K) \to \mathbf{St}(\mathcal{C}, J)$.*

Proof. One implication is obvious: if F^* maps K-stacks to J-stacks, in particular it maps K-sheaves to J-sheaves (see Proposition 2.3.1), and thus F is (J, K)-continuous. Now suppose instead that F is (J, K)-continuous, and consider a K-stack $\mathbb{E} : \mathcal{D}^{op} \to \mathbf{CAT}$: we have to show that $\mathbb{E} \circ F^{op} : \mathcal{C}^{op} \to \mathbf{CAT}$ is a J-stack, i.e. that for every J-covering sieve $m : S \rightarrowtail \text{よ}(X)$ the functor

$$\mathbf{Ind}_{\mathcal{C}}(\text{よ}(X), \mathbb{E} \circ F^{op}) \xrightarrow{-\circ m} \mathbf{Ind}_{\mathcal{C}}(S, \mathbb{E} \circ F^{op})$$

is an equivalence (see Definition 2.3.2). To do so, we exploit the 2-adjunction $s_K \mathrm{Lan}_{F^{op}} \dashv (- \circ F^{op}) i_K$: the functor $- \circ m$ translates into the functor

$$L'_S : \mathbf{St}(\mathcal{D}, K)(s_K \mathrm{Lan}_{F^{op}}(\text{よ}(X)), \mathbb{E}) \to \mathbf{St}(\mathcal{D}, K)(s_K \mathrm{Lan}_{F^{op}}(S), \mathbb{E})$$

acting by precomposition with $- \circ s_K \mathrm{Lan}_{F^{op}}(m)$. The 2-functors s_K and $\mathrm{Lan}_{F^{op}}$ act on presheaves respectively like a_K and $\mathrm{lan}_{F^{op}}$, and thus $s_K \mathrm{Lan}_{F^{op}} \simeq \mathrm{a}_K \mathrm{lan}_{F^{op}}$. But by Lemma 1.2.2 the functor F is (J, K)-continuous if and only if each arrow

$$\mathrm{a}_K \mathrm{lan}_{F^{op}}(m) \simeq s_K \mathrm{Lan}_{F^{op}}(m)$$

is an isomorphism of sheaves, and thus L'_S is an equivalence. □

Remark 4.3.1. One can also prove directly that stacks are preserved by the direct image along continuous functors as follows. We showed in Section 2.3 that a pseudonatural transformation $S \Rightarrow \mathbb{E} \circ F^{op}$ can be interpreted as *descent data* $(U_y, \alpha_{y,z})_{y \in S}$, i.e. the given of objects $U_y \in \mathbb{E}(F(Y))$ for every $y : Y \to X$ in $S(Y)$, and of a family of isomorphisms $\alpha_{y,z} : \mathbb{E}(F(z))(U_y) \simeq U_{yz}$ of $\mathbb{E}(F(Z))$ for every y in S and every $z : Z \to Y$, satisfying suitable compatibility properties. Since F is continuous it is in particular cover-preserving, and hence $F(S)$ generates a K-covering sieve R over $F(X)$ in $\mathcal{D}$. Now, starting from the descent data $(U_y, \alpha_{y,z})$, one can obtain data $(U'_f, \alpha'_{f,g})_{f \in R}$ for $\mathbb{E}$. Indeed, any arrow $f \in R$ is of the kind $F(y)h$ for some $y \in S$, and thus we set $U'_f = \mathbb{E}(h)(U_y)$, and the isomorphisms α' are then defined in the obvious way: the fact that this yields well-defined descent data for $\mathbb{E}$ can be proved exploiting the site-theoretic description of (J, K)-continuous functors in Definition 1.2.1. Since $\mathbb{E}$ is a stack, the descent data $(U'_f, \alpha'_{f,g})_{f \in R}$ admit a 'gluing' $U' \in \mathbb{E}(F(X))$, which is also a gluing for the original descent datum $(U_y, \alpha_{y,z})$ of the indexed category $\mathbb{E} \circ F^{op}$. A similar argument works for morphisms of descent data, proving that the functor $- \circ m$ in the previous proposition is in fact an equivalence.

We conclude that (J, K)-continuous functors (thus in particular morphisms of sites) induce an adjunction between categories of stacks:

Corollary 4.3.2. *Consider a (J, K)-continuous functor $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$: it induces a 2-adjunction*

$$\mathbf{St}^s(\mathcal{C}, J) \underset{\mathbf{St}(F)_*}{\overset{\mathbf{St}(F)^*}{\underset{\perp}{\rightleftarrows}}} \mathbf{St}^s(\mathcal{D}, K),$$

to which we shall refer simply by $\mathbf{St}(F)$*. The 2-functor* $\mathbf{St}(F)_*$ *is called the* direct image of stacks along F *and acts as the precomposition*

$$F^* := (- \circ F^{op}) : \mathbf{Ind}_{\mathcal{D}} \to \mathbf{Ind}_{\mathcal{C}};$$

In terms of fibrations, a stack $q : \mathcal{E} \to \mathcal{D}$ *is mapped by* $\mathbf{St}(F)_*$ *to its strict pseudopullback* $p : \mathcal{P} \to \mathcal{C}$ *along* F*. The left adjoint* $\mathbf{St}(F)^*$ *is the* inverse image of stacks along F *and acts as the composite*

$$\mathbf{St}^s(\mathcal{C}, J) \xrightarrow{i_J} \mathbf{Ind}^s_{\mathcal{C}} \xrightarrow{\mathrm{Lan}_{F^{op}}} \mathbf{Ind}^s_{\mathcal{D}} \xrightarrow{s_K} \mathbf{St}^s(\mathcal{D}, K),$$

where s_K *denotes the stackification functor and* $\mathrm{Lan}_{F^{op}}$ *can be computed as in Proposition 4.2.1. In terms of fibrations, a stack* $p : \mathcal{P} \to \mathcal{C}$ *is mapped by* $\mathbf{St}(F)^*$ *to the stackification of its inverse image* $\mathrm{Lan}_{F^{op}}([p])$ *along* F*, which can be computed as in Proposition 4.2.3.*

In particular, any geometric morphism $f : \mathscr{F} \to \mathscr{E}$ *induces a pair of adjoint functors, denoted by* $\mathbf{St}(f)$*:*

$$\mathbf{St}^s(\mathscr{E}) \underset{\mathbf{St}(f)_*}{\overset{\mathbf{St}(f)^*}{\rightleftarrows}} \mathbf{St}^s(\mathscr{F}) \; , \quad \mathbf{St}(f)^* \dashv \mathbf{St}(f)_*.$$

Proof. The fact that $s_K \circ \mathrm{Lan}_{F^{op}} \circ i_J$ is adjoint to $\mathbf{St}(F)_*$ is a standard argument about adjunctions (see Lemma D.5). Proposition 4.3.1 shows that the precomposition $- \circ F^{op}$ maps stacks to stacks when F is (J, K)-continuous, and justifies the description of $\mathbf{St}(F)_*$ in $\mathcal{C}$-indexed terms; its description in fibrational terms as a pseudopullback comes instead from Proposition 4.1.1. The last claim holds since any geometric morphism $f : \mathscr{F} \to \mathscr{E}$ is induced by the morphism of sites $f^* : (\mathscr{E}, J^{can}_{\mathscr{E}}) \to (\mathscr{F}, J^{can}_{\mathscr{F}})$. □

As already mentioned in Section 2.4, an important consequence of this is the fact that all canonical fibrations are stacks, if those of toposes are:

Corollary 4.3.3. *Consider a site* $(\mathcal{C}, J)$ *and set* $\mathscr{E} = \mathbf{Sh}(\mathcal{C}, J)$*. Suppose that the canonical fibration* $\mathcal{S}_{(\mathscr{E}, J^{can}_{\mathscr{E}})}$ *of* $\mathscr{E}$ *is a* $J^{can}_{\mathscr{E}}$*-stack: then the canonical fibration* $\mathcal{S}_{(\mathcal{C},J)}$ *of* $(\mathcal{C}, J)$ *is a* J*-stack.*

Proof. One can check immediately that the direct image of $\mathcal{S}_{(\mathscr{E}, J^{can}_{\mathscr{E}})}$ along the morphism of sites

$$(\mathcal{C}, J) \xrightarrow{\ell_J} (\mathscr{E}, J^{can}_{\mathscr{E}}),$$

is the canonical fibration $\mathcal{S}_{(\mathcal{C},J)}$. But ℓ_J is $(J, J^{can}_{\mathscr{E}})$-continuous and the canonical fibration $\mathcal{S}_{(\mathscr{E}, J^{can}_{\mathscr{E}})}$ is a $J^{can}_{\mathscr{E}}$-stack, thus $\mathcal{S}_{(\mathcal{C},J)}$ is a J-stack. □

In a similar way to morphisms of sites, comorphisms of sites also induce an adjunction between categories of stacks, this time by restricting the action of the right Kan extension:

Proposition 4.3.4. *Consider a comorphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$: it induces a 2-adjunction*

$$\mathbf{St}^s(\mathcal{D}, K) \underset{(C_F^{\mathbf{St}})_*}{\overset{(C_F^{\mathbf{St}})^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{St}^s(\mathcal{C}, J),$$

to which we shall refer by $C_F^{\mathbf{St}}$. The right adjoint $(C_F^{\mathbf{St}})_*$ acts by restriction of the right Kan extension $\mathrm{Ran}_{F^{op}}$ (see Proposition 4.2.1) to stacks; on the other hand, the left adjoint $(C_F^{\mathbf{St}})^*$ acts as the composite 2-functor

$$\mathbf{St}^s(\mathcal{D}, K) \xrightarrow{i_K} \mathbf{Ind}^s_{\mathcal{D}} \xrightarrow{F^*} \mathbf{Ind}^s_{\mathcal{C}} \xrightarrow{s_J} \mathbf{St}^s(\mathcal{C}, J),$$

where $F^* := (- \circ F^{op})$ and s_J is the stackification functor. Moreover, there is an equivalence

$$(C_F^{\mathbf{St}})^* \circ s_K \cong s_J \circ F^*.$$

Proof. We only need to show F is a comorphism of sites if and only if the 2-functor $\mathrm{Ran}_{F^{op}} : \mathbf{Ind}_{\mathcal{C}} \to \mathbf{Ind}_{\mathcal{D}}$ restricts to a 2-functor $(C_F^{\mathbf{St}})_* : \mathbf{St}(\mathcal{C}, J) \to \mathbf{St}(\mathcal{D}, K)$: then standard considerations about adjoints (see Lemma D.5 below) imply that $(C_F^{\mathbf{St}})^* := s_J \circ F^* \circ i_K$ is left adjoint to $(C_F^{\mathbf{St}})_*$.

Suppose first that $\mathrm{Ran}_{F^{op}}$ maps J-stacks to K-stacks: then in particular it maps J-sheaves to K-sheaves and hence F is a comorphism by condition (ii) of Definition 1.2.2. Conversely, suppose that F is a comorphism, consider a J-stack $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{Cat}$, a K-sieve $m_S : S \rightarrowtail よ(D)$ and the diagram

$$\begin{array}{ccc} \mathbf{Ind}_{\mathcal{D}}(よ(D), \mathrm{Ran}_{F^{op}}(\mathbb{D})) & \xrightarrow{\sim} & \mathbf{Ind}_{\mathcal{C}}(よ(C) \circ F^{op}, \mathbb{D}) \\ {\scriptstyle -\circ m_S}\downarrow & & \downarrow{\scriptstyle -\circ(m_S \circ F^{op})} \\ \mathbf{Ind}_{\mathcal{D}}(S, \mathrm{Ran}_{F^{op}}(\mathbb{D})) & \xrightarrow{\sim} & \mathbf{Ind}_{\mathcal{C}}(S \circ F^{op}, \mathbb{D}) \end{array},$$

where the horizontal equivalences come from the adjunction $F^* \dashv \mathrm{Ran}_{F^{op}}$. By condition (iv) of Definition 1.2.2 the precomposition $F^* := (- \circ F^{op}) : [\mathcal{D}^{op}, \mathbf{Set}] \to [\mathcal{C}^{op}, \mathbf{Set}]$ maps any K-dense monomorphism to a J-dense monomorphism: in other words, if $m_S : S \rightarrowtail よ(D)$ is a K-sieve, then $F^*(m_S) = (\circ(m_S \circ F^{op})) : S \circ F^{op} \rightarrowtail よ(C) \circ F^{op}$ is J-dense. Since $F^*(m_S)$ is J-dense and $\mathbb{D}$ is a J-stack, the vertical arrow on the right is an equivalence, hence thus the one on the left is also an equivalence and $\mathrm{Ran}_{F^{op}}(\mathbb{D})$ is a K-stack.

The last claim is justified by the fact that both $(C_F^{\mathbf{St}})^* \circ s_K$ and $s_J \circ F^*$ are left adjoint to $i_K \circ (C_F^{\mathbf{St}})_* \cong \mathrm{Ran}_{F^{op}} \circ i_J$. □

Corollary 4.3.5. *Consider a (J, K)-continuous comorphism of sites*

$$F : (\mathcal{C}, J) \to (\mathcal{D}, K).$$

It induces a triple of 2-adjoints

$$\mathbf{St}^s(\mathcal{C}, J) \;\begin{array}{c} \xrightarrow{(C_F^{\mathbf{St}})_!} \\ \perp \\ \xleftarrow{(C_F^{\mathbf{St}})^*} \\ \perp \\ \xrightarrow[(C_F^{\mathbf{St}})_*]{} \end{array}\; \mathbf{St}^s(\mathcal{D}, K) \ ,$$

where $(C_F^{\mathbf{St}})_*$ *acts by restriction of* $\mathrm{Ran}_{F^{op}}$*,* $(C_F^{\mathbf{St}})^* = \mathbf{St}(F)_*$ *acts by restriction of* F^* *and* $(C_F^{\mathbf{St}})_! = \mathbf{St}(F)^*$ *is the composite 2-functor*

$$\mathbf{St}^s(\mathcal{C}, J) \xrightarrow{i_J} \mathbf{Ind}_{\mathcal{C}}^s \xrightarrow{\mathrm{Lan}_{F^{op}}} \mathbf{Ind}_{\mathcal{D}}^s \xrightarrow{s_K} \mathbf{St}^s(\mathcal{D}, K).$$

Proof. This is an immediate consequence of the previous results: the action of $\mathrm{Ran}_{F^{op}}$ restricts to stacks since F is a comorphism, while that of F^* restricts to stacks since F is continuous. □

4.4 Change of base for sheaves

In Section 2.5 we introduced the truncation-inclusion adjunction

$$\mathbf{Sh}(\mathcal{C}, J) \;\underset{j_J}{\overset{t_J}{\leftrightarrows}}\; \mathbf{St}^s(\mathcal{C}, J)$$

to describe the connection between J-sheaves and J-stacks. The same functors relate the action of morphisms and comorphisms of sites on stacks to their action on sheaves, as follows:

Proposition 4.4.1. *Consider a continuous functor* $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$*. Then the inverse image* $\mathbf{Sh}(F)^* : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ *is isomorphic to the composite functor*

$$\mathbf{Sh}(\mathcal{C}, J) \xhookrightarrow{j_J} \mathbf{St}^s(\mathcal{C}, J) \xrightarrow{\mathbf{St}(F)^*} \mathbf{St}^s(\mathcal{D}, K) \xrightarrow{t_K} \mathbf{Sh}(\mathcal{D}, K).$$

In particular, denoting by $t_{\mathcal{C}} \dashv j_{\mathcal{C}}$ *the truncation-inclusion adjunction in the case of the trivial topology on* $\mathcal{C}$*, the functor* $\mathrm{lan}_{F^{op}} : [\mathcal{C}^{op}, \mathbf{Set}] \to [\mathcal{D}^{op}, \mathbf{Set}]$ *is isomorphic to the composite*

$$[\mathcal{C}^{op}, \mathbf{Set}] \xhookrightarrow{j_{\mathcal{C}}} \mathbf{Ind}_{\mathcal{C}}^s \xrightarrow{\mathrm{Lan}_{F^{op}}} \mathbf{Ind}_{\mathcal{D}}^s \xrightarrow{t_{\mathcal{D}}} [\mathcal{D}^{op}, \mathbf{Set}].$$

Proof. Since the 2-functor $\mathbf{St}(F)_* : \mathbf{St}(\mathcal{D}, K) \to \mathbf{St}(\mathcal{C}, J)$ acts by precomposition with F^{op}, its action restricts to that of $\mathbf{Sh}(F)_*$ on sheaves: but then we can consider the diagram

$$\begin{array}{ccccc} \mathbf{Sh}(\mathcal{D}, K) & \underset{j_K}{\overset{t_K}{\leftrightarrows}} & \mathbf{St}^s(\mathcal{D}, K) & \underset{\mathbf{St}(F)_*}{\overset{\mathbf{St}(F)^*}{\leftrightarrows}} & \mathbf{St}^s(\mathcal{C}, J) \\ & & & & \uparrow j_J \\ & & \xrightarrow{\qquad \mathbf{Sh}(F)_* \qquad} & & \mathbf{Sh}(\mathcal{C}, J) \end{array} ,$$

and by applying Lemma D.5 it follows that the left adjoint $\mathbf{Sh}(F)^*$ is isomorphic to the composite $t_K \circ \mathbf{St}(F)^* \circ j_J$. □

Similarly, the geometric morphism induced by a comorphism of sites is obtained by truncating the adjunction between categories of stacks as follows:

Proposition 4.4.2. *Consider a comorphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$. The inverse image $C_F^* : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ of the geometric morphism induced by F is isomorphic to the composite functor*

$$\mathbf{Sh}(\mathcal{D}, K) \xhookrightarrow{j_K} \mathbf{St}^s(\mathcal{D}, K) \xrightarrow{(C_F^{\mathbf{St}})^*} \mathbf{St}^s(\mathcal{C}, J) \xrightarrow{t_J} \mathbf{Sh}(\mathcal{C}, J).$$

Proof. Notice that the two inclusions

$$\mathbf{Sh}(\mathcal{C}, J) \xhookrightarrow{\iota_J} [\mathcal{C}^{op}, \mathbf{Set}] \xhookrightarrow{j_{\mathcal{C}}} \mathbf{Ind}_{\mathcal{C}}, \ \mathbf{Sh}(\mathcal{C}, J) \xhookrightarrow{j_J} \mathbf{St}(\mathcal{C}, J) \xhookrightarrow{i_J} \mathbf{Ind}_{\mathcal{C}}$$

are equal, and thus their left adjoints are isomorphic: $\mathrm{a}_J \circ t_{\mathcal{C}} \cong \tau_J \circ s_J$. But then the following holds:

$$\begin{aligned} t_J \circ (C_F^{\mathbf{St}})^* \circ j_K &= t_J \circ s_J \circ F^* \circ i_K \circ j_K \\ &\cong \mathrm{a}_J \circ t_{\mathcal{C}} \circ F^* \circ j_{\mathcal{D}} \circ \iota_K. \end{aligned}$$

As we know, the 2-functor $F^* : \mathbf{Ind}_{\mathcal{D}} \to \mathbf{Ind}_{\mathcal{C}}$ restricts to presheaves: if we adopt (with a slight abuse of notation) the same symbol for the restriction $F^* : [\mathcal{D}^{op}, \mathbf{Set}] \to [\mathcal{C}^{op}, \mathbf{Set}]$, we have thus the isomorphism $F^* \circ j_{\mathcal{D}} \cong j_{\mathcal{C}} \circ F^*$, and hence

$$\mathrm{a}_J \circ t_{\mathcal{C}} \circ F^* \circ j_{\mathcal{D}} \circ \iota_K \cong \mathrm{a}_J \circ t_{\mathcal{C}} \circ j_{\mathcal{C}} \circ F^* \circ \iota_K$$

Finally, since $j_{\mathcal{C}}$ is fully faithful with left adjoint $t_{\mathcal{C}}$, the composite $t_{\mathcal{C}} \circ j_{\mathcal{C}} \cong$ is isomorphic to the identity functor of $[\mathcal{C}^{op}, \mathbf{Set}]$, and therefore we may conclude that

$$t_J \circ (C_F^{\mathbf{St}})^* \circ j_K \cong \mathrm{a}_J \circ F^* \circ \iota_K \cong C_F^*.$$

□

Finally, we can also study the change of base functors for sheaves from a fibrational point of view:

Lemma 4.4.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, K a Grothendieck topology on $\mathcal{D}$ and $\ell_K : \mathcal{D} \to \mathbf{Sh}(\mathcal{D}, K)$ be the canonical functor. Then the functor $\mathrm{a}_K \circ \mathrm{lan}_{F^{op}} : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{D}, K)$ admits the following fibrational description: for any presheaf P on $\mathcal{C}$,*

$$\mathrm{a}_K \circ \mathrm{lan}_{F^{op}}(P) = \mathrm{colim}(\ell_K \circ F \circ \pi_P).$$

In other words, $\mathrm{a}_K \circ \mathrm{lan}_{F^{op}}(P)$ coincides with the discrete part of the K-comprehensive factorization of the composite functor $F \circ \pi_P$.

Proof. From $P \simeq \mathrm{colim}(よ_{\mathcal{C}} \circ \pi_P)$ it follows that

$$\begin{aligned} \mathrm{a}_K \circ \mathrm{lan}_{F^{op}}(P) &\simeq \mathrm{a}_K \circ \mathrm{lan}_{F^{op}}(\mathrm{colim}(よ_{\mathcal{C}} \circ \pi_P)) \\ &\simeq \mathrm{colim}(\mathrm{a}_K \circ \mathrm{lan}_{F^{op}} \circ よ_{\mathcal{C}} \circ \pi_P) \\ &\simeq \mathrm{colim}(\mathrm{a}_K \circ よ_{\mathcal{D}} \circ F \circ \pi_P) \\ &\simeq \mathrm{colim}(\ell_K \circ F \circ \pi_P). \end{aligned}$$

The last claim is just the definition of K-comprehensive factorization (see Definition 2.5.1). □

Proposition 4.4.4. *Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a (J, K)-continuous functor: the inverse image $\mathbf{Sh}(F)^* : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ induced by F (which is $(C_F^{\mathbf{St}})_!$ when F is a continuous comorphism of sites) acts by mapping any J-sheaf P on $\mathcal{C}$, seen as a discrete J-stack $\pi_P : \int P \to \mathcal{C}$, to the discrete part of the K-comprehensive factorization of the composite functor $F \circ \pi_P$.*

In particular, if F is a (J, K)-continuous comorphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ then the sheaf $(C_F)_!(P) = \mathbf{Sh}(F)^*(P)$ corresponds to the second component of the (terminally connected, local homeomorphism)-factorization of the geometric morphism $C_F \circ \prod_P$:

$$\begin{array}{ccc} \mathbf{Sh}(\mathcal{C}, J)/P & \xrightarrow{\textit{term. conn.}} & \mathbf{Sh}(\mathcal{D}, K)/(C_F)_!(P) \\ {\scriptstyle \Pi_P}\downarrow & & \downarrow{\scriptstyle \Pi_{(C_F)_!(P)}} \\ \mathbf{Sh}(\mathcal{C}, J) & \xrightarrow[C_F]{} & \mathbf{Sh}(\mathcal{D}, K) \end{array} .$$

Proof. Both $\mathbf{Sh}(F)^*$ and $(C_F)_!$ are defined as the composite

$$\mathbf{Sh}(\mathcal{C}, J) \xhookrightarrow{\iota_J} [\mathcal{C}^{op}, \mathbf{Set}] \xrightarrow{\mathrm{lan}_{F^{op}}} [\mathcal{D}^{op}, \mathbf{Set}] \xrightarrow{\mathrm{a}_K} \mathbf{Sh}(\mathcal{D}, K),$$

and we showed in the previous result that $\mathrm{a}_K \circ \mathrm{lan}_{F^{op}}$ acts by mapping any presheaf P to the discrete part of the K-comprehensive factorization of $F \circ \pi_P$. We also mentioned in Section 2.5 that the K-comprehensive factorization of a continuous comorphism of sites p induces the (terminally connected, local homeomorphism)-factorization of the corresponding geometric morphism C_p: therefore, from the K-comprehensive factorization of the continuous comorphism of sites

$$(\textstyle\int P, J_P) \xrightarrow{\pi_P} (\mathcal{C}, J) \xrightarrow{F} (\mathcal{D}, K)$$

as

$$(\textstyle\int P, J_P) \xrightarrow{\bar{F}} (\int (C_F)_!(P), J_{(C_F)_!(P)}) \xrightarrow{\pi_{(C_F)_!(P)}} (\mathcal{D}, K),$$

we can deduce that the (terminally connected, local homeomorphism)-factorization of the geometric morphism $C_{F \circ \pi_P} \cong C_F \circ C_{\pi_P}$ is given by

$$\mathbf{Sh}(\textstyle\int P, J_P) \xrightarrow{C_{\bar{F}}} \mathbf{Sh}(\int (C_F)_!(P), J_{(C_F)_!(P)}) \xrightarrow{C_{\pi_{(C_F)_!(P)}}} \mathbf{Sh}(\mathcal{D}, K).$$

Finally, from Theorem 5.2.1 we recall that $\mathbf{Sh}(\int P, J_P) \simeq \mathbf{Sh}(\mathcal{C}, J)/P$ and $C_{\pi_P} \cong \prod_P$, and similar identities hold for the K-sheaf $(C_F)_!(P)$. □

We have in particular the following description of direct and inverse image, when topologies are trivial:

Proposition 4.4.5. *Consider a functor $F : \mathcal{C} \to \mathcal{D}$, and two presheaves $P : \mathcal{C}^{op} \to \mathbf{Set}$ and $Q : \mathcal{D}^{op} \to \mathbf{Set}$ with associated fibrations $\pi_P : \int P \to \mathcal{C}$ and $\pi_Q : \int Q \to \mathcal{D}$.*

- *The direct image of Q along F is computed as the strict pullback of π_Q along F:*

$$\begin{array}{ccc} \int(F^*(Q)) & \longrightarrow & \int Q \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \pi_Q} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

- *Consider the fibration $r : (1_\mathcal{D} {\downarrow} F \circ \pi_P) \to \mathcal{D}$, and denote its class of r-vertical arrows by S: the fibration associated to the presheaf $\mathrm{Lan}_{F^{op}}(P)$ is computed as the localization $\bar{r} : (1_\mathcal{D} {\downarrow} F \circ \pi_P)[S^{-1}] \to \mathcal{D}$, and the fibration associated to $\mathrm{lan}_{F^{op}}(P)$ by taking the second component of the comprehensive factorization of $\bar{r}$:*

$$\begin{array}{ccccc} (1_\mathcal{D} {\downarrow} F \circ \pi_P) & & \xrightarrow{\quad\quad r \quad\quad} & & \mathcal{D} \\ & \searrow & & \nearrow{\scriptstyle \bar{r}} & \big\uparrow {\scriptstyle \pi_{\mathrm{lan}_{F^{op}}(P)}} \ \text{compr. fact.} \\ & & (1_\mathcal{D} {\downarrow} F \circ \pi_P)[S^{-1}] & & \\ & & & \searrow & \\ & & & & \int(\mathrm{lan}_{F^{op}}(P)). \end{array}$$

Proof. The description of the direct image follows directly from its fibrational definition as a pullback. Notice that the pseudopullback of π_P along F is in general a Street fibration, but one can show immediately that it is equivalent to the strict pullback of π_P along F, which is instead a Grothendieck fibration.

The description of $\mathrm{Lan}_{F^{op}}(P)$ is the one appearing in Proposition 4.2.3: $\mathrm{Lan}_{F^{op}}(P)$ is computed by localizing $(1_\mathcal{D} {\downarrow} F \circ \pi_P)$ with respect to the class of its vertical arrows whose component in $\int P$ is cartesian. But, since every arrow in $\int P$ is cartesian, for P is discrete, we conclude that $\bar{r}$ is the fibration associated with $\mathrm{lan}_{F^{op}}(P)$. Finally, we know by Proposition 4.4.1 that the fibration of $\mathrm{lan}_{F^{op}}$ can be recovered as the truncation of $\bar{r}$, and by Corollary 2.5.4 the truncation functor $t_\mathcal{D} : \mathbf{Ind}_\mathcal{D} \to [\mathcal{D}^{op}, \mathbf{Set}]$ acts on fibrations by mapping to the second component of the comprehensive factorization. □

Chapter 5

Giraud toposes

This chapter explores the idea that fibrations are most proficiently studied, in a topos-theoretic setting, when they are considered as continuous comorphisms of sites. Since this approach stems from Giraud's article [12], it seems fitting to adopt the name *Giraud topology* for the topology associated to a fibration over a site, and of *Giraud topos* for the associated topos (which Giraud called *classifying topos* loc. cit.). In the first section we shall introduce Giraud topologies along with some basic properties, first presented in [6]. The second section is dedicated to the particular case of dependent product functors, which provide a first example of the efficacy of the use of comorphisms of sites and fibrations in the context of relative topos theory: the material belongs to Section 3 of [7].

5.1 Fibrations and comorphisms of sites

Definition 5.1.1 [12, §2.2]. Consider a site $(\mathcal{C}, J)$ and a fibration $p : \mathcal{D} \to \mathcal{C}$: we call *Giraud topology for* p the smallest topology $J_{\mathcal{D}}$ over $\mathcal{D}$ making p into a comorphism of sites (see Proposition 1.2.3). The $\mathbf{Sh}(\mathcal{C}, J)$-topos

$$C_p : \mathbf{Sh}(\mathcal{D}, J_{\mathcal{D}}) \to \mathbf{Sh}(\mathcal{C}, J)$$

will be called the *classifying topos of the fibration* p, or its *Giraud topos*, and it will be denoted by $\mathrm{Gir}_J(p)$. If $\mathcal{D}$ is the Grothendieck fibration associated to a $\mathcal{C}$-indexed category $\mathbb{D}$ we will adopt the notations $J_{\mathbb{D}}$ and $\mathrm{Gir}_J(\mathbb{D})$. In particular, for a representable presheaf $よ(X)$ we introduced in Example 1.2.1 the notation J_X as a shorthand for $J_{よ(X)}$.

For a generic functor $p : \mathcal{D} \to \mathcal{C}$, the smallest topology M^p_J on $\mathcal{D}$ making p into a comorphism of sites may not have an explicit characterization: in full generality, the best we can do is presenting M^p_J using the pullback-stable family (i.e. the *coverage*) of generating sieves S_D mentioned in Proposition 1.2.3. However, if p is a fibration, its Giraud topology $J_{\mathcal{D}}$ admits a fully explicit characterization:

Proposition 5.1.1 [6, Theorem 3.13]. *Given a fibration $p : \mathcal{D} \to \mathcal{C}$, a sieve R is $J_{\mathcal{D}}$-covering if and only if the collection of cartesian arrows in it is sent to a J-covering family through p.*

Although we are speaking about Giraud *toposes*, a size issue needs to be addressed, because there is no reason for the category $\mathrm{Gir}_J(\mathcal{D})$ to be a topos, and in particular for the site $(\mathcal{D}, J_{\mathcal{D}})$ to be small or even small-generated. We will ignore this problem as long as it will not hinder the tractation; in the context of the *fundamental adjunction* (Chapter 6) we will avoid size issues altogether by restricting to a suitable class of fibrations, called essentially J-small (see Definition 6.3.1). However, it is important to remark that in the context of discrete fibrations there are no problems of size:

Lemma 5.1.2 [8, Lemma 2.10.8]. *Given a small-generated site $(\mathcal{C}, J)$ and a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$, the site $(\int P, J_P)$ is small-generated.*

Proof. Denote by $\mathcal{A} \hookrightarrow \mathcal{C}$ a small J-dense full subcategory of $\mathcal{C}$. Consider the full subcategory $\mathcal{B} \hookrightarrow \int P$ whose objects are pairs (A, U) with A in $\mathcal{A}$: then $\mathcal{B}$ as a set of objects and is locally small, therefore it is small. Now consider an object (X, U) of $\int P$. Since $\mathcal{A} \hookrightarrow \mathcal{C}$ is J-dense, there exists a J-covering family $\{y_i : A_i \to X \mid i \in I\}$ such that each A_i belongs to $\mathcal{A}$. By Proposition 5.1.1 the family $\{y_i : (A_i, P(y_i)(U)) \to (X, U) \mid i \in I\}$ is J_P-covering, and thus $\mathcal{B}$ is J_P-dense in $\int P$. □

There are many results showing how fibrations interact well with the theory of comorphisms. First of all, fibrations *and* their morphisms naturally become *continuous* comorphisms of sites with respect to Giraud topologies:

Proposition 5.1.3 [6, Theorem 4.44 and Corollary 4.47]. *Consider a fibration $p : \mathcal{D} \to \mathcal{C}$ and a topology J over $\mathcal{C}$. Then p is a $(J_{\mathcal{D}}, J)$-continuous comorphism of sites. More generally, given two fibrations $p : \mathcal{D} \to \mathcal{C}$ and $q : \mathcal{E} \to \mathcal{C}$ and a morphism of fibrations $(F, \varphi) : [p] \to [q]$ between them, then F is a $(J_{\mathcal{D}}, J_{\mathcal{E}})$-continuous comorphism of sites.*

This entails that the functor

$$\mathfrak{G} : \mathbf{Cat}/\mathcal{C} \to \mathbf{Com}/(\mathcal{C}, J)$$

introduced in Section 1.2 restricts to small fibrations providing a functor

$$\mathfrak{G} : \mathbf{Fib}^s_{\mathcal{C}} \to \mathbf{Com}_{cont}/(\mathcal{C}, J),$$

and that a fibration is mapped to its Giraud topos via the composite

$$\mathbf{Fib}^s_{\mathcal{C}} \xrightarrow{\mathfrak{G}} \mathbf{Com}_{cont}/(\mathcal{C}, J) \xrightarrow{C_{(-)}} \mathbf{EssTopos}/\mathbf{Sh}(\mathcal{C}, J).$$

Giraud topologies can be also used to study some properties of the fibration at hand. An example of this is the following result, which is a broad generalization of Proposition 2.1, item (3) of [12]:

Proposition 5.1.4 [8, Proposition 2.11.3]. *Consider a subcanonical site* $(\mathcal{C}, J)$ *and a cloven fibration* $p : \mathcal{D} \to \mathcal{C}$*: then* p *is a prestack if and only if its Giraud topology* $J_{\mathcal{D}}$ *is subcanonical.*

Proof. We start by supposing that $\mathcal{D}$ is a prestack. Consider a $J_{\mathcal{D}}$-covering family $R = \{f_i : \mathrm{dom}(f_i) \to D \mid i \in I\}$: then $p(R) = \{p(f_i)\}$ is a J-covering family of $p(D)$. We will call S the $J_{p(D)}$-covering family $\{p(f_i) : [p(f_i)] \to [1_{p(D)}]\ i \in I\}$ for $[1_{p(D)}]$ in $\mathcal{C}/p(D)$. Consider now some other object X of $\mathcal{D}$: a matching family for R and よ(X) is the given, for any $i \in I$, of an arrow $\alpha_i : \mathrm{dom}(f_i) \to X$, subject to the condition that for every span of arrows h and k such that $f_i h = f_j k$ it holds that $\alpha_i h = \alpha_j k$.

We start by considering the family of arrows $\{p(\alpha_i) : \mathrm{dom}(p(f_i)) \to p(X)\}$. It is easy to verify that they are a matching family for よ$(p(X))$ and $p(R)$, and hence since J is subcanonical there exists a unique $\beta : p(D) \to p(X)$ such that $p(\alpha_i) = \beta p(f_i)$.

Using β we can now 'move over' $p(D)$ by considering the Hom-functor $\mathrm{Hom}((D, 1_{p(D)}), \mathbb{D}(\beta)(X, 1_{p(X)})) : (\mathcal{C}/p(D))^{op} \to \mathbf{Set}$. We will now build a matching family for it and S. To do so, notice first that f_i and $\widehat{p(f_i)}_D$ are both cartesian lifts of $p(f_i)$: therefore, there is a unique canonical isomorphism $\rho_i : \mathrm{dom}(\widehat{p(f_i)}_D) \to \mathrm{dom}(f_i)$ comparing them. There is also a canonical isomorphism $\sigma_i : \mathrm{dom}(\alpha_i) \to \mathrm{dom}(\widehat{p(\alpha_i)}_X)$. So we can consider the composite arrows $\gamma_i := \chi_{\beta, p(f_i), X} \sigma_i \rho_i$, and a lengthy calculation shows that they constitute a matching family for $\mathrm{Hom}((D, 1_{p(D)}), \mathbb{D}(\beta)(X, 1_{p(X)}))$ and S: if $\mathcal{D}$ is a prestack they admit an amalgamation $\gamma : \mathrm{dom}(\widehat{1_{p(D)}}_D) \to \mathrm{dom}(\widehat{\theta_{\beta,X}}_{\mathrm{dom}(\widehat{\beta}_X)})$. Now, if we consider the arrow α defined as the composite

$$D \xrightarrow{\widehat{1_{p(D)}}_D^{-1}} \mathrm{dom}(\widehat{1_{p(D)}}_D) \xrightarrow{\gamma} \mathrm{dom}(\widehat{\theta_{\beta,X}}_{\mathrm{dom}(\widehat{\beta}_X)}) \xrightarrow{\widehat{\theta_{\beta,X}}_{\mathrm{dom}(\widehat{\beta}_X)}} \mathrm{dom}(\widehat{\beta}_X) \xrightarrow{\widehat{\beta}_X} X$$

it is again lenghty but straightforward to see that it provides an amalgamation for the matching family $\{\alpha_i\}$, proving that よ(X) is a $J_{\mathcal{D}}$-sheaf and hence that $J_{\mathcal{D}}$ is subcanonical.

Conversely, suppose $J_{\mathcal{D}}$ is subcanonical and consider a J_X-covering sieve S for it and $\mathrm{Hom}((A, \alpha), (B, \beta))$. Without loss of generality we may assume that $S = \{f_i : [f_i] \to [1_X] \mid i \in I\}$ covers the terminal object of $\mathcal{C}/X$, by Corollary 2.3.5, so that the matching family consists of arrows $\gamma_i : \mathrm{dom}(\widehat{\alpha f_i}_A) \to \mathrm{dom}(\widehat{\beta f_i}_B)$ satisfying suitable compatibility conditions. We can now consider the $J_{\mathcal{D}}$-covering family $R = \{\widehat{\alpha f_i}_A \mid i \in I\}$ over A, and the arrows $\mu_i := \widehat{\beta f_i}_B \gamma_i : \mathrm{dom}(\widehat{\alpha f_i}_A) \to B$. A calculation shows that it is a matching family for R and よ(B), and thus it admits a unique amalgamation $\gamma : A \to B$, which also proves to be a unique amalgamation for the original matching family $\{\gamma_i\}$. Hence all Hom-functors are sheaves and we conclude that $\mathcal{D}$ is a prestack. □

Example 5.1.1. The request for the fibration p to be a prestack with a subcanonical Giraud topology is not enough for J to be subcanonical. Indeed, consider $\mathcal{C}$ to be the arrow category $t : 0 \to 1$, $\mathcal{D}$ the empty category and $p : \mathcal{D} \to \mathcal{C}$ the unique functor. Notice that $\mathcal{D}$ corresponds to the discrete $\mathcal{C}$-indexed category $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ with constant value the empty category. Now consider on $\mathcal{C}$ the topology J whose only non-trivial sieve is the singleton $\{t\}$: it is not subcanonical, since よ(0) is not a J-sheaf. $\mathbb{D}$ is a sheaf for this topology, and hence it is a J-stack; moreover, the induced topology K on $\mathcal{D}$ is the only topology on the empty category, which is trivially subcanonical.

5.2 Dependent product functors

As announced in the introduction, a nice description of dependent product functors is a first example of the fruitful application of fibrations and comorphisms of sites to relative topos theory. In Appendix A we will return to the dependent product functor from an elementary point of view.

We have already mentioned in Section 1.2 that an arrow $e : E \to E'$ in a topos $\mathscr{E}$ induces an essential geometric morphism

$$\mathscr{E}/E \underset{\prod_e}{\overset{\sum_e}{\rightleftarrows}} \mathscr{E}/E', \quad \textstyle\sum_e \dashv e^* \dashv \prod_e, \quad e^* : \mathscr{E}/E' \to \mathscr{E}/E$$

where the inverse image e^* is defined by pullback of arrows along e. The direct image $\prod_e$ is called the *dependent product along* e, while the essential image $\sum_e$ is called the *dependent sum along* e; in particular, $\sum_e$ acts by post-composition with e. When $e = !_E : E \to 1_{\mathscr{E}}$ is the unique arrow to the terminal object of $\mathscr{E}$, the adjoint triple above is denoted by

$$\mathscr{E}/E \underset{\prod_E}{\overset{\sum_E}{\rightleftarrows}} \mathscr{E}. \quad \textstyle\sum_E \dashv E^* \dashv \prod_E, \quad E^* : \mathscr{E} \to \mathscr{E}/E$$

The behaviour of $\prod_e$ is usually described in the literature using the internal logical structure of the topos. The following result, which sums up the content of Section 3 of [7], shows instead that from a site-theoretic perspective dependent products arise quite simply from discrete fibrations, seen as comorphisms of sites.

Theorem 5.2.1. *Consider a site $(\mathcal{C}, J)$ and a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ with corresponding Grothendieck fibration $\pi_P : \int P \to \mathcal{C}$, and denote by J_P its*

Giraud topology. The composite functor

$$A : \textstyle\int P \xrightarrow{\pi_P} \mathcal{C} \xrightarrow{\ell_J} \mathbf{Sh}(\mathcal{C}, J) \xrightarrow{\mathrm{a}_J(P)^*} \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P)$$

is flat and J_P-continuous, and it induces an adjoint equivalence of toposes

$$L_P^J : \mathbf{Sh}(\textstyle\int P, J_P) \underset{\longrightarrow}{\overset{\sim}{\underset{\top}{\longleftarrow}}} \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P) : R_P^J$$

- *In the context of presheaves, i.e. when J is the trivial topology over $\mathcal{C}$, the functor $R_P : [\mathcal{C}^{op}, \mathbf{Set}]/P \to [(\int P)^{op}, \mathbf{Set}]$ is defined on an arrow $[h : H \to P]$ by*
$$R_P[h](X, U) = h_X^{-1}(U)$$
for each (X, U) in $\int P$.
Its left adjoint $L_P : [(\int P)^{op}, \mathbf{Set}] \to [\mathcal{C}^{op}, \mathbf{Set}]/P$ is defined for a presheaf $W : (\int P)^{op} \to \mathbf{Set}$ as
$$L_P(W)(X) := \bigsqcup_{U \in P(X)} W(X, U),$$
with the structural morphism $L_P(W) \to P$ defined by projecting each component $W(X, U)$ to the element U. The definition of R_P and L_P on arrows is straightforward.

- *The functor $R_P^J : \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P) \to \mathbf{Sh}(\int P, J_P)$ acts on an object $[h : H \to \mathrm{a}_J(P)]$ of $\mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P)$ as*
$$R_P^J[h](X, U) = R_P[\eta_P^*(h)](X, U) \cong \{x \in H(X) \mid h_X(x) = (\eta_P)_X(U)\},$$
where $\eta_P : P \to \mathrm{a}_J(P)$ is the unit of the adjunction $\mathrm{a}_J \dashv \iota_J$.
Its left adjoint $L_P^J : \mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P)$ is described for a presheaf $W : \int P^{op} \to \mathbf{Set}$ as
$$L_P^J(W) := \mathrm{colim}(\textstyle\int W \xrightarrow{\pi_W} \int P \xrightarrow{A} \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P)) = \underset{x \in W(X,U)}{\mathrm{colim}} \ell_J(X)$$
and the structural morphism
$$L_P^J(W) \to \mathrm{a}_J(P)$$
is the arrow induced via the colimit's universal property by the cocone
$$\mathrm{a}_J(\ulcorner U \urcorner) : \ell_J(X) \to \mathrm{a}_J(P),$$
where each $\ulcorner U \urcorner : よ(X) \to P$ corresponds to $U \in P(X)$ via the Yoneda lemma.

Moreover, the equivalence R_P^J is pseudonatural in P. More explicitly, given an arrow $f : P \to Q$ in $\mathbf{Sh}(\mathcal{C}, J)$, the corresponding comorphism of fibrations $\int f : \int P \to \int Q$ induces a geometric morphism $C_{\int f} : \mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\int Q, J_Q)$ which is essentially isomorphic to the dependent product functor $\prod_{\mathrm{a}_J(f)} : \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P) \to \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(Q)$ via the equivalences above: that is, the square

$$\begin{array}{ccc} \mathbf{Sh}(\int P, J_P) & \xrightarrow{C_{\int f}} & \mathbf{Sh}(\int Q, J_Q) \\ R_P^J \uparrow \wr \downarrow L_P^J & & R_Q^J \uparrow \wr \downarrow L_Q^J \\ \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P) & \xrightarrow[\prod_{\mathrm{a}_J(f)}]{} & \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(Q) \end{array}$$

is commutative up to natural isomorphism. In particular, the geometric morphism $C_{\pi_P} : \mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\mathcal{C}, J)$ is essentially isomorphic to the dependent product functor $\prod_{\mathrm{a}_J(P)} : \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P) \to \mathbf{Sh}(\mathcal{C}, J)$, i.e. the following triangle is commutative up to canonical isomorphism:

$$\begin{array}{ccc} \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P) & \overset{\sim}{\longrightarrow} & \mathbf{Sh}(\int P, J_P) \\ & {\scriptstyle \prod_{\mathrm{a}_J(P)}} \searrow \quad \swarrow {\scriptstyle C_{\pi_P}} & \\ & \mathbf{Sh}(\mathcal{C}, J) & \end{array} .$$

Proof. The proof of the equivalence in the presheaf case (i.e. when J is the trivial topology) can be found in [21, Proposition A1.1.7] and the equivalence for a generic J in [6, Section 6.7]. In particular, one can verify explicitly that the square

$$\begin{array}{ccc} [\mathcal{C}^{op}, \mathbf{Set}]/P & \underset{L_P}{\overset{R_P}{\rightleftarrows}}\ (\top) & [(\int P)^{op}, \mathbf{Set}] \\ \mathrm{a}_P \downarrow \dashv \uparrow \iota_P & & \mathrm{a}_{J_P} \downarrow \dashv \uparrow \iota_{J_P} \\ \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P) & \underset{L_P^J}{\overset{R_P^J}{\rightleftarrows}}\ (\top) & \mathbf{Sh}(\int P, J_P) \end{array} \tag{5.1}$$

is commutative, where the adjunction $\mathrm{a}_P \dashv \iota_P$ is defined on $[h : H \to P]$ as $\mathrm{a}_P([h]) := [\mathrm{a}_J(h)]$ (cf. Proposition A.6).

To prove the pseudonaturality, we start from the presheaf case. For any presheaf $W : (\int Q)^{op} \to \mathbf{Set}$, it holds that

$$\begin{aligned} (f^* \circ L_Q)(W)(X) &\cong \coprod_{U \in P(X)} W(X, f_X(U)) \\ &= L_P \circ (- \circ (\textstyle\int f)^{op})(W)(X) \end{aligned}$$

naturally in $X \in \mathcal{C}$, therefore

$$L_P \circ (- \circ (\textstyle\int f)^{op}) = f^* \circ L_Q, \; R_Q \circ \prod_f = \mathrm{ran}_{(\int f)^{op}} \circ R_P. \tag{5.2}$$

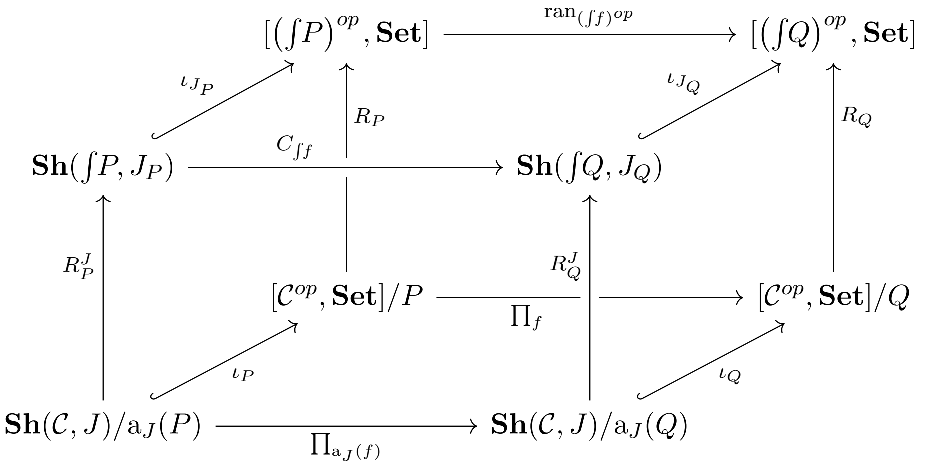
By Lemma 1.2.4 the morphism of fibrations $\int f : \int P \to \int Q$ is a comorphism of sites with respect to Giraud topologies. Notice also that the square

$$\begin{array}{ccc} (\int P, J^{tr}_{\int P}) & \xrightarrow{\int f} & (\int Q, J^{tr}_{\int Q}) \\ {\scriptstyle 1_{\int P}}\uparrow & & \uparrow{\scriptstyle 1_{\int P}} \\ (\int P, J_P) & \xrightarrow{\int f} & (\int Q, J_Q) \end{array}$$

of comorphisms is obviously commutative (where $J^{tr}_{\mathcal{A}}$ denotes the trivial topology on a category $\mathcal{A}$) and thus it induces the commutative square

$$\begin{array}{ccc} [(\int P)^{op}, \mathbf{Set}] & \xrightarrow{\mathrm{ran}_{(\int f)^{op}}} & [(\int Q)^{op}, \mathbf{Set}] \\ \uparrow & & \uparrow \\ \mathbf{Sh}(\int P, J_P) & \xrightarrow{C_{\int f}} & \mathbf{Sh}(\int Q, J_Q) \end{array} \tag{5.3}$$

of geometric morphisms. Combining this with Equation (5.2), Square (5.1) and Proposition A.6 we conclude that the cube



is commutative, and the commutativity of the front square states precisely the pseudonaturality in our claim.

Finally, by taking f equal to the arrow $!_P : P \to 1_{\mathbf{Sh}(\mathcal{C},J)}$ we obtain the last claim (notice that $\int(1_{\mathbf{Sh}(\mathcal{C},J)}) \simeq \mathcal{C}$). □

Remark 5.2.1. It can be easily checked that if P is a J-sheaf then the functor $L_P : [(\int P)^{op}, \mathbf{Set}] \to [\mathcal{C}^{op}, \mathbf{Set}]/P$ takes values in J-sheaves [28, Exercise III.8(b)], whence the functor L^J_P is the restriction of L_P along the canonical inclusions

$$\iota_{J_P} : \mathbf{Sh}(\int P, J_P) \hookrightarrow [(\int P)^{op}, \mathbf{Set}], \ \iota_P : \mathbf{Sh}(\mathcal{C}, J)/P \hookrightarrow [\mathcal{C}^{op}, \mathbf{Set}]/P.$$

In particular, if P and Q are J-sheaves, the dependent product $\prod_f$ is computed regardless of the topological *datum* of J. This is due to the fact that, in this case, R_P^J and L_Q^J are restrictions of R_P and L_Q respectively. Thus one can derive an explicit formula of the dependent product for sheaves:

Corollary 5.2.2. *Consider a site $(\mathcal{C}, J)$, two J-sheaves P and Q and an arrow $f : P \to Q$. For any arrow $h : H \to P$ in $\mathbf{Sh}(\mathcal{C}, J)$ and for any object X in $\mathcal{C}$, we have*

$$\prod\nolimits_f [h](X) = \{(q, \underline{x}) \mid q \in Q(X),\ \underline{x} \in A_f^h(X)\}$$

(with the canonical projection to Q), where $A_f^h(X)$ is the set

$$\{\underline{x} \in \prod_{\substack{g:Y\to X \\ p\in f_Y^{-1}(Q(g)(q))}} h_Y^{-1}(p) \mid \forall\gamma \text{ s.t. } \mathrm{cod}(\gamma) = \mathrm{dom}(g),\ H(\gamma)(x_{g,p}) = x_{g\circ\gamma, P(\gamma)(p)}\}$$

Proof. First of all, by Theorem 5.2.1, for any arrow $f : P \to Q$ of presheaves we have the chain of equivalences

$$\begin{aligned} \prod\nolimits_{\mathrm{a}_J(f)} &\cong L_Q^J \circ (C_{\int f})_* \circ R_P^J \\ &\cong \mathrm{a}_Q \circ L_Q \circ \mathrm{ran}_{(\int f)^{op}} \circ R_P \circ i_P \\ &\cong \mathrm{a}_Q \circ \prod\nolimits_f^{\mathrm{pr}} \circ i_P, \end{aligned}$$

where $\prod_f^{\mathrm{pr}}$ is the dependent product at the level of presheaves. The dependent product $\prod_f$ is induced by the comorphism of sites $\int f$, so in particular $\prod_f^{\mathrm{pr}}$ is simply the right Kan extension functor $\mathrm{ran}_{(\int f)^{op}}$. Its value $\mathrm{ran}_{(\int f)^{op}}(R_P^J[h])$ can be computed (see e.g. [21, Example A4.1.4]) as

$$\mathrm{ran}_{(\int f)^{op}}(R_P^J[h])(X, q) = \lim\left(((X, q)\downarrow(\textstyle\int f)^{op}) \to (\textstyle\int P)^{op} \xrightarrow{R_P^J[h]} \mathbf{Set} \right).$$

The nodes of the diagram are arrows $g : (Y, f_Y(p)) \to (X, q)$ in $\int Q$, so they are in fact indexed by couples $g : Y \to X$, $p \in P(Y)$ such that $f_Y(p) = Q(g)(q)$; the edges of the diagram are the arrows $\gamma : Y' \to Y$ in $\mathcal{C}$ such that $g' = g \circ \gamma$ and $P(\gamma)(p) = p'$. Notice that γ is actually an arrow $(Y', f_{Y'}(p')) \to (Y, f_Y(p))$ by the latter conditions and the naturality of f. Recalling that $R_P^J([h])(Y, p) = h_Y^{-1}(p)$ for J-sheaves, the above limit is the set $A_f^h(X)$ in the statement of the corollary. Finally, the coproduct of all these fibres yields the desired expression of the dependent product $\prod_f[h]$. □

We conclude this section by providing a technical lemma that we will exploit later, stating that given a J-covering sieve $R \rightarrowtail \text{よ}(X)$, the site $(\int R, J_R)$ is Morita-equivalent to $(\mathcal{C}/X, J_X)$:

Lemma 5.2.3. *Given a site* $(\mathcal{C}, J)$ *and a* J*-sieve* $m_R : R \rightarrowtail よ(X)$*, then* $\int m_R : (\int R, J_R) \to (\mathcal{C}/X, J_X)$ *induces an equivalence of toposes*

$$C_{\int m_R} : \mathbf{Sh}(\textstyle\int R, J_R) \xrightarrow{\sim} \mathbf{Sh}(\mathcal{C}/X, J_X).$$

Proof. This is a corollary of Theorem 5.2.1, since the square

$$\begin{array}{ccc} \mathbf{Sh}(\int R, J_R) & \xrightarrow{\sim} & \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(R) \\ {\scriptstyle C_{\int m_R}}\downarrow & & \wr\downarrow{\scriptstyle \prod_{\mathrm{a}_J(m_R)}} \\ \mathbf{Sh}(\mathcal{C}/X, J_X) & \xrightarrow{\sim} & \mathbf{Sh}(\mathcal{C}, J)/\ell_J(X) \end{array}$$

commutes up to natural isomorphism, where $\prod_{\mathrm{a}_J(m_R)}$ is an equivalence since $\mathrm{a}_J(m_R)$ is invertible. □

Chapter 6

The fundamental adjunction

This chapter is devoted to showing that fibrations and stacks over a base site $(\mathcal{C}, J)$ are adjoint to toposes over the topos $\mathbf{Sh}(\mathcal{C}, J)$, the so-called *fundamental adjunction* of [8]. This result is a generalization of the classical presheaf-bundle adjunction for topological spaces, and it is an instance of the widespread alternative point of view in mathematics between 'indexed structures' and 'fibred structures'.

In Section 6.1 we begin by recalling the presheaf-bundle adjunction for topological spaces, along with some technical results: some further results about étale maps of topological spaces will appear later in Appendix C. Section 6.2 deals with the adjoints to the Grothendieck construction functor, especially in relation with Giraud topologies. Section 6.3 introduces the fundamental adjunction, and Section 6.4 shows that the canonical stack of the base site acts as a dualizing object for the fundamental adjunction. Finally, the last section introduces the notion of relative site motivated by the previous results.

All results in this chapter, unless stated otherwise, appear in Chapter 5, Section 6.2 and Chapter 8 of [8].

6.1 The presheaf-bundle adjunction for topological spaces

In the present section we recall the adjunction $\Lambda \dashv \Gamma$ between the topos of presheaves over a topological space X and the category of bundles over X, as a motivating example for the fundamental adjunction. We also take an opportunity to analyse some aspects of the adjunction that pave the way for later developments in the study of preorder sites (see Section 7.2).

Given a topological space X, denote by $\mathcal{O}(X)$ its poset of open subsets, the topos $\mathbf{Psh}(X) := [\mathcal{O}(X)^{op}, \mathbf{Set}]$ is called the category of presheaves (of sets) over X. The topos of sheaves for the canonical open cover topology $J^{can}_{\mathcal{O}(X)}$ on $\mathcal{O}(X)$ is denoted by $\mathbf{Sh}(X)$. On the other hand, if $\mathbf{Top}$ is the

1-category of topological spaces, the 1-categorical slice $\mathbf{Top}/X$ is called the category of *bundles* over X.

Since $\mathcal{O}(X)$ is a poset, there is at most one arrow $i : V \hookrightarrow U$ for any two V and U in $\mathcal{O}(X)$ (cf. the beginning of Section 7.2): therefore, for any presheaf P over X and any $s \in P(U)$ its image $P(i)(s)$ is usually denoted just by $s_{|V}$. We recall that for a presheaf P over X the *stalk of* P *at* $x \in X$ is defined as the colimit of sets $P_x := \operatorname{colim}_{x\in U\in\mathcal{O}(X)} P(U)$. For any $s \in P(U)$ its equivalence class in P_x is called *germ of* s *at* x and it is denoted s_x: then $s \in P(U)$ and $t \in P(V)$ have the same germ at $x \in U \cap V$ if and only if there exists $W \subseteq U \cap V$ such that $x \in W$ and $s_{|W} = t_{|W}$. Of course, any arrow $h : P \to Q$ of presheaves induces an arrow $h_x : P_x \to Q_x$ by the universal property of colimits.

There is an adjunction (cf. [28, Sections II.4, II.5, II.6])

$$\mathbf{Psh}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Top}/X \quad (\Lambda \dashv \Gamma) .$$

The functor Λ maps a presheaf P to its *bundle of germs*, i.e. the canonical projection map $\pi_P : E_P = \coprod_{x\in X} P_x \to X$; on arrows Λ acts by mapping $h : P \to Q$ to $\Lambda_h := \coprod_{x\in X} h_x : \coprod_{x\in X} P_x \to \coprod_{x\in X} Q_x$. For any $s \in P(U)$ we can define a map $\dot{s} : U \to E_P$ sending a point $x \in U$ to the germ s_x: since

$$\dot{s}(U) \cap \dot{t}(V) = \bigcup_{\substack{W\subseteq U\cap V\\ r=s|_W=t|_W}} \dot{r}(W),$$

for any $s \in P(U)$ and $t \in P(V)$, the sets of the form $\dot{s}(U)$ are the basis for a topology over E_P. With this topology one can show that π_P and all the functions of kind Λ_h or $\dot{s}$ are continuous.

The functor Γ is the *local sections* functor, assigning to a bundle $p : E \to X$ the presheaf Γ_p which sends each open set U of X to the set $\Gamma_p(U)$ of continuous maps $s : U \to E$ such that $p \circ s = i_U : U \hookrightarrow X$: these are called the *sections* of p defined on U. More compactly, $\Gamma_p := \mathbf{Top}/X([i_{(-)}], [p]) : \mathcal{O}(X)^{op} \to \mathbf{Set}$. The functor Λ takes values in the full subcategory $\mathbf{Etale}(X)$ of étale bundles on X (that is, the local homeomorphisms to X), while the functor Γ takes values in the full subcategory $\mathbf{Sh}(X)$ of $\mathbf{Psh}(X)$; in fact, the adjunction $\Lambda \dashv \Gamma$ restricts to an equivalence between $\mathbf{Etale}(X)$ and $\mathbf{Sh}(X)$.

Remark 6.1.1. In particular, the adjunction $\Lambda \dashv \Gamma$ presents the space E_P as a colimit of topological spaces over X, since for any bundle $q : Y \to X$ we have

$$\begin{aligned}\mathbf{Top}/X([\pi_P], [q]) &\simeq \mathbf{Psh}(X)(P, \mathbf{Top}/X([i_{(-)}], [q]))\\ &\simeq [(\textstyle\int P)^{op}, \mathbf{Set}](1, \mathbf{Top}/X([i_{\pi_P(-)}], [q])),\end{aligned}$$

where the last equivalence is a consequence of Proposition 3.2.1. Thus E_P is the colimit of the diagram $D : \int P \to \mathbf{Top}/X$ mapping every (U, s) to the inclusion $U \hookrightarrow X$.

We will now provide some pointfree results related to the adjunction $\Lambda \dashv \Gamma$. The first result states that set-theoretic sections for a bundle of germs are topological sections if and only if they are locally so:

Proposition 6.1.1. *Let X be a topological space, U an open set of X, P a presheaf on X and $s : U \to \coprod_{x\in X} P_x$ a map such that $p_P \circ s = i_U$. Then the following conditions are equivalent:*

(i) s is continuous;

(ii) there is an open covering $\{U_i \hookrightarrow U \mid i \in I\}$ of U such that for each $i \in I$ there is $t_i \in P(U_i)$, $s|_{U_i} = \dot{t}_i$.

Proof. We use the fact that the collection of subsets $\dot{t}(V)$ for $V \in \mathcal{O}(X)$ and $t \in P(V)$ are an open covering and a basis of E_P. Notice that given $z \in s^{-1}(\dot{t}(V)) \subseteq U$, then there must be some $v \in V$ such that $s(z) = t_v$. This implies that $z = \pi_P(s(z)) = \pi_P(t_v) = v$, so that $s^{-1}(\dot{t}(V)) \subseteq V$, and $s(z) = t_z$.

First suppose that s is continuous: then the collection of subsets of the form $Z := s^{-1}(\dot{t}(V))$ is an open covering of U, and we have shown above that $s|_Z = t|_Z$: thus condition (ii) is satisfied.

Conversely, assume (ii). Since the subsets $\dot{t}(V)$ form a basis, showing the continuity of s is equivalent to verifying that the inverse image under s of any of these open sets is open. We will prove this by showing that $s^{-1}(\dot{t}(V))$ contains an open neighbourhood of each of its points. Consider thus $y \in s^{-1}(\dot{t}(V))$: we already know that $s(y) = t_y$, but by our hypothesis there is also an $i \in I$ such that $y \in U_i$ and $s(y) = (t_i)_y$. Since $(t_i)_y = t_y$ there is an open neighbourhood W_i of y such that $W_i \subseteq U_i \cap V$ and $t_i|_{W_i} = t|_{W_i}$. So W_i is an open neighbourhood of y contained in $s^{-1}(\dot{t}(V))$, and thus $s^{-1}(\dot{t}(V))$ is open. □

The following technical lemma will allow us to show that sections or local homeomorphisms are always open maps:

Lemma 6.1.2. *Let $s : U \to E$ be a section of a local homeomorphism $p : E \to X$ on an open set U of X. Then, any open subset V of U can be covered by a family $\{V_i \mid i \in I\}$ of open subsets $V_i \subseteq V$ such that for each $i \in I$, $s(V_i)$ is contained in an open subset A_i of E such that $p|_{A_i} : A_i \to p(A_i)$ is an homeomorphism onto an open subset $p(A_i)$ of X.*

Proof. Since p is a local homeomorphism, there is an open covering $\{A_i \subseteq E \mid i \in I\}$ of E such that for each i, $p|_{A_i} : A_i \to p(A_i)$ is an homeomorphism onto an open set $p(A_i)$ of X. By taking $V_i = s^{-1}(A_i) \cap V$, we thus obtain an open covering of V satisfying the required condition. □

Corollary 6.1.3. *Consider a topological space X and P in $\mathbf{Psh}(X)$: then any section of the étale bundle $E_P \to X$ is an open map.*

Proof. Consider an open V of X and a section $s : V \to E_P$: given a covering $\{V_i \mid i \in I\}$ of V as in the previous lemma, $s(V) = \bigcup_{i\in I} s(V_i)$, and each $s(V_i)$ is open as it is the inverse image of the open set $V_i = p(s(V_i)) \subseteq p(A_i)$ along the local homeomorphism $p|_{A_i} : A_i \to p(A_i)$. □

Finally, we recall another technical result about open maps:

Lemma 6.1.4. *If a continuous map $e : E \to X$ of topological spaces is open, then $e^{-1} : \mathcal{O}(X) \to \mathcal{O}(E)$ admits a left adjoint $e_!$, which acts by mapping $U \subseteq E$ to $e(U) \subseteq X$.*

We now provide two results which hint that the topological presheaf-bundle adjunction is essentially point-free in nature; we defer all remarks about them after their proofs. The first result shows that the bundle of germs for a presheaf $P \in \mathbf{Psh}(X)$ is a topological site of presentation for the Giraud topos $\mathrm{Gir}_{J^{can}_{\mathcal{O}(X)}}(P)$ (see Definition 5.1.1), giving a topos-theoretic motivation to the relevance of the étale bundle of a presheaf:

Proposition 6.1.5. *Let X be a topological space and P a presheaf P on it. Consider the category $\int P$ and endow it with its Giraud topology J_P. In particular, $\{(U_i, s_{|U_i}) \hookrightarrow (U, s) \mid i \in I\}$ is J_P-covering if and only if $\{U_i \mid i \in I\}$ is an open covering of U. Consider the bundle of germs E_P: then the functor $f_P : \int P \to \mathcal{O}(E_P)$ defined by $f_P(U, s) := \dot{s}(U)$ is a morphism and comorphism of sites which induces an equivalence of toposes*

$$C_{f_P} : \mathbf{Sh}(\textstyle\int P, J_P) \simeq \mathbf{Sh}(E_P).$$

Moreover, this equivalence is compatible with the canonical geometric morphisms to $\mathbf{Sh}(X)$, and it is natural in P.

Proof. To prove that f_P is a morphism of sites we use the characterization of [6, Definition 3.2], which was divided in four conditions:

(i) f_P sends covering families to covering families: indeed, if $\{(U_i, s_{|U_i})\}$ is an open cover of (U, s) then $U = \bigcup_{i\in I} U_i$ and thus $\dot{s}(U) = \bigcup_{i\in I} \dot{s_{|U_i}}(U_i)$.

(ii) for any W in $\mathcal{O}(E_P)$ there is a covering family $W_i \subseteq W$ such that for each i there exists a (U_i, s_i) in $\int P$ such that $W_i \subseteq \dot{s}_i(U_i)$. This is trivial, because since the sets $\dot{s}(U)$ form a basis for E_P there is a family of (U_i, s_i) such that $W = \bigcup_{i\in I} \dot{s}_i(U_i)$.

(iii) for any pair (U, s), (V, t) in $\int P$ and every $W \subseteq \dot{s}(U) \cap \dot{t}(V)$ there exist an open covering $\{W_i \mid i \in I\}$ of W and open subsets $U_i \subseteq U \cap V$ such that $s_{|U_i} = t_{|U_i} = r_i$ and $W_i \subseteq \dot{r_i}(U_i)$. Since $\dot{s}(U) \cap \dot{t}(V) = \bigcup_{Z\in I} \dot{s_{|Z}}(Z)$, where $I = \{Z \in \mathcal{O}(X) \mid Z \subseteq U \cap V,\ s_{|Z} = t_{|Z}\}$, we can take the U_i's above as the opens Z, set $r_Z := s_{|Z}$ and $W_Z := W \cap \dot{r_Z}(Z)$.

(iv) The fourth condition concerns parallel pairs of arrows and is trivial in this case.

To check that f_P is a comorphism of sites, consider (U, s) in $\int P$ and an open covering of $\dot{s}(U)$: without loss of generality we may assume it to be of the form $\{\dot{t_i}(V_i) \mid i \in I\}$ for (V_i, t_i) in $\int P$. We want to show that there is a J_P-covering family $\{(U_j, s_j)\}$ of (U, s) such that its image through f_P is contained in the sieve generated by $\{\dot{t_i}(V_i) \mid i \in I\}$: that is, every $\dot{s_j}(U_j)$ is contained in some $\dot{t_i}(V_i)$. First of all, notice that the equality $\bigcup_{i\in I} \dot{t_i}(V_i) = \dot{s}(U)$ implies immediately that $\bigcup_{i\in I} V_i = U$. Now consider any $v \in V_i$: since $(t_i)_v = (s_{|V_i})_v$ there must be an open neighbourhood $W_i(v) \subseteq V_i$ of v such that $(t_i)_{|W_i(v)} = s_{|W_i(v)}$. Since each V_i is covered by the $W_i(v)$, it follows that they also cover U:, but then the objects $(W_i(v), (t_i)_{|W_i(v)})$ are a J_P-covering of (U, s) satisfying the requirement above.

To prove that the induced geometric morphism C_{f_P} is an equivalence, we will exploit [6, Proposition 7.18], that states that if f_P is a morphism and comorphism of sites $f_P : (\int P, J_P) \to (\mathcal{O}(E_P), J^{can}_{\mathcal{O}(E_P)})$ then it induces an equivalence of toposes $\mathbf{Sh}(E_P) \simeq \mathbf{Sh}(\int P, J_P)$ if and only if it is J_P-full and $J^{can}_{\mathcal{O}(E_P)}$-dense. Both notions are defined in [6, Definition 3.2]. The J_P-fullness is trivial, while $J^{can}_{\mathcal{O}(E_P)}$-density reduces to the $\dot{s}(U)$ constituting a basis for E_P.

Finally, the canonical geometric morphism $\mathbf{Sh}(E_P) \to \mathbf{Sh}(X)$ is induced by the morphism of sites $\pi_P^{-1} : (\mathcal{O}(X), J^{can}_{\mathcal{O}(X)}) \to (\mathcal{O}(E_P), J^{can}_{\mathcal{O}(E_P)})$, but since $\pi_P : E_P \to X$ is a local homeomorphism it is open (see Proposition 2.4.4 of [3]), and therefore π_P^{-1} admits a left adjoint $\pi_P(-)$, which by [6, Proposition 3.14] is a comorphism of sites such that $\mathbf{Sh}(\pi_P^{-1}) \cong C_{\pi_P(-)}$.

If we consider an arrow of presheaves $h : P \to Q$ then Λ_h is a local homeomorphism, since π_Q and $\pi_P = \pi_Q \circ \Lambda_h$ are, and hence it is also open. Once again, this implies that the canonical geometric morphism $\mathbf{Sh}(\Lambda_h^{-1})$: $\mathbf{Sh}(E_P) \to \mathbf{Sh}(E_Q)$ is also induced by the comorphism of sites $\Lambda_h(-)$: $\mathcal{O}(E_P) \to \mathcal{O}(E_Q)$. Since the diagrams of comorphisms on the left commute, they induce the commutative diagrams of geometric morphisms on the right, and so we conclude the naturality in P of the equivalence obtained:

$$\begin{array}{ccc} \int P & \xrightarrow{f_P} & \mathcal{O}(E_P) \\ & \searrow & \downarrow{\scriptstyle \pi_P(-)} \\ & & \mathcal{O}(X) \end{array} \qquad \begin{array}{ccc} \mathbf{Sh}(\int P, J_P) & \xrightarrow{\simeq} & \mathbf{Sh}(E_P) \\ & \searrow & \downarrow{\scriptstyle Sh(\pi_P)} \\ & & \mathbf{Sh}(X) \end{array}$$

$$\begin{array}{ccc} \int P & \xrightarrow{f_P} & E_P \\ {\scriptstyle \int h}\downarrow & & \downarrow{\scriptstyle \Lambda_h} \\ \int Q & \xrightarrow{f_Q} & E_Q \end{array} \qquad \begin{array}{ccc} \mathbf{Sh}(\int P, J_P) & \xrightarrow{\simeq} & \mathbf{Sh}(E_P) \\ {\scriptstyle \int h}\downarrow & & \downarrow{\scriptstyle C_{\Lambda_h}} \\ \mathbf{Sh}(\int Q, J_Q) & \xrightarrow{\simeq} & \mathbf{Sh}(E_Q) \end{array}$$

□

Corollary 6.1.6. *The two functors*

$$[\mathcal{O}(X)^{op}, \mathbf{Set}] \xrightarrow{\Lambda} \mathbf{Top}/X \xrightarrow{\mathbf{Sh}(-)} \mathbf{Topos}/\mathbf{Sh}(X)$$

and

$$[\mathcal{O}(X)^{op}, \mathbf{Set}] \xrightarrow{\int(-)} \mathbf{Fib}_{\mathcal{O}(X)} \xrightarrow{\mathfrak{G}} \mathbf{Com}/(\mathcal{O}(X), J^{can}_{\mathcal{O}(X)}) \xrightarrow{C_{(-)}} \mathbf{Topos}/\mathbf{Sh}(X)$$

are naturally isomorphic.

We now focus on the sheafification functor $\mathbf{Psh}(X) \to \mathbf{Sh}(X)$, which, as mentioned in the beginning of the section, is equivalent to the composite $\Gamma \circ \Lambda$. The next result shows that it is essentially localic in spirit.

Proposition 6.1.7. *Let X be a topological space, U an open set of X, P a presheaf on X. Then the continuous sections on U for the bundle $\Lambda(P)$ are in a natural bijective correspondence with the homomorphisms of frames $f : \mathcal{O}(E_P) \to \mathcal{O}(U)$ such that $f \circ \pi_P^{-1} = i_U^{-1} = (-) \cap U$. In other words, there is a natural isomorphism of sheaves*

$$\mathbf{Top}/_1 X(-, E_P) \simeq \mathbf{Locale}/_1 \mathcal{O}(X)(\mathcal{O}(-), \mathcal{O}(E_P)) : \mathcal{O}(X)^{op} \to \mathbf{Set},$$

Moreover, the natural isomorphism is also natural on P.

Proof. Given a continuous map $s : U \to E_P$ such that $\pi_P \circ s = i_U$, the frame homomorphism $s^{-1} : \mathcal{O}(E_P) \to \mathcal{O}(U)$ satisfies $s^{-1} \circ \pi^{-1} = i_U^{-1} = (-) \cap U$. Conversely, we can show that any frame homomorphism $f : \mathcal{O}(E_P) \to \mathcal{O}(U)$ satisfying $f \circ \pi^{-1} = (-) \cap U$ is actually of the form s^{-1} for a unique continuous section s. First, we notice that the value of a continuous section s at a point x is uniquely determined by the basic open sets $\dot{t}(V)$ which contain $s(x)$, since if $s(x) \in \dot{t}(V)$ then $s(x) = t_x$. In other words, the value of s at x is uniquely determined by the basic open sets $\dot{t}(V)$ such that $x \in s^{-1}(\dot{t}(V))$. Replacing s^{-1} by f we get the recipe for defining the continuous section s_f associated with a frame homomorphism f as above, for we set $s_f(x)$ equal to t_x for any $t \in P(V)$ such that $x \in f(\dot{t}(V))$. Let us show that this definition is well-posed. Firstly, given $x \in U$, the collection of pairs (V, t) such that $x \in f(\dot{t}(V))$ is non-empty since, as $\mathcal{O}(E_P)$ is covered by the subsets of the form $\dot{t}(V)$ and f is a frame homomorphism to $\mathcal{O}(U)$, the open U is covered by the subsets of the form $f(\dot{t}(V))$. Secondly, for any such (V, t) it holds that $f(\dot{t}(V)) \subseteq f(\pi_P^{-1}(V)) = V \cap U$, thus $f(\dot{t}(V)) \subseteq V$ and t_x is well-defined. Lastly, we have to check that for any (V, t) and (V', t') such that $x \in f(\dot{t}(V)) \cap f(\dot{t'}(V'))$ it holds that $t_x = t'_x$. We have that

$$\dot{t}(V) \cap \dot{t'}(V') = \bigcup_{\substack{W \subseteq V \cap V' \\ r = t|_W = t'|_W}} \dot{r}(W),$$

and since f is a frame homomorphism it follows that

$$f(\dot{t}(V)) \cap f(\dot{t'}(V')) = \bigcup_{\substack{W \subseteq V \cap V' \\ r = t|_W = t'|_W}} f(\dot{r}(W)).$$

Therefore, there are an open set $W \subseteq V \cap V'$ and some $r \in P(W)$ such that $x \in f(\dot{r}(W)) \subseteq W$ and $r = t|_W = t'|_W$; thence $t_x = r_x = t'_x$ as required. The continuity of the map s_f follows from Proposition 6.1.1 since, by definition, it is locally represented by section of P, while its uniqueness follows from the above remark about its values being determined by the open sets containing them. Finally, it is immediate to check that for $V \subseteq U$ and any $h : P \to Q$ the squares

$$\begin{array}{ccc} \Gamma(\Lambda_P)(U) & \xrightarrow{\sim} & \mathbf{Locale}/_1\mathcal{O}(X)(\mathcal{O}(U), \mathcal{O}(E_P)) \\ \downarrow & & \downarrow{\scriptstyle -\circ(-\cap V)} \\ \Gamma(\Lambda_P)(V) & \xrightarrow{\sim} & \mathbf{Locale}/_1\mathcal{O}(X)(\mathcal{O}(V), \mathcal{O}(E_P)), \end{array}$$

$$\begin{array}{ccc} \Gamma\Lambda_P & \xrightarrow{\sim} & \mathbf{Locale}/_1\mathcal{O}(X)(\mathcal{O}(-), \mathcal{O}(E_P)) \\ {\scriptstyle \Gamma\Lambda_h}\downarrow & & \downarrow{\scriptstyle -\circ\Lambda_h^{-1}} \\ \Gamma\Lambda_Q & \xrightarrow{\sim} & \mathbf{Locale}/_1\mathcal{O}(X)(\mathcal{O}(-), \mathcal{O}(E_Q)), \end{array}$$

are commutative. The first expresses the naturality of the isomorphism built above, and the second the naturality in P. □

Corollary 6.1.8. *The two functors*

$$\mathbf{Psh}(X) \xrightarrow{\Lambda} \mathbf{Top}/X \xrightarrow{\Gamma} \mathbf{Psh}(X)$$

and

$$\mathbf{Psh}(X) \xrightarrow{\Lambda} \mathbf{Top}/X \xrightarrow{\mathcal{O}(-)} \mathbf{Locale}/_1\mathcal{O}(X) \xrightarrow{[f] \mapsto \mathbf{Locale}/_1\mathcal{O}(X)(\mathcal{O}(-),[f])} \mathbf{Psh}(X)$$

are naturally isomorphic.

Remarks 6.1.2. (i) Concretely, the condition $f \circ \pi^{-1} = (-) \cap U$, which is the dual of $\pi_P \circ s = i_U$, means that for any open set Z of X,

$$\bigvee_{\substack{V \subseteq Z \\ t \in P(V)}} f(\dot{t}(V)) = Z \cap U.$$

(ii) It is true more generally that, for any topological space X, the open sets functor $\mathcal{O} : \mathbf{Top} \to \mathbf{Locale}$ induces an equivalence of categories between $\mathbf{Etale}/X$ and the category $\mathbf{LH}/X$, where $\mathbf{LH}$ is the category of local homeomorphisms of locales to X and local homeomorphisms between them, which is a full subcategory of $\mathbf{Loc}/X$ (cf. [21, Lemma C1.3.2(iii)-(v)]).

(iii) Proposition 6.1.7 hints to the independence of the sheafification of a presheaf P in $\mathbf{Psh}(X)$ from the points of X, since it describes sheafification as a hom functor of locales. The same localic nature of the sheafification process is shown by Proposition 6.1.5; in particular, the functor $f_P : \int P \to E_P$ extends to an isomorphism of locales $\mathrm{Id}_{J_P}(\int P) \xrightarrow{\sim} \mathcal{O}(E_P)$ (see Section 7.2 and in particular Remark 7.2.3), which is exactly the isomorphism induced by the equivalence of localic toposes $\mathbf{Sh}(E_P) \simeq \mathbf{Sh}(\int P, J_P)$.

(iv) Consider P in $\mathbf{Psh}(X)$ and the diagram

$$D : \textstyle\int P \to \mathbf{Com}/(\mathcal{O}(X), J^{can}_{\mathcal{O}(X)})$$

which maps every object (U, s) in $\int P$ to the comorphism of sites $i_U(-) : (\mathcal{O}(U), J^{can}_{\mathcal{O}(U)}) = (\mathcal{O}(X)/U, (J^{can}_{\mathcal{O}(X)})_U) \to (\mathcal{O}(X), J^{can}_{\mathcal{O}(X)})$. The arrows $\dot{s}(-) : (\mathcal{O}(U), J^{can}_{\mathcal{O}(U)}) \to (\mathcal{O}(E_P), J^{can}_{\mathcal{O}(E_P)})$ provide a cocone under D. We will show in the next section that $(\int P, J_P)$ is the colimit of D: therefore we have a unique comorphism of sites $(\int P, J_P) \to \mathcal{O}(E_P)$ induced by the universal property of colimits, which is exactly the functor f_P of Proposition 6.1.5. Finally, we will see in Section 6.3 that when moving to toposes of sheaves over $\mathbf{Sh}(X)$, the colimit cocone for $(\int P, J_P)$ is preserved: but since $\mathbf{Sh}(\int P, J_P) \simeq \mathbf{Sh}(E_P)$, we can conclude that $\mathbf{Sh}(E_P)$ is the colimit in the category of toposes over $\mathbf{Sh}(X)$ for the diagram $\int P \to \mathbf{Topos}/\mathbf{Sh}(X)$, $(U, s) \mapsto [\mathbf{Sh}(i_U) : \mathbf{Sh}(U) \to \mathbf{Sh}(X)]$.

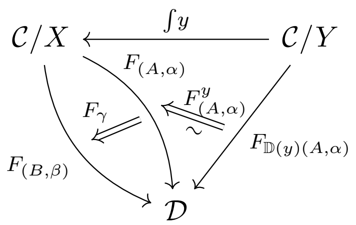
6.2 The adjoints to the Grothendieck construction

We proved in Chapter 2 that the 2-functor $\mathcal{G} : \mathbf{Ind}_{\mathcal{C}} \to \mathbf{cFib}_{\mathcal{C}}$ is an equivalence of 2-categories: we dedicate this section to showing that by embedding the codomain $\mathbf{cFib}_{\mathcal{C}}$ into $\mathbf{CAT}/\mathcal{C}$ the equivalence extends to an adjunction, and that similar adjunctions arise when considering the inclusion of fibrations into comorphisms of sites using Giraud topologies. In particular, since the passage $\mathbb{D} \mapsto \mathcal{G}(\mathbb{D})$ is in fact a colimit, this will imply that the Giraud site associated to a $\mathcal{C}$-indexed category is a colimit in the category of sites and (continuous) comorphisms.

From now on we shall consider the diagram $\mathcal{C}/- : \mathcal{C} \to \mathbf{cFib}_{\mathcal{C}}$ mapping X to $p_X : \mathcal{C}/X \to \mathcal{C}$ and $y : Y \to X$ to $\int y : \mathcal{C}/Y \to \mathcal{C}/X$; the corresponding diagram $\mathcal{C} \to \mathbf{Ind}_{\mathcal{C}}$ operates by mapping X to the presheaf よ(X), and y to the arrow of presheaves よ(y). The first thing to remark is that the fibred Yoneda lemma (Proposition 2.1.10) presents a cloven fibration $p : \mathcal{D} \to \mathcal{C}$ as a $\mathbb{D}$-weighted colimit in $\mathbf{cFib}_{\mathcal{C}}$ of the diagram $\mathcal{C}/- : \mathcal{C} \to \mathbf{cFib}_{\mathcal{C}}$:

$$\mathbf{cFib}_{\mathcal{C}}(\mathcal{D}, \mathcal{X}) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}, \mathbb{X}) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}, \mathbf{cFib}_{\mathcal{C}}(\mathcal{C}/-, \mathcal{X}))$$

considering the image of the identity of $\mathcal{D}$ via this equivalence, we obtain the following colimit cocone:



We recall that the morphism of fibrations $(F_{(A,\alpha)}, \varphi_{(A,\alpha)}) \in \mathbf{cFib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D})$ is defined as follows: $F_{(A,\alpha)}[y] := \mathrm{dom}(\widehat{\alpha y}_A)$, for $z : [yz] \to [y]$ we set $F_{(A,\alpha)}(z) := \lambda_{\alpha y,z,A}$, and finally $\varphi_{(A,\alpha)}([y]) := \theta^{-1}_{\alpha y,A}$. To define $F_\gamma : F_{(A,\alpha)} \Rightarrow F_{(B,\beta)}$, notice that the image of $\gamma\widehat{\alpha y}_A$ via p factors through the image of $\widehat{\beta y}_B$: thus there is a unique $F_\gamma([y]) : \mathrm{dom}(\widehat{\alpha y}_A) \to \mathrm{dom}(\widehat{\beta y}_B)$ induced by the property of cartesian arrows. Finally, to define the components of $F^y_{(A,\alpha)}$, notice that for $[z]$ in $\mathcal{C}/Y$ it holds that $F_{(A,\alpha)}\textstyle\int y([z]) = \mathrm{dom}(\widehat{\alpha yz}_A)$ and $F_{\mathbb{D}(y)(A,\alpha)}([z]) = \mathrm{dom}(\widehat{\theta_{\alpha y,A} z}_{\mathrm{dom}(\widehat{\alpha y})_A})$: thus we define $F^y_{(A,\alpha)}([z]) := \chi^{-1}_{\alpha y,z,A}$.

Remark 6.2.1. The fact that any cloven fibration is a colimit of the fibrations $\mathcal{C}/X$ is an evident generalization of the fundamental result that any presheaf is a colimit of representable presheaves (since $\int よ(X) \simeq \mathcal{C}/X$).

We now introduce the adjoints to $\mathcal{G}$. The following result introduces the Grothendieck construction as part of an adjoint triple:

Proposition 6.2.1. *Denote by* $\Lambda_{\mathbf{CAT}/\mathcal{C}}$ *the 2-functor*

$$\mathbf{Ind}_{\mathcal{C}} \xrightarrow{\mathcal{G}} \mathbf{cFib}_{\mathcal{C}} \xrightarrow{\mathrm{For}} \mathbf{CAT}/\mathcal{C}.$$

It admits a left adjoint $\mathfrak{L}$*, defined on objects as*

$$\mathfrak{L} : [F : \mathcal{D} \to \mathcal{C}] \mapsto [(-\downarrow F) : \mathcal{C}^{op} \to \mathbf{CAT}].$$

Moreover, if $\mathcal{C}$ *is small, then the 2-functor*

$$\Lambda_{\mathbf{CAT}/\mathcal{C}} : \mathbf{Ind}_{\mathcal{C}} \xrightarrow{\mathcal{G}} \mathbf{CAT}/\mathcal{C}$$

also admits a right adjoint $\Gamma_{\mathbf{CAT}/\mathcal{C}}$*, defined on objects as*

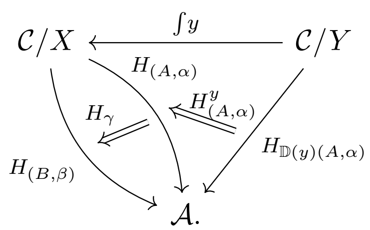
$$\Gamma_{\mathbf{CAT}/\mathcal{C}} : [F : \mathcal{D} \to \mathcal{C}] \mapsto [\mathbf{CAT}/\mathcal{C}(\mathcal{C}/-, [F]) : \mathcal{C}^{op} \to \mathbf{CAT}].$$

Proof. These results are mentioned in [31]. One reference where both are proved in a wider context using coend calculus is [18, Proposition 2.1, Definition 2.5 and Proposition 3.2]. The restriction on the size of $\mathcal{C}$ is needed so that $\Gamma_{\mathbf{CAT}/\mathcal{C}}([F])$ is a locally small $\mathcal{C}$-indexed category. □

Since the right adjoint to $\mathcal{G}$ is a contravariant hom-functor, we obtain in particular the following result:

Corollary 6.2.2. *Consider a small category $\mathcal{C}$ and a cloven fibration $\mathcal{D} \to \mathcal{C}$ with corresponding pseudofunctor $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$: then $\mathcal{D}$ is the $\mathbb{D}$-weighted colimit of the diagram $\mathcal{C}/- : \mathcal{C} \to \mathbf{CAT}/\mathcal{C}$, with colimit cocone given by the functors $F_{(A,\alpha)}$.*

Remark 6.2.2. Consider a cocone under the diagram $\mathcal{C}/-$ with vertex $\mathcal{A}$:



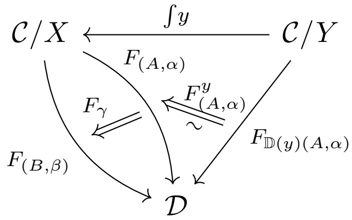
Let us write explicitly the (essentially unique) functor $h : \mathcal{D} \to \mathcal{A}$ induced by the universal property of colimits. Knowing that the colimit cocone is given by the functors $F_{(A,\alpha)}$, and that for every D in $\mathcal{D}$ there is a canonical isomorphism $D \simeq F_{(D,1_{p(D)})}([1_{p(D)}])$, the request that $H_{(A,\alpha)} \simeq hF_{(A,\alpha)}$ forces the definition of h. Indeed, $h(D) \simeq hF_{(D,1_{p(D)})}([1_{p(D)}]) \simeq H_{(D,1_{p(D)})}([1_{p(D)}])$, and hence we can set

$$h(D) := H_{(D,1_{p(D)})}([1_{p(D)}]).$$

The definition of the action of h on arrows is trickier, but in fact there is only one possible canonical way of defining, given $g : D \to E$ in $\mathcal{D}$, an arrow $h(g) : h(D) \to h(E)$. If we denote by $v_g : D \to \mathrm{dom}(\widehat{p(g)}_E)$ the unique arrow such that $\widehat{p(g)}_E v_g = g$ and $p(v_g) = \theta_{p(g),E}$, we have an arrow $v_g : (D, 1_{p(D)}) \to (\mathrm{dom}(\widehat{p(g)}_E), \theta_{p(g),E}) = \mathbb{D}(p(g))(E, 1_{p(E)})$ in $\mathbb{D}(p(D))$; then $h(g)$ is the composite arrow

$$H_{(D,1_{p(D)})}([1_{p(D)}]) \xrightarrow{H_{v_g}([1_{p(D)}])} H_{\mathbb{D}(p(g))(E,1_{p(E)})} \xrightarrow{H^{p(g)}_{(E,1_{p(E)})}([1_{p(D)}])} H_{(E,1_{p(E)})}([p(g)]) \xrightarrow{H_{(E,1_{p(E)})}(p(g))} H_{(E,1_{p(E)})}([1_{p(E)}]).$$

So far we have only dealt with raw categories: let us now study the interaction of the adjoints to $\mathcal{G}$ with Giraud topologies by endowing $\mathcal{C}$ with a topology J. For any cloven fibration $p : \mathcal{D} \to \mathcal{C}$, every leg $F_{(A,\alpha)}$ of the colimit cocone



is a $(J_X, J_{\mathcal{D}})$-continuous comorphism by Proposition 5.1.3, therefore the cocone lives in the slice category $\mathbf{Com}_{cont}/(\mathcal{C}, J)$. What is relevant is that it is still a colimit cocone, as an immediate consequence of the following lemma:

Lemma 6.2.3. *With the notations of Remark 6.2.2, and considering topologies J on $\mathcal{C}$ and T on $\mathcal{A}$:*

(i) Suppose that the legs $H_{(A,\alpha)}$ and $q : \mathcal{A} \to \mathcal{C}$ are comorphisms of sites: then h is a comorphism of sites $(\mathcal{D}, J_{\mathcal{D}}) \to (\mathcal{A}, T)$;

(ii) Suppose that all the legs $H_{(A,\alpha)}$ are $(J_{p(D)}, T)$-continuous functors: then h is a $(J_{\mathcal{D}}, T)$-continuous functor.

Proof. (i) Take D in $\mathcal{D}$ and $R \in T(h(D))$: since $D = F_{(D,1_{p(D)})}([1_{p(D)}])$ and $hF_{(D,1_{p(D)})} \simeq H_{(D,1_{p(D)})}$, which is a comorphism of sites, there is a sieve $S \in J_{p(D)}([1_{p(D)}])$ such that $H_{(D,1_{p(D)})}(S) \subseteq R$. But now we recall that a continuous functor between sites is always cover-preserving [6, Proposition 4.13]: in particular $F_{(D,1_{p(D)})} : (\mathcal{C}/p(D), J_{p(D)}) \to (\mathcal{D}, J_{\mathcal{D}})$ is so, and thus the sieve $S' = \langle F_{(D,1_{p(D)})}(S)\rangle$ is $J_{\mathcal{D}}$-covering. But then $h(S') \subseteq R$, up to isomorphism, and hence h is a comorphism of sites.

(ii) Consider a T-sheaf $W : \mathcal{A}^{op} \to \mathbf{Set}$: we wish to show that $W \circ h^{op} : \mathcal{D}^{op} \to \mathbf{Set}$ is a $J_{\mathcal{D}}$-sheaf. To do so we exploit Lemma 6.3.2 stating that $W \circ h^{op}$ is a $J_{\mathcal{D}}$-sheaf if and only if every composite $W \circ h^{op} \circ F^{op}_{(A,\alpha)}$ is a J_X-sheaf: but this is true because $hF_{(A,\alpha)} \simeq H_{(A,\alpha)}$, which is a continuous.

□

This entails immediately the following results:

Corollary 6.2.4. *Consider a small category $\mathcal{C}$ and the four 2-functors*

$$\Lambda_{\mathbf{Com}/(\mathcal{C},J)} : \mathbf{Ind}_{\mathcal{C}} \xrightarrow{\mathcal{G}} \mathbf{cFib}_{\mathcal{C}} \xrightarrow{\mathfrak{G}} \mathbf{Com}/(\mathcal{C}, J),$$

$$\Lambda_{\mathbf{Com}_{cont}/(\mathcal{C},J)} : \mathbf{Ind}_{\mathcal{C}} \xrightarrow{\mathcal{G}} \mathbf{cFib}_{\mathcal{C}} \xrightarrow{\mathfrak{G}} \mathbf{Com}_{cont}/(\mathcal{C}, J),$$

$$\Gamma_{\mathbf{Com}/(\mathcal{C},J)} : \mathbf{Com}/(\mathcal{C}, J) \xrightarrow{[p]\mapsto \mathbf{Com}/(\mathcal{C},J)(\mathcal{C}/-,[p])} \mathbf{Ind}_{\mathcal{C}},$$

$$\Gamma_{\mathbf{Com}_{cont}/(\mathcal{C},J)} : \mathbf{Com}_{cont}/(\mathcal{C}, J) \xrightarrow{[p]\mapsto \mathbf{Com}_{cont}/(\mathcal{C},J)(\mathcal{C}/-,[p])} \mathbf{Ind}_{\mathcal{C}} :$$

then there are 2-adjunctions

$$\mathbf{Ind}_{\mathcal{C}} \underset{\Gamma_{\mathbf{Com}/(\mathcal{C},J)}}{\overset{\Lambda_{\mathbf{Com}/(\mathcal{C},J)}}{\perp}} \mathbf{Com}/(\mathcal{C},J)\ , \qquad \mathbf{Ind}_{\mathcal{C}} \underset{\Gamma_{\mathbf{Com}_{cont}/(\mathcal{C},J)}}{\overset{\Lambda_{\mathbf{Com}_{cont}/(\mathcal{C},J)}}{\perp}} \mathbf{Com}_{cont}/(\mathcal{C},J)\ .$$

In particular, given a cloven fibration $p : \mathcal{D} \to \mathcal{C}$, the site $(\mathcal{D}, J_{\mathcal{D}})$ is the $\mathbb{D}$-weighted diagram of the functor $\mathfrak{G} \circ (\mathcal{C}/-) : \mathcal{C} \to \mathbf{Fib}_{\mathcal{C}} \to \mathbf{Com}_{cont}/(\mathcal{C}, J)$. Moreover, the forgetful functors

$$\mathbf{Com}_{cont}/(\mathcal{C}, J) \to \mathbf{Com}/(\mathcal{C}, J) \to \mathbf{CAT}/\mathcal{C}.$$

reflect and preserve the $\mathbb{D}$-weighted colimit of $\mathfrak{G} \circ (\mathcal{C}/-)$.

Proof. Consider a comorphism of sites $q : (\mathcal{E}, T) \to (\mathcal{C}, J)$. By the previous lemma, the functor

$$\mathbf{Ind}_{\mathcal{C}}(\mathbb{D}, \mathbf{CAT}/\mathcal{C}(\mathcal{C}/-, [q])) \to \mathbf{CAT}/\mathcal{C}(\mathcal{G}(\mathbb{D}), [q])$$

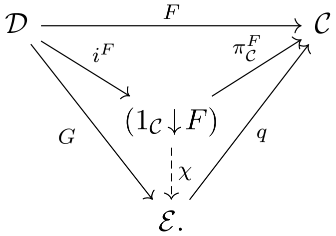
restricts to a functor

$$\mathbf{Ind}_{\mathcal{C}}(\mathbb{D}, \mathbf{Com}/(\mathcal{C}, J)(\mathcal{C}/-, [q])) \to \mathbf{Com}/(\mathcal{C}, J)(\mathcal{G}(\mathbb{D}), [q]),$$

and its quasi-inverse (precomposition with the colimit cocone) obviously restricts to **Com** too. The same considerations hold for $\mathbf{Com}_{cont}$. □

The adjunction $\mathfrak{L} \dashv \Lambda_{\mathbf{CAT}/\mathcal{C}}$, on the other hand, offers a point of view on fibrations of generalized elements. Indeed, the composite 2-functor $\mathcal{G} \circ \mathfrak{L}$ maps any functor $F : \mathcal{D} \to \mathcal{C}$ to its fibration of generalized elements $(1_{\mathcal{C}} {\downarrow} F) \to \mathcal{C}$ (see [6, Section 3.4.4]), which by the adjunction satisfies the following universal property:

Corollary 6.2.5. *Let $F : \mathcal{D} \to \mathcal{C}$ be a functor. Then the fibration $\pi^F_{\mathcal{C}}$: $(1_{\mathcal{C}} {\downarrow} F) \to \mathcal{C}$ satisfies the following universal property: given a factorization of F through a fibration q, i.e. a functor $G : \mathcal{D} \to \mathcal{E}$ such that $q \circ G \cong F$, there is a unique morphism of fibrations $\chi : \pi^F_{\mathcal{C}} \to q$ such that $G = \chi \circ i^F$: as in the diagram*



6.3 The fundamental adjunction

We are now ready to formulate our *fundamental adjunction*, which extends the adjunctions of the previous sections by showing that for a small-generated site $(\mathcal{C}, J)$ and a cloven Street fibration $p : \mathcal{D} \to \mathcal{C}$ the Giraud topos $\mathrm{Gir}_J(p)$ is a $\mathbb{D}$-weighted colimit in the category of Grothendieck toposes over $\mathbf{Sh}(\mathcal{C}, J)$.

In the previous section we have considered, for each object (A, α) in the fibre $\mathbb{D}(X)$, the canonical functor

$$F_{(A,\alpha)} : \mathcal{C}/X \to \mathcal{D},$$

and we have shown that the functors of this form are the legs of a colimit cocone of categories. Since each functor $F_{(A,\alpha)}$ is a morphism of fibrations, it is also a continuous comorphisms of sites with respect to Giarud's topologies (cf. Proposition 5.1.3) and thus it induces a geometric morphism $C_{F_{(A,\alpha)}}$: $\mathbf{Sh}(\mathcal{C}/X, J_X) \to \mathrm{Gir}_J(\mathcal{D})$. To ease the notation, we will denote each $C_{F_{(A,\alpha)}}$ with $C_{(A,\alpha)}$, and for every arrow $\gamma : (A, \alpha) \to (B, \beta)$ of $\mathbb{D}(X)$ we will denote by $C_\gamma : C_{(B,\beta)} \Rightarrow C_{(A,\alpha)}$ the induced natural transformation. Our goal is to show that the collection of the geometric morphisms $C_{(A,\alpha)}$ provides a colimit cocone of toposes.

Let us now provide two technical results that will come in handy later. The first result is about the family of functors $C^*_{(A,\alpha)}$:

Proposition 6.3.1. *The functors*

$$- \circ (F_{(A,\alpha)})^{op} : [\mathcal{D}^{op}, \mathbf{Set}] \to [(\mathcal{C}/X)^{op}, \mathbf{Set}]$$

are jointly conservative, and the same holds for the functors $C^*_{(A,\alpha)}$.

Proof. Since the functors $C^*_{(A,\alpha)}$ act by restricting the functors $-\circ(F_{(A,\alpha)})^{op}$, the second claim follows from the first one. So consider $r : H \Rightarrow K$ in $[\mathcal{D}^{op}, \mathbf{Set}]$ and suppose that for every X in $\mathcal{C}$ and every (A, α) in $\mathcal{D}(X)$ the arrow $r \circ (F_{(A,\alpha)})^{op}$ is invertible. In particular, notice that

$$(r \circ F_{(D,1_{p(D)})})([1_{p(D)}]) := r(\mathrm{dom}(\widehat{1_{p(D)}}_D)) :$$

but $\mathrm{dom}(\widehat{1_{p(D)}}_D)$ is canonically isomorphic to D, since both $\widehat{1_{p(D)}}_D$ and 1_D are cartesian lifts for $1_{p(D)}$, and hence in particular the arrows $(r \circ F_{(D,1_{p(D)})})([1_{p(D)}])$ and $r(D)$ are canonically isomorphic. This implies that all components of the natural transformation r are bijective, and hence r is invertible. □

The second result shows that the property of being a $J_{\mathcal{D}}$-sheaf over $\mathcal{D}$ can be checked 'locally', by moving to the slice categories $\mathcal{C}/X$:

Lemma 6.3.2. *A presheaf $W : \mathcal{D}^{op} \to$ **Set** is a $J_{\mathcal{D}}$-sheaf if and only if for every X in $\mathcal{C}$ and every (A, α) in $\mathbb{D}(X)$ the presheaf $W \circ (F_{(A,\alpha)})^{op}$ is a J_X-sheaf.*

Proof. Of course if W is a $J_{\mathcal{D}}$-sheaf then $W \circ (F_{(A,\alpha)})^{op} =: C^*_{(A,\alpha)}(W)$ is a J_X-sheaf. Conversely, suppose that all composites as above are J_X-sheaves: we will build, from a matching family for W and a $J_{\mathcal{D}}$-covering family R over D, a matching family for $W \circ (F_{(D,1_{p(D)})})^{op}$, and then show that if it admits an amalgamation so does the first one. Consider a $J_{\mathcal{D}}$-covering family $R = \{f_i : \mathrm{dom}(f_i) \to D \mid i \in I\}$, i.e. the datum of $\alpha_i \in W(\mathrm{dom}(f_i))$ satisfying the usual compatibility condition. By definition of $J_{\mathcal{D}}$-covering family, all the arrows f_i are cartesian and the projection $\{p(f_i)\}$ is a J-covering family for $p(D)$: we can then lift it to a $J_{p(D)}$-covering family $S = \{p(f_i) : [p(f_i)] \to [1_{p(D)}] \mid i \in I\}$ in $\mathcal{C}/p(D)$. Now, notice that f_i and $\widehat{p(f_i)}_D$ are both cartesian lifts of $p(f_i)$, and therefore there are canonical isomorphisms $\gamma_i : \mathrm{dom}(\widehat{p(f_i)}_D) \to \mathrm{dom}(f_i)$ between their domains. We can define $\beta_i \in W(\mathrm{dom}(\widehat{p(f_i)}_D)) = (W \circ F_{(D,1_{p(D)})})^{op}([p(f_i)])$ as $\beta_i = W(\gamma_i)(\alpha_i)$: it is now immediate to check that it is a matching family for S and the composite $W \circ (F_{(D,1_{p(D)})})^{op}$. Since $W \circ (F_{(D,1_{p(D)})})^{op}$ is a $J_{p(D)}$-sheaf, that matching family admits a unique amalgamation $\beta \in W(\mathrm{dom}(\widehat{1_{p(D)}}_D))$. Finally, since $\widehat{1_{p(D)}}_D : \mathrm{dom}(\widehat{1_{p(D)}}_D) \to D$ is an isomorphism, we can consider the element $\alpha := W(\widehat{1_{p(D)}}_D^{-1})(\beta) \in W(D)$, which provides an amalgamation for the matching family $\{\alpha_i\}$. ☐

Remark 6.3.1. Notice that in both results we could restrict to those functors of the kind $F_{(D,1_{p(D)})}$. This is an indication that these propositions, duly reformulated, would hold also in the setting of Grothendieck fibrations.

What we will show in a moment is that the colimit $(\mathcal{D}, J_{\mathcal{D}}) \simeq \mathrm{colim}^{\mathbb{D}}_{ps}\, \mathcal{C}/-$ in $\mathbf{Com}/(\mathcal{C}, J)$, which we have introduced in the previous section, is preserved by the pseudofunctor $\mathbf{Com}/(\mathcal{C}, J) \xrightarrow{C_{(-)}} \mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J)^{co}$. To prove the universal property of colimits we will resort to the adjoint functor theorem for toposes:

Theorem 6.3.3. *A functor $F : \mathscr{E} \to \mathscr{F}$ between toposes is a left adjoint if and only if it preserves arbitrary colimits.*

However, we must deal with the size issue already mentioned in Section 5.1: nothing grants that for an arbitrary fibration $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ the category $\mathrm{Gir}_J(\mathbb{D})$ is a Grothendieck topos, even if the base site is small. To circumvent this, we propose the following definition:

Definition 6.3.1. Given a small-generated site $(\mathcal{C}, J)$, a $\mathcal{C}$-indexed category $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ is essentially J-small if the Giraud site $(\mathcal{G}(\mathbb{D}), J_{\mathbb{D}})$ is small-generated. We will denote by $\mathbf{Ind}^J_{\mathcal{C}}$ and $\mathbf{St}^J(\mathcal{C}, J)$ the categories of essentially J-small fibrations and J-stacks.

Remarks 6.3.2. (i) From the point of view of fibrations, we can define a fibration $p : \mathcal{D} \to \mathcal{C}$ to be essentially J-small if and only if the site $(\mathcal{D}, J_{\mathcal{D}})$ is small-generated.

(ii) Lemma 5.1.2 proved that if $(\mathcal{C}, J)$ is small-generated then every discrete fibration $\int P \to \mathcal{C}$ is essentially J-small. We can extend the proof technique to any small fibration over $\mathcal{C}$ as follows. If we denote by $\mathcal{A}$ the small J-dense subcategory of $\mathcal{C}$, every object X in $\mathcal{C}$ admits a J-covering family $y_i : A_i \to X$ whose domains all lie in $\mathcal{A}$. Then consider the full subcategory $\mathcal{B} \hookrightarrow \mathcal{G}(\mathbb{D})$ whose objects are of the form (A, U), where A is an object of $\mathcal{A}$ and $U \in \mathbb{D}(A)$: it is a small category, and every object (X, U) in $\mathcal{G}(\mathbb{D})$ admits a $J_{\mathbb{D}}$-covering family $(A_i, \mathbb{D}(y_i)(U)) \to (X, U)$ of objects in $\mathcal{B}$ (where we used the explicit description of $J_{\mathbb{D}}$ given in Proposition 5.1.1). Therefore, $\mathcal{B}$ is a small $J_{\mathbb{D}}$-dense subcategory of $\mathcal{G}(\mathbb{D})$.

(iii) We shall see later in Corollary 6.5.3 that for every small-generated site $(\mathcal{C}, J)$ the canonical stack $\mathcal{S}_{(\mathcal{C},J)}$ is essentially J-small.

We can now prove that the Giraud topos of an essentially J-small fibration is a weighted colimit of étale toposes:

Theorem 6.3.4. *Given an essentially J-small cloven fibration $p : \mathcal{D} \to \mathcal{C}$ with corresponding $\mathcal{C}$-indexed category $\mathbb{D}$, the topos $\mathrm{Gir}_J(p)$ is the $\mathbb{D}$-weighted colimit of the diagram*

$$L : \mathcal{C} \xrightarrow{\mathcal{C}/-} \mathbf{cFib}_{\mathcal{C}} \xrightarrow{\mathfrak{G}} \mathbf{Com}/(\mathcal{C}, J) \xrightarrow{C_{(-)}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J).$$

More explicitly, for any $\mathbf{Sh}(\mathcal{C}, J)$-topos $\mathscr{E}$ there is an equivalence between

$$\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)\,(\mathrm{Gir}_J(p), \mathscr{E})$$

and

$$\mathbf{Ind}_{\mathcal{C}}\left(\mathbb{D}, \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)\left(\mathbf{Sh}(\mathcal{C}/(-), J_{(-)}), \mathscr{E}\right)\right),$$

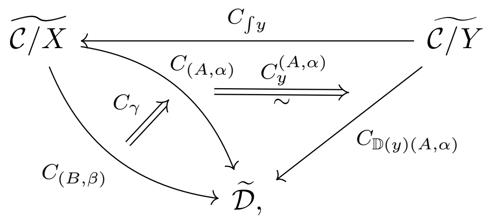
which moreover is pseudonatural in $\mathscr{E}$.

Proof. In this proof we will adopt the shorthand $\widetilde{\mathcal{A}} := \mathbf{Sh}(\mathcal{A}, K)$. There is no risk of confusion, since we will not consider different topologies over a same category.

Let us start by considering the pseudonatural equivalence in Yoneda lemma and compose it with $C_{(-)} \circ \mathfrak{G}$:

$$\mathbb{D}(X) \xrightarrow{\sim} \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D}) \to \mathbf{Topos}^{co}/\widetilde{\mathcal{C}}(\widetilde{\mathcal{C}/X}, \widetilde{\mathcal{D}})$$

We obtain a transformation, which is pseudonatural in X, acting by mapping (A, α) in $\mathbb{D}(X)$ to the geometric morphism $C_{(A,\alpha)}$, and $\gamma : (A, \alpha) \to (B, \beta)$ to the natural transformation $C_\gamma : C_{(B,\beta)} \Rightarrow C_{(A,\alpha)}$. The pseudonaturality condition implies the existence for any $y : Y \to X$ in $\mathcal{C}$ of natural isomorphisms $C_y^{(A,\alpha)} : C_{(A,\alpha)} C_{\int y} \cong C_{\mathbb{D}(y)(A,\alpha)}$ satisfying the usual compatibility conditions. This provides us with a $\mathbb{D}$-weighted cocone under L with vertex $\widetilde{\mathcal{D}}$, which we call $\underline{C}$:



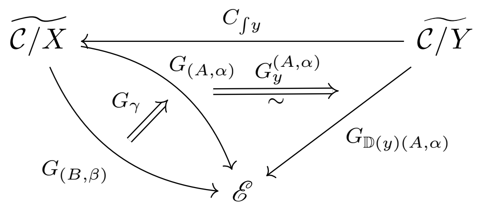
where we are omitting all structural geometric morphisms to $\widetilde{\mathcal{C}}$.

We want to show that $\underline{C}$ is indeed the colimit cocone. More explicitly, we consider the functor

$$(- \circ \underline{C}) : \mathbf{Topos}^{co}/\widetilde{\mathcal{C}}\left(\widetilde{\mathcal{D}}, \mathscr{E}\right) \to \mathbf{Ind}_{\mathcal{C}}\left(\mathbb{D}, \mathbf{Topos}^{co}/\widetilde{\mathcal{C}}\left(\widetilde{\mathcal{C}/-}, \mathscr{E}\right)\right)$$

which starting from $(F, \varphi) : \widetilde{\mathcal{D}} \to \mathscr{E}$ composes it with the legs of the cocone $\underline{C}$: then $\underline{C}$ is a colimit cocone if the functor $(- \circ \underline{C})$ is an equivalence (pseudonatural in $\mathscr{E}$).

To build a quasi-inverse for $(- \circ \underline{C})$, we start by considering a $\mathbb{D}$-weighted cone $\underline{G}$ under L with vertex $\mathscr{E}$:



In the following, we will work with inverse images of geometric morphisms: this makes things a bit easier, since the behaviour of $C^*_{(A,\alpha)}$ is simply precomposition with the functor $(F_{(A,\alpha)})^{op}$. Therefore, we want to build from the data of the cocone $\underline{G}$ an inverse image functor $H : \mathscr{E} \to \widetilde{\mathcal{D}}$, so that its composition with the cocone $\underline{C}$ results again in $\underline{G}$.

Consider an object E of $\mathscr{E}$. We recall that for every D in $\mathcal{D}$ it holds that $D \simeq F_{(D,1_{p(D)})}([1_{p(D)}])$, from which we infer that

$$H(E)(D) \simeq H(E)F_{(D,1_{p(D)})}([1_{p(D)}]).$$

Since we want $H(E)F_{(D,1_{p(D)})} = C^*_{(A,\alpha)}(H(E))$ to be isomorphic to $G^*_{(D,1_{p(D)})}(E)$, we can set $H(E)(D) := G^*_{(D,1_{p(D)})}(E)([1_{p(D)}])$. The definition of $H(E)$ is a bit more intricate on arrows. For $r : D' \to D$ in $\mathcal{D}$, if we consider the cartesian lift $\widehat{p(r)}_D$, then by cartesianity r must factor through it with a unique

$v_r : D' \to \mathrm{dom}(\widehat{p(r)}_D)$ such that moreover $p(v_r) = \theta_{p(r),D}$, and this provides an arrow $v_r : (D', 1_{p(D')}) \to (\mathrm{dom}(\widehat{p(r)}_D), \theta_{p(r),D}) = \mathbb{D}(p(r))(D, 1_{p(D)})$. Then we can define $H(E)(r)$ as the following composite arrow:

$$
\begin{array}{ccc}
G^*_{(D,1_{p(D)})}(E)[1_{p(D)}] & \xrightarrow{G^*_{(D,1_{p(D)})}(E)(p(r))} & G^*_{(D,1_{p(D)})}(E)([p(r)]) \\
 & & \| \\
\Big\downarrow{\scriptstyle H(E)(r)} & & C^*_{\int p(r)} G^*_{(D,1_{p(D)})}(E)([1_{p(D')}]) \\
 & & \Big\downarrow{\scriptstyle G^{(D,1_{p(D)})}_{p(r)}([1_{p(D')}])} \\
G^*_{(D',1_{p(D')})}(E)([1_{p(D')}]) & \xleftarrow[G_{v_r}(E)([1_{p(D')}])]{} & G^*_{(\mathrm{dom}(\widehat{p(r)}_D),\theta_{p(r),D})}(E)([1_{p(D')}])
\end{array}
.
$$

A computation shows that $H(E)$ is a functor. It is easier to define H on arrows: given $g : E \to E'$ in $\mathscr{E}$, $H(g) : H(E) \to H(E')$ is defined componentwise, by setting for D in $\mathcal{D}$ the arrow $H(g)(D)$ as

$$G^*_{(D,1_{p(D)})}(g)([1_{p(D)}]) : G^*_{(D,1_{p(D)})}(E)([1_{p(D)}]) \to G_{(D,1_{p(D)})}(E')([1_{p(D)}]).$$

This defines a natural transformation $H(E) \Rightarrow H(E')$, and the association $g \mapsto H(g)$ is functorial.

Now, by the very definition of $H(E)$ it holds that for every X in $\mathcal{C}$ and every (B, β) we have $C^*_{(B,\beta)} H(E) \simeq G_{(B,\beta)}(E)$. All these are sheaves, and by applying Lemma 6.3.2 we have that $H(E)$ is a $J_{\mathcal{D}}$-sheaf.

So far we have built a functor $H : \mathscr{E} \to \widetilde{\mathcal{D}}$ such that the cone $\underline{G}$ is essentially equivalent to the composite of H with the cone $\underline{C}$: we want it to be the inverse image of a geometric morphism. But this follows from Lemma D.3, since all the functors $C_{(A,\alpha)}$ and $C_{(A,\alpha)} \circ H \simeq G_{(A,\alpha)}$ preserve finite limits and arbitrary colimits, and since the $C_{(A,\alpha)}$ are jointly conservative, it follows that H must also preserve finite limits and arbitrary colimits. By the adjoint functor theorem for Grothendieck toposes we conclude that H is the inverse image of a geometric morphism $\widetilde{\mathcal{D}} \to \mathscr{E}$. It is also immediate to check that it is a morphism in the slice $\mathbf{Topos}^{co}/\widetilde{\mathcal{C}}$, thus we have defined the behaviour of the functor

$$\mathbf{Ind}_{\mathcal{C}}(\mathbb{D}, \mathbf{Topos}^{co}/\widetilde{\mathcal{C}}(\widetilde{\mathcal{C}/-}, \mathscr{E})) \to \mathbf{Topos}^{co}/\widetilde{\mathcal{C}}(\widetilde{\mathcal{D}}, \mathscr{E})$$

on objects. To define it on arrows, we start with two cocones $\underline{G}$ and $\underline{G'}$, corresponding to inverse images functors $H : \mathscr{E} \to \widetilde{\mathcal{D}}$ and $H' : \mathscr{E} \to \widetilde{\mathcal{D}}$. An arrow $\xi : \underline{G} \Rrightarrow \underline{G'}$ is a modification, and we want to build from it a natural trasformation $\eta : H' \Rightarrow H$ of geometric morphisms. To define η, notice that ξ corresponds to the given for every (A, α) in $\mathbb{D}(X)$ of a natural transformation $\xi_{(A,\alpha)} : G'_{(A,\alpha)} \Rightarrow G_{(A,\alpha)}$ of geometric morphisms (the direction is reversed by the ^{co}), satisfying some naturality conditions. We can set $\eta(E) : H'(E) \Rightarrow H(E) : \mathcal{D}^{op} \to \mathbf{Set}$ componentwise as

$$\eta(E)(D) := G'^{*}_{(D,1_{p(D)})}(E)([1_{p(D)}]) \xrightarrow{\xi_{(D,1_{p(D)})}(E)([1_{p(D)}])} G^{*}_{(D,1_{p(D)})}(E)([1_{p(D)}])$$

From the definition of modification, $\eta(E)(D)$ is natural in both components, and thus provides a 2-cell of geometric morphisms $\eta : H' \Rightarrow H$. $\square$

Remark 6.3.3. If $\mathscr{E} = \mathbf{Sh}(\mathcal{A}, T)$ and all the geometric morphisms $G_{(A,\alpha)}$ are induced by continuous comorphisms of sites, then the morphism $H : \mathrm{Gir}_J(p) \to \mathscr{E}$ is induced, up to isomorphism, by the continuous comorphism of sites $h : \mathcal{D} \to \mathcal{A}$ defined in the proof of Corollary 6.2.2.

As a corollary we have that Giraud toposes for presheaves are canonically seen as conical colimits:

Corollary 6.3.5. *Consider a small-generated site* $(\mathcal{C}, J)$. *For any presheaf* $P : \mathcal{C}^{op} \to \mathbf{Set}$, *the topos* $\mathrm{Gir}_J(P)$ *is the conical pseudocolimit of the diagram*

$$D : \textstyle\int P \xrightarrow{p_P} \mathcal{C} \xrightarrow{\mathcal{C}/-} \mathbf{cFib}_{\mathcal{C}} \xrightarrow{\mathfrak{G}} \mathbf{Com}/(\mathcal{C}, J) \xrightarrow{C_{(-)}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J).$$

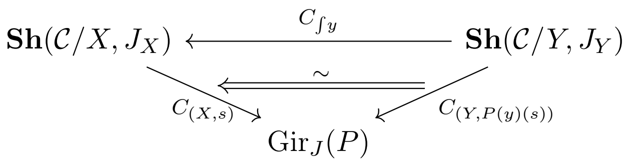
In particular, if P *is the terminal presheaf we have that* $\mathbf{Sh}(\mathcal{C}, J)$ *is the conical pseudocolimit of the diagram* $C_{(-)}\mathfrak{G}(\mathcal{C}/-)$.

If we consider the 1-category $\mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J)$, *where geometric morphisms are identified up to equivalence, then* $\mathrm{Gir}_J(P)$ *is also the 1-colimit of* D *in* $\mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J)$.

Proof. The first claim follows from Corollary 3.2.2, which states that conification is possible for pseudocolimits with a discrete weight, so that

$$\mathrm{Gir}_J(P) \simeq \mathrm{colim}^P_{ps}(C_{(-)}\mathfrak{G}(\mathcal{C}/-)) \simeq \mathrm{colim}_{ps}(C_{(-)}\mathfrak{G}(\mathcal{C}/-)p_P).$$

Its colimit cocone is that of the geometric morphisms



indexed by the objects (X, s) in $\int P$. If in particular we consider the terminal presheaf $1 : \mathcal{C}^{op} \to \mathbf{Set}$, it is immediate to see that the corresponding Grothendieck fibration is the identity functor of $\mathcal{C}$, and thus $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathrm{colim}_{ps}(C_{(-)}\mathfrak{G}(\mathcal{C}/-))$.

For the last part, consider a 1-cocone $G_{(X,s)} : \mathbf{Sh}(\mathcal{C}/X, J_X) \to \mathscr{F}$ under the diagram D. In the proof of Theorem 6.3.4 the definition of the induced geometric morphism $H : \mathrm{Gir}_J(P) \to \mathscr{F}$ was forced by the conditions $G_{(X,s)} \simeq HC_{(X,s)}$. If we want to build a 1-colimit we must strenghten them to the equalities $G_{(X,s)} = HC_{(X,s)}$, and this determines H uniquely up to isomorphism: thus $\mathrm{Gir}_J(P)$ is a 1-colimit of toposes. $\square$

Remark 6.3.4. We have already said in Chapter 3 that the two colimits $\mathrm{colim}^{\mathbb{D}}_{ps} D$ and $\mathrm{colim}_{ps}(D \circ p_{\mathbb{D}})$ are in general different, if $\mathbb{D}$ is not discrete. We provide an explicit example using Giraud toposes. Take $\mathcal{C} = \mathbb{1}$, the terminal category, and endow it with the trivial topology: then $\mathbf{Sh}(\mathbb{1}, J^{tr}_{\mathbb{1}}) \simeq \mathbf{Set}$ and $D : \mathbb{1} \to \mathbf{Topos}^{co}/\mathbf{Set}$ maps the unique object of $\mathbb{1}$ to $\mathbf{Set}$. Consider now the category $\mathbb{2}$ that has two objects 0 and 1 and an arrow $t : 0 \to 1$ between them: the unique functor $p : \mathbb{2} \to \mathbb{1}$, is a fibration and the Giraud topology over $\mathbb{2}$ is the trivial topology, hence $\mathrm{colim}^{\mathbb{2}}_{ps}(D) \simeq \mathbf{Sh}(\mathbb{2}, J_{\mathbb{2}}) \simeq [\mathbb{2}, \mathbf{Set}]$. On the other hand, $Dp : \mathbb{2} \to \mathbf{Topos}^{co}/\mathbf{Set}$ maps the two objects of $\mathbb{2}$ to the topos $\mathbf{Set}$ and the unique arrow between them to the identity geometric morphism of $\mathbf{Set}$, thus $\mathrm{colim}_{ps}(D \circ p) \simeq \mathbf{Set}$.

Finally, from Theorem 6.3.4 we can deduce the *fundamental adjunction* between essentially J-small cloven fibrations over $\mathcal{C}$ and toposes over $\mathbf{Sh}(\mathcal{C}, J)$:

Corollary 6.3.6. *For any small-generated site* $(\mathcal{C}, J)$*, the two pseudofunctors*

$$\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{cFib}^J_{\mathcal{C}} \xrightarrow{\mathfrak{G}} \mathbf{Com}/(\mathcal{C}, J) \xrightarrow{C_{(-)}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)$$

defined by

$$\left[[p : \mathcal{D} \to \mathcal{C}] \xrightarrow{(F,\varphi)} [q : \mathcal{E} \to \mathcal{C}]\right] \mapsto \left[[\mathrm{Gir}_J(p)] \xrightarrow{(C_F, C_\varphi)} [\mathrm{Gir}_J(q)]\right]$$

and

$$\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Ind}^J_{\mathcal{C}}$$

defined by

$$[E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)] \mapsto \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), [E]) : \mathcal{C}^{op} \to \mathbf{CAT},$$

up to the 2-equivalence $\mathbf{Ind}^J_{\mathcal{C}} \simeq \mathbf{cFib}^J_{\mathcal{C}}$ *are the components of a 2-adjunction*

$$\mathbf{cFib}^J_{\mathcal{C}} \underset{\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}}{\overset{\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}}{\rightleftarrows}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) \quad (\perp)$$

Finally, as the 2-adjunction $\Lambda_{\mathbf{CAT}/\mathcal{C}} \dashv \Gamma_{\mathbf{CAT}/\mathcal{C}}$ can be formulated for sites and continuous comorphisms, so the 2-adjunction $\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} \dashv \Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}$ can also be formulated on a smaller class of toposes. Indeed, since $\mathfrak{G}$ maps all fibrations in $\mathbf{cFib}_{\mathcal{C}}$ and their morphisms to continuous comorphisms of sites, it follows that the geometric morphisms in the image of $\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}$ are all essential geometric morphisms, and thus the colimit cocone lives in $\mathbf{EssTopos}$. Moreover, if we go back to the proof of Theorem 6.3.4 and supposed that all the legs $G^*_{(A,\alpha)}$ of the cone of the inverse images preserve arbitrary limits, H does too by Lemma D.3, which is to say that if all the legs $G_{(A,\alpha)}$ are essential geometric morphisms then the induced functor H is an essential geometric morphism. This proves the following result:

Corollary 6.3.7. *There is a 2-adjunction*

$$\mathbf{cFib}_{\mathcal{C}}^{J} \underset{\Gamma_{\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)}}{\overset{\Lambda_{\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)}}{\perp}} \mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)$$

where, for a cloven fibration $p : \mathcal{D} \to \mathcal{C}$ with corresponding $\mathcal{C}$-indexed category $\mathbb{D}$,

$$\Lambda_{\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)}(p) := [C_{p_{\mathcal{D}}} : \mathrm{Gir}_J(P) \to \mathbf{Sh}(\mathcal{C},J)]$$

and $\Gamma_{\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)}$ is defined by mapping an essential geometric morphism $E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C},J)$ to the $\mathcal{C}$-indexed category

$$\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)(\mathbf{Sh}(\mathcal{C},J),[E]) : \mathcal{C}^{op} \to \mathbf{CAT}.$$

Remark 6.3.5. This could also be derived from the adjunction $C_{(-)} \dashv (-)_!$ between $\mathbf{EssTopos}^{co}$ and $\mathbf{Com}_{cont}$ which we introduced in Corollary 1.3.2. Indeed, since $C_{(-)}$ is a left adjoint it preserves weighted pseudocolimits: thus from $(\mathcal{G}(\mathbb{D}), J_{\mathbb{D}}) \simeq \mathrm{colim}_{ps}^{\mathbb{D}}(\mathcal{C}/-)$ it follows that

$$\mathrm{Gir}_J(\mathbb{D}) := \mathbf{Sh}(\mathcal{G}(\mathbb{D}), J_{\mathbb{D}}) \simeq \mathrm{colim}_{ps}^{\mathbb{D}} \mathbf{Sh}(\mathcal{C}/-, J(-))$$

in $\mathbf{EssTopos}^{co}$. Moreover, the functor $C_{(-)}$ restricts to a functor of slice categories

$$C_{(-)} : \mathbf{Com}_{cont}/(\mathcal{C},J) \to \mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J),$$

so the diagram $\mathbf{Sh}(\mathcal{C}/-, J_{(-)})$ has image in the slice $\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)$. Finally, since the forgetful functor $\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J) \to \mathbf{EssTopos}^{co}$ reflects colimits, the topos $\mathrm{Gir}_J(\mathbb{D})$ is also the colimit of $\mathbf{Sh}(\mathcal{C}/-, J_{(-)})$ in $\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)$.

The codomain of the local section functors Γ can also be restricted:

Lemma 6.3.8. *Given a site $(\mathcal{C},J)$, the functor*

$$\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J) \to \mathbf{cFib}_{\mathcal{C}}$$

factors through $\mathbf{St}(\mathcal{C},J)$. The same holds for $\Gamma_{\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C},J)}$.

Proof. Indeed, consider X in $\mathcal{C}$, $m_R : R \rightarrowtail よ(X)$ in $J(X)$ and any $\mathbf{Sh}(\mathcal{C},J)$-topos $\mathscr{E}$: then the diagram

$$\begin{array}{ccc} \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \Gamma(\mathscr{E})) & \xrightarrow{\sim} & \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)(\mathbf{Sh}(\mathcal{C}/X, J_X), \mathscr{E}) \\ {\scriptstyle -\circ \int m_R} \Big\downarrow & & \Big\downarrow {\scriptstyle -\circ C_{\int m_R}} \wr \\ \mathbf{Fib}_{\mathcal{C}}(\int R, \Gamma(\mathscr{E})) & \xrightarrow{\sim} & \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)(\mathbf{Sh}(\int R, J_R), \mathscr{E}) \end{array}$$

shows that the functor $- \circ \int m_R$ is an equivalence, and hence $\Gamma(\mathscr{E})$ is a J-stack. $\square$

The previous lemma entails that we may restrict the fundamental adjunction to stacks, by applying Lemma D.5:

Proposition 6.3.9. *There are 2-adjunctions*

$$\mathbf{St}^J(\mathcal{C}, J) \underset{\Gamma}{\overset{}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J),$$

$$\mathbf{St}^J(\mathcal{C}, J) \underset{\Gamma'}{\overset{}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C}, J),$$

where

- *in both cases the left adjoint acts by mapping a J-stack $\mathbb{D} : \mathcal{C}^{op} \to \mathbf{CAT}$ to its classifying topos $C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), J_{\mathbb{D}}) \to \mathbf{Sh}(\mathcal{C}, J)$;*
- *the right adjoint Γ maps a $\mathbf{Sh}(\mathcal{C}, J)$-topos $\mathscr{E}$ to the J-stack*
$$\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), \mathscr{E}) : \mathcal{C}^{op} \to \mathbf{CAT};$$
- *the right adjoint Γ' maps an essential $\mathbf{Sh}(\mathcal{C}, J)$-topos $\mathscr{E}$ to the J-stack*
$$\mathbf{EssTopos}^{co}/\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), \mathscr{E}) : \mathcal{C}^{op} \to \mathbf{CAT}.$$

6.4 The canonical fibration as a dualizing object

A consequence of the fundamental adjunction is that Giraud toposes are equivalent to particular categories of morphisms of fibrations:

Corollary 6.4.1. *Consider a site $(\mathcal{C}, J)$ and a $\mathcal{C}$-indexed category $\mathbb{D}$: then*

$$\mathrm{Gir}_J(\mathbb{D}) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}, \mathcal{S}^V_{(\mathcal{C},J)})^{op} \simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}^V, \mathcal{S}_{(\mathcal{C},J)}).$$

Proof. The notation $\mathbb{D}^V$ was introduced in Example 2.1.1(v). We also use the shorthand $\widetilde{\mathcal{A}}$ for any sheaf topos $\mathbf{Sh}(\mathcal{A}, T)$. If we denote by $\widetilde{\mathcal{C}}[\mathbb{O}]$ the object classifier over $\widetilde{\mathcal{C}}$ (which exists by [21, Example B3.2.9]), then the following chain of natural equivalences holds:

$$\begin{aligned}
\mathrm{Gir}_J(\mathbb{D}) :&= \mathbf{Sh}(\mathcal{G}(\mathbb{D}), J_{\mathbb{D}}) \\
&\simeq \mathbf{Topos}/\widetilde{\mathcal{C}}(\widetilde{\mathcal{G}(\mathbb{D})}, \widetilde{\mathcal{C}}[\mathbb{O}]) \\
&\simeq \mathbf{Topos}^{co}/\widetilde{\mathcal{C}}(\widetilde{\mathcal{G}(\mathbb{D})}, \widetilde{\mathcal{C}}[\mathbb{O}])^{op} \\
&\simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}, \mathbf{Topos}^{co}/\widetilde{\mathcal{C}}(\widetilde{\mathcal{C}/-}, \widetilde{\mathcal{C}}[\mathbb{O}]))^{op} \\
&\simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}, \widetilde{\mathcal{C}/-}^{V})^{op} \\
&\simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}^V, \mathcal{S}_{(\mathcal{C},J)}),
\end{aligned}$$

where we exploited the fact that $\mathbf{Ind}_{\mathcal{C}}(\mathbb{D}^V, \mathbb{E}) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}, \mathbb{E}^V)^{op}$ (which is immediate to check) and that by definition $\mathcal{S}_{(\mathcal{C},J)} := \widetilde{\mathcal{C}/-}$. ☐

Remark 6.4.1. This result generalizes Proposition 2.3 of [12], where it is formulated for lex stacks over a lex site.

As a corollary we obtain an alternative proof of the fact that the canonical fibration $\mathcal{S}_{(\mathcal{C},J)}$ is a stack:

Corollary 6.4.2. *The canonical fibration of a site is a stack.*

Proof. Consider a site $(\mathcal{C}, J)$ and a J-covering sieve $m_R : R \rightarrowtail よ(X)$: Lemma 5.2.3 showed that

$$\mathbf{Sh}(\textstyle\int R, J_R) \simeq \mathbf{Sh}(\mathcal{C}/X, J_X)$$

via the geometric morphism C_{m_R}. Therefore we have a commutative square

$$\begin{array}{ccc} \mathbf{Sh}(\int R, J_R) & \overset{\sim}{\longrightarrow} & \mathbf{Ind}_{\mathcal{C}}(R^V, \mathcal{S}_{(\mathcal{C},J)}) \\ \wr\uparrow & & \uparrow{\scriptstyle -\circ m_R} \\ \mathbf{Sh}(\mathcal{C}/X, J_X) & \overset{\sim}{\longrightarrow} & \mathbf{Ind}_{\mathcal{C}}(よ(X)^V, \mathcal{S}_{(\mathcal{C},J)}) \end{array}$$

Finally, it is enough to notice that since both R and $よ(X)$ are discrete they are left unchanged by $(-)^V$, i.e. $R \simeq R^V$ and $よ(X) \simeq よ(X)^V$, to conclude the proof. ☐

The equivalence $\mathrm{Gir}_J(\mathbb{D}) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}^V, \mathcal{S}_{(\mathcal{C},J)})$ gives us a way of seeing $J_{\mathbb{D}}$-sheaves as gluing of local data, since a $J_{\mathbb{D}}$-sheaf $W : \mathcal{G}(\mathbb{D})^{op} \to \mathbf{Set}$ corresponds to a morphism of Grothendieck fibrations $\mathcal{G}(\mathbb{D}^V) \to \mathcal{G}(\mathcal{S}_{(\mathcal{C},J)})$, i.e. to the following data:

- for every X in $\mathcal{C}$ and U in $\mathbb{D}(X)$, a J_X-sheaf $H_U : (\mathcal{C}/X)^{op} \to \mathbf{Set}$;
- for every U in $\mathbb{D}(X)$ and every pair of arrows $y : Y \to X$ in $\mathcal{C}$ and $a : \mathbb{D}(y)(U) \to V$ in $\mathbb{D}(Y)$, a morphism of presheaves $h_{(y,a)} : H_V \to H_U \circ (\int y)^{op}$ such that the association $(y, a) \mapsto h_{(y,a)}$ is functorial and moreover whenever a is invertible then $h_{(y,a)}$ is invertible.

A similar description of arrows can be given.

Corollary 6.4.1 also implies that the canonical stack $\mathcal{S}_{(\mathcal{C},J)}$ is a *dualizing object* between the two categories $\mathbf{Ind}_{\mathcal{C}}$ and $\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)$, because it allows us to express both Γ and Λ as hom-functors in $\mathcal{S}_{(\mathcal{C},J)}$ as the two following equivalences show:

$$\begin{aligned} \Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}(\mathscr{E}) &:= \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), \mathscr{E}) \\ &\simeq \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)(\mathcal{S}_{(\mathcal{C},J)}(-), \mathscr{E}), \\ \Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}(\mathbb{D}) &:= \mathrm{Gir}_J(\mathbb{D}) \\ &\simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}^V, \mathcal{S}_{(\mathcal{C},J)}). \end{aligned}$$

6.5 Relative toposes

Given an internal category $\mathbb{D}$ of $\mathbf{Sh}(\mathcal{C}, J)$, its Giraud topos $\mathrm{Gir}_J(\mathbb{D}) := \mathbf{Sh}(\mathcal{G}(\mathbb{D}), J_{\mathbb{D}})$ is the topos of internal presheaves for $\mathbb{D}$:

$$\mathrm{Gir}_J(\mathbb{D}) \simeq [\mathbb{D}^{op}, \mathbf{Sh}(\mathcal{C}, J)].$$

This appears for instance as Proposition C2.5.4 of [21]. Corollary 6.4.1 provides the external intuition for that: if we think of $\mathcal{S}_{(\mathcal{C},J)}$ as the embodiment of the topos $\mathbf{Sh}(\mathcal{C}, J)$ in terms of stacks over $\mathcal{C}$, then we have an immediate parallelism between the equivalence above and

$$\mathrm{Gir}_J(\mathbb{D}) \simeq \mathbf{Ind}_{\mathcal{C}} \left(\mathbb{D}^V, \mathcal{S}_{(\mathcal{C},J)}\right).$$

Thinking of Giraud toposes as toposes of presheaves over a fibration is the starting point to develop relative topos theory. We provide here some first results in this direction, from Chapter 8 of [8].

Definition 6.5.1. Let $(\mathcal{C}, J)$ be a small-generated site. A *relative presheaf topos* over $\mathbf{Sh}(\mathcal{C}, J)$ is a topos of the form $C_{p_{\mathbb{D}}} : \mathrm{Gir}_J(\mathbb{D}) = \mathbf{Sh}(\mathcal{G}(\mathbb{D}), J_{\mathbb{D}}) \to \mathbf{Sh}(\mathcal{C}, J)$, where $\mathbb{D}$ is a $\mathcal{C}$-indexed category.

Remark 6.5.1. In the interest of maximal generality, we do not require $\mathbb{D}$ to be a stack on $(\mathcal{C}, J)$, nor to be the $\mathcal{C}$-indexing of an internal category in $\mathbf{Sh}(\mathcal{C}, J)$, since indexed categories simultaneously generalize both notions.

As any Grothendieck topos is a subtopos of a presheaf topos, so we define relative toposes as subtoposes of relative presheaf toposes. We have already mentioned in the introductory chapter that subtoposes of $\mathbf{Sh}(\mathcal{C}, J)$ correspond to the Grothendieck topologies K on $\mathcal{C}$ such that $J \subseteq K$, and this motivates the following definition:

Definition 6.5.2. Let $(\mathcal{C}, J)$ be a small-generated site. A *relative site* over $(\mathcal{C}, J)$ is a pair $(\mathbb{D}, K)$, where $\mathbb{D}$ is a $\mathcal{C}$-indexed category and K is a Grothendieck topology on $\mathcal{G}(\mathbb{D})$ containing the Giraud topology $J_{\mathbb{D}}$. Any relative site corresponds to a site $(\mathcal{G}(\mathbb{D}), K)$ endowed with a structure comorphism of sites $p_{\mathbb{D}} : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C}, J)$.

Remark 6.5.2. There are some size issues involved in the notion of relative site. We do not require smallness hypotheses in our definition as, for technical reasons, it is convenient to be able to work also with large presentation sites. One can resolve such issues either by working with respect to a bigger Grothendieck universe, or by showing that the relevant sites under consideration are in fact small-generated (for instance, one can show that the canonical site of a geometric morphism, in the sense of Definition 6.5.4, is small-generated).

Trivial relative sites are those such that the Grothendieck topology K coincides with Giraud's topology $J_{\mathbb{D}}$: as in the classical setting, they yield relative presheaf toposes. Accordingly, arbitrary relative sites yield arbitrary subtoposes of relative presheaf toposes:

Definition 6.5.3. Let $(\mathcal{C}, J)$ be a small-generated site. A *relative topos* over $\mathbf{Sh}(\mathcal{C}, J)$ is a Grothendieck topos $\mathscr{E}$ together with a geometric morphism $E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$.

Of course, every relative site $(\mathbb{D}, K)$ over $(\mathcal{C}, J)$ yields a relative topos over $\mathbf{Sh}(\mathcal{C}, J)$, by considering the geometric morphism

$$C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), K) \to \mathbf{Sh}(\mathcal{C}, J).$$

To prove the opposite implication that any relative topos is the topos of sheaves for a relative site, we need first to build a fibration from any geometric morphism $f : \mathscr{F} \to \mathscr{E}$. In fact this was already done in Example 2.1.1(ii), where the $\mathscr{E}$-indexed category $\mathbb{I}_f$ associated with f is defined on objects by mapping E of $\mathscr{E}$ to the slice topos $\mathscr{F}/f^*(E)$, with its transition morphisms being the obvious pullback functors. In fact, this can be done more generally. Consider two sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ and a (J, K)-continuous functor $A : \mathcal{C} \to \mathcal{D}$: then we can define a $\mathcal{C}$-indexed category $\mathbb{I}_A$ by mapping any X in $\mathcal{C}$ to the slice $\mathbf{Sh}(\mathcal{D}, K)/\ell_K(A(X))$, while the transition morphisms are pullback functors. Notice that the fibration $\mathbb{I}_f$ defined from the geometric morphism f is now a particular instance of this, where we take as continuous functors the morphism of sites $f^* : (\mathscr{E}, J^{can}_{\mathscr{E}}) \to (\mathscr{F}, J^{can}_{\mathscr{F}})$. Every fibration of this form is in fact a stack:

Proposition 6.5.1. *Let $A : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a (J, K)-continuous functor: then the fibration $\mathbb{I}_A$ defined above is a J-stack. In particular, for every geometric morphism $f : \mathscr{F} \to \mathscr{E}$ the $\mathscr{E}$-indexed category $\mathbb{I}_f$ of Example 2.1.1(ii) is a $J^{can}_{\mathscr{E}}$-stack.*

Proof. Notice that $\mathbb{I}_A$ corresponds to the composite pseudofunctor

$$\mathcal{C}^{op} \xrightarrow{A^{op}} \mathcal{D}^{op} \xrightarrow{\mathcal{S}_{(\mathcal{D},K)}} \mathbf{CAT},$$

where $\mathcal{S}_{(\mathcal{D},K)}$ denotes the canonical stack for the site $(\mathcal{D}, K)$ (see Definition 2.4.1 and Theorem 2.4.1).Then we can exploit the notion of direct image of fibrations, introduced in Section 4.1, and the fact that the direct image along a continuous functor maps stacks to stacks (Proposition 4.3.1), to conclude that $\mathbb{I}_A$ is a J-stack. □

For a geometric morphism $f : \mathscr{F} \to \mathscr{E}$, the fibration associated to $\mathbb{I}_f$ is made as follows: objects over E in $\mathscr{E}$ are arrows $[u : U \to f^*(E)]$ of $\mathscr{F}$, and morphisms $(e, a) : [v : V \to f^*(E')] \to [u : U \to f^*(E)]$ are indexed by two arrows $e : E' \to E$ and $a : V \to U$ making the diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & U \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ f^*(E') & \xrightarrow[f^*(e)]{} & f^*(E) \end{array}$$

commutative. Cartesian arrows of $\mathcal{G}(\mathbb{I}_f)$ are characterized as those such that the square above is a pullback square.

Notice that the fibration $\mathcal{G}(\mathbb{I}_f)$ corresponds in fact to the comma category $(1_{\mathscr{F}} \downarrow f^*)$. We have already met this kind of category in Theorem 1.4.4, where we showed that any geometric morphism induced by a morphism of sites $A : (\mathcal{C}, J) \to (\mathcal{D}, K)$ can be described as the geometric morphism induced by the fibration $(1_{\mathcal{D}} \downarrow A) \to \mathcal{C}$ upon endowing the domain with a suitable Grothendieck topology $\bar{K}$. Applying that in our specific case, we obtain the following result:

Theorem 6.5.2. *Let $f : \mathscr{F} \to \mathscr{E}$ be a geometric morphism. There exists a Grothendieck topology J_f over $\mathcal{G}(\mathbb{I}_f)$ such that the toposes $\mathscr{F}$ and $\mathbf{Sh}(\mathcal{G}(\mathbb{I}_f), J_f)$ are equivalent as $\mathscr{E}$-toposes: more specifically, a family $\{(e_i, a_i) : [v_i : V_i \to f^*(E_i)] \to [u : U \to f^*(E)] \mid i \in I\}$ is J_f-covering if and only if the family $\{e_i : E_i \to E \mid i \in I\}$ is epimorphic in $\mathscr{E}$.*

Thus the geometric morphism $f : \mathscr{F} \to \mathscr{E}$ presents $\mathscr{F}$ as a topos of relative sheaves over the stack *$\mathbb{I}_f$:*

$$\mathscr{F} \simeq \mathbf{Sh}(\mathcal{G}(\mathbb{I}_f), J_f) =: \mathbf{Sh}_{\mathscr{E}}(\mathbb{I}_f, J_f).$$

We call the Grothendieck topology J_f the relative topology *of f.*

Proof. We apply Theorem 1.4.4 to the morphism of sites

$$f^* : (\mathscr{E}, J_{\mathscr{E}}^{can}) \to (\mathscr{F}, J_{\mathscr{F}}^{can}).$$

The category $(1_{\mathscr{F}} \downarrow f^*)$ can be endowed with a topology $\overline{J_{\mathscr{F}}^{can}}$ such that $\pi_{\mathscr{E}} : (1_{\mathscr{F}} \downarrow f^*) \to \mathscr{E}$ is a comorphism of sites and $\pi_{\mathscr{F}} : (1_{\mathscr{F}} \downarrow f^*) \to \mathscr{F}$ induces an equivalence of toposes making the diagram

$$\begin{array}{ccccc}
\mathscr{F} & \overset{\sim}{\longrightarrow} & \mathbf{Sh}(\mathscr{F}, J_{\mathscr{F}}^{can}) & \overset{\sim}{\longrightarrow} & \mathbf{Sh}((1_{\mathscr{F}} \downarrow f^*), \overline{J_{\mathscr{F}}^{can}}) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbf{Sh}(f^*)} & \swarrow{\scriptstyle C_{\pi_{\mathscr{E}}}} & \\
\mathscr{E} & \overset{\sim}{\longrightarrow} & \mathbf{Sh}(\mathscr{E}, J_{\mathscr{E}}^{can}) & &
\end{array}$$

is commutative. Setting $J_f := \overline{J_{\mathscr{F}}^{can}}$ concludes the proof. ☐

With a similar technique we can show that the $\mathcal{S}_{(\mathcal{C},J)}$ is an alternative site of presentation for $\mathbf{Sh}(\mathcal{C}, J)$:

Corollary 6.5.3. *Consider an essentially small site $(\mathcal{C}, J)$, then the canonical stack $\pi_{(\mathcal{C},J)} : \mathcal{S}_{(\mathcal{C},J)} \to \mathcal{C}$ induces an equivalence of toposes*

$$C_{\pi_{(\mathcal{C},J)}} : \mathbf{Sh}(\mathcal{S}_{(\mathcal{C},J)}, J_{\mathcal{S}_{(\mathcal{C},J)}}) \xrightarrow{\sim} \mathbf{Sh}(\mathcal{C}, J),$$

where $J_{\mathcal{S}_{(\mathcal{C},J)}} := \overline{J_{\mathbf{Sh}(\mathcal{C},J)}^{can}}$ in the notation of Theorem 1.4.4.

Proof. We recall that $\pi_{(\mathcal{C},J)} : \mathcal{S}_{(\mathcal{C},J)} \to \mathcal{C}$ was defined as the fibration $\pi : (1_{\mathbf{Sh}(\mathcal{C},J)} {\downarrow} \ell_J) \to \mathcal{C}$, with π being the canonical projection onto $\mathcal{C}$ (cf. Definition 2.4.1). The conclusion follows from Theorem 1.4.4, by considering the following diagram:

$$\begin{array}{ccc}
\mathbf{Sh}(\mathcal{C},J) & \overset{\sim}{\longrightarrow} & \mathbf{Sh}((1_{\mathbf{Sh}(\mathcal{C},J)} {\downarrow} \ell_J), \overline{J^{can}_{\mathbf{Sh}(\mathcal{C},J)}}) \\
& {\scriptstyle \mathbf{Sh}(\ell_J)} \searrow^{\sim} & \downarrow {\scriptstyle C_{\pi_{(\mathcal{C},J)}}} \\
& & \mathbf{Sh}(\mathbf{Sh}(\mathcal{C},J), J^{can}_{\mathbf{Sh}(\mathcal{C},J)}).
\end{array}$$

□

We can consider, more specifically, geometric morphisms induced by arbitrary morphisms of sites:

Definition 6.5.4. Given a morphism of sites $A : (\mathcal{C},J) \to (\mathcal{D},K)$, the *relative site of* A is the site $(\mathbb{I}_A, J^K_A)$ (where J^K_A is equal to the topology $\overline{K}$ of Theorem 1.4.4), together with the canonical projection functor $\pi_A : \mathbb{I}_A \to \mathcal{C}$, which is a comorphism of sites $(\mathbb{I}_A, J^K_A) \to (\mathcal{C},J)$. In particular, for a geometric morphism $f : \mathscr{F} \to \mathscr{E}$, regarded as a morphism of sites $f^* : (\mathscr{E}, J^{can}_{\mathscr{E}}) \to (\mathscr{F}, J^{can}_{\mathscr{F}})$, we call the site $(\mathbb{I}_{f^*}, J^{J^{can}_{\mathscr{F}}}_{f^*})$ the *relative site of* f.

By applying again Theorem 1.4.4, we can generalize Theorem 6.5.2 to morphisms of sites as follows:

Theorem 6.5.4. *Let* $A : (\mathcal{C},J) \to (\mathcal{D},K)$ *be a morphism of small-generated sites. Then the geometric morphism* $\mathbf{Sh}(A)$ *induced by* A *coincides with the structure geometric morphism* C_{π_A} *associated with the relative site* $(\mathbb{I}_A, J^K_A)$.

Corollary 6.5.5. *Every relative topos* $f : \mathscr{F} \to \mathbf{Sh}(\mathcal{C},J)$ *is of the form* $C_{p_{\mathbb{D}}}$ *for some relative site* $p_{\mathbb{D}} : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C},J)$.

Proof. It is sufficient to apply the previous result by seeing f as induced by the morphism of sites

$$(\mathcal{C},J) \xrightarrow{\ell_J} (\mathbf{Sh}(\mathcal{C},J), J^{can}_{\mathbf{Sh}(\mathcal{C},J)}) \xrightarrow{f^*} (\mathscr{F}, J^{can}_{\mathscr{F}}).$$

□

Chapter 7

The fundamental adjunction in the discrete setting

In the present chapter, we specialize the fundamental adjunction to presheaves: unless stated otherwise, all the results in this chapter stem from Chapter 6 of [8]. We begin in the first section by restricting the adjunction to presheaves and sheaves, obtaining a general form of the topological presheaf-bundle adjunction that holds for all sites; after that, we consider the particular case of preorder sites and locales, for which the topos-theoretic data of the fundamental adjunction vanishes and the situation can be described at the level of sites. The general techniques used to study the preorder case are stated to a general class of sites and (co)morphisms in the final section of the chapter.

7.1 The presheaf-bundle adjunction for sites

Suppose that $\mathcal{C}$ is small: then, given any presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$, its discrete fibration $\int P$ is again a small category. Since the image of $\mathcal{G}$ in $\mathbf{CAT}/\mathcal{C}$ is contained in the sub-2-category of strictly commutative triangles, in particular presheaves are sent through $\mathcal{G}$ to the 1-categorical slice $\mathbf{Cat}/_1\mathcal{C}$. With this in mind, we can begin by building an adjunction between presheaves over $\mathcal{C}$ and categories over $\mathcal{C}$ as follows:

Proposition 7.1.1. *Consider a small category $\mathcal{C}$. The adjunction of Proposition 6.2.1 induces an adjunction*

$$[\mathcal{C}^{op}, \mathbf{Set}] \underset{\Gamma_{\mathbf{Cat}/_1\mathcal{C}}}{\overset{\Lambda_{\mathbf{Cat}/_1\mathcal{C}}}{\rightleftarrows}} \mathbf{Cat}/_1\mathcal{C} \quad (\Lambda_{\mathbf{Cat}/_1\mathcal{C}} \dashv \Gamma_{\mathbf{Cat}/_1\mathcal{C}})$$

where:

- $\mathbf{Cat}/_1\mathcal{C}$ *is the 1-categorical slice of* $\mathbf{Cat}$ *over* $\mathcal{C}$ *(see Definition 1.4.1);*

- $\Lambda_{\mathbf{Cat}/_1\mathcal{C}}$ *is the restriction of* $\Lambda_{\mathbf{CAT}/\mathcal{C}}$*: it maps a presheaf* P *to the functor* $\int P \to \mathcal{C}$*, and an arrow* $g : P \to Q$ *of presheaves to the functor* $\int g : \int P \to \int Q$*;*

- $\Gamma_{\mathbf{Cat}/_1\mathcal{C}}$ *maps a functor* $p : \mathcal{D} \to \mathcal{C}$ *to the presheaf*

$$\mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/-, \mathcal{D}) : \mathcal{C}^{op} \to \mathbf{Set},$$

  and acts accordingly on arrows.

All objects of $[\mathcal{C}^{op}, \mathbf{Set}]$ *are fixed points of the adjunction, while the fixed points of* $\mathbf{Cat}/_1\mathcal{C}$ *are exactly the discrete Grothendieck fibrations over* $\mathcal{C}$*.*

Proof. First of all, we remark that the hypothesis that both $\mathcal{C}$ and $\mathcal{D}$ are small is needed to ensure that the image of $\Gamma_{\mathbf{Cat}/_1\mathcal{C}}$ is in $[\mathcal{C}^{op}, \mathbf{Set}]$.

Given a presheaf P in $[\mathcal{C}^{op}, \mathbf{Set}]$ and a functor $p : \mathcal{D} \to \mathcal{C}$, we want to build a natural isomorphism

$$[\mathcal{C}^{op}, \mathbf{Set}](P, \mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/-, [p])) \simeq \mathbf{Cat}/_1\mathcal{C}(\textstyle\int P, [p]).$$

Consider an arrow of presheaves $g : P \to \mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/-, [p])$: it is the given for each X in $\mathcal{C}$ of a map $g_X : P(X) \to \mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/X, [p])$, so that for each $y : Y \to X$ in $\mathcal{C}$ and $s \in P(X)$ the naturality condition $g_X(s) \circ \int y = g_Y(P(y)(s))$ (we recall that $\int y : \mathcal{C}/Y \to \mathcal{C}/X$ acts by post-composition with y). Starting from g, we can define a functor $\bar{g} : \int P \to \mathcal{D}$ by setting $\bar{g}(X, s) = g_X(s)([1_X])$, and for every $y : Y \to X$ in $\mathcal{C}$ and $s \in P(X)$

$$\left[(Y, P(y)(s)) \xrightarrow{y} (X, s)\right] \mapsto$$

$$\mapsto \left[g_Y(P(y)(s))([1_Y]) = g_X(s)([y]) \xrightarrow{g_X(s)(y)} g_X(s)([1_X])\right].$$

Using the fact that each $g_X(s)$ is a functor over the base $\mathcal{C}$, it follows that $\bar{g}$ is also a functor over the base category $\mathcal{C}$.

Conversely, consider a functor $h : \int P \to \mathcal{D}$ over $\mathcal{C}$, an object X in $\mathcal{C}$ and $s \in P(X)$: we can define a functor $\tilde{h}_X(s) : \mathcal{C}/X \to \mathcal{D}$ over $\mathcal{C}$ by setting

$$\left[[yz] \xrightarrow{z} [y]\right] \mapsto \left[h(Z, P(yz)(s)) \xrightarrow{h(z)} h(Y, P(y)(s))\right],$$

and this provides for each X a map $\tilde{h}_X : P(X) \to \mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/X, [p])$. The identity $\tilde{h}_Y(P(y)(s)) = \tilde{h}_X(s) \circ \int y$ is easily verified, therefore the maps $\tilde{h}_X$ are the components of a natural transformation $h : P \to \mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/-, [p])$. The two associations $g \mapsto \bar{g}$ and $h \mapsto \tilde{h}$ provide the two components of the natural isomorphism we wanted, showing that $\Lambda_{\mathbf{Cat}/_1\mathcal{C}} \dashv \Gamma_{\mathbf{Cat}/_1\mathcal{C}}$.

For a presheaf P, the unit of the adjunction is the arrow $\eta_P : P \to \Gamma\Lambda(P)$ such that for every X in $\mathcal{C}$ the function $(\eta_P)_X : P(X) \to \mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/X, \int P)$

maps any $s \in P(X)$ to the functor $\mathcal{C}/X \to \int P$ defined on objects as $[y : Y \to X] \mapsto (Y, P(y)(s))$. One immediately verifies that η_P is in fact the isomorphism $P \xrightarrow{\sim} \mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/-, \int P)$ given by fibred Yoneda lemma, and hence every presheaf P is a fixed point.

For any $p : \mathcal{D} \to \mathcal{C}$ the counit $\varepsilon_{[p]} : \int(\mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/-, [p])) \to [p]$ is defined by mapping a pair $(X, F : \mathcal{C}/X \to \mathcal{D})$ to $F([1_X])$. If $\varepsilon_{[p]}$ is invertible the category $\mathcal{D}$ is isomorphic to the discrete fibration $\int(\mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/-, [p]))$. On the other hand, suppose that p is a discrete fibration, which is to say that up to isomorphism $\mathcal{D}$ is equivalent to the category of elements $\int P$ for some presheaf P over $\mathcal{C}$. Notice that, since all the arrows in a discrete fibration are cartesian, any functor in $\Gamma(\int P)(X) = \mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/X, \int P)$ is in fact a morphism of fibrations. Thus we have that

$$\Gamma(\textstyle\int P) = \mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/-, \int P) \simeq \mathbf{Fib}_{\mathcal{C}}(よ(-), P) \simeq P,$$

again by fibred Yoneda's lemma, implying $\Lambda\Gamma(P) \simeq \int P$: we conclude that, if $\mathcal{D}$ is a discrete Grothendieck fibration, it is a fixed point of the adjunction.
□

We now move to the topos-theoretic level. We begin by remarking that in general the hom category between two toposes is *not* a set, and hence we have to refine the definition of the right adjoint $\Gamma_{\mathbf{Topos}/\mathbf{Sh}(\mathcal{C},J)^{co}}$ if we want it to take values in $[\mathcal{C}^{op}, \mathbf{Set}]$. This is rather immediate:

Definition 7.1.1. We call a geometric morphism $F : \mathscr{F} \to \mathbf{Sh}(\mathcal{C}, J)$ *small relative to* $\mathbf{Sh}(\mathcal{C}, J)$ if for any J-sheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ the geometric morphisms $\mathbf{Sh}(\mathcal{C}, J)/P \to \mathscr{F}$ over $\mathbf{Sh}(\mathcal{C}, J)$ form a set (up to equivalence of geometric morphisms): more compactly, if the category

$$\mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}, J)/P, \mathscr{F})$$

is essentially small. We denote by $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ the full subcategory of the 1-category $\mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J)$ whose objects are the small geometric morphisms relative to $\mathbf{Sh}(\mathcal{C}, J)$.

Remark 7.1.1. In fact, one can reduce to checking the smallness of all categories $\mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}, J)/\ell_J(X), \mathscr{F})$ for X an object of $\mathcal{C}$, because the topos $\mathbf{Sh}(\mathcal{C}, J)/P \simeq \mathbf{Sh}(\int P, J_P)$ is a conical colimit of toposes of the form $\mathbf{Sh}(\mathcal{C}/X, J_X) \simeq \mathbf{Sh}(\mathcal{C}, J)/\ell_J(X)$, by Corollary 6.3.5.

The adjunction between presheaves and bundles over a topological space (see Section 6.1) restricts to an equivalence between sheaves and étale bundles over the space. In a similar way, the presheaf-bundle adjunction for sites induces an equivalence between the category $\mathbf{Sh}(\mathcal{C}, J)$ and a special class of $\mathbf{Sh}(\mathcal{C}, J)$-toposes, namely those that are étale:

Definition 7.1.2. A geometric morphism $F : \mathscr{F} \to \mathscr{E}$ is said to be *étale*, or a *local homeomorphism* , if there is some E in $\mathscr{E}$ such that F is isomorphic to the dependent product geometric morphism $\prod_E : \mathscr{E}/E \to \mathscr{E}$. We denote by $\mathbf{Topos}^{\acute{e}tale}/_1\mathbf{Sh}(\mathcal{C}, J)$ the 1-category of étale $\mathbf{Sh}(\mathcal{C}, J)$-toposes.

Étale toposes are the way in which one can externalize objects of a topos $\mathscr{E}$ as toposes over $\mathscr{E}$, and significantly this process is full and faithful:

Lemma 7.1.2. *Consider a topos $\mathscr{E}$: then the functor $\mathscr{E} \to \mathbf{Topos}/_1\mathscr{E}$ of 1-categories mapping each E in $\mathscr{E}$ to $\prod_E : \mathscr{E}/E \to \mathscr{E}$ and each $g : X \to Y$ to $\prod_g : \mathscr{E}/X \to \mathscr{E}/Y$ presents $\mathscr{E}$ as the full subcategory of étale geometric morphisms over $\mathscr{E}$. More explicitly, there is an isomorphism*

$$\mathscr{E}(X, Y) \simeq \mathbf{Topos}/_1\mathscr{E}(\mathscr{E}/X, \mathscr{E}/Y).$$

Proof. This is a well known result, but let us sketch the correspondence for later use. Starting from $g : X \to Y$, we consider the functor $\prod_g$. Viceversa, consider the arrow $\Delta_Y : Y \to Y \times Y$ in $\mathscr{E}$: it is also an arrow $\Delta_Y : 1_{\mathscr{E}/Y} \to Y^*(Y)$ of $\mathscr{E}/Y$, where $Y^* : \mathscr{E} \to \mathscr{E}/Y$ is the usual functor mapping any Z to $[\pi : Y \times Z \to Y]$. Now take any $F : \mathscr{E}/X \to \mathscr{E}/Y$: if we consider the arrow $F^*(\Delta_Y) : 1_{\mathscr{E}/X} \to X^*(Y)$, i.e. $F^*(\Delta_Y) : [1_X] \to [X \times Y \to X]$, it is an arrow $\langle 1, g\rangle : X \to X \times Y$ of $\mathscr{E}$ and this provides our arrow $g : X \to Y$. □

Remarks 7.1.2. (i) It follows that the category $\mathbf{Topos}^{\acute{e}tale}/_1\mathbf{Sh}(\mathcal{C}, J)$ embeds fully inside $\mathbf{Topos}_1/\mathbf{Sh}(\mathcal{C}, J)$, and moreover, by definition, every étale geometric morphism is essential and thus $\mathbf{Topos}^{\acute{e}tale}/_1\mathbf{Sh}(\mathcal{C}, J)$ embeds faithfully into $\mathbf{EssTopos}/_1\mathbf{Sh}(\mathcal{C}, J)$.

(ii) Every local homeomorphism to $\mathbf{Sh}(\mathcal{C}, J)$ is small relative to $\mathbf{Sh}(\mathcal{C}, J)$, because the geometric morphisms over $\mathbf{Sh}(\mathcal{C}, J)$ from a local homeomorphisms $\mathbf{Sh}(\mathcal{C}, J)/P$ to a local homeomorphism $\mathbf{Sh}(\mathcal{C}, J)/Q$ correspond precisely to the arrows $P \to Q$ in $\mathbf{Sh}(\mathcal{C}, J)$.

(iii) Suppose that $\mathscr{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$. Since $\mathbf{Sh}(\int P, J_P) \simeq \mathbf{Sh}(\mathcal{C}, J)/\mathrm{a}_J(P)$ by Theorem 5.2.1, the previous lemma is telling us that a geometric morphism $\mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\int Q, J_Q)$ is presented by an arrow $\mathrm{a}_J(P) \to \mathrm{a}_J(Q)$. The fact that we can replace geometric morphisms with arrows in the topos is very important, since it means that we can 'hide' the topos-theoretic content of the adjunction and work at a lower level of complexity. The same happens for topological spaces (see Section 6.1), where the right adjoint Γ can be described as a hom functor at the level of topological spaces, and more generally for preordered categories (see Section 7.2).

We can now restrict the fundamental adjunction to presheaves, generalizing to all sites the topological presheaf-bundle adjunction of Section 6.1:

Proposition 7.1.3. *Consider a small-generated site $(\mathcal{C}, J)$.*

(i) There is an adjunction of 1-categories

$$[\mathcal{C}^{op}, \mathbf{Set}] \underset{\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}}{\overset{\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}}{\rightleftarrows}} \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J).$$

The functor $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ *is the restriction of* $\Lambda_{\mathbf{Topos}/\mathbf{Sh}(\mathcal{C},J)^{co}}$*. It maps a presheaf* P *to* $[C_{p_P} : \mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\mathcal{C}, J)]$ *and an arrow* $g : P \to Q$ *to* $C_{\int g} : \mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\int Q, J_Q)$*. By Theorem 5.2.1 we can rephrase this as* $P \mapsto [\prod_{a_J(P)} : \mathbf{Sh}(\mathcal{C}, J)/a_J(P) \to \mathbf{Sh}(\mathcal{C}, J)]$ *and* $g \mapsto \prod_{a_J(g)} : \mathbf{Sh}(\mathcal{C}, J)/a_J(P) \to \mathbf{Sh}(\mathcal{C}, J)/a_J(Q)$

The functor $\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ *acts like a Hom-functor by mapping an object* $[F : \mathscr{F} \to \mathbf{Sh}(\mathcal{C}, J)]$ *of* $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ *to the presheaf*

$$\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}, J)/\ell_J(-), \mathscr{F}) : \mathcal{C}^{op} \to \mathbf{Set}.$$

(ii) The image of $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ *factors through* $\mathbf{Topos}^{\acute{e}tale}/\mathbf{Sh}(\mathcal{C}, J)$*, and the image of* $\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ *factors through* $\mathbf{Sh}(\mathcal{C}, J)$*;*

(iii) for any J*-sheaf* Q *it holds that* $\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}([\prod_Q]) \simeq Q$*, implying that the fixed points of* $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ *are precisely the étale geometric morphisms, while those of* $[\mathcal{C}^{op}, \mathbf{Set}]$ *are* J*-sheaves. In particular, the composite functor* $\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ *is naturally isomorphic to*

$$i_J \circ a_J : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J) \to [\mathcal{C}^{op}, \mathbf{Set}];$$

(iv) the adjunction $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)} \dashv \Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}$ *restricts to an equivalence*

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Topos}^{\acute{e}tale}/_1\mathbf{Sh}(\mathcal{C}, J).$$

Proof. Let us once again adopt the notation $\widetilde{\mathcal{C}} := \mathbf{Sh}(\mathcal{C}, J)$ for the sake of brevity, and let us drop the subscripts for Λ and Γ.

(i) This is just a restriction of the adjunction appearing in Theorem 6.3.4: indeed, the equivalence of Hom-categories

$$\mathbf{Topos}^{co}/\widetilde{\mathcal{C}}(\Lambda(P), \mathscr{E}) \simeq \mathbf{Ind}_{\mathcal{C}}(P, \Gamma(\mathscr{E}))$$

appearing there restricts to a bijection of Hom-sets (up to equivalence of geometric morphisms)

$$\mathbf{Topos}^s/_1\widetilde{\mathcal{C}}(\Lambda(P), \mathscr{E}) \simeq [\mathcal{C}^{op}, \mathbf{Set}](P, \Gamma(\mathscr{E}))$$

because $\Lambda(P)$ is a 1-colimit of toposes by Corollary 6.3.5.

(ii) The fact that the image of Λ factors through $\mathbf{Topos}^{étale}/\widetilde{\mathcal{C}}$ is true by definition. The image of Γ is contained in $\widetilde{\mathcal{C}}$ as a consequence of Lemma 6.3.8, once recalled that a presheaf P is a J-stack if and only if it is a J-sheaf (see Proposition 2.3.1).

(iii) Considering an étale geometric morphism $[\prod_Q : \widetilde{\mathcal{C}}/Q \to \widetilde{\mathcal{C}}]$, then

$$\Gamma([\textstyle\prod_Q]) := \mathbf{Topos}^s/_1\widetilde{\mathcal{C}}(\widetilde{\mathcal{C}}/\ell_J(-), \widetilde{\mathcal{C}}/Q) \simeq \widetilde{\mathcal{C}}(\ell_J(-), Q) \simeq Q$$

This implies that $\Lambda\Gamma([\prod_Q]) \simeq \Lambda(Q) \simeq [\prod_Q]$, and hence $[\prod_Q]$ is a fixed point for $\Lambda \dashv \Gamma$. Conversely, if $[F : \mathscr{F} \to \widetilde{\mathcal{C}}]$ is a fixed point then it is isomorphic to $[\prod_{\mathrm{a}_J(\Gamma([F]))}]$ and hence it is étale. The identity above also implies that $\Gamma\Lambda(P) \simeq \Gamma([\prod_{\mathrm{a}_J(P)}]) \simeq \mathrm{a}_J(P)$, and in particular that a presheaf P is a fixed point for $\Lambda \dashv \Gamma$ if and only if it is a J-sheaf.

(iv) It follows from restricting the adjunction $\Lambda \dashv \Gamma$ to its fixed points.

□

7.2 The fundamental adjunction for preorders and locales

We now specialize the two adjunctions

$$[\mathcal{C}^{op}, \mathbf{Set}] \underset{\Gamma_{\mathbf{Cat}/_1\mathcal{C}}}{\overset{\Lambda_{\mathbf{Cat}/_1\mathcal{C}}}{\rightleftarrows}} \mathbf{Cat}/_1\mathcal{C}\ , \quad [\mathcal{C}^{op}, \mathbf{Set}] \underset{\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}}{\overset{\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}}{\rightleftarrows}} \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$$

to the case when $\mathcal{C}$ is a preorder. The existence of a presheaf-bundle adjunction in this context was already hinted by the multiple point-free results of Section 6.1. We recall that a preorder can always be interpreted as a *preorder category* by setting, for every pair of objects X and Y of $\mathcal{C}$, that $\mathrm{Hom}(Y, X) = \{*\}$ if and only if $Y \leq X$. Moreover, functors between preorders are exactly the order preserving maps. This implies that the category of small preorders is a full sub-2-category **Preord** of **Cat**, whose 0-cells are small preorder categories.

The first remark is that the functor $\Lambda_{\mathbf{Cat}/_1\mathcal{C}} : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Cat}/_1\mathcal{C}$, which acts as the Grothendieck construction, factors through $\mathbf{Preord}/\mathcal{C}$:

Lemma 7.2.1. *Consider a preorder $\mathcal{C}$ and a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$: then $\int P$ is a preorder.*

Proof. Consider (X, s) and (Y, t) in $\int P$. A morphism $(Y, t) \to (X, s)$ is indexed by an arrow $y : Y \to X$ such that $P(y)(s) = t$. Hence if $\mathcal{C}$ is a preorder there is at most one such arrow, when $Y \leq X$ and $t = s_{|Y}$, and thus $\int P$ is a preorder. □

In particular, any slice $\mathcal{C}/X$ is a preorder, since it is equivalent to $\int よ(X)$: as a matter of fact, $\mathcal{C}/X$ is precisely the down-set $X\downarrow = \{Y \in \mathcal{C} \mid Y \leq X\}$, with the induced ordering. The following is now an immediate corollary:

Proposition 7.2.2. *For any small preorder category $\mathcal{C}$, there is an adjunction*

$$[\mathcal{C}^{op}, \mathbf{Set}] \underset{\Gamma_{\mathbf{Preord}/_1\mathcal{C}}}{\overset{\Lambda_{\mathbf{Preord}/_1\mathcal{C}}}{\rightleftarrows}} \mathbf{Preord}/_1\mathcal{C} \quad (\perp)$$

where:

- *the notation* $\mathbf{Preord}/_1\mathcal{C}$ *is that of Definition 1.4.1;*
- *both functors* $\Lambda_{\mathbf{Preord}/_1\mathcal{C}}$ *map a presheaf over* $\mathcal{C}$ *to the order-preserving map* $\int P \to \mathcal{C}$*;*
- $\Gamma_{\mathbf{Preord}/_1\mathcal{C}}$ *is a contravariant hom-functor, defined for* $p : \mathcal{D} \to \mathcal{C}$ *as* $\Gamma([p]) := \mathbf{Preord}/_1\mathcal{C}(\mathcal{C}/-, [p])$.

Proof. We can consider the natural equivalence

$$[\mathcal{C}^{op}, \mathbf{Set}](P, \mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/-, [p])) \simeq \mathbf{Cat}/_1\mathcal{C}(\textstyle\int P, [p]),$$

where $P : \mathcal{C}^{op} \to \mathbf{Set}$ and we choose $p : \mathcal{D} \to \mathcal{C}$ to be a functor between preorders: since $\int P$ and every category $\mathcal{C}/X$ are preorders,

$$\mathbf{Cat}/_1\mathcal{C}(\mathcal{C}/-, [p]) = \mathbf{Preord}/_1\mathcal{C}(\mathcal{C}/-, [p])$$

and

$$\mathbf{Cat}/_1\mathcal{C}(\textstyle\int P, [p]) = \mathbf{Preord}/_1\mathcal{C}(\textstyle\int P, [p]),$$

the natural isomorphism above becomes

$$[\mathcal{C}^{op}, \mathbf{Set}](P, \mathbf{Preord}/_1\mathcal{C}(\mathcal{C}/-, [p])) \simeq \mathbf{Preord}/_1\mathcal{C}(\textstyle\int P, [p]).$$

□

If we want to study the fixed points of $\Lambda_{\mathbf{Preord}/_1\mathcal{C}} \dashv \Gamma_{\mathbf{Preord}/_1\mathcal{C}}$ we need to isolate among all morphisms of preorders $Q \to \mathcal{C}$ those that correspond to discrete fibrations, and those that correspond to discrete stacks. To do so we exploit the following technical result:

Lemma 7.2.3. *A functor* $F : \mathcal{A} \to \mathcal{B}$ *is a cloven Grothendieck fibration if and only if for every* A *in* $\mathcal{A}$ *the functor*

$$F_A : \mathcal{A}/A \to \mathcal{B}/F(A)$$

induced by restriction of F *has a right adjoint* F^A *satisfying the identity* $F_A F^A = \mathrm{id}_{\mathcal{B}/F(A)}$. *Moreover,* F *is a discrete fibration if and only if the identity* $F^A F_A = \mathrm{id}_{\mathcal{A}/A}$ *is also verified.*

Proof. The first consequence is well known and can be found for instance in [13, Theorem 2.10]. In particular, F^A acts by mapping any object $[h : X \to F(A)]$ of $\mathcal{B}/F(A)$ to its cartesian lift $[\bar{h} : \bar{X} \to A]$ in $\mathcal{A}/A$. In particular, F is discrete if and only if for every $h : X \to F(A)$ there is exactly one arrow $\bar{h} : \bar{X} \to A$ such that $F(\bar{h}) = h$. This implies that for any arrow $y : Y \to A$ it must hold that $F^A F_A([y]) = \overline{F(y)} = y$, since their images via F_A are both $[F(y)]$. This means that F^A is also a left inverse for F_A. The converse is obvious. □

Corollary 7.2.4. *A monotone map $f : P \to \mathcal{C}$ of preorders is a discrete fibration if and only if for every a in P the monotone map*

$$f_a : a \downarrow \to f(a) \downarrow$$

defined by restriction of f admits a pseudoinverse $f^a : f(a) \downarrow \to a \downarrow$.

Finally, let us suppose that $\mathcal{C}$ is endowed with a Grothendieck topology J, and consider a monotone map of preorders $f : P \to \mathcal{C}$ satisfying the equivalent conditions of Corollary 7.2.4: the corresponding presheaf $\bar{P} : \mathcal{C}^{op} \to \mathbf{Set}$ is defined by mapping each X in $\mathcal{C}$ to the set

$$\bar{P}(X) := \{a \in P \mid f(a) = X\};$$

moreover, for $Y \leq X$ the induced map $\bar{P}(X) \to \bar{P}(Y)$ maps every $a \in \bar{P}(X)$ to the element $f^a(Y)$ of $\bar{P}(Y)$: this because $f^a(Y)$ is (the domain of) the cartesian lift of $Y \leq X$ with codomain a. Expressing the condition for $\bar{P}$ to be a J-sheaf, in terms of matching families and amalgamations, we obtain the characterization of discrete stacks over a preorder site:

Definition 7.2.1. Consider a preorder site $(\mathcal{C}, J)$. We shall call a monotone map of preorders $f : P \to \mathcal{C}$ *étale* if it satisfies the equivalent conditions of Corollary 7.2.4. More explicitly, if for every $a \in P$ there is a monotone map $f^a : f(a) \downarrow \to a \downarrow$ such that for every $b \leq a$ and $x \leq f(a)$ one has $f^a(f(b)) \simeq b$ and $f(f^a(x)) \simeq x$.

We shall call f *J-étale* if it a discrete J-stack. Explicitly, if for every sieve $S \in J(X)$ and every set of elements

$$\{a_Y \in P \mid Y \in S\}$$

satisfying $f(a_Y) = Y$ and such that whenever $Z \leq Y$ then $a_Z = f^{a_Y}(Z)$, there is a unique $a \in P$ such that $f(a) = X$ and $a_Y = f^a(Y)$.

We shall denote by $\mathbf{Etale}(\mathcal{C})$ the full sub-2-category of $\mathbf{Preord}/\mathcal{C}$ of étale posets over $\mathcal{C}$, and by $\mathbf{Etale}(\mathcal{C}, J)$ the full subcategory of $\mathbf{Etale}(\mathcal{C})$ of J-étale preorders over $\mathcal{C}$.

Remark 7.2.1. Our definition of étale map $f : P \to \mathcal{C}$ of preorders is a mild generalization of Definition 5 in [17], where étale maps are defined for posets. Notice that in [17] f is seen as a map $P^{op} \to \mathcal{C}^{op}$, and thus isomorphisms of the upper segments $x \uparrow$ and $f(x) \uparrow$ are required.

The following result is now completely tautological:

Proposition 7.2.5. *The adjunction of Proposition 7.2.2 restricts to the equivalences*

$$[\mathcal{C}^{op}, \mathbf{Set}] \simeq \mathbf{Etale}(\mathcal{C}), \ \mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Etale}(\mathcal{C}, J).$$

We now specialize to preorder categories the presheaf-bundle adjunction

$$\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)} \dashv \Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}.$$

In this case all the relevant data already live at the level of sites, and this simplifies in particular the description of a_J; however, in order to do so we will exploit frames and locales. A *frame* is a distributive lattice with all joins, and such that meets distribute over infinite joins, i.e.

$$a \wedge \left(\bigvee_{i \in I} b_i\right) = \bigvee_{i \in I} (a \wedge b_i).$$

A morphism of frames is an operation-preserving map: with this definition frames form the (2-)category **Frame**. If we see a frame F as a preorder category, its canonical topology corresponds with the join-cover topology: for $a \in F$, a family $\{a_i \leq a \mid i \in I\}$ is J_F^{can}-covering if $\bigvee_{i \in I} a_i = a$. Moreover, homomorphisms of frames are exactly the morphisms of sites with respect to the canonical topologies (see Lemma 7.2.11).

The category **Locale** of *locales* is defined as $\mathbf{Frame}^{op}$. In particular, for a locale L it is custom to denote the corresponding frame by $\mathcal{O}(L)$, while if $g : L \to M$ is an arrow of locales its corresponding morphism of frames is denoted by $g^{-1} : \mathcal{O}(M) \to \mathcal{O}(L)$. This notation reflects the point of view that locales generalize topological spaces, and frames their topologies. Extending the definition of topos of sheaves of a topological space, for a locale L we define its topos of sheaves as $\mathbf{Sh}(L) := \mathbf{Sh}(\mathcal{O}(L), J^{can}_{\mathcal{O}(L)})$. Similarly, for an arrow of locales $g : L \to M$ we denote by $\mathbf{Sh}(g) : \mathbf{Sh}(L) \to \mathbf{Sh}(M)$ the induced geometric morphism $\mathbf{Sh}(g^{-1}) : \mathbf{Sh}(\mathcal{O}(L), J^{can}_{\mathcal{O}(L)}) \to \mathbf{Sh}(\mathcal{O}(M), J^{can}_{\mathcal{O}(M)})$.

A *localic topos* is any topos of sheaves for a locale. Their 2-category is denoted by **LocTopos**. The following result, which describes the connection between locales and toposes, will be of capital importance in the sequel.

Proposition 7.2.6 [28, Chapter IX, Proposition 6.2]. *The 2-functor*

$$\mathbf{Sh}(-) : \mathbf{Locale} \to \mathbf{Topos}$$

is 2-full and faithful: that is, there is a pseudonatural equivalence

$$\mathbf{Locale}(L, M) \simeq \mathbf{Topos}(\mathbf{Sh}(L), \mathbf{Sh}(M)).$$

In particular,

$$\mathbf{Locale} \simeq \mathbf{LocTopos}.$$

In the following we shall also exploit ideals. Given a site $(\mathcal{C}, J)$, a *J-ideal* is a collection $\mathcal{I}$ of objects of $\mathcal{C}$ such that the following conditions hold:

(i) if X belongs to $\mathcal{I}$ and there is an arrow $Y \to X$ then Y belongs to $\mathcal{I}$;

(ii) for any X in $\mathcal{C}$, if there exists $S \in J(X)$ such that for every arrow in f the object $\mathrm{dom}(f)$ belongs to $\mathcal{I}$, then X belongs to $\mathcal{I}$.

In particular, every object X of $\mathcal{C}$ generates a principal J-ideal $\langle X \rangle_J$, which is the smallest J-ideal containing X: an element Y of $\mathcal{C}$ belongs to $\langle X \rangle_J$ if and only if it admits a J-covering $\{W_i \to Y\}$ such that every W_i has an arrow to X. The set of J-ideals of a small site is denoted by $\mathrm{Id}_J(\mathcal{C})$, and when $\mathcal{C}$ is a preorder it is an alternative presentation site for $\mathbf{Sh}(\mathcal{C}, J)$:

Lemma 7.2.7 [4, Theorem 3.1]. *Consider a preorder site $(\mathcal{C}, J)$. Then $\mathrm{Id}_J(\mathcal{C})$ is a frame when ordered by inclusion, and*

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathrm{Id}_J(\mathcal{C})).$$

In particular, every topos of sheaves for a preorder site is localic.

The passage from preorders to locales of ideals also transforms continuous comorphisms into morphisms of locales. More explicitly, if $A : (\mathcal{D}, K) \to (\mathcal{C}, J)$ is a continuous comorphism of preorder sites, the inverse image map $A^{-1}(-)$ restricts to a morphism of frames $g_A^{-1} : \mathrm{Id}_J(\mathcal{C}) \to \mathrm{Id}_K(\mathcal{D})$ such that the geometric morphisms C_A and $Sh(g_A)$ are equivalent.

One can easily see that for a locale L endowed with the canonical topology, a principal J_L^{can}-ideal $\langle a \rangle_J$ is just the down-set $a\downarrow$. Moreover, all ideals are principal, since one can check easily that $\mathcal{I} = (\bigvee \mathcal{I})\downarrow$, and this has the consequence that there is an isomorphism of locales

$$\mathrm{Id}_{J_L^{can}}(L) \simeq L.$$

Next, consider the homomorphism of frames $- \wedge a : L \to a\downarrow$ and denote by $i_a : a\downarrow \hookrightarrow L$ the corresponding arrow of locales. An *open sublocale* of L is any arrow to L isomorphic to one of the kind i_a. The following lemma shows that, given a preorder site $(\mathcal{C}, J)$, for every X in $\mathcal{C}$ the comorphism $\mathcal{C}/X \to \mathcal{C}$ induces an open sublocale inclusion $\mathrm{Id}_{J_X}(\mathcal{C}/X) \to \mathrm{Id}_J(\mathcal{C})$:

Lemma 7.2.8. *Consider a preorder site $(\mathcal{C}, J)$ and an object X in $\mathcal{C}$. The fibration $\mathcal{C}/X \to \mathcal{C}$ induces a morphism of locales $\mathrm{Id}_{J_X}(\mathcal{C}/X) \to \mathrm{Id}_J(\mathcal{C})$ which is an open sublocale inclusion.*

More explicitly, denote by $\mathrm{Sub}(\langle X \rangle_J)$ the collection of sub-J-ideals of the J-principal ideal $\langle X \rangle_J$: it corresponds to the open sublocale $(\langle X \rangle_J)\downarrow$ of the locale $\mathrm{Id}_J(\mathcal{C})$. Then $\mathrm{Id}_{J_X}(\mathcal{C}/X)$ and $\mathrm{Sub}(\langle X \rangle_J)$ are isomorphic as sublocales of $\mathrm{Id}_J(\mathcal{C})$.

Proof. First of all, we remark that under the equivalence $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathrm{Id}_J(\mathcal{C}))$, every representable $\ell_J(X)$ corresponds to the subterminal object $\ell(\langle X \rangle_J)$ of $\mathbf{Sh}(\mathrm{Id}_J(\mathcal{C}))$ (cf. Remark 2.6 (a) of [4]). Therefore

$$\mathbf{Sh}(\mathrm{Id}_{J_X}(\mathcal{C}/X)) \simeq \mathbf{Sh}(\mathcal{C}/X, J_X) \simeq \mathbf{Sh}(\mathcal{C}, J)/\ell_J(X) \simeq \mathbf{Sh}(\mathrm{Id}_J(\mathcal{C}))/\ell(\langle X \rangle_J).$$

The latter topos, which is still localic, is the topos of sheaves for the locale of its subterminal objects. But subterminal objects of $\mathbf{Sh}(\mathrm{Id}_J(\mathcal{C}))/\ell(\langle X \rangle_J)$ correspond to the subobjects of $\ell(\langle X \rangle_J)$, which in turn correspond to the sublocales of $\langle X \rangle_J$, and thus we have

$$\mathbf{Sh}(\mathrm{Id}_{J_X}(\mathcal{C}/X)) \simeq \mathbf{Sh}(\mathrm{Sub}(\langle X \rangle_J)),$$

implying $\mathrm{Id}_{J_X}(\mathcal{C}/X) \simeq \mathrm{Sub}(\langle X \rangle_J)$. □

Remark 7.2.2. Let us denote by $p : \mathcal{C}/X \to \mathcal{C}$ the usual fibration, then the isomorphism of the previous lemma can be made explicit as follows:

$$R : \mathrm{Sub}(\langle X \rangle_J) \to \mathrm{Id}_{J_X}(\mathcal{C}/X),\ R(\mathcal{K}) := p^{-1}(\mathcal{K}) = \{[Y \leq X] \mid Y \in \mathcal{K}\};$$

$$R^{-1} : \mathrm{Id}_{J_X}(\mathcal{C}/X) \to \mathrm{Sub}(\langle X \rangle_J),\ R^{-1}(\mathcal{I}) := \bigcup_{\substack{\mathcal{J} \in \mathrm{Id}_J(\mathcal{C}),\\ \mathcal{J} \subseteq \langle X \rangle_J \\ p^{-1}(\mathcal{J}) \subseteq \mathcal{I}}} \mathcal{J}.$$

Finally, let us recall that there is a notion of *étale morphism of locales*. A morphism $f : L \to M$ of locales is said to be étale if there exist elements x_i in L such that $\bigvee_i x_i = \top_L$, and elements $y_i \in M$ such that f restricts to isomorphisms of open sublocales $x_i \downarrow \xrightarrow{\sim} y_i \downarrow$ (see [3, Definition 1.7.1]). When denoting by $\mathbf{Locale}^{étale}$ the category of locales and their étale morphisms, the equivalence of Proposition 7.2.6 induces an equivalence

$$\mathbf{Locale}^{étale}/F(L, M) \simeq \mathbf{Topos}^{étale}/_1\mathbf{Sh}(F)(\mathbf{Sh}(L), \mathbf{Sh}(M))$$

for any three locales F, L and M.

With all these ingredients, we are now ready to formulate the presheaf-bundle adjunction for preorder sites:

Proposition 7.2.9. *For any small preorder site $(\mathcal{C}, J)$ there is an adjunction*

$$[\mathcal{C}^{op}, \mathbf{Set}] \underset{\Gamma_{\mathbf{Locale}/_1 \mathrm{Id}_J(\mathcal{C})}}{\overset{\Lambda_{\mathbf{Locale}/_1 \mathrm{Id}_J(\mathcal{C})}}{\rightleftarrows}} \mathbf{Locale}/_1 \mathrm{Id}_J(\mathcal{C}), \quad (\bot)$$

where:

- $\Lambda_{\mathbf{Locale}/_1 \mathrm{Id}_J(\mathcal{C})}$ *maps a presheaf* $P : \mathcal{C}^{op} \to \mathbf{Set}$ *to the morphism of locales* $g_{\pi_P} : \mathrm{Id}_{J_P}(\int P) \to \mathrm{Id}_J(\mathcal{C})$ *whose corresponding morphism of frames is* π_P^{-1} *(induced by the* (J_P, J)*-continuous comorphism of sites* $\pi_P : \int P \to \mathcal{C}$*);*

- $\Gamma_{\mathbf{Locale}/_1 \mathrm{Id}_J(\mathcal{C})}$ *acts as a contravariant hom-functor, mapping a morphism of locales* $g : L \to \mathrm{Id}_J(\mathcal{C})$ *to*

  $$\mathbf{Locale}/_1 \mathrm{Id}_J(\mathcal{C})(\mathrm{Sub}(\langle - \rangle_J), L) : \mathcal{C}^{op} \to \mathbf{Set}.$$

  Explicitly, the value of $\Gamma(L)$ *at* $X \in \mathcal{C}$ *is the set of sections of* g *over the open sublocale* $\mathrm{Sub}(\langle X \rangle_J) \hookrightarrow \mathrm{Id}_J(\mathcal{C})$*, i.e. the locale morphisms* $\mathrm{Sub}(\langle X \rangle_J) \to L$ *making the following diagram commutative:*

  $$\begin{array}{ccc} \mathrm{Sub}(\langle X \rangle_J) & \longrightarrow & L \\ & \searrow & \downarrow{\scriptstyle g} \\ & & \mathrm{Id}_J(\mathcal{C}). \end{array}$$

Moreover, the composite functor $\Gamma_{\mathbf{Locale}/_1 \mathrm{Id}_J(\mathcal{C})} \circ \Lambda_{\mathbf{Locale}/_1 \mathrm{Id}_J(\mathcal{C})}(P)$ *corresponds to the sheafification* $\mathrm{a}_J(P)$*: for every* X *in* $\mathcal{C}$ *we have*

$$\mathrm{a}_J(P)(X) := \mathbf{Locale}/_1 \mathrm{Id}_J(\mathcal{C})(\mathrm{Sub}(\langle X \rangle_J), \mathrm{Id}_{J_P}(\textstyle\int P)).$$

The adjunction restricts to an equivalence

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Locale}^{\acute{e}tale} / \mathrm{Id}_J(\mathcal{C}).$$

Proof. We can start by remarking that the functor

$$\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)} : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$$

takes values inside the smaller slice category $\mathbf{LocTopos}/_1\mathbf{Sh}(\mathcal{C}, J)$: this is true since every site $(\int P, J_P)$ is a preorder site, and hence the topos $\mathbf{Sh}(\int P, J_P)$ is localic. On the other hand, given a localic topos $\mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$, the legs of every cocone in $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), \mathscr{E})$, and the geometric morphism $\mathbf{Sh}(\int P, J_P) \to \mathscr{E}$ induced by the universal property of colimits, will be geometric morphisms of localic toposes too. This means that the pseudonatural equivalence

$$\begin{aligned}[][\mathcal{C}^{op}, \mathbf{Set}](P, \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), \mathscr{E})) \simeq \\ \simeq \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\textstyle\int P, J_P), \mathscr{E})\end{aligned}$$

restricts to an equivalence

$$\begin{aligned}[][\mathcal{C}^{op}, \mathbf{Set}](P, \mathbf{LocTopos}/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), \mathscr{E})) \simeq \\ \simeq \mathbf{LocTopos}/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\textstyle\int P, J_P), \mathscr{E})\end{aligned}$$

and thus we have an adjunction

$$[\mathcal{C}^{op}, \mathbf{Set}] \underset{\Gamma_{\mathbf{LocTopos}/_1\mathbf{Sh}(\mathcal{C},J)}}{\overset{\Lambda_{\mathbf{LocTopos}/_1\mathbf{Sh}(\mathcal{C},J)}}{\rightleftarrows}} \mathbf{LocTopos}/_1\mathbf{Sh}(\mathcal{C}, J).$$

Since $(\mathcal{C}, J)$ is a preorder site we have

$$\mathbf{LocTopos}/_1\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{LocTopos}/_1\mathbf{Sh}(\mathrm{Id}_J(\mathcal{C})) \simeq \mathbf{Locale}/_1\,\mathrm{Id}_J(\mathcal{C})$$

and we can conclude that there is an equivalence

$$[\mathcal{C}^{op}, \mathbf{Set}](P, \mathbf{Locale}/_1\,\mathrm{Id}_J(\mathcal{C})(\mathrm{Id}_{J_{(-)}}(\mathcal{C}/-), L)) \simeq \\ \simeq \mathbf{Locale}/_1\,\mathrm{Id}_J(\mathcal{C})(\mathrm{Id}_{J_P}(\textstyle\int P), L)$$

proving that $\Lambda_{\mathbf{Locale}/_1\,\mathrm{Id}_J(\mathcal{C})} \dashv \Gamma_{\mathbf{Locale}/_1\,\mathrm{Id}_J(\mathcal{C})}$. Lemma 7.2.8 provides us the isomorphism $\mathrm{Id}_{J_{(-)}}(\mathcal{C}/-) \simeq \mathrm{Sub}(\langle -\rangle_J)$, justifying the description of $\Gamma_{\mathbf{Locale}/_1\,\mathrm{Id}_J(\mathcal{C})}$ in the claim of the theorem.

Finally, the chain of known equivalences

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Topos}^{\acute{e}tale}/_1\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Locale}^{\acute{e}tale}/\,\mathrm{Id}_J(\mathcal{C})$$

provides the restriction of the adjunction to sheaves. □

When $(\mathcal{C}, J) = (L, J_L^{can})$ with L a locale, we can exploit the isomorphism $\mathrm{Id}_{J_L^{can}}(L) \cong L$ to formulate the adjunction above in the localic case:

Corollary 7.2.10. *For any locale L there is an adjunction*

$$[\mathcal{O}(L)^{op}, \mathbf{Set}] \underset{\Gamma_{\mathbf{Locale}/_1 L}}{\overset{\Lambda_{\mathbf{Locale}/_1 L}}{\rightleftarrows}} \mathbf{Locale}/_1 L \ , \quad (\Lambda \dashv \Gamma)$$

where:

- $\Lambda_{\mathbf{Locale}/_1 L}$ *maps a presheaf $P : \mathcal{O}(L)^{op} \to \mathbf{Set}$ to the morphism of locales $g_{\pi_P} : \mathrm{Id}_{J_P}(\int P) \to L$, where $\pi_P : \int P \to \mathcal{C}$ and J_P is Giraud's topology induced by the canonical topology J_L^{can}: g_{π_P} corresponds to the homomorphism of frames $\pi_P^{-1} : L \to \mathrm{Id}_{J_P}(\int P)$;*

- $\Gamma_{\mathbf{Locale}/_1 L}$ *acts as a contravariant hom-functor, mapping a morphism of locales $g : M \to L$ to*

  $$\mathbf{Locale}/_1 L((-)\downarrow, M) : \mathcal{O}(L)^{op} \to \mathbf{Set}.$$

  Explicitly, the value of $\Gamma(M)$ at a certain $a \in L$ is the set of sections of $g : M \to L$ over the open sublocale $a\downarrow \hookrightarrow L$.

Moreover, the composite $\Gamma_{\mathbf{Locale}/_1 L}\Lambda_{\mathbf{Locale}/_1 L}(P)$ corresponds to the sheafification $\mathrm{a}_{J_L^{can}}(P)$, and for any a in L we have

$$\mathrm{a}_{J_L^{can}}(P)(a) := \mathbf{Locale}/_1 L(a\downarrow, \mathrm{Id}_{J_P}(\textstyle\int P)).$$

Remarks 7.2.3.

(i) In Section 7.3 we shall describe a general context in which the fundamental adjunction restricts from toposes to sites and their morphisms or comorphisms, mimicking the behaviour of the adjunction in the preorder case.

(ii) The existence of the adjunction of Corollary 7.2.10 is implicit in the results of [3, Section 2.6] and in the exercises from 9 to 12 of [28, Chapter IX], though in both cases the focus is on the equivalence between sheaves over the locale and étale mappings to it. We also remark that, relating the equivalence $\mathbf{Sh}(L) \simeq \mathbf{Etale}/L$ to $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Etale}(\mathcal{C}, J)$, which holds for any preorder site $(\mathcal{C}, J)$, we can conclude that a monotone map of preorders $f : P \to \mathcal{C}$ is J-étale if and only if the corresponding homomorphism of frames $f^{-1} : \mathrm{Id}_J(\mathcal{C}) \to \mathrm{Id}_{J_P}(\int P)$ is an étale arrow of locales.

(iii) We can now see how exactly the topological framework connects to the localic adjunction of the previous corollary. In Proposition 6.1.7 we have seen that for a topological space X and a presheaf $P \in \mathbf{Psh}(X)$ it holds that, for any open U of X,

$$\mathrm{a}_J(P)(U) \simeq \mathbf{Locale}/\mathcal{O}(X)(\mathcal{O}(U), \mathcal{O}(E_P)),$$

where E_P is the étale bundle associated to P. By the last result however we have that

$$\mathrm{a}_J(P)(U) \simeq \mathbf{Locale}/\mathcal{O}(X)(\mathcal{O}(U), \mathrm{Id}_{J_P}(\textstyle\int P)),$$

where $(\int P, J_P)$ is the Giraud site associated to P. The apparent difference is immediately explained by noticing that the map

$$\mathrm{Id}_{J_P}(\textstyle\int P) \to \mathcal{O}(E_P),\ \mathcal{I} \mapsto \displaystyle\bigcup_{(U,s)\in\mathcal{I}} \dot{s}(U)$$

provides an isomorphism between the two locales $\mathcal{O}(E_P)$ and $\mathrm{Id}_{J_P}(\int P)$. This is in fact an extension of the map f_P we defined in Proposition 6.1.5, and indeed that result entails the isomorphism $\mathrm{Id}_{J_P}(\int P) \simeq \mathcal{O}(E_P)$, since the equivalence

$$\mathbf{Sh}(E_P) \simeq \mathbf{Sh}(\textstyle\int P, J_P)$$

of localic toposes restrict to an isomorphism of their locales of subterminal objects. Thus, even though classically one defines the sheafification of $P \in \mathbf{Psh}(X)$ as the local sections of the étale bundle $E_P \to X$, these considerations show explicitly that there is no need to work in the topological context, since all the relevant information lives at the localic level. Finally, we remark that this last consideration is essentially the content of [3, Proposition 2.5.4], where nonetheless there is no explicit reference to the locale $\mathrm{Id}_{J_P}(\int P)$ but instead to the locale of closed subpresheaves of P.

We conclude this section by remarking that our description of the sheafification in the localic case is naturally site-theoretic, and that in the case of topological spaces sheafification can also be described using comorphisms of sites. First of all, the following holds:

Lemma 7.2.11. *Consider two frames L and M. Frame homomorphisms $L \to M$ correspond to morphisms of sites $(L, J_L^{can}) \to (M, J_M^{can})$. In other words, the 2-functor*

$$\mathbf{Frame} \xhookrightarrow{L \mapsto (L, J_L^{can})} \mathbf{Site}$$

is 2-fully faithful.

Proof. Since a frame is a category with finite limits, a functor $A : L \to M$ is a morphism of sites if and only if it preserves finite limits and is cover-preserving: this means that it preserves finite meets and arbitrary joins, i.e. it is a homomorphism of frames. □

This implies that for a locale L, an element a of L and a presheaf $P \in \mathbf{Psh}(L)$ we have

$$\begin{aligned} \mathrm{a}(P)(a) &= \mathbf{Locale}/L(a\downarrow, \mathrm{Id}_{J_P}(\textstyle\int P)) \\ &= (L, J_L^{can})/\mathbf{Site}((\mathrm{Id}_{J_P}(\textstyle\int P), J^{can}_{\mathrm{Id}_{J_P}(\int P)}), (a\downarrow, J^{can}_{a\downarrow})). \end{aligned}$$

Suppose now that $L = \mathcal{O}(X)$ for a topological space X. Proposition 6.1.3 showed that the local sections of the étale bundle $\pi : E_P \to X$ are open maps, and using Lemma 6.1.4 we have that a section $s : U \to E_P$ (triangle on the left) induces *two* commutative triangles:

$$\begin{array}{ccc} U & \xrightarrow{s} & E_P \\ & \searrow^{i_U} & \downarrow \pi \\ & & X \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathcal{O}(U) & \xleftarrow{s^{-1}} & \mathcal{O}(E_P) \\ & \nwarrow^{i_U^{-1}} & \uparrow \pi^{-1} \\ & & \mathcal{O}(X) \end{array}, \quad \begin{array}{ccc} \mathcal{O}(U) & \xrightarrow{s_!} & \mathcal{O}(E_P) \\ & \searrow^{(i_U)_!} & \downarrow \pi_! \\ & & \mathcal{O}(X) \end{array}$$

We already know that the middle triangle is a triangle of morphisms of sites, and by applying [6, Proposition 3.14], we have that the triangle on the right is a diagram of comorphisms of sites. In fact, more can be said:

Proposition 7.2.12. *For any section $s : U \to E_P$, the comorphism*

$$s_! : (\mathcal{O}(U), J^{can}_{\mathcal{O}(U)}) \to (\mathcal{O}(E_P), J^{can}_{\mathcal{O}(E_P)})$$

is $(J^{can}_{\mathcal{O}(U)}, J^{can}_{\mathcal{O}(E_P)})$-continuous.

Proof. Consider a sheaf W in $\mathbf{Sh}(E_P)$: we want to show that $W \circ s_!^{op}$ is a sheaf over U. To do so, consider a family of opens $\{V_i \subseteq U \mid i \in I\}$ and take for every i an element $x_i \in W \circ s_!^{op}(V_i) = W(s(V_i))$ so that for every open $Z \subseteq V_i \cap V_j$ one has $x_{i|s(Z)} = x_{j|s(Z)}$: we will prove that the elements

x_i are also a matching family for W on the open covering $\{s(V_i) \mid i \in I\}$, and since W is a sheaf this will imply the existence of an amalgamation $x \in W(\cup_i s(V_i)) = W(s(\cup_i V_i))$. To see that the x_i's are a matching family for W, we can restrict to considering the basic opens $\dot{r}(Z)$ of E_P. First of all, if $\dot{r}(Z) \subseteq s(V_i)$ and $z \in Z$, there exists $y \in V_i$ such that $r_z = s(y)$, and thus $z = \pi(r_z) = \pi s(y) = y$: therefore not only $Z \subseteq V_i$, but actually $\dot{r} = s_{|Z}$. If $\dot{r}(Z) \subseteq s(V_i) \cap s(V_j)$, we have that $x_{i|\dot{r}(Z)} = x_{i|s(Z)} = x_{j|s(Z)} = x_{j|\dot{r}(Z)}$, where the second equality holds by the matching condition for the x_i's. This proves that the family of the x_i is a matching family for the sheaf W and the covering $\{s(V_i) \mid i \in I\}$, and thus it admits the amalgamation $x \in W(\cup s(V_i))$ we needed. □

Lemma 6.1.7 showed that *any* frame homomorphism $f : \mathcal{O}(E_P) \to \mathcal{O}(U)$ satisfying $i_U^{-1} = f \circ \pi^{-1}$ is of the form s^{-1} for some section $s : U \to E_P$: this implies that it admits a left adjoint $s_!$ which by the last result is a $(J^{can}_{\mathcal{O}(U)}, J^{can}_{\mathcal{O}(E_P)})$-continuous comorphism of sites. Conversely, consider a $(J^{can}_{\mathcal{O}(U)}, J^{can}_{\mathcal{O}(E_P)})$-continuous comorphism of sites $B : \mathcal{O}(U) \to \mathcal{O}(E_P)$: then by [6, Proposition 4.11(iii)] it is cover-preserving, i.e. preserves arbitrary colimits, and thus it admits a right adjoint which is a morphism of sites $f : \mathcal{O}(E_P) \to \mathcal{O}(U)$. We can thus conclude the following:

Proposition 7.2.13. *Consider a topological space X, an open subset $U \subseteq X$ and a presheaf $P \in \mathbf{Psh}(X)$: then there are natural isomorphisms*

$$\begin{aligned}
\mathrm{a}_J(P)(U) &:= \mathbf{Top}/_1 X(U, E_P) \\
&\simeq \mathbf{Locale}/\mathcal{O}(X)(\mathcal{O}(U), \mathcal{O}(E_P)) \\
&\simeq (\mathcal{O}(X), J^{can}_{\mathcal{O}(X)})/_1\mathbf{Site}((\mathcal{O}(E_P), J^{can}_{\mathcal{O}(E_P)}), (\mathcal{O}(U), J^{can}_{\mathcal{O}(U)})) \\
&\simeq \mathbf{Com}_{cont}/_1(\mathcal{O}(X), J^{can}_{\mathcal{O}(X)})((\mathcal{O}(U), J^{can}_{\mathcal{O}(U)}), (\mathcal{O}(E_P), J^{can}_{\mathcal{O}(E_P)})).
\end{aligned}$$

Remark 7.2.4. The last line presents a slight abuse of notation. We are considering the slice category of continuous comorphisms of sites $\mathcal{O}(U) \to \mathcal{O}(E_P)$ over $\mathcal{O}(X)$, but the functor $\pi_! : \mathcal{O}(E_P) \to \mathcal{O}(X)$ does not belong to $\mathbf{Com}_{cont}$ since it is not continuous in general. On the other hand, the comorphism $(i_U)_!$ is continuous, since it is precisely the discrete fibration $p_U : \mathcal{O}(X)/U \to \mathcal{O}(X)$.

7.2.1 The fundamental adjunction in the language of internal locales

The presheaf-bundle adjunction for locales also admits a formulation in the language of internal locales, which has the perk that the base locale is 'absorbed' by the topos we are working in.

First of all, the notion of frame can obviously be interpreted in any topos. The part of the theory regarding the finitary structure is easily interpreted.

An internal bounded meet-semilattice in a topos $\mathscr{E}$ will be an object L of $\mathscr{E}$ provided with three arrows

$$1 \xrightarrow{\top} L,\ 1 \xrightarrow{\bot} L,\ L \times L \xrightarrow{\wedge} L,$$

the interpretations of respectively the top element, the bottom element and the meet operation, making the obvious diagrams commutative. As per the infinitary operation of arbitrary joins, it can be interpreted as an arrow

$$\bigvee : \mathscr{P}(L) \to L,$$

where $\mathscr{P}(L) = \Omega^L$ is the power object of L (see Appendix A): in the internal logic of $\mathscr{E}$, we think of the arrow $\bigvee$ as mapping any $S \subseteq L$ to its join $\bigvee S \in L$. By writing suitable commutative diagrams expressing the usual axioms, we obtain the notion of an *internal frame* L in $\mathscr{E}$. A *homomorphism of internal frames* $f : L \to M$ is an arrow commuting with all the operations. We shall denote by $\mathbf{Frame}(\mathscr{E})$ the category of internal frames of a topos $\mathscr{E}$, and by $\mathbf{Loc}(\mathscr{E}) = \mathbf{Frame}(\mathscr{E})^{op}$ its *category of internal locales*.

The following result, first appeared as Proposition 2 in [22, Chapter VI, §3] (cf. also [21, Section C1.6]), shows that internal locales of a localic topos can be externalized:

Proposition 7.2.14. *For any locale L, there is an equivalence of categories*

$$H : \mathbf{Locale}/_1 L \xrightarrow{\sim} \mathbf{Loc}(\mathbf{Sh}(L)).$$

In particular, to a morphism of frames $f : L \to M$ is associated the internal locale

$$H(f) : L^{op} \to \mathbf{Set},\ H(f)(a) = \{m \in M \mid m \leq f(a)\}.$$

Using the equivalence H we can reformulate the adjunction of Proposition 7.2.2. To do so, take a preorder site $(\mathcal{C}, J)$ and consider the composite

$$\bar{\Lambda} : [\mathcal{C}^{op}, \mathbf{Set}] \xrightarrow{\Lambda_{\mathbf{Locale}/1\,\mathrm{Id}_J(\mathcal{C})}} \mathbf{Locale}/_1\, \mathrm{Id}_J(\mathcal{C}) \xrightarrow{H} \mathbf{Loc}\,(\mathbf{Sh}(\mathcal{C}, J)) :$$

the first functor maps a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ to the morphism of locales $g_{p_P} : \mathrm{Id}_{J_P}(\int P) \to \mathrm{Id}_J(\mathcal{C})$, i.e. to the homomorphism of frames $p_P^{-1} : \mathrm{Id}_J(\mathcal{C}) \to \mathrm{Id}_{J_P}(\int P)$; the second functor maps g_{p_P} to a presheaf $\bar{\Lambda}(P) : \mathcal{C}^{op} \to \mathbf{Set}$ which is an internal locale of $\mathbf{Sh}(\mathcal{C}, J)$. By the definition of the functor H, we have that for Z in $\mathcal{C}$

$$\bar{\Lambda}(P)(Z) := \{\mathcal{I} \in \mathrm{Id}_{J_P}(\textstyle\int P) \mid p_P(\mathcal{I}) \subseteq \langle Z \rangle_J\},$$

where we use the fact that under the equivalence $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathrm{Id}_J(\mathcal{C}))$ the element Z corresponds to the principal J-ideal $\langle Z \rangle_J$. In particular, when

$P = よ(X)$ we can exploit Lemma 7.2.8 to obtain the following chain of natural isomorphisms:

$$\begin{aligned}\bar{\Lambda}(P)(Z) &:= \{\mathcal{I} \in \mathrm{Id}_{J_X}(\mathcal{C}/X) \mid p_X(\mathcal{I}) \subseteq \langle Z \rangle_J\} \\ &\simeq \{\mathcal{I} \in \mathrm{Id}_J(\mathcal{C}) \mid \mathcal{I} \subseteq \langle X \rangle_J \cap \langle Z \rangle_J\} \\ &= \mathrm{Sub}(\langle X \rangle_J \cap \langle Z \rangle_J).\end{aligned}$$

We end up with the following result:

Proposition 7.2.15. *Given a preorder site $(\mathcal{C}, J)$, there is an adjunction*

$$[\mathcal{C}^{op}, \mathbf{Set}] \underset{\Gamma_{\mathbf{Loc}(\mathbf{Sh}(\mathcal{C},J))}}{\overset{\Lambda_{\mathbf{Loc}(\mathbf{Sh}(\mathcal{C},J))}}{\rightleftarrows}} \mathbf{Loc}(\mathbf{Sh}(\mathcal{C}, J)) \qquad (\perp)$$

which acts as follows:

- *The functor $\Lambda_{\mathbf{Loc}(\mathbf{Sh}(\mathcal{C},J))}$ maps a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ to the internal locale acting for any X in $\mathcal{C}$ as*

  $$\Lambda_{\mathbf{Loc}(\mathbf{Sh}(\mathcal{C},J))}(P)(X) = \bar{\Lambda}(P)(X) = \{\mathcal{I} \in \mathrm{Id}_{J_P}(\textstyle\int P) \mid \pi_P(\mathcal{I}) \subseteq \langle X \rangle_J\}$$

- *The functor $\Gamma_{\mathbf{Loc}(\mathbf{Sh}(\mathcal{C},J))}$ acts by mapping $L \in \mathbf{Loc}(\mathbf{Sh}(\mathcal{C}, J))$ to the presheaf*

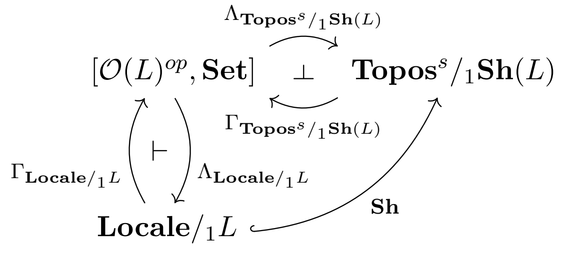
  $$\Gamma_{\mathbf{Loc}(\mathbf{Sh}(\mathcal{C},J))}(L) : \mathcal{C}^{op} \to \mathbf{Set}$$

  which acts as a contravariant hom-functor of internal locales:

  $$\Gamma_{\mathbf{Loc}(\mathbf{Sh}(\mathcal{C},J))} : X \mapsto \mathbf{Loc}(\mathbf{Sh}(\mathcal{C}, J))(\mathrm{Sub}(\langle - \rangle_J \cap \langle X \rangle_J), L).$$

7.3 A site-restrictibility condition for the fundamental adjunction

In Section 7.2 we have shown that, given a locale L, its presheaf-bundle adjunction $\Lambda_{\mathbf{Topos}^s/\mathbf{Sh}(L)} \dashv \Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(L)}$ induces an adjunction $\Lambda_{\mathbf{Locale}/_1 L} \dashv \Gamma_{\mathbf{Locale}/_1 L}$ (Corollary 7.2.10): we may think of this as the site-theoretic restriction of the *topos*-theoretic presheaf-bundle adjunction. We represent both adjunctions in the same diagram, as follows, where $\mathbf{Sh}$ is instead the functor mapping any morphism of locales $M \to L$ to its corresponding geometric morphism $\mathbf{Sh}(M) \to \mathbf{Sh}(L)$:



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Analysing the proof of the results in Section 7.2, essentially what made it possible to obtain from the topos-theoretic adjunction the site-theoretic one is the fact that the left adjoint $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(L)}$ factors through the functor $\mathbf{Sh} : \mathbf{Locale}/L \to \mathbf{Topos}^s/_1\mathbf{Sh}(L)$, which moreover is full and faithful: this is essentially an application of Lemma D.4. Therefore, for any special class of sites and (co)morphisms satisfying the same restrictibility property of Λ we can apply the same strategy and obtain the following results:

Corollary 7.3.1. *Consider a site $(\mathcal{C}, J)$ and suppose that there is a functor $\mathcal{A} \to (\mathcal{C}, J)/\mathbf{Site}$ from any category $\mathcal{A}$ such that the following hold:*

- *The composite functor $\mathcal{A}^{op} \to ((\mathcal{C}, J)/\mathbf{Site})^{op} \to \mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J)$ factors through $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ so that $i : \mathcal{A}^{op} \to \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ is full and faithful;*
- *there is a functor $\bar{\Lambda} : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathcal{A}$ such that $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)} \simeq i\bar{\Lambda}$.*

$$\begin{array}{ccc} ((\mathcal{C}, J)/\mathbf{Site})^{op} & \xrightarrow{\mathbf{Sh}(-)} & \mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J) \\ \uparrow & & \uparrow \\ \mathcal{A}^{op} & \underset{i}{\hookrightarrow} & \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J) \end{array}$$

$$\begin{array}{ccc} [\mathcal{C}^{op}, \mathbf{Set}] & \xrightarrow{\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}} & \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J) \\ & \searrow^{\bar{\Lambda}} & \uparrow i \\ & & \mathcal{A}^{op} \end{array}$$

Then $\bar{\Lambda}$ admits a right adjoint $\bar{\Gamma}$, and $\mathrm{a}_J(P) \simeq \bar{\Gamma}\bar{\Lambda}(P)$.

Corollary 7.3.2. *Consider a site $(\mathcal{C}, J)$ and suppose that there is a functor $\mathcal{A} \to \mathbf{Com}/(\mathcal{C}, J)$ from any category $\mathcal{A}$ such that the following hold:*

- *The composite $\mathcal{A} \to \mathbf{Com}/(\mathcal{C}, J) \to \mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J)$ factors through $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ so that $i : \mathcal{A} \to \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ is full and faithful;*
- *there is a functor $\bar{\Lambda} : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathcal{A}$ such that $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)} \simeq i\bar{\Lambda}$.*

$$\begin{array}{ccc} \mathbf{Com}/(\mathcal{C}, J) & \xrightarrow{C_{(-)}} & \mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J) \\ \uparrow & & \uparrow \\ \mathcal{A} & \underset{i}{\hookrightarrow} & \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J) \end{array}$$

$$\begin{array}{ccc} [\mathcal{C}^{op}, \mathbf{Set}] & \xrightarrow{\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}} & \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J) \\ & \searrow^{\bar{\Lambda}} & \uparrow i \\ & & \mathcal{A}. \end{array}$$

Then $\bar{\Lambda}$ admits a right adjoint $\bar{\Gamma}$, and $\mathrm{a}_J(P) \simeq \bar{\Gamma}\bar{\Lambda}(P)$.

Remark 7.3.1. Consider a preorder site $(\mathcal{C}, J)$: we can take as the functor $\mathcal{A} \to (\mathcal{C}, J)/\textbf{Site}$ the functor

$$\mathrm{Id}_J(\mathcal{C})/\textbf{Frame} \to (\mathrm{Id}_J(\mathcal{C}), J^{can}_{\mathrm{Id}_J(\mathcal{C})})/\textbf{Site}$$

mapping $f : \mathrm{Id}_J(\mathcal{C}) \to L$ to the same arrow seen as a morphism of sites $f : (\mathrm{Id}_J(\mathcal{C}), J^{can}_{\mathrm{Id}_J(\mathcal{C})}) \to (L, J^{can}_L)$ (cf. Lemma 7.2.11), we obtain again the adjunctions of Proposition 7.2.2 and Corollary 7.2.10.

Chapter 8

A site-theoretic interpretation of sheafification

The presheaf-bundle adjunction for sites of Section 7.1 shows that for any essentially small site $(\mathcal{C}, J)$ the sheafification functor $\mathrm{a}_J : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ can be described as the composite

$$[\mathcal{C}^{op}, \mathbf{Set}] \xrightarrow{\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}} \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J) \xrightarrow{\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C},J)}} [\mathcal{C}^{op}, \mathbf{Set}].$$

I.e. for any presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ and X in $\mathcal{C}$ we have

$$\mathrm{a}_J(P)(X) \simeq \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/X, J_X), \mathbf{Sh}(\textstyle\int P, J_P)).$$

The present chapter combines this description of a_J above with the site-theoretic presentation of relative geometric morphisms developed in Section 1.4. Elements of the sheafification can be understood using morphisms and comorphisms of sites, and in particular we can interpret them as 'locally matching families' of morphisms of fibrations. In the last section we will focus on topological spaces, and show the connections between the various possible interpretations of the sheafification functor. The content of this chapter corresponds to Section 6.4 of [8].

8.1 Sheafification via morphisms of sites

We begin by describing the elements in $\mathrm{a}_J(P)(X)$ using morphisms of sites, via the following 1-categorical variation of Theorem 1.4.3:

Proposition 8.1.1. *Consider a comorphism of sites $p : (\mathcal{D}, K) \to (\mathcal{C}, J)$ and a geometric morphism $E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$. Denote by A_E the morphism of sites $(\mathcal{C}, J) \to (\mathscr{E}, J^{can}_{\mathscr{E}})$ corresponding to E. Geometric morphisms in* $\mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J)([E], [C_p])$*, i.e. equivalence classes of geometric morphisms*

F making the diagram

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{F} & \mathbf{Sh}(\mathcal{D}, K) \\ {\scriptstyle E}\downarrow & \swarrow {\scriptstyle C_p} & \\ \mathbf{Sh}(\mathcal{C}, J) & & \end{array}$$

commutative up to isomorphism, are in bijective correspondence with equivalence classes of morphisms of sites

$$A_F : (\mathcal{D}, K) \to (\mathscr{E}, J_{\mathscr{E}}^{can})$$

such that there is a natural transformation $\bar{\varphi} : A_F \Rightarrow A_E \circ p$ *inducing another natural transformation* $\tilde{\varphi} : F^* \circ \mathrm{a}_J \Rightarrow E^* \circ \mathrm{a}_J \circ \mathrm{lan}_{p^{op}}$ *so that the composite*

$$F^* \circ \mathrm{a}_K \circ p^* \xrightarrow{\tilde{\varphi} \circ p^*} E^* \circ \mathrm{a}_J \circ \mathrm{lan}_{p^{op}} \circ p^* \xrightarrow{E^* \circ \mathrm{a}_J \circ \varepsilon} E^* \circ \mathrm{a}_J$$

is invertible (where ε *is the counit of* $\mathrm{lan}_{p^{op}} \dashv p^*$*).*

Proof. Theorem 1.4.3 showed the equivalence

$$\mathbf{Topos} /\!\!/ \mathbf{Sh}(\mathcal{C}, J)([E], [C_p]) \simeq \mathbf{Site}((\mathcal{D}, K), (\mathscr{E}, J_{\mathscr{E}}^{can}))/E^* \ell_J p,$$

stating that a geometric morphism $F : \mathscr{E} \to \mathbf{Sh}(\mathcal{D}, K)$ endowed with a natural transformation $\varphi : F^* \circ C_p^* \Rightarrow E^*$ corresponds to a morphism $A_F : (\mathcal{D}, K) \to (\mathscr{E}, J_{\mathscr{E}}^{can})$ of sites endowed with a natural transformation $\bar{\varphi} : A_F \Rightarrow A_E \circ p$. Using Remark 1.4.2, we see that φ is invertible if and only if $\bar{\varphi} : A_F \Rightarrow A_E p$ satisfies the condition in the claim. Finally, let us consider objects of

$$\mathbf{Topos}/_1 \mathbf{Sh}(\mathcal{C}, J)([E], [C_p]) :$$

two equivalent geometric morphisms $F \cong F' : \mathscr{E} \to \mathbf{Sh}(\mathcal{D}, K)$ correspond to equivalent morphisms of sites $A_F \cong A_{F'} : (\mathcal{D}, K) \to (\mathscr{E}, J_{\mathscr{E}}^{can})$, and if $C_p F \cong E$ then there exists *some* invertible 2-cell $\varphi : F^* \circ C_p^* \overset{\sim}{\Longrightarrow} E^*$, and thus *some* $\bar{\varphi}$ as claimed. ☐

Using Remark 1.4.2, we have that $\bar{\varphi}$ satisfies the claim of the result if and only if for every presheaf $H : \mathcal{C}^{op} \to \mathbf{Set}$, if we consider for every X in $\mathcal{C}$ the collection of elements $x \in H(X)$ (with $\ulcorner x \urcorner : よ(X) \to H$ the corresponding arrow of $[\mathcal{C}^{op}, \mathbf{Set}]$) and the collection of arrow $y : p(D) \to X$, the arrows $\alpha_{x,y}$ defined as

$$A_F(D) \xrightarrow{\bar{\varphi}(D)} A_E(p(D)) \xrightarrow{A_E(y)} A_E(X) = E^* \ell_J(X) \xrightarrow{E^* \mathrm{a}_J(\ulcorner x \urcorner)} E^* \mathrm{a}_J(H)$$

form a colimit cocone in $\mathscr{E}$. We can condense these considerations in the following description of sheafification in terms of equivalence classes of morphisms of sites:

Proposition 8.1.2. *Consider a site $(\mathcal{C}, J)$ and a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ with corresponding Grothendieck fibration $p : \int P \to \mathcal{C}$. For an object X in $\mathcal{C}$, denote by $B_X : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}/X, J_X)$ the flat J-continuous functor associated to $C_{p_X} : \mathbf{Sh}(\mathcal{C}/X, J_X) \to \mathbf{Sh}(\mathcal{C}, J)$: it acts by mapping any Y in $\mathcal{C}$ to the sheaf $\ell_J(Y) \circ p_X^{op}$, and accordingly on arrows. The set $\mathrm{a}_J(P)(X)$ is isomorphic to the set of equivalence classes of morphisms of sites*

$$A : (\textstyle\int P, J_P) \to (\mathbf{Sh}(\mathcal{C}/X, J_X), J^{can}_{\mathbf{Sh}(\mathcal{C}/X, J_X)})$$

such that there is a natural transformation

$$\varphi : A \Rightarrow B_X p$$

satisfying the following: for every presheaf $H : \mathcal{C}^{op} \to \mathbf{Set}$, if we consider for every Y in $\mathcal{C}$ the collection of arrows $y : \text{よ}(Y) \to H$ of $[\mathcal{C}^{op}, \mathbf{Set}]$ and for every (Z, s) in $\int P$ the collection of arrows $z : Z \to Y$, the composites

$$\alpha_{y,z} : A(Z, s) \xrightarrow{\varphi(Z,s)} B_X(Z) \xrightarrow{B_X(z)} B_X(Y) \xrightarrow{B_X(y)} B_X(H)$$

form a colimit cocone in $\mathbf{Sh}(\mathcal{C}/X, J_X)$.

8.2 Sheafification via comorphisms of sites

Elements in $\mathrm{a}_J(P)(X) = \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/X, J_X), \mathbf{Sh}(\int P, J_P))$ can also be described using J-equivalent comorphisms of sites. First of all, we recall that any geometric morphism $H : \mathbf{Sh}(\mathcal{C}/X, J_X) \to \mathbf{Sh}(\int P, J_P)$ over the base topos $\mathbf{Sh}(\mathcal{C}, J)$ is a local homeomorphism, and thus it is essential. Since both its domain and codomain are induced from continuous comorphisms of sites, we apply Proposition 1.4.2 to obtain the following:

Corollary 8.2.1. *Consider a site $(\mathcal{C}, J)$ and a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$. Denote by $p : \int P \to \mathcal{C}$ and $p_X : \mathcal{C}/X \to \mathcal{C}$ the relevant fibrations. Giving a geometric morphism H such that the triangle*

$$\begin{array}{ccc} \mathbf{Sh}(\mathcal{C}/X, J_X) & \xrightarrow{H} & \mathbf{Sh}(\int P, J_P) \\ & {\scriptstyle C_X} \searrow \quad \swarrow {\scriptstyle C_p} & \\ & \mathbf{Sh}(\mathcal{C}, J) & \end{array}$$

is commutative (up to equivalence) is the same as giving a continuous comorphism of sites $B : (\mathcal{C}/X, J_X) \to ([(\int P)^{op}, \mathbf{Set}], \widehat{J_P})$, up to J_P-equivalence (see Section 1.3), endowed with a natural transformation $\tau : B \Rightarrow p^* \text{よ}_{\mathcal{C}} p_X$ such that the composite

$$\bar{\tau} : \mathrm{lan}_{p^{op}} B \xRightarrow{\mathrm{lan}_{p^{op}} \circ \tau} \mathrm{lan}_{p^{op}} p^* \text{よ}_{\mathcal{C}} p_X \xRightarrow{\varepsilon' \circ \text{よ}_{\mathcal{C}} p_X} \text{よ}_{\mathcal{C}} p_X$$

is sent by a_J *to an isomorphism, where* ε' *is the counit of* $\mathrm{lan}_{p^{op}} \dashv p^*$*. In particular, for any* X *in* $\mathcal{C}$ *it follows that* $a_J(P)(X)$ *is isomorphic to the set of pairs* (B, τ) *as above, where* B *is chosen up to* J*-equivalence.*

Remark 8.2.1. Recall that for $W : (\int P)^{op} \to \mathbf{Set}$, (Y, U) in $\int P$ and Z in $\mathcal{C}$ then $\mathrm{lan}_{q^{op}}(W)(Z) := \mathrm{colim}_{\varphi: Z \to q(Y,U)} W(Y, U)$. We can describe the natural transformation $\bar{\tau} := (\varepsilon' \circ よ_{\mathcal{C}} p_X)(\mathrm{lan}_{q^{op}} \circ \tau)$ componentwise as follows: set $[w : W \to X]$ in $\mathcal{C}/X$, so that the component $\bar{\tau}([w])$ is an arrow $\mathrm{lan}_{q^{op}} B([w]) \to よ_{\mathcal{C}}(W)$ of $[\mathcal{C}^{op}, \mathbf{Set}]$. Its component at Z in $\mathcal{C}$ is the unique arrow

$$\mathrm{colim}_{\varphi: Z \to q(Y,U)} B([w])(Y, U) \to \mathrm{colim}_{\varphi: Z \to q(Y,U)} \mathcal{C}(Y, W) \to \mathcal{C}(Z, W)$$

induced by colimit property from the cocone whose leg indexed by $\varphi : Z \to q(Y, U)$ is the arrow

$$B([w])(Y, U) \xrightarrow{\tau([w])(Y,U)} \mathcal{C}(Y, W) \xrightarrow{- \circ \varphi} \mathcal{C}(Z, W).$$

8.3 Sheafification via matching families of comorphisms of sites

A third approach to the sheafification, which is more geometric in spirit, allows to present any

$$H \in \mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/A, J_A), \mathbf{Sh}(\textstyle\int P, J_P)) \simeq a_J(P)(A)$$

'locally' by morphisms of fibrations: that is, one can consider a J-covering family $\{g_u : D_u \to A \mid u \in U\}$ such that restricting H to each of the toposes $\mathbf{Sh}(\mathcal{C}/D_u, J_{D_u})$ results in a geometric morphism induced by a morphism of fibrations $\mathcal{C}/D_u \to \int P$. This point of view connects with the definition of the elements of $a_J(P)(A)$ as locally matching families of elements of P, which can be found for instance in [6, Proposition 2.19]. The argument goes as follows:

- we express the topos $\mathbf{Sh}(\int P, J_P)$ as a colimit of local homeomorphisms of the kind $\mathbf{Sh}(\mathcal{C}, J)/\ell_J(p(X, s))$: by pulling back the colimit along H we obtain an expression of $\mathbf{Sh}(\mathcal{C}/A, J_A)$ as a colimit of local homeomorphisms of the kind $\mathbf{Sh}(\mathcal{C}, J)/B(X, s)$;
- each of the toposes $\mathbf{Sh}(\mathcal{C}, J)/B(X, s)$ in turn can be expressed as a colimit of local homeomorphisms of the kind $\mathbf{Sh}(\mathcal{C}, J)/\ell_J(Y_\alpha)$;
- finally, each topos $\mathbf{Sh}(\mathcal{C}, J)/\ell_J(Y_\alpha)$ can be covered by étale toposes $\mathbf{Sh}(\mathcal{C}, J)/\ell_J(D_u)$ such that the composite geometric morphisms

$$\mathbf{Sh}(\mathcal{C}, J)/\ell_J(D_u) \to \mathbf{Sh}(\mathcal{C}, J)/\ell_J(A)$$

and
$$\mathbf{Sh}(\mathcal{C},J)/\ell_J(D_u) \to \mathbf{Sh}(\mathcal{C},J)/\ell_J(p(X,s))$$
are induced by arrows $g_u : D_u \to A$ and $g'_u : D_u \to p(X,s)$, and thus by morphisms between the corresponding fibrations.

First of all, we combine Theorem 5.2.1 and Corollary 6.3.5 to obtain the following:

Lemma 8.3.1. *Consider a site $(\mathcal{C},J)$ and a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$: then the cocone*

$$\begin{array}{ccc} \mathbf{Sh}(\mathcal{C},J)/\ell_J(p(Y,P(y)(s))) & \xrightarrow{\prod_{\ell_J(y)}} & \mathbf{Sh}(\mathcal{C},J)/\ell_J(p(X,s)) \\ & \searrow^{\prod_{\mathrm{a}_J(\ulcorner s \urcorner よ(y))}} & \downarrow \prod_{\mathrm{a}_J(\ulcorner s \urcorner)} \\ & & \mathbf{Sh}(\mathcal{C},J)/\mathrm{a}_J(P) \end{array}$$

is a colimit cocone, where each (X,s) is an object of $\int P$ and $\ulcorner s \urcorner : よ(X) \to P$ is the arrow corresponding to $s \in P(X)$ by Yoneda's lemma.

If we take a geometric morphism H making the triangle

$$\begin{array}{ccc} \mathbf{Sh}(\mathcal{C},J)/\ell_J(A) & \xrightarrow{H} & \mathbf{Sh}(\mathcal{C},J)/\mathrm{a}_J(P) \\ & {\scriptstyle \prod_{\ell_J(A)}} \searrow \quad \swarrow {\scriptstyle \prod_{\mathrm{a}_J(P)}} & \\ & \mathbf{Sh}(\mathcal{C},J) & \end{array}$$

commutative, by Lemma 7.1.2 there exists $h : \ell_J(A) \to \mathrm{a}_J(P)$ such that $H \cong \prod_h$. The inverse image H^* corresponds to pulling back along h: in particular, we shall consider the pullbacks

$$\begin{array}{ccc} B(X,s) & \xrightarrow{c(X,s)} & \ell_J(p(X,s)) \\ {\scriptstyle b(X,s)}\downarrow & \lrcorner & \downarrow {\scriptstyle \mathrm{a}_J(\ulcorner s \urcorner)} \\ \ell_J(A) & \xrightarrow{h} & \mathrm{a}_J(P). \end{array}$$

Thus pulling back the colimit in Lemma 8.3.1 along H, we obtain a colimit cocone

$$\begin{array}{ccc} \mathbf{Sh}(\mathcal{C},J)/B(Y,P(y)(s)) & \longrightarrow & \mathbf{Sh}(\mathcal{C},J)/B(X,s) \\ & \searrow^{\prod_{b(Y,P(y)(s))}} & \downarrow \prod_{b(X,s)} \\ & & \mathbf{Sh}(\mathcal{C},J)/\ell_J(A). \end{array}$$

By applying again Lemma 8.3.1, we can express each of the étale toposes $\mathbf{Sh}(\mathcal{C},J)/B(X,s)$ as a colimit whose legs are of the kind

$$\mathbf{Sh}(\mathcal{C},J)/\ell_J(Y_\alpha) \xrightarrow{\prod_\alpha} \mathbf{Sh}(\mathcal{C},J)/B(X,s)$$

for all arrows $\alpha : \ell_J(Y_\alpha) \to B(X,s)$. Notice that by composing colimit cocones we obtain a jointly epic family of arrows

$$\ell_J(Y) \xrightarrow{\alpha} B(X,s) \xrightarrow{b(X,s)} \ell_J(A).$$

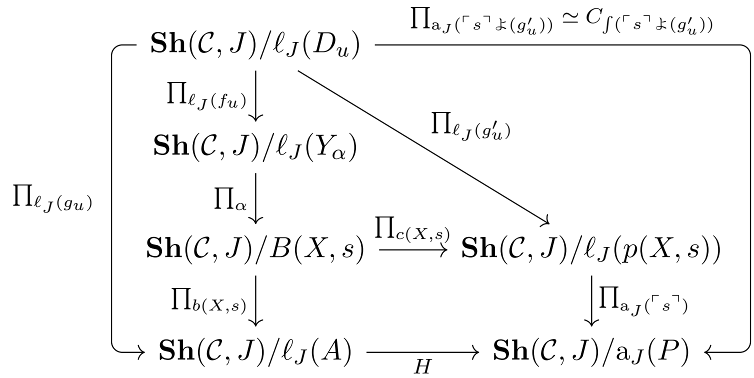
Finally, we exploit the following result:

Lemma 8.3.2 [6, Proposition 2.5]. *Consider a site $(\mathcal{C}, J)$ and a morphism $\alpha : \ell_J(X) \to \ell_J(Y)$: there exist two families of arrows $\{f_u : D_u \to X \mid u \in U\}$ and $\{g_u : D_u \to Y \mid u \in U\}$ of $\mathcal{C}$ such that $\{f_u \mid u \in U\}$ is J-covering, $\ell_J(g_u) = \alpha \circ \ell_J(f_u)$ and for every span $h : W \to D_u$, $k : W \to D_v$ in $\mathcal{C}$ such that $f_u h = f_v k$ we have $\ell_J(g_u h) = \ell_J(g_v k)$.*

This lemma states that every arrow between representables is 'locally induced' at the level of the presentation site. We can apply this lemma to the composite arrows

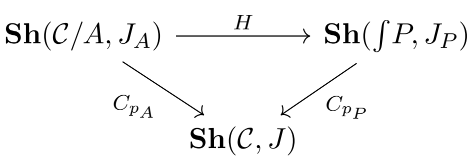
$$\ell_J(Y) \xrightarrow{\alpha} B(X,s) \xrightarrow{b(X,s)} \ell_J(A), \;\; \ell_J(Y) \xrightarrow{\alpha} B(X,s) \xrightarrow{c(X,s)} \ell_J(p(X,s)) :$$

by refining enough the J-covering family $\{f_u : D_u \to Y_\alpha\}$ we conclude that there exist two families $\{g_u : D_u \to A\}$ and $\{g'_u : D_u \to X\}$, all indexed by a set U, such that $c(X,s) \circ \alpha \circ \ell_J(f_u) = \ell_J(g'_u)$ and $b(X,s) \circ \alpha \circ \ell_J(f_u) = \ell_J(g_u)$. The diagram



is commutative, hence for every arrow g_u the composite $H \circ \prod_{\ell_J(g_u)}$ is equivalent to the functor $C_{\int(\ulcorner s \urcorner \text{よ}(g'_u))}$, i.e. it is presented by a comorphism of sites. Moreover, the family of all arrows $\ell_J(g_u)$ is jointly epic (since the families of the arrows $b(X,s)$, α and $\ell_J(f_u)$ all are), and hence the family $\{g_u\}$ is J-covering. We can thus sum up our conclusions as follows:

Proposition 8.3.3. *Consider a site $(\mathcal{C}, J)$, an object A of $\mathcal{C}$, a presheaf $P : \mathcal{C}^{op} \to$ **Set** and a geometric morphism H making the diagram*



commutative (up to equivalence). There is a J-covering family $\{g_u : D_u \to A \mid u \in U\}$ and a family of morphisms of fibrations $\{G_u : \mathcal{C}/D_u \to \int P \mid u \in U\}$ such that each composite geometric morphism

$$\mathbf{Sh}(\mathcal{C}/D_u, J_{D_u}) \xrightarrow{C_{\int g_u}} \mathbf{Sh}(\mathcal{C}/A, J_A) \xrightarrow{H} \mathbf{Sh}(\textstyle\int P, J_P)$$

is equivalent to the geometric morphism

$$C_{G_u} : \mathbf{Sh}(\mathcal{C}/D_u, J_{D_u}) \to \mathbf{Sh}(\textstyle\int P, J_P).$$

Of course, a geometric morphism H admits multiple presentations in this way, since one can choose different families $\{f_u\}$, $\{g_u\}$ and $\{g'_u\}$ to present it at the level of sites. Starting instead from the comorphisms G_u, one can recover H quite easily: indeed, the geometric morphisms $C_{G_u} : \mathbf{Sh}(\mathcal{C}/D_u, J_{D_u}) \to \mathbf{Sh}(\int P, J_P)$ form a cocone under the diagram indexed by the elements of the J-covering family $\{g_u\}$, which induces a geometric morphism from their colimit, which is $\mathbf{Sh}(\mathcal{C}/A, J_A)$, to $\mathbf{Sh}(\int P, J_P)$.

8.4 Sheafification for a presheaf over a topological space

We conclude this chapter by collecting the various possible descriptions of the sheafification in the context of topological spaces. Consider a topological space X and a presheaf $P : \mathcal{O}(X)^{op} \to \mathbf{Set}$, and denote its sheafification by $\mathrm{a}(P)$. For U in $\mathcal{O}(X)$ we have

$$\begin{aligned}
\mathrm{a}(P)(U) &= \mathbf{Top}/X(U, E_P) && (1)\\
&\simeq \mathbf{Locale}/\mathcal{O}(X)(\mathcal{O}(U), \mathcal{O}(E_P)) && (2)\\
&\simeq (\mathcal{O}(X), J^{can}_{\mathcal{O}(X)})/\mathbf{Site}((\mathcal{O}(E_P), J^{can}_{\mathcal{O}(E_P)}), (\mathcal{O}(U), J^{can}_{\mathcal{O}(U)})) && (3)\\
&\simeq \mathbf{Com}_{cont}/(\mathcal{O}(X), J^{can}_{\mathcal{O}(X)})((\mathcal{O}(U), J^{can}_{\mathcal{O}(U)}), (\mathcal{O}(E_P), J^{can}_{\mathcal{O}(E_P)})) && (4)\\
&= \mathbf{Topos}^s/\mathbf{Sh}(X)(\mathbf{Sh}(U), \mathbf{Sh}(E_P)) && (5)\\
&\simeq \mathbf{Topos}^s/\mathbf{Sh}(X)(\mathbf{Sh}(U), \mathbf{Sh}(\textstyle\int P, J_P)) && (5^*)\\
&\simeq \mathbf{Site}^{\dagger}((\textstyle\int P, J_P), (\mathcal{O}(U), J^{can}_{\mathcal{O}(U)})) && (6)\\
&\simeq \mathbf{Com}^{J_P}_{cont}((\mathcal{O}(U), J^{can}_{\mathcal{O}(U)}), ([(\textstyle\int P)^{op}, \mathbf{Set}], \widehat{J_P}))/^{\bullet} p^* \text{よ}_{\mathcal{O}(X)} p_U && (7)\\
&\simeq \text{Locally matching families of } \mathbf{Fib}_{\mathcal{O}(X)}(\mathcal{O}(-), \textstyle\int P) && (8)
\end{aligned}$$

where the symbol † denotes the morphisms of sites

$$A : (\textstyle\int P, J_P) \to (\mathcal{O}(U), J^{can}_{\mathcal{O}(U)})$$

satisfying the condition

($\dagger$) there is a natural transformation $A \Rightarrow i_U^{-1} \circ p$, where

$$i_U^{-1} := (- \cap U) : \mathcal{O}(X) \to \mathcal{O}(U)$$

and $p : \int P \to \mathcal{O}(X)$, whose induced natural transformation $\varphi : \mathbf{Sh}(A)^* \mathrm{a}_{J_P} \Rightarrow \mathbf{Sh}(i_U)^* \mathrm{a}_{J^{can}_{\mathcal{O}(X)}} \mathrm{lan}_{p^{op}}$ at the level of geometric morphism is such that the composite

$$\begin{array}{c} \mathbf{Sh}(A)^* \circ \mathrm{a}_{J_P} \circ p^* \\ \Big\Downarrow {\scriptstyle \varphi \circ p^*} \\ \mathbf{Sh}(i_U)^* \circ \mathrm{a} \circ \mathrm{lan}_{p^{op}} \circ p^* \\ \Big\Downarrow {\scriptstyle \mathbf{Sh}(i_U)^* \circ \mathrm{a} \circ \varepsilon} \\ \mathbf{Sh}(i_U)^* \circ \mathrm{a} \end{array}$$

is invertible (where ε is the counit of $\mathrm{lan}_{p^{op}} \dashv p^*$) (this results from Proposition 8.1.2).

On the other hand, the symbol $\bullet$ states that we consider J_P-equivalence classes of continuous comorphisms of sites $B : \mathcal{O}(U) \to [(\int P)^{op}, \mathbf{Set}]$ with a natural transformation $\tau : B \Rightarrow p^* よ_{\mathcal{O}(X)} p_U$ satisfying the requirement

($\bullet$) : the composite

$$\mathrm{lan}_{p^{op}} \circ B \xrightarrow{\mathrm{lan}_{p^{op}} \circ \tau} \mathrm{lan}_{p^{op}} \circ p^* \circ よ_{\mathcal{C}} \circ p_X \xrightarrow{\varepsilon' \circ よ_{\mathcal{C}} \circ p_X} よ_{\mathcal{C}} \circ p_X$$

is sent by a to an isomorphism, where ε' is the counit of $\mathrm{lan}_{p^{op}} \dashv p^*$ (see Corollary 8.2.1).

Let us recapitulate how we can go from one form to the other.

The connection between the first four items was already partially explored at the end of Section 7.2, but we briefly sum it up. To go from (1) to (2) and (3), we map a section $s : U \to E_P$ to $s^{-1} : \mathcal{O}(E_P) \to \mathcal{O}(U)$ (where morphisms of sites and homomorphisms of frames coincide by Lemma 7.2.11); going from (2) and (3) to (1), i.e. reconstructing the section $s : U \to E_P$ from a homomorphism of frames $\mathcal{O}(E_P) \to \mathcal{O}(U)$, is the content of Proposition 6.1.7. Mapping s to $s_! = s(-) : \mathcal{O}(U) \to \mathcal{O}(E_P)$ provides instead a way to go from (1) to (4). Finally, a continuous comorphism of sites in (4) admits a right adjoint which is a morphism of sites, and so we go back to (3).

By applying either $\mathbf{Sh}(-)$ or $C_{(-)}$ we can go from (1), (2), (3), (4) to (5); conversely, one can easily go back from (5) to (2) by restricting a geometric morphism $\mathbf{Sh}(U) \to \mathbf{Sh}(E_P)$ to the locales of subterminal objects.

To go from (5) to (5*) and viceversa it is enough to exploit the equivalence $\mathbf{Sh}(\int P, J_P) \simeq \mathbf{Sh}(E_P)$ of Proposition 6.1.5.

To go from (5*) to (6), we can simply restrict the inverse image of a geometric morphism $H : \mathbf{Sh}(U) \to \mathbf{Sh}(\int P, J_P)$ to representables; viceversa, a morphism of sites $\int P \to \mathbf{Sh}(U)$ induces a geometric morphism $\mathbf{Sh}(U) \to \mathbf{Sh}(\int P, J_P)$. The connection between (5*) and (7) is the content of Corollary 8.2.1, while the connection between (5*) and (8) was sketched in Section 8.3.

To conclude, we sketch the explicit connection between the first items and the last items, without going through the categories of geometric morphisms.

If we start from (1), i.e. from a section $s : U \to E_P$, it is easy to verify that the corresponding morphism of sites of (6)

$$A_s : \textstyle\int P \to \mathcal{O}(U)$$

is defined by mapping any object (V, t) of $\int P$ to the open $s^{-1}(\dot{t}(V)) \subseteq U$. The inclusions

$$s^{-1}(\dot{t}(V)) = \{x \in U \cap V \mid s(x) = t_x\} \subseteq U \cap V$$

provide the components of a natural transformation $A_s \Rightarrow i_U^{-1} p$ satisfying (†). Viceversa, consider a morphism of sites

$$A : (\textstyle\int P, J_P) \to (\mathcal{O}(U), J^{can}_{\mathcal{O}(U)})$$

satisfying (†): we can define a homomorphism of frames $f_A : \mathcal{O}(E_P) \to \mathcal{O}(U)$ by setting the image of basic opens as

$$f_A(\dot{t}(V)) := \bigcup \{W \subseteq U \mid W \subseteq A(V, t)\}$$

and requiring that f_A preserve arbitrary unions. This allows us to go from (6) to (2).

Finally, let us connect (1) and (7). Starting from a section $s : U \to E_P$, we consider the continuous comorphism of sites $s(-) : \mathcal{O}(U) \to \mathcal{O}(E_P) \simeq \mathrm{Id}_{J_P}(\int P)$ of item (4). By considering the corresponding geometric morphism $\mathbf{Sh}(U) \to \mathbf{Sh}(\mathrm{Id}_{J_P}(\int P))$ and applying to it Proposition 1.3.3 we obtain that s corresponds to the continuous comorphism of sites

$$\mathcal{O}(U) \xrightarrow{R} \mathbf{Sh}(\textstyle\int P, J_P) \hookrightarrow [(\int P)^{op}, \mathbf{Set}],$$

where in particular R maps an open $V \subseteq U$ to the union of subterminals

$$\bigcup_{\substack{W \subseteq V, \\ r \in P(W) \\ \text{s.t. } s_W = \dot{r}}} \ell_{J_P}(W, r).$$

Conversely, start from $B : \mathcal{O}(U) \to [(\int P)^{op}, \mathbf{Set}]$ a continuous comorphism of sites as in item (7), it induces a geometric morphism $F : \mathbf{Sh}(U) \to \mathbf{Sh}(\int P, J_P)$ whose inverse image acts by mapping a J_P-sheaf $H : (\int P)^{op} \to \mathbf{Set}$ to the $J^{can}_{\mathcal{O}(U)}$-sheaf

$$F^*(H) : \mathcal{O}(U)^{op} \to \mathbf{Set}, \ F^*(H)(V) := [(\textstyle\int P)^{op}, \mathbf{Set}](B(V), H).$$

By restricting to subterminals we obtain a frame homomorphism $\mathrm{Id}_{J_P}(\int P) \simeq \mathcal{O}(E_P) \to \mathcal{O}(U)$, i.e. an element of (2) and (3).

Appendices

Appendix A

Dependent product in elementary toposes

Dependent products in elementary toposes can be built in various ways (see the introduction of [7] for a short survey on the matter). Since their existence for a category $\mathscr{E}$ with finite limits is equivalent to $\mathscr{E}$ being locally cartesian closed, and hence cartesian closed, their construction relies heavily on higher-order constructions. The present appendix presents the content of Section 2 of [7], which proposes a description of dependent products based on reducing the use of higher-order tools to the bare minimum, as we exploit mainly finite limits, images and power objects.

Let us set the following notations for this chapter. We shall call $\mathscr{E}$ the base topos, and we shall consider an arrow $f : P \to Q$ of $\mathscr{E}$: we wish to describe the dependent product

$$\prod_f : \mathscr{E}/P \to \mathscr{E}/Q$$

by computing its value $\prod_f[h]$ at an object $[h : H \to P]$ of $\mathscr{E}/P$.

We begin by recalling the construction of dependent products for the topos $\mathscr{E} = \mathbf{Set}$ (cf. also [28, Theorem I.9.3]). In this case dependent products are described quite easily, as the name suggests, as products of a family of indexed sets. First of all, a P-set $h : H \to P$ can be seen as the P-indexed family of sets $\{h^{-1}(p)\}_{p\in P}$: notice that this comes from the categorical equivalence $\mathbf{Set}/P \simeq \mathbf{Set}^P$, which is a particular instance of Theorem 5.2.1. It follows that the dependent product $\prod_f[h]$ is the Q-set $\{\prod_{f(p)=q} h^{-1}(p)\}_{q\in Q}$, while its 'glued' form is the set $\coprod_{q\in Q}\prod_{f(p)=q} h^{-1}(p) \to Q$.

To generalize this construction to an arbitrary elementary topos, we need to describe it with a suitable formula in the internal language. Notice that the description above contains set-indexed products and coproducts: what makes the generalization possible is the fact that all terms are 'bounded', which allows us to represent them in terms of power objects and finitary products.

For a set W denote by $\mathscr{P}(W)$ its powerset. For a family of sets $H_i \subseteq H$ indexed by $i \in I$ it holds that

$$\prod_{i\in I} H_i = \{w \in \mathscr{P}(H \times I) \mid \forall\, i \in I\ \exists!\ x \in H((x,i) \in w)$$
$$\wedge\ \forall i \in I\ \forall\ x \in H((x,i) \in w \Rightarrow x \in H_i)\},$$

and thus in particular

$$\prod_{f(p)=q} h^{-1}(p) = \{w \in \mathscr{P}(H \times f^{-1}(q)) \mid \forall\ p \in f^{-1}(q)\ \exists!\ x \in H((x,p) \in w),$$
$$\forall\ p \in f^{-1}(q)\ \forall x \in H((x,p) \in w \Rightarrow h(x) = p))\}.$$

The element w belongs to a set parameterized by q: since $f^{-1}(q) \subseteq P$, we can remove this dependence with the equivalent formulation

$$\prod_{f(p)=q} h^{-1}(p) = \{w \in \mathscr{P}(H \times P) \mid \forall p \in P(f(p) = q \Rightarrow \exists!\ x \in H((x,p) \in w)),$$
$$\forall\ p \in P\ \forall\ x \in H((x,p) \in w \Rightarrow h(x) = p \wedge f(p) = q)\}.$$

We glue all these fibres together along the indexing given by Q and obtain the following internal expression for the dependent product.

Proposition A.1. *The dependent product $\prod_f[h]$ is defined by the internal language formula*

$$\textstyle\prod_f[h] = \coprod_{q\in Q} \prod_{p\in f^{-1}(q)} h^{-1}(p)$$
$$= \{(q,w) \in Q \times \mathscr{P}(H \times P) \mid \forall p \in P\ (f(p) = q \Rightarrow \exists! x \in H((x,p) \in w)),$$
$$\forall p \in P\ \forall x \in H((x,p) \in w \Rightarrow h(x) = p \wedge f(p) = q)\}$$

with structural morphism the projection onto the Q-component.

For a sketch of the proof see [7, Proposition 1.1]. The set-theoretic proof is constructive and thus holds in any topos, and it relies on the functional completeness of the internal language of a topos, i.e. the possibility to define arrows element-wise. Our goal now is to extract from the logical description above a more categorical construction of the dependent product.

First of all, we recall that in a finitely complete category $\mathscr{E}$ 'the' *power object* of an object X is an object $\mathscr{P}(X)$ together with a subobject $\in_X \rightarrowtail X \times \mathscr{P}(X)$ such that for any subobject $n : N \rightarrowtail X \times Y$ there is a unique $n' : Y \to \mathscr{P}(X)$ for which there is a pullback square

$$\begin{array}{ccc} N & \longrightarrow & \in_X \\ {\scriptstyle n}\downarrow & & \downarrow \\ X \times Y & \xrightarrow{1\times n'} & X \times \mathscr{P}(X). \end{array}$$

If $\mathscr{E}$ is well-powered (that is, the collection of subobjects of any given object is a set) this means that $\mathscr{P}(X)$ is the representing object for the functor $\mathrm{Sub}(X \times -) : \mathscr{E}^{op} \to \mathbf{Set}$. The arrow n' is the *classifying arrow* for n. The subobject $\in_X$ is to be thought internally as the collection of pairs (x, S) where $x \in X$, $S \subseteq X$ and $x \in S$; the arrow n' sends y to $\{x' \mid (x', y) \in N\}$, so that N is indeed the pullback of $\in_X$ along $1 \times n'$. In particular, we denote by $\{\cdot\}_X : X \to \mathscr{P}(X)$ the classifying arrow for the diagonal subobject $\Delta_X : X \rightarrowtail X \times X$.

The power-object construction can be made into a contravariant functor $\mathscr{P} : \mathscr{E}^{op} \to \mathscr{E}$. In the internal language, for any $\omega : X \to Y$ the arrow $\mathscr{P}(\omega) : \mathscr{P}(Y) \to \mathscr{P}(X)$ operates as the inverse image. It is well known that a finitely complete category with all power objects is an elementary topos [21, Sections A2.1-A2.3]; in particular, the subobject classifier is $\Omega = \mathscr{P}(1_{\mathscr{E}})$.

Remark A.1. In $\mathscr{E}/Q$, the power object of $[k : K \to Q]$ is the equalizer of the two arrows

$$Q \times \mathscr{P}(K) \xrightarrow{\pi_{\mathscr{P}(K)}} \mathscr{P}(K),$$

$$Q \times \mathscr{P}(K) \xrightarrow{(\mathscr{P}(k)\circ\{\cdot\}_Q)\times 1_{\mathscr{P}(K)}} \mathscr{P}(K) \times \mathscr{P}(K) \xrightarrow{\wedge_K} \mathscr{P}(K),$$

where $\wedge_K : \mathscr{P}(K) \times \mathscr{P}(K) \to \mathscr{P}(K)$ represents internally the intersection of subobjects (see [28, Theorem IV.7.1]). The domain of $\mathscr{P}([k])$ is denoted by $\mathscr{P}_Q(k)$ and is described in the internal language as

$$\mathscr{P}_Q(k) = \{(q, S) \in Q \times \mathscr{P}(K) \mid S \subseteq k^{-1}(q)\}.$$

To obtain a categorical construction of $\prod_f [h]$, we split its formulaic description in Proposition A.1 into smaller pieces: more precisely, we will find three subobjects of $Q \times \mathscr{P}(H \times P)$, which we shall call $\forall_{f\times}(S)$, T_1^f and T_2^h such that $\prod_f [h]$ is their intersection.

The first piece we are interested in is described internally by the formula

$$\{(q, w) \in Q \times \mathscr{P}(H \times P) \mid \forall p \in P\ (f(p) = q \Rightarrow \exists! x \in H((x, p) \in w))\} :$$

we shall call it $\forall_{f\times 1}(S)$, as it can be presented as the image of a certain object S via an *external* $\forall$ *functor*. First of all, we define

$$S := \{(p, w) \in P \times \mathscr{P}(H \times P) \mid \exists! x \in H((x, p) \in w)\},$$

a subobject of $P \times \mathscr{P}(H \times P)$ which expresses the functionality of w in the variable p. In turn, S is obtained starting from the subobject

$$e_P^H := \in_{H\times P} \rightarrowtail H \times P \times \mathscr{P}(H \times P) :$$

its classifying arrow, denoted by $\varphi : P \times \mathscr{P}(H \times P) \to \mathscr{P}(H)$, acts internally as $(p, w) \mapsto \{x \in H \mid (x, p) \in w\}$. Then notice that internally S is the collection of pairs (p, w) such that $\varphi(p, w)$ is a singleton, and thus is built as the pullback

$$\begin{array}{ccc} S & \longrightarrow & H \\ \downarrow & \lrcorner & \downarrow \{\cdot\}_H \\ P \times \mathscr{P}(H \times P) & \xrightarrow{\varphi} & \mathscr{P}(H). \end{array} \tag{A.1}$$

Now, in the above description of $\prod_f [h]$ this functionality of w in p is required for all the p's in the fibres of f. To address this, we can exploit the (external) $\forall_g$ functor: in general, for a subobject $A \rightarrowtail X$ and an arrow $g : X \to Y$,

$$\forall_g(A) = \{y \in Y \mid g^{-1}(y) \subseteq A\}.$$

In particular, for $f \times 1_{\mathscr{P}(H\times P)} : P \times \mathscr{P}(H \times P) \to Q \times \mathscr{P}(H \times P)$, we obtain the object

$$\forall_{f\times 1}(S) = \{(q, w) \in Q \times \mathscr{P}(H \times P) \mid \forall p \in P\,(f(p) = q \Rightarrow \exists! x \in H((x, p) \in w))\}.$$

Next, we intersect $\forall_{f\times 1}(S)$ with

$$T_1^f = \{(q, w) \in Q \times \mathscr{P}(H \times P) \mid \forall\, p \in P\ \forall\, x \in H\ ((x, p) \in w \Rightarrow f(p) = q)\},$$

which expresses a "fibre condition" relating elements of w and q: in fact, recall that w should be thought of as a tuple in $\prod_{p\in f^{-1}(q)} h^{-1}(p)$. The subobject T_1^f can be described by the equivalent condition $w \subseteq \pi_P^{-1} f^{-1}(q)$ and thus it coincides with $\mathscr{P}_Q(f \circ \pi_P : H \times P \to Q)$. Alternatively $T_1^f = \forall_\tau(W_1)$, where τ is the obvious arrow

$$\tau : Q \times \in_{H\times P} \rightarrowtail Q \times H \times P \times \mathscr{P}(H \times P) \to Q \times \mathscr{P}(H \times P)$$

and

$$\begin{array}{ccc} W_1 & \rightarrowtail & H \times P \times \mathscr{P}(H \times P) \\ \downarrow & \lrcorner & \downarrow \langle f\pi_P, 1_{H\times P\times \mathscr{P}(H\times P)}\rangle \\ Q \times \in_{H\times P} & \xrightarrow{1_Q \times e_P^H} & Q \times H \times P \times \mathscr{P}(H \times P), \end{array}$$

with π_P being the canonical projection $H \times P \times \mathscr{P}(H \times P) \to P$, is the subobject of quadruples (q, x, p, w) such that $(x, p) \in w$ and $q = f(p)$.

A third subobject we are interested in is

$$T_2^h = \{(q, w) \in Q \times \mathscr{P}(H \times P) \mid \forall\, p \in P\ \forall\, x \in H\ ((x, p) \in w \Rightarrow h(x) = p)\},$$

stating that whenever $(x, p) \in w$, (x, p) also belongs to the graph of h. Similarly to T_1^f, we also have that $T_2^h = \forall_\tau(W_2)$, where

$$\begin{array}{ccc} W_2 & \rightarrowtail & Q \times H \times \mathscr{P}(H \times P) \\ \downarrow & \lrcorner & \downarrow 1_Q \times \langle 1_H, h\rangle \times 1_{\mathscr{P}(H\times P)} \\ Q \times \in_{H\times P} & \xrightarrow{1_Q \times e_P^H} & Q \times H \times P \times \mathscr{P}(H \times P) \end{array}$$

is the subobject of quadruples (q, x, p, w) such that $(x, p) \in w$ and $p = h(x)$.

Thus $\prod_f[h] \simeq \forall_{f\times 1}(S) \cap T_1^f \cap T_2^h$ in the internal language of the topos: we now give a purely categorical proof of the fact that the object $\forall_{f\times 1}(S) \cap T_1^f \cap T_2^h$, with its canonical projection to Q, satisfies the universal property of the object $\prod_f[h]$. The proof reduces to the study of certain subobjects of $H \times P \times K$, which will serve as a 'bridge' between arrows $[f^*(k) : f^*(K) \to P] \to [h]$ and $[k : K \to Q] \to \prod_f[h]$. Here is how they come into play: an arrow $\alpha : [f^*(k)] \to [h]$ is precisely an arrow $\alpha : f^*(K) \to H$ in $\mathscr{E}$ satisfying the slice condition $h \circ \alpha = f^*(k)$; identifying this arrow with its graph $\langle \alpha, 1\rangle : f^*(K) \rightarrowtail H \times f^*(K)$ and regarding $f^*(K)$ as a subobject of $P \times K$ via the monomorphism $\langle f^*(k), k^*(f)\rangle : f^*(K) \rightarrowtail P \times K$, we obtain a subobject $\langle m_H, m_P, m_K\rangle : M \rightarrowtail H \times P \times K$ satisfying the following properties:

(i) $h \circ m_H = m_P$, i.e. it is a morphism in the slice topos $\mathscr{E}/P$;

(ii) there is an isomorphism $\bar{m} : M \xrightarrow{\sim} f^*(K)$ such that $f^*(k) \circ \bar{m} = m_P$ and $k^*(f) \circ \bar{m} = m_K$, i.e. M represents the graph of an arrow $f^*(K) \to H$.

In other words, the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\ \sim\ } & f^*(K) \\ {\scriptstyle \langle m_H, m_K\rangle}\downarrow & \searrow^{\langle m_H, m_P, m_K\rangle} & \downarrow{\scriptstyle \langle \alpha, f^*(k), k^*(f)\rangle} \\ H \times K & \xrightarrow[\langle 1_H, h\rangle \times 1_K]{} & H \times P \times K. \end{array} \tag{A.2}$$

On the other hand, *any* arrow $\beta : K \to \mathscr{P}(H \times P)$ corresponds to a subobject M of $H \times P \times K$, which can be described, using the internal language, as the collection of triples (x, p, u) such that $(x, p) \in \beta(u)$ and, categorically, as the following pullback:

$$\begin{array}{ccc} M & \longrightarrow & \in_{H\times P} \\ {\scriptstyle \langle m_H, m_P, m_K\rangle}\downarrow & \lrcorner & \downarrow \\ H \times P \times K & \xrightarrow[1_H \times 1_P \times \beta]{} & H \times P \times \mathscr{P}(H \times P) \end{array} \tag{A.3}$$

Lemma A.2. *Consider $\langle k, \beta\rangle : K \to Q \times \mathscr{P}(H \times P)$ and M classified by β as in square (A.3).*

(i) $\langle k, \beta\rangle$ factors through T_1^f if and only if $\langle m_H, m_P, m_K\rangle : M \rightarrowtail H \times P \times K$ factors through $1_H \times \langle f^*(k), k^*(f)\rangle : H \times f^*(K) \rightarrowtail H \times P \times K$;

(ii) $\langle k, \beta\rangle$ factors through T_2^h if and only if $h \circ m_H = m_P$;

(iii) $\langle k, \beta\rangle$ factors through $\forall_{f\times 1}(S)$ if and only if there is a morphism $\alpha : f^*(K) \to H$ such that $\langle \alpha, 1\rangle : f^*(K) \rightarrowtail H \times f^*(K)$ is the pullback of M along $1_H \times \langle f^*(k), k^*(f)\rangle$.

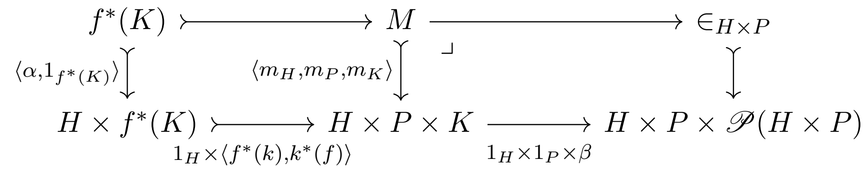
Proof. We preliminarily observe that we have the following rectangle, whose internal squares are both pullbacks and whose lower composite arrow is τ:

$$\begin{array}{ccccc} M & \overset{\langle m_H, m_P, m_K\rangle}{\rightarrowtail} & H \times P \times K & \longrightarrow & K \\ \downarrow & \lrcorner & \downarrow {\scriptstyle \langle k\circ\pi_K, 1_H\times 1_P\times\beta\rangle} & \lrcorner & \downarrow {\scriptstyle \langle k,\beta\rangle} \\ Q\times \in_{H\times P} & \underset{1_Q\times e^H_P}{\rightarrowtail} & Q\times H\times P\times \mathscr{P}(H\times P) & \longrightarrow & Q\times \mathscr{P}(H\times P) \end{array}$$

(i) $\langle k, \beta\rangle$ factors through $T^f_1 = \forall_\tau(W_1)$ if and only if its pullback along τ factors through $W_1 \rightarrowtail Q\times \in_{H\times P}$. By the universal property of the pullback square defining W_1, $\tau^*(\langle k, \beta\rangle)$ factors through $W_1 \rightarrowtail Q\times \in_{H\times P}$ if and only if the composite arrow $\langle k \circ \pi_K, 1_H \times 1_P \times \beta\rangle \circ \langle m_H, m_P, m_K\rangle = \langle k \circ m_K, m_H, m_P, \beta \circ m_K\rangle : M \to Q \times H \times P \times \mathscr{P}(H\times P)$ factors through $\langle f\circ\pi_P, 1_{P\times\mathscr{P}(H\times P)}\rangle : H\times P\times\mathscr{P}(H\times P) \rightarrowtail Q \times H \times P \times \mathscr{P}(H \times P)$. Now, if such a factorization exists then it is necessarily equal to $\langle m_H, m_P, \beta \circ m_K\rangle : M \to H \times P \times \mathscr{P}(H \times P)$, and this arrow satisfies the required property if and only if, denoting by π_Q the canonical projection $Q\times H\times P\times\mathscr{P}(H\times P) \to Q$, $\pi_Q\circ\langle f\circ \pi_P, 1_{P\times\mathscr{P}(H\times P)}\rangle\circ\langle m_H, m_P, \beta\circ m_K\rangle = \pi_Q\circ\langle k\circ m_K, m_H, m_P, \beta\circ m_K\rangle$. But this holds precisely when $f \circ m_P = k \circ m_K$, i.e. $\langle m_H, m_P, m_K\rangle$ factors through $1_H \times \langle f^*(k), k^*(f)\rangle : H \times f^*(K) \rightarrowtail H \times P \times K$.

(ii) Similarly to (i), $\langle k, \beta\rangle$ factors through $T^h_2 = \forall_\tau(W_2)$ if and only if its pullback along τ factors through $W_2 \rightarrowtail Q\times \in_{H\times P}$, equivalently if and only if the arrow $\langle k \circ m_K, m_H, m_P, \beta \circ m_K\rangle$ factors through $1_Q \times \langle 1_H, h\rangle\times 1_{\mathscr{P}(H\times P)} : Q\times H\times\mathscr{P}(H\times P) \to Q\times H\times P\times\mathscr{P}(H\times P)$. Now, if such a factorization exists it is necessarily equal to $\langle k\circ m_K, m_H, \beta\circ m_K\rangle$, and this arrow satisfies the required property if and only if, denoting by π'_P the canonical projection $Q \times H \times P \times \mathscr{P}(H \times P) \to P$, $\pi'^P \circ \langle k \circ m_K, m_H, m_P, \beta \circ m_K\rangle = \pi'^P \circ (1_Q \times \langle 1_H, h\rangle \times 1_{\mathscr{P}(H\times P)}) \circ \langle k \circ m_K, m_H, \beta \circ m_K\rangle$, i.e. $m_P = h \circ m_H$.

(iii) $\langle k, \beta\rangle$ factors through $\forall_{f\times 1}(S)$ if and only if its pullback along $f \times 1$, i.e. $\langle f^*(k), \beta \circ k^*(f)\rangle : f^*(K) \to P \times \mathscr{P}(H \times P)$, factors through $S \rightarrowtail P\times\mathscr{P}(H\times P)$; this happens if and only if there is some $\alpha : f^*(K) \to H$ such that $\{\cdot\}_H \circ \alpha = \varphi \circ \langle f^*(k), \beta \circ k^*(f)\rangle$ (α is also unique since $\{\cdot\}_H$ is monic, cf. [21, Corollary A2.2.3]). Now, the arrows $\{\cdot\}_H \circ \alpha$ and $\varphi \circ \langle f^*(k), \beta \circ k^*(f)\rangle$ are equal if and only if they classify the same subobject of $H \times f^*(K)$. It is immediate to see that $\{\cdot\}_H \circ \alpha$ classifies $\langle \alpha, 1_{f^*(K)}\rangle : f^*(K) \rightarrowtail H \times f^*(K)$, and that $\varphi \circ \langle f^*(k), \beta \circ k^*(f)\rangle$ classifies the pullback of $\in_{H\times P} \rightarrowtail H \times P \times \mathscr{P}(H \times P)$ along $1_H \times \langle f^*(k), \beta \circ k^*(f)\rangle$, which coincides (by the pullback lemma) with the pullback of $\langle m_H, m_P, m_K\rangle$ along $1_H \times \langle f^*(k), k^*(f)\rangle$. So $\{\cdot\}_H \circ \alpha = \varphi \circ \langle f^*(k), \beta \circ k^*(f)\rangle$ if and only if the left-hand square in the following diagram is a pullback:



□

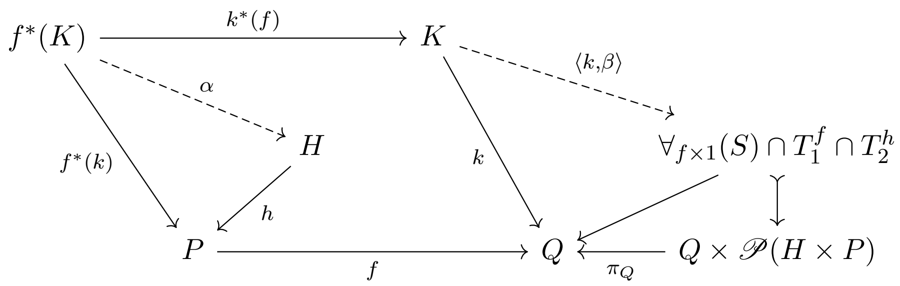
Notice that the factorization of $\langle k,\beta\rangle$ through $\forall_{f\times 1}(S)$ alone grants the existence of α. Using the internal language, we can express this condition as the requirement that for every $p\in P$, if $f(p)=k(u)$ then there exists a unique x such that $(x,p)\in\beta(u)$. In fact, $\alpha : f^*(K)\to H$ assigns to each such pair (p,u) that single $x\in H$. On the other hand, T_1^f and T_2^h provide the fibre-like conditions which α and β must satisfy.

We can conclude the following:

Theorem A.3. *Let $h : H\to P$ be an object of $\mathscr{E}/P$. Then, with the above notation, $\prod_f[h]\cong\forall_{f\times 1}(S)\cap T_1^f\cap T_2^h$. More specifically, for any object $k : K\to Q$ of $\mathscr{E}/Q$, there is a natural bijective correspondence between the arrows $[f^*(k) : f^*(K)\to P]\to[h]$ in $\mathscr{E}/P$ and $[k : K\to Q]\to\prod_f[h]$ in $\mathscr{E}/Q$. This correspondence sends*

- *an arrow $\alpha : [f^*(k)]\to[h]$ in $\mathscr{E}/P$ to the arrow $\langle k,\beta\rangle : [k]\to\forall_{f\times 1}(S)\cap T_1^f\cap T_2^h\rightarrowtail Q\times\mathscr{P}(H\times P)$ in $\mathscr{E}/Q$, where $\beta : K\to\mathscr{P}(H\times P)$ is the classifying arrow of the graph of α, regarded as a subobject of $H\times P\times K$;*
- *an arrow $\langle k,\beta\rangle : [k]\to\forall_{f\times 1}(S)\cap T_1^f\cap T_2^h\rightarrowtail Q\times\mathscr{P}(H\times P)$ in $\mathscr{E}/Q$ to the arrow $\alpha : [f^*(k)]\to[h]$ in $\mathscr{E}/P$ whose graph in $\mathscr{E}$ is the subobject of $H\times P\times K$ classified by β.*

Proof. We will show that a subobject $\langle m_H,m_P,m_K\rangle : M\rightarrowtail H\times P\times K$ makes diagram (A.2) commutative, i.e. corresponds to an arrow $\alpha : [f^*(k)]\to[h]$ in $\mathscr{E}/P$, if and only if its classifying arrow $\beta : K\to\mathscr{P}(H\times P)$ is such that $\langle k,\beta\rangle$ factors through $\forall_{f\times 1}(S)\cap T_1^f\cap T_2^h$:



Lemma A.2(ii) says that $\langle k,\beta\rangle$ factors through T_2^h if and only if M satisfies condition (i) for diagram (A.2). Lemma A.2(i) tells us that $\langle k,\beta\rangle$ factors through T_1^f if and only if $\langle m_H, m_P, m_K\rangle : M \rightarrowtail H \times P \times K$ factors through $1_H \times \langle f^*(k), k^*(f)\rangle : H \times f^*(K) \rightarrowtail H \times P \times K$, while by Lemma A.2(iii) $\langle k, \beta\rangle$ factors through $\forall_{f\times 1}(S)$ if and only if its pullback along $1_H \times \langle f^*(k), k^*(f)\rangle : H \times f^*(K) \rightarrowtail H \times P \times K$ is isomorphic to $\langle \alpha, 1\rangle : f^*(K) \rightarrowtail H \times f^*(K)$. Therefore, β factors through T_1^f and $\forall_{f\times 1}(S)$ if and only if condition (ii) for diagram (A.2) is satisfied. So we can conclude that $\beta : K \to \forall_{f\times 1}(S) \cap T_1^f \cap T_2^h$ as a morphism of $\mathscr{E}/Q$ corresponds to a unique morphism $\alpha : [f^*(k)] \to [h]$ in $\mathscr{E}/P$ and viceversa. The naturality of this correspondence is immediate, as all the arrows involved in it are defined by universal properties.

□

The pervasive appearance of $\forall$ in the previous results is not a big step towards an elementary treatment of the dependent product, but it can be reduced to more basic structures, namely limits and power objects. To do so we must recall the definition of the *covariant power-object functor* (see [21, pag. 92]): it is defined as $\mathscr{P}$ on objects, but it sends an arrow $f : Y \to X$ to the arrow $\exists f : \mathscr{P}(Y) \to \mathscr{P}(X)$ classifying the image of $\in_Y \rightarrowtail Y \times \mathscr{P}(Y) \xrightarrow{f\times 1} X \times \mathscr{P}(Y)$. The notation $\exists f$ is justified by the fact that $S \subseteq Y$ is sent to $\{f(y) \mid y \in S\} = \{x \in X \mid \exists y \in S\ (f(y) = x)\}$. Using the covariant power object functor, $\forall$ can be described as follows:

Proposition A.4. *Let $f : Y \to X$ be an arrow and $i : A \rightarrowtail Y$ a subobject in an elementary topos $\mathscr{E}$. Then $\forall_f(A) \simeq A'$ (as subobjects of X), where A' is defined by the following pullback square:*

$$\begin{array}{ccccc} A' & & \longrightarrow & & \mathscr{P}(A) \\ \downarrow & \lrcorner & & & \downarrow \exists i \\ X & \xrightarrow[\{\cdot\}_X]{} & \mathscr{P}(X) & \xrightarrow[\mathscr{P}(f)]{} & \mathscr{P}(Y) \end{array}$$

More explicitly, the vertical arrow $\exists i : \mathscr{P}(A) \to \mathscr{P}(Y)$ is the classifying arrow of the composite subobject

$$\in_A \rightarrowtail A \times \mathscr{P}(A) \xrightarrow{i\times 1_{\mathscr{P}(A)}} Y \times \mathscr{P}(A)$$

and the horizontal arrow $\mathscr{P}(f) \circ \{\cdot\}_X : X \to \mathscr{P}(Y)$ is the classifying arrow of the graph of f.

Proof. Using the internal language, we have:

$$A' = \{(x, N) \in X \times \mathscr{P}(A) \mid f^{-1}(x) = i(N)\} \cong \{x \in X \mid f^{-1}(x) \subseteq i(A)\} = \forall_f(A).$$

□

Distinguishing T_1^f and T_2^h we were able to isolate what exactly causes the induced arrow $\alpha : f^*(K) \to H$ to yield a morphism in the slice topos

$\mathscr{E}/P$; yet, we could have treated the intersection of T_1^f and T_2^h as a whole from the very beginning: being both $T_1^f = \forall_\tau(W_1)$ and $T_2^h = \forall_\tau(W_2)$, their intersection is $\forall_\tau(W_1 \cap W_2)$ (this holds since $\forall$ is a right adjoint and thus commutes with intersections). There is yet another very simple formulation for $T_1^f \cap T_2^h$, in terms of a power object in the topos $\mathscr{E}/Q$:

Proposition A.5. *As a subobject of $Q \times \mathscr{P}(H \times P)$, $T_1^f \cap T_2^h$ is isomorphic to the composite monomorphism*

$$\mathscr{P}_Q(H \xrightarrow{f \circ h} Q) \overset{m}{\rightarrowtail} Q \times \mathscr{P}(H) \xrightarrow{1_Q \times \exists(\langle 1,h \rangle)} Q \times \mathscr{P}(H \times P).$$

Proof. By Remark A.1 we have that

$$\mathscr{P}_Q(H \xrightarrow{f \circ h} Q) = \{(q,S) \in Q \times \mathscr{P}(H) \mid S \subseteq (f \circ h)^{-1}(q)\},$$

and thus

$$\begin{aligned}
&(1_Q \times \exists(\langle 1,h \rangle)) \circ m = \\
&\quad = \{(q,w) \in Q \times \mathscr{P}(H \times P) \mid (\exists S)((q,S) \in \mathscr{P}_Q([fh]) \wedge w = \langle 1,h \rangle(S))\} \\
&\quad = \{(q,w) \in Q \times \mathscr{P}(H \times P) \mid \forall (x,p) \in w \, (p = h(x) \wedge f(p) = q)\} \\
&\quad = T_1^f \cap T_2^h.
\end{aligned}$$

□

To conclude, we describe the interaction between dependent products and subtoposes $\mathscr{F} \hookrightarrow \mathscr{E}$. In general, any geometric morphism $\psi : \mathscr{F} \to \mathscr{E}$ induces, for each $E \in \mathscr{E}$, a geometric morphism $\psi_E : \mathscr{F}/\psi^*(E) \to \mathscr{E}/E$ making the following diagram commutative:

$$\begin{array}{ccc}
\mathscr{F} & \xrightarrow{\psi} & \mathscr{E} \\
{\scriptstyle \Pi^{\mathscr{F}}_{\psi^*(E)}} \uparrow & & \uparrow {\scriptstyle \Pi^{\mathscr{E}}_E} \\
\mathscr{F}/\psi^*(E) & \xrightarrow[\psi_E]{} & \mathscr{E}/E.
\end{array}$$

The inverse image ψ_E^* is defined as $\psi_E^*([h]) := [\psi^*(h)]$, while the direct image ψ_{E*} sends an object $k : K \to \psi^*(E)$ to the pullback of $\psi_*(k)$ along the unit $\eta_E : E \to \psi_*\psi^*(E)$ [21, Example A4.1.3]. In particular, when ψ is an inclusion ψ_E is also an inclusion [6, Example 5.18]. Let us now consider a subtopos $a \dashv i : \mathscr{F} \hookrightarrow \mathscr{E}$. For any arrow $f : P \to Q$, the square

$$\begin{array}{ccc}
\mathscr{E}/P & \xleftarrow{f^*} & \mathscr{E}/Q \\
{\scriptstyle (i_P)^*} \downarrow & & \downarrow {\scriptstyle (i_Q)^*} \\
\mathscr{F}/i^*(P) & \xleftarrow[(i^*(f))^*]{} & \mathscr{F}/i^*(Q)
\end{array}$$

of inverse images is commutative, since a preserves pullbacks. Therefore, the square of direct images if also commutative. Notice that if P and Q lie in $\mathscr{F}$, $\mathrm{a}(P) \cong P$, $\mathrm{a}(Q) \cong Q$ and $(i_P)_*$, $(i_Q)_*$ are the canonical inclusion functors induced by the embedding of $\mathscr{F}$ into $\mathscr{E}$.

Summarizing, we have the following result:

Proposition A.6. *Let $i : \mathscr{F} \hookrightarrow \mathscr{E}$ be a subtopos and $f : P \to Q$ an arrow in $\mathscr{E}$. Then*

$$\begin{array}{ccc} \mathscr{E}/P & \xrightarrow{\prod_f^{\mathscr{E}}} & \mathscr{E}/Q \\ {\scriptstyle (i_P)_*}\uparrow & & \uparrow{\scriptstyle (i_Q)_*} \\ \mathscr{F}/\mathrm{a}(P) & \xrightarrow[\prod_{\mathrm{a}(f)}^{\mathscr{F}}]{} & \mathscr{F}/\mathrm{a}(Q) \end{array} \tag{A.4}$$

is a commutative diagram of geometric morphisms.

In particular, if $f : P \to Q$ is a morphism in $\mathscr{F}$, then the dependent product $\prod_f^{\mathscr{F}} : \mathscr{F}/P \to \mathscr{F}/Q$ is the restriction of $\prod_f^{\mathscr{E}} : \mathscr{E}/P \to \mathscr{E}/Q$ along the canonical inclusions $\mathscr{F}/P \hookrightarrow \mathscr{E}/P$ and $\mathscr{F}/Q \hookrightarrow \mathscr{E}/Q$.

Appendix B

Some results on Grothendieck universes

In this appendix we recap some results about Grothendieck universes, our main interest being the study of geometric morphisms between toposes of sheaves valued in different universes. The original results in this appendix appear in [8], and have been developed in view of an application to the study of the dychotomy between 'petit' and 'gros' toposes.

Universes were first introduced in the appendix of Exposé I in [1], but another standard reference is [26]. First of all, a *Grothendieck universe* is a set $\mathcal{U}$ satisfying the following four axioms:

(i) if $x \in \mathcal{U}$ and $y \in x$ then $y \in \mathcal{U}$;

(ii) if $x, y \in \mathcal{U}$ then $\{x, y\} \in \mathcal{U}$;

(iii) if $x \in \mathcal{U}$ then $\mathscr{P}(x) \in \mathcal{U}$;

(iv) if $I \in \mathcal{U}$, for any map $f : I \to \mathcal{U}$ then $\bigcup_{i \in I} f(i) \in \mathcal{U}$.

Note that this is called *pre-universe* in [26], and a universe is a pre-universe that contains the set ω of von Neumann finite ordinals. This latter condition is equivalent to asking that a universe is not empty, as remarked after Proposition 7 in [1, Appendix]. In short, a universe is a set closed under the usual set theoretic operations that can be performed on its elements. Universes can be used to avoid the dichotomy set/class, when dealing with size issues in category theory: instead of small sets, one can speak about $\mathcal{U}$*-small* sets, i.e. sets that are isomorphic to an element of $\mathcal{U}$. If one is given a set which is not $\mathcal{U}$-small, i.e. which is $\mathcal{U}$*-large*, one can suppose that there is a wider universe $\mathcal{V}$ containing both said set and $\mathcal{U}$, and thus 'widen the horizon'. We recall though that the assumption that every set be contained in a universe is a powerful set-theoretic axiom, which implies the existence of a strongly inaccessible cardinal containing every other chosen cardinal.

A category $\mathcal{C}$ is a *$\mathcal{U}$-category* if for every pair of objects Y and X of $\mathcal{C}$ the hom-set $\mathcal{C}(Y, X)$ is $\mathcal{U}$-small; the category $\mathcal{C}$ is *locally $\mathcal{U}$-small category* in the terminology of [26]. In a similar fashion, one says that $\mathcal{C}$ is *$\mathcal{U}$-small* if its set of morphisms is $\mathcal{U}$-small.

Denote by $\mathbf{Set}_{\mathcal{U}}$ the topos of $\mathcal{U}$-small sets: then it is a $\mathcal{U}$-category. Given a further universe $\mathcal{V}$ such that $\mathcal{U} \in \mathcal{V}$ (by axiom (ii) we have also $\mathcal{U} \subseteq \mathcal{V}$) we have an obvious full and faithful inclusion

$$\mathbf{Set}_{\mathcal{U}} \hookrightarrow \mathbf{Set}_{\mathcal{V}}.$$

For every category $\mathcal{C}$ we can consider its $\mathcal{U}$-presheaves, i.e. the contravariant functors $\mathcal{C}^{op} \to \mathbf{Set}_{\mathcal{U}}$: if $\mathcal{C}$ is $\mathcal{U}$-small then $[\mathcal{C}^{op}, \mathbf{Set}_{\mathcal{U}}]$ is locally $\mathcal{U}$-small. If $\mathcal{U} \in \mathcal{V}$, a $\mathcal{U}$-category $\mathcal{C}$ is also a $\mathcal{V}$-category and we have again a full and faithful inclusion

$$[\mathcal{C}^{op}, \mathbf{Set}_{\mathcal{U}}] \hookrightarrow [\mathcal{C}^{op}, \mathbf{Set}_{\mathcal{V}}].$$

A site $(\mathcal{C}, J)$ is called a *$\mathcal{U}$-site* if it admits a J-dense full subcategory that is $\mathcal{U}$-small (see [1, Definitions 3.0.1 and 3.0.2]); alternatively one could say that $(\mathcal{C}, J)$ is *$\mathcal{U}$-small generated*. In particular, the site is $\mathcal{U}$-small if $\mathcal{C}$ is a $\mathcal{U}$-small category. One can speak of $\mathcal{U}$-sheaves on a $\mathcal{U}$-site, thus obtaining the topos $\mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J)$. A *$\mathcal{U}$-topos* is a $\mathcal{U}$-category equivalent to a category of $\mathcal{U}$-sheaves for a $\mathcal{U}$-site. Moreover, in analogy with the presheaf case, given an inclusion of universes $\mathcal{U} \subseteq \mathcal{V}$ one can consider the $\mathcal{U}$-site $(\mathcal{C}, J)$ as a $\mathcal{V}$-site, thus producing a full and faithful inclusion of sheaf toposes

$$\mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J) \hookrightarrow \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J).$$

Finally, we recall that all usual properties of toposes are true when one works with universes. For instance, a topos of $\mathcal{U}$-sheaves is closed under $\mathcal{U}$-small limits and colimits. This implies that when one considers the sheafification functor $[\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J)$, which is defined by colimits, it commutes with the expansion of universes: that is, we are left with the two (essentially) commutative squares

$$\begin{array}{ccccccc}
\mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J) & \hookrightarrow & [\mathcal{C}^{op}, \mathbf{Set}_{\mathcal{V}}] & & \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J) & \xleftarrow{\mathrm{a}_J} & [\mathcal{C}^{op}, \mathbf{Set}_{\mathcal{V}}] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J) & \hookrightarrow & [\mathcal{C}^{op}, \mathbf{Set}_{\mathcal{U}}] & & \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J) & \xleftarrow{\mathrm{a}_J} & [\mathcal{C}^{op}, \mathbf{Set}_{\mathcal{U}}]
\end{array}$$

This is essentially the content of [1, Exposé II, Proposition 3.6].

As we have anticipated above, our main interest is to study the behaviour of geometric morphisms with respect to the change of universe. In the following we will always assume $\mathcal{U} \in \mathcal{V}$ to be two universes and $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ to be $\mathcal{U}$-sites. Consider two geometric morphisms $F : \mathbf{Sh}_{\mathcal{U}}(\mathcal{D}, K) \to \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J)$

and $G : \mathbf{Sh}_{\mathcal{V}}(\mathcal{D}, K) \to \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J)$: we will say that G is an extension of F, or that F is a restriction of G, if the two squares

$$\begin{array}{ccccccc} \mathbf{Sh}_{\mathcal{V}}(\mathcal{D}, K) & \xrightarrow{G_*} & \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J) & \quad & \mathbf{Sh}_{\mathcal{V}}(\mathcal{D}, K) & \xleftarrow{G^*} & \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathbf{Sh}_{\mathcal{U}}(\mathcal{D}, K) & \xrightarrow{F_*} & \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J) & & \mathbf{Sh}_{\mathcal{U}}(\mathcal{D}, K) & \xleftarrow{F^*} & \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J) \end{array}$$

are commutative up to natural isomorphism. For instance, we can restate our previous considerations about a_J by saying that $\mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{op}, \mathbf{Set}_{\mathcal{V}}]$ restricts to $\mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{op}, \mathbf{Set}_{\mathcal{U}}]$. This holds more in general for any geometric morphism induced by a morphism or a comorphism of sites:

Proposition B.1. *Let $\mathcal{U} \subseteq \mathcal{V}$ be universes and $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be $\mathcal{U}$-sites.*

(i) Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of $\mathcal{U}$-sites. Then the F is also a morphism of $\mathcal{V}$-sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ and the geometric morphism $\mathbf{Sh}_{\mathcal{U}}(F) : \mathbf{Sh}_{\mathcal{U}}(\mathcal{D}, K) \to \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J)$ is (up to isomorphism) the restriction of $\mathbf{Sh}_{\mathcal{V}}(F) : \mathbf{Sh}_{\mathcal{V}}(\mathcal{D}, K) \to \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J)$.

(ii) Let $G : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of $\mathcal{U}$-sites. Then G is also a comorphism of $\mathcal{V}$-sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$, and the geometric morphism $(C_G)_{\mathcal{U}} : \mathbf{Sh}_{\mathcal{U}}(\mathcal{D}, K) \to \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J)$ is given by the restriction of $(C_G)_{\mathcal{V}} : \mathbf{Sh}_{\mathcal{V}}(\mathcal{D}, K) \to \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J)$.

Proof. (i) Since both $\mathbf{Sh}_{\mathcal{U}}(F)_*$ and $\mathbf{Sh}_{\mathcal{V}}(F)_*$ act as the precomposition $- \circ F^{op}$, evidently the former is the restriction of the latter. Moreover, [1, Exposé II, Proposition 1.5] shows that F is (J, K)-continuous as a $\mathcal{U}$-functor if and only if it is (J, K)-continuous as a $\mathcal{V}$-functor, and that $\mathbf{Sh}_{\mathcal{U}}(F)^*$ is the restriction of $\mathbf{Sh}_{\mathcal{V}}(F)^*$. So far we have the two (essentially) commutative squares

$$\begin{array}{ccccccc} \mathbf{Sh}_{\mathcal{V}}(\mathcal{D}, K) & \xrightarrow{Sh_{\mathcal{V}}(F)_*} & \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J) & \quad & \mathbf{Sh}_{\mathcal{V}}(\mathcal{D}, K) & \xleftarrow{Sh_{\mathcal{V}}(F)^*} & \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathbf{Sh}_{\mathcal{U}}(\mathcal{D}, K) & \xrightarrow{Sh_{\mathcal{U}}(F)_*} & \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J) & & \mathbf{Sh}_{\mathcal{U}}(\mathcal{D}, K) & \xleftarrow{Sh_{\mathcal{U}}(F)^*} & \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J) \end{array} .$$

We are left with considerations on the flatness of F: that is, we want to know whether $\mathbf{Sh}_{\mathcal{U}}(F)^*$ preserves finite limits if and only if $\mathbf{Sh}_{\mathcal{V}}(F)^*$ does. Since $\mathbf{Sh}_{\mathcal{U}}(F)^*$ is a restriction of $\mathbf{Sh}_{\mathcal{V}}(F)^*$ one implication is obvious. For the converse we can resort to the definition of morphism of sites provided in [6, Definition 3.2]: F is a morphism of sites as a $\mathcal{U}$-functor if and only if it satisfies the four requirements of said definition, which are site-theoretic and thus are independent from the universe of choice.

(ii) Again, since $(C_G)^*_{\mathcal{U}}$ and $(C_G)^*_{\mathcal{V}}$ both act as the precomposition $- \circ G^{op}$, the former is a restriction of the latter. Finally, [1, Exposé II, Proposition 2.3(4)] shows that $(C_G)_{\mathcal{U}*}$ is the restriction of $(C_G)_{\mathcal{V}*}$.

□

The previous proposition is based on the consideration that the properties 'being a morphism of sites' and 'being a comorphism of sites' can be formulated entirely at the level of the sites. We can state this as a general principle:

Metatheorem B.2. *Consider a morphism (resp. comorphism) of $\mathcal{U}$-sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$: any property P of $\mathbf{Sh}_{\mathcal{U}}(F)$ (resp. $(C_F)_{\mathcal{U}}$) that is site-theoretic is stable under extension.*

Proof. Suppose that $\mathbf{Sh}_{\mathcal{U}}(F)$ satisfies a property P if and only if the morphism of sites F satisfies a site-theoretic property Q: that is, Q can be expressed in terms of $\mathcal{U}$-small families of arrows and objects of the $\mathcal{U}$-sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$. Now consider any wider universe $\mathcal{V} \ni \mathcal{U}$: since $\mathcal{U}$-small sets are $\mathcal{V}$-small, if F satisfies Q as a morphism of $\mathcal{U}$-sites it obviously satisfies Q as a morphism of $\mathcal{V}$-sites, and thus $\mathbf{Sh}_{\mathcal{V}}(F)$ satisfies P. □

Notice that the converse may not hold: if the sites are not $\mathcal{U}$-small, we may have that F satisfies a property for $\mathcal{V}$-sites (for instance, involving a sufficiently large set of morphisms) without it satisfying the same property for $\mathcal{U}$-sites.

On the matter of restrictibility of properties, we can start by remarking that some constructions on sheaves are preserved and reflected by the inclusion $\mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J) \hookrightarrow \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J)$. For instance, it is known that a $\mathcal{U}$-small diagram D of $\mathcal{U}$-sheaves in $\mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J)$, has as co-/limit a $\mathcal{U}$-sheaf, and it coincides with the co-/limit calculated in $\mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J)$. In general, we will call *operation* any process through which we can associate to a diagram in a category a further object, such as the calculation of limits and colimits. We will say that an operation is *stable under restriction* if, whenever $\mathcal{U} \in \mathcal{V}$, the choice of $\mathcal{U}$-small diagrams of $\mathcal{U}$-sheaves yields a $\mathcal{U}$-sheaf as output. Now suppose that a a geometric morphism $G : \mathbf{Sh}_{\mathcal{V}}(\mathcal{D}, K) \to \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J)$ admits a restriction $F : \mathbf{Sh}_{\mathcal{U}}(\mathcal{D}, K) \to \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J)$, and suppose that G satisfies some property P which can be characterized by operations that are stable under restriction. Then of course F satisfies the same property, because the action of F on $\mathcal{U}$-sheaves corresponds (up to isomorphism) to that of G, and the property P, when applied to $\mathcal{U}$-small diagrams of $\mathcal{U}$-sheaves, yields again $\mathcal{U}$-sheaves. We end up with the following metatheorem:

Metatheorem B.3. *Consider a geometric morphism $G : \mathbf{Sh}_{\mathcal{V}}(\mathcal{D}, K) \to \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J)$: any property of G that can be formulated in terms of operations that are stable under restriction is stable under restriction.*

An immediate consequence of these two metatheorems are the two following results:

Proposition B.4. *Consider two universes $\mathcal{U} \in \mathcal{V}$, two $\mathcal{U}$-sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, and two geometric morphisms $F : \mathbf{Sh}_{\mathcal{U}}(\mathcal{D}, K) \to \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J)$ and $G : \mathbf{Sh}_{\mathcal{V}}(\mathcal{D}, K) \to \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J)$ such that F is a restriction of G. Then the following holds:*

(i) if G is an embedding, F is an embedding;

(ii) if G is a surjection, F is a surjection;

(iii) if G is essential, F is essential;

(iv) if G is local (i.e. G_* has a fully faithful right adjoint $G^!$), F is local; moreover, $F^!$ is the restriction of $G^!$ to $\mathcal{U}$-sheaves.

Proof. (i) The geometric morphism G is an inclusion if and only if G_* is fully faithful. Since the two functors $\mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J) \hookrightarrow \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J)$ and $\mathbf{Sh}_{\mathcal{U}}(\mathcal{D}, K) \hookrightarrow \mathbf{Sh}_{\mathcal{V}}(\mathcal{D}, K)$ are also fully faithful, this immediately forces F_* to be fully faithful and thus F is an embedding.

(ii) G is a surjection if and only if G^* is faithful. By applying an argument similar to that of the previous item, we immediately conclude that F^* is faithful and hence F is a surjection.

(iii) G is essential if and only if G^* has a further left adjoint, and this happens if and only if G^* preserves $\mathcal{V}$-small colimits: but this implies that F^* preserves $\mathcal{U}$-small colimits, and thus F is essential.

(iv) G_* has a right adjoint if and only if G_* preserves arbitrary limits: we then have the same argument as in the previous item. Notice now that the right adjoint $G^!$ of G_* an be defined as follows: for $P \in \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J)$, $G^!(P) : \mathcal{D}^{op} \to \mathbf{Set}$, $G^!(P)(D) := \mathbf{Sh}_{\mathcal{V}}(\mathcal{C}, J)(G_*(\ell_J(D)), P)$. If $P \in \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J)$, a quick computation shows that

$$G^!(P)(D) \simeq \mathbf{Sh}_{\mathcal{U}}(\mathcal{C}, J)(F_*(\ell_J(D)), P) = F^!(P)(D),$$

and thus $F^!$ restricts $G^!$. Finally, the considerations on the full faithfulness of $F^!$ are the same as those in the previous items.

□

Proposition B.5. *Consider a morphism of $\mathcal{U}$-sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$:*

(i) $\mathbf{Sh}_{\mathcal{U}}(F)$ is a surjection if and only if $\mathbf{Sh}_{\mathcal{V}}(F)$ is a surjection.

(ii) $\mathbf{Sh}_{\mathcal{U}}(F)$ is an embedding if and only if $\mathbf{Sh}_{\mathcal{V}}(F)$ is an embedding.

Consider a comorphism of $\mathcal{U}$-sites $F : (\mathcal{D}, J) \to (\mathcal{C}, J)$:

(iii) $(C_F)_\mathcal{U}$ *is a surjection if and only if* $(C_F)_\mathcal{V}$ *is a surjection.*

(iv) $(C_F)_\mathcal{U}$ *is an embedding if and only if* $(C_F)_\mathcal{V}$ *is an embedding.*

Proof. All the restrictions were proven in the previous result. Conversely:

(i) $\mathbf{Sh}_\mathcal{U}(F)$ is a surjection if and only if F is cover-reflecting, by [6, Theorem 6.3(i)]: since said property is site-theoretic, $\mathbf{Sh}_\mathcal{V}(F)$ is also a surjection.

(ii) By [6, Theorem 6.3(iii)], $\mathbf{Sh}_\mathcal{U}(F)$ is an embedding if and only if $F : (\mathcal{C}, J_F) \to (\mathcal{D}, K)$ is a weakly dense morphism of sites, where J_F is the topology over $\mathcal{C}$ of those sieves that are sent by F to K-covering families. In turn, [6, Proposition 5.5] formulates the weak denseness condition purely in site-theoretic terms, and thus $\mathbf{Sh}_\mathcal{V}(F)$ is also an embedding.

(iii) [6, Proposition 7.1] provides a purely site-theoretic description for $(C_F)_\mathcal{U}$ to be surjective, and thus it is stable under extension.

(iv) [6, Proposition 7.6] characterizes the property of $(C_F)_\mathcal{U}$ to be an inclusion quantifying over arrows of the site $(\mathcal{D}, K)$ and the presheaf topos $[\mathcal{D}^{op}, \mathbf{Set}_\mathcal{U}]$: therefore, said property is still satisfied when F is considered as a comorphism of $\mathcal{V}$-sites.

□

Appendix C

Conditions for a generic map to be étale

Given a topological space X and a function

$$\pi : E \to X,$$

we will analyse in this appendix some set-theoretic conditions under which π is an étale map. We have already mentioned in Section 6.1 that étale spaces come as colimits of their local sections: and indeed, it is the choice of those sections that we want to be continuous that forces all the relevant structure onto π. This is made very explicit from the following result:

Proposition C.1. *[3, Lemma 2.4.7] Consider a local homeomorphism $\pi : E \to X$: then the topology on E is the final topology making all its local sections continuous.*

Explicitly put, if we consider the family of continuous local sections of π, the topology on E is the *finest* topology making them continuous. As we can see, the topology on E is canonically defined once we know which sections are the continuous ones. Moreover, for a local homeomorphism $\pi : E \to X$ the two following properties always hold:

(i) E is covered by its local sections, i.e. the family of continuous sections of π is jointly epic;

(ii) the continuous sections of π are open.

We will see in a moment that assuming these two properties for a certain class of sections of π is enough to build a topology $\tau_\pi^{\mathcal{S}}$ on E such that π is a local homeomorphism with respect to it. After that, we will confront $\tau_\pi^{\mathcal{S}}$ with another topology $\sigma_\pi^{\mathcal{S}}$, which provides us with a subbase for the étale topology. Finally, we will consider necessary and sufficient conditions for a map to be étale. The content of this appendix is essentially that of Sections 6.1.1 and 6.1.2 of [8].

In general, for an open subset U of X we shall call *local section of* π *at* U any map $s : U \to E$ making the diagram

$$\begin{array}{ccc} U & \xrightarrow{\;s\;} & E \\ & \underset{i_U}{\searrow} & \downarrow{\scriptstyle \pi} \\ & & X \end{array}$$

commutative. We will denote by $Sec(\pi)$ the set of pairs (U, s), where $U \in \mathcal{O}(X)$ and $s : U \to E$ is a local section of π. Notice in particular that every section s of π is injective, since $\pi s = i_U$ is injective.

First of all, we consider on E the topology of Proposition C.1:

Proposition C.2. *Consider a topological space* X*, a map* $\pi : E \to X$ *of sets and a collection* $\mathcal{S} \subseteq Sec(\pi)$ *of local sections of* π*. The collection*

$$\tau_\pi^{\mathcal{S}} := \{W \subseteq E \mid \forall (U, s) \in \mathcal{S},\ s^{-1}(W) \in \mathcal{O}(U)\}$$

is a topology on E*, and it is the finest topology making all the local sections in* $\mathcal{S}$ *continuous. Moreover,* $\tau_\pi^{\mathcal{S}}$ *makes* π *continuous.*

Proof. Consider one local section $s : U \to E$ of π: the set

$$\tau_{(U,s)} := \{W \subseteq E \mid s^{-1}(W) \in \mathcal{O}(U)\}$$

provides a topology over E, which is the finest topology making s continuous. It follows that the finest topology making all the sections $s \in \mathcal{S}$ continuous will be the intersection of the topologies $\tau_{(U,s)}$ over all possible choices of $(U, s) \in \mathcal{S}$, which is

$$\bigcap_{(U,s)\in\mathcal{S}} \tau_{(U,s)} = \{W \subseteq E \mid \forall\ (U, s) \in \mathcal{S},\ s^{-1}(W) \in \mathcal{O}(U))\},$$

i.e. the topology $\tau_\pi^{\mathcal{S}}$ above. Finally, to prove that π is continuous, consider an open $V \in \mathcal{O}(X)$: then for every $(U, s) \in \mathcal{S}$, we have that $s^{-1}\pi^{-1}(V) = i_U^{-1}(V) := V \cap U$, which is open, and thus $\pi^{-1}(V) \in \tau_\pi^{\mathcal{S}}$. □

Let us consider the relevant properties for the sections of an étale map which we listed at the beginning of the section. First of all, joint surjectivity of $\mathcal{S}$ is connected to the openness of π:

Lemma C.3. *Consider* $\pi : E \to X$ *and* $\mathcal{S} \subseteq Sec(\pi)$ *as in Lemma C.2: if the local sections in* $\mathcal{S}$ *are jointly surjective over* E *then* π *is open.*

Proof. Consider any $W \in \tau_\pi^{\mathcal{S}}$ and any $(U, s) \in \mathcal{S}$. Notice that if $x \in s^{-1}(W)$, then $s(x) \in W$ and thus $x = \pi s(x) \in \pi(W)$: this implies that

$$\bigcup_{(U,s)\in\mathcal{S}} s^{-1}(W) \subseteq \pi(W)$$

for every $W \in \tau_\pi^{\mathcal{S}}$. Conversely, take $x \in \pi(W)$, i.e. $x = \pi(w)$ for some $w \in W$: since the local sections in $\mathcal{S}$ are jointly surjective, there is some $(U, s) \in \mathcal{S}$ and $u \in U$ such that $s(u) = w$, which implies $x = \pi s(u) = u \in s^{-1}(W)$. This proves the opposite inclusion, and since all the $s^{-1}(W)$ are open also $\pi(W)$ is open. □

Let us now turn our attention to openness of the sections. Since all the sections in $\mathcal{S}$ should be open with respect to $\tau_\pi^{\mathcal{S}}$, we should in particular require for every $(U, s) \in \mathcal{S}$ that $s(U)$ be open in E. Spelling out this request explicitly we obtain the following condition:

[†] for every (U, s), $(V, t) \in \mathcal{S}$ the set

$$t^{-1}(s(U)) = \{x \in V \cap U \mid s(x) = t(x)\}$$

is open in X.

Notice that [†] is in fact the generalization of a well known property of local homeomorphisms (see for instance [3, Lemma 2.4.6]), the *local equality condition* for sections: given two different local sections (U, s) and (U, t) of π and an element $x \in U$ such that $s(x) = t(x)$, there exists an open neighbourhood $W \subseteq U$ of x such that $s_{|W} = t_{|W}$.

Lemma C.4. *Consider $\pi : E \to X$ and $\mathcal{S} \subseteq Sec(\pi)$ as in Lemma C.2. If $\mathcal{S}$ satisfies [†] then its sections satisfy the local equality condition; the converse holds if $\mathcal{S}$ is closed under subsections, i.e. if $(U, s) \in \mathcal{S}$ and $W \subseteq U$ is open then $(W, s_{|W}) \in \mathcal{S}$.*

Proof. Suppose that all sections in $\mathcal{S}$ are open and consider two sections (U, s) and (U, t) and $x \in U$ such that $s(x) = t(x)$: then $x \in t^{-1}(s(U))$, which is an open subset of U such that $s_{|t^{-1}(s(U))} = t_{|t^{-1}(s(U))}$, and hence the local equality condition is satisfied.

Conversely, consider two sections (U, s) and (V, t) and an element $x \in t^{-1}(s(U))$: by the local equality condition, applied to the sections $(V \cap U, s_{|V \cap U})$ and $(V \cap U, t_{|V \cap U})$ of $\mathcal{S}$, there exists some open neighbourhood $Z \subseteq V \cap U$ of x such that $t_{|Z} = s_{|Z}$. This implies that $Z \subseteq t^{-1}(s(U))$ and hence $t^{-1}(s(W))$ is open in X. Since this holds for all choices of (U, s) and (V, t) we conclude that the sections of $\mathcal{S}$ are open. □

We are now ready to show that if $\mathcal{S}$ satisfies [†] and is jointly surjective then π can be made into a local homeomorphism:

Proposition C.5. *Consider a map $\pi : E \to X$, where X is a topological space, and a family of local sections $\mathcal{S} \subseteq Sec(\pi)$. Suppose that*

(i) the local sections of $\mathcal{S}$ are jointly surjective, i.e. for every $w \in E$ there exist $(U, s) \in \mathcal{S}$ and $x \in U$ such that $w = s(x)$;

(ii) $\mathcal{S}$ *satisfies* [†]*, i.e. for every two sections* (U, s) *and* (V, t) *in* $\mathcal{S}$ *the set* $t^{-1}(s(U)) = s^{-1}(t(V)) = \{x \in U \cap V \mid s(x) = t(x)\}$ *is open in* X*.*

If we endow E *with the topology* $\tau_\pi^{\mathcal{S}}$ *of Lemma C.2, the map* π *is a local homeomorphism, and the continuous sections of* π *are precisely those that are gluings of subsections of sections in* $\mathcal{S}$*.*

Proof. Consider an element $w \in E$: we want an open neighbourhood $W \in \tau_\pi^{\mathcal{S}}$ of w such that $\pi(W)$ is open and $\pi_{|W}$ is a homeomorphism.

Since the local sections of $\mathcal{S}$ are jointly surjective, there exists $(U, s) \in \mathcal{S}$ such that $w \in \mathrm{Im}(s)$. Since the sections of $\mathcal{S}$ are open, $s(U)$ is an open neighbourhood of w. Finally, consider the arrow

$$\pi_{|s(U)} : s(U) \to \pi s(U) = U :$$

it is continuous and open since it restricts π, which is continuous and open (by Lemma C.3); it is also bijective, since it is the inverse of $s : U \xrightarrow{\sim} s(U)$. Thus $\pi_{|s(U)}$ is a homeomorphism.

Finally, consider any continuous section $(V, t) \in Sec(\pi)$ of π, i.e. such that t is continuous; since π is a local homeomorphism, t is also open by Corollary 6.1.3. If we consider the open subset $t(V) \subseteq E$, by the joint surjectivity of the sections in $\mathcal{S}$ we have that there exist $(U_i, s_i) \in \mathcal{S}$ for $i \in I$ such that $t(V) \subseteq \bigcup_i s_i(U_i)$. If we consider the opens $Z_i := s_i^{-1}(t(V)) = \{x \in U_i \cap V \mid t(x) = s_i(x)\} \subseteq U_i \cap V$, we have that $(Z_i, s_{i|Z_i}) = (Z_i, t_{|Z_i})$ is a subsection both of (U_i, s_i) and (V, t). Now, for every $v \in V$ there exists $i \in I$ and $x \in U_i$ such that $t(v) = s_i(x)$: applying π this implies that $v = x$, thus $v \in U_i$ and $t(v) = s_i(v)$, so that $v \in Z_i$. This tells us that V is covered by the opens Z_i, and thus the section $t : V \to E$ is indeed the gluing of sections $t_{|Z_i} : Z_i \to E$, which are subsections of the sections (U_i, s_i) of $\mathcal{S}$. □

Remark C.1. The family $\mathcal{S}$ may be completely arbitrary, but there is no loss in generality if we consider it closed under subsections and gluings, since such a closure does not affect the induced topology $\tau_\pi^{\mathcal{S}}$. Indeed, suppose that $(U, s) \in \mathcal{S}$ and call $\mathcal{S}' := \mathcal{S} \cup \{(W, s_{|W})\}$ for a subsection $(W, s_{|W})$ of (U, s): then evidently $\tau_\pi^{\mathcal{S}'} \subseteq \tau_\pi^{\mathcal{S}}$. Conversely if we take $V \in \tau_\pi^{\mathcal{S}}$, we have that $(s_{|W})^{-1}(V) = s^{-1}(V) \cap W$, which is open in W, and hence $V \in \tau_\pi^{\mathcal{S}'}$, so that $\tau_\pi^{\mathcal{S}} = \tau_\pi^{\mathcal{S}'}$. Now consider $(U_i, s_i) \in \mathcal{S}$ such that for every i, j we have $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$, call $U = \cup_i U_i$ and $s : U \to E$ the map obtained by gluing the maps s_i. Call $\mathcal{S}' = \mathcal{S} \cup \{(U, s)\}$. Of course, $\tau_\pi^{\mathcal{S}'} \subseteq \tau_\pi^{\mathcal{S}}$. Conversely, consider $W \in \tau_\pi^{\mathcal{S}}$: in particular, for every i we have $s_i^{-1}(W)$ open in X, and hence $s^{-1}(W) = \cup_i s_i^{-1}(W)$ is open in X. Thus $W \in \tau_\pi^{\mathcal{S}}$, and again $\tau_\pi^{\mathcal{S}'} = \tau_\pi^{\mathcal{S}}$.

Consider a topology $\mathcal{O}(E)$ on E making π continuous, and denote by $Cont(\pi, \mathcal{O}(E)) \subseteq Sec(\pi)$ the class of local sections of π that are continuous when E is endowed with $\mathcal{O}(E)$. The last line of Proposition C.5 tells us that $Cont(\pi, \tau_\pi^{\mathcal{S}})$ is the closure of $\mathcal{S}$ under subsections and gluings. By the previous remark, we have that

$$\tau_\pi^{\mathcal{S}} = \tau_\pi^{Cont(\pi, \tau_\pi^{\mathcal{S}})}.$$

Now, suppose that the topology $\mathcal{O}(E)$ over E already makes π into a local homeomorphism: then Proposition C.1 states that

$$\mathcal{O}(E) = \tau_\pi^{Cont(\pi, \mathcal{O}(E))};$$

if we are given a smaller family of sections $\mathcal{S} \subseteq Cont(\pi, \mathcal{O}(E))$, we have for sure that

$$\mathcal{O}(E) = \tau_\pi^{Cont(\pi, \mathcal{O}(E))} \subseteq \tau_\pi^{\mathcal{S}},$$

and thus $\tau_\pi^{\mathcal{S}}$ is the *biggest* topology on E making π into a local homeomorphism with all the sections in $\mathcal{S}$ continuous.

Since $\tau_\pi^{\mathcal{S}}$ is an upper bound for the topologies making π a local homeomorphism, we can wonder which topology could be the lower bound. Such a topology σ must in particular contain as opens all sets $s(U)$ for (U, s). This leads us to consider on E the topology $\sigma_\pi^{\mathcal{S}}$, generated by the subbase

$$\mathcal{B}_{\mathcal{S}} = \{s(U) \subseteq E \mid (U, s) \in \mathcal{S}\}.$$

An open set W of $\sigma_\pi^{\mathcal{S}}$ is either E or a union of finite intersections of elements of $\mathcal{B}_{\mathcal{S}}$, i.e. something of the form

$$W = \bigcup_{i \in I} \left(\bigcap_{j=1}^{n_i} s_{i,j}(U_{i,j}) \right)$$

for $(U_{i,j}, s_{i,j}) \in \mathcal{S}$. This topology is closer in definition with those usually considered when studying structural sheaves and spectra for algebraic theories (we shall recall some examples later).

The first thing that we have to remark is that the continuity of π is not granted, when E is endowed with $\sigma_\pi^{\mathcal{S}}$. For instance, consider the set $X = \{0, 1\}$ with the discrete topology, $E = \{0, 1\}$ and take $\pi : E \to X$ to be the identity map; then take $\mathcal{S}$ to consist only of the section $\{0\} \hookrightarrow \{0, 1\}$. It is immediate to see that $\sigma_\pi^{\mathcal{S}} = \{\emptyset, \{0\}, \{0, 1\}\}$, and hence π is not continuous. On the other hand, continuity of the sections in $\mathcal{S}$ is directly related with the openness condition [†] and the inclusion of $\sigma_\pi^{\mathcal{S}}$ into $\tau_\pi^{\mathcal{S}}$:

Lemma C.6. *The following are equivalent:*

(i) the sections in $\mathcal{S}$ are continuous when E is endowed with $\sigma_\pi^{\mathcal{S}}$;

(ii) $\mathcal{S}$ satisfies [†], i.e. the sections in $\mathcal{S}$ are open with respect to $\tau_\pi^{\mathcal{S}}$;

(iii) $\sigma_\pi^{\mathcal{S}} \subseteq \tau_\pi^{\mathcal{S}}$.

Proof. (i)⇒(ii). Suppose that every section in $\mathcal{S}$ is continuous, and consider two sections (U, s) and (V, t): since $s(U) \in \mathcal{B}_{\mathcal{S}}$, we have that $t^{-1}(s(U))$ is open in X. As this holds for all choices of (U, s) and (V, t) in $\mathcal{S}$, we conclude that $\mathcal{S}$ satisfies [†].

(ii)⇒(iii). Consider a section $(U, s) \in \mathcal{S}$ and the corresponding basic open $s(U) \in \mathcal{B}_{\mathcal{S}}$. Since [†] holds, for every section $(V, t) \in \mathcal{S}$ the subset $t^{-1}(s(U))$ is open in X, and thus $s(U) \in \tau_\pi^{\mathcal{S}}$ by definition of $\tau_\pi^{\mathcal{S}}$. We have $\mathcal{B}_{\mathcal{S}} \subseteq \tau_\pi^{\mathcal{S}}$, which implies $\sigma_\pi^{\mathcal{S}} \subseteq \tau_\pi^{\mathcal{S}}$.

(iii)⇒ (i). Consider a section $(U, s) \in \mathcal{S}$ and any open $W \in \sigma_\pi^{\mathcal{S}}$: since $\sigma_\pi^{\mathcal{S}} \subseteq \tau_\pi^{\mathcal{S}}$, the subset $s^{-1}(W)$ is open in X by definition of $\tau_\pi^{\mathcal{S}}$. Since this holds for every choice of W, the section (U, s) is continuous. □

The opposite inclusion of topologies, on the other hand, does not seem to enjoy a similar nice characterization, but it is necessary for $\mathcal{S}$ to be jointly epic:

Lemma C.7. *The sections in $\mathcal{S}$ are jointly epic if and only if the opens in $\mathcal{B}_{\mathcal{S}}$ cover E. If this holds, $\tau_\pi^{\mathcal{S}} \subseteq \sigma_\pi^{\mathcal{S}}$.*

Proof. The first consideration is tautological. Consider $W \in \tau_\pi^{\mathcal{S}}$: by definition, for every section $(U, s) \in \mathcal{S}$ we have $s^{-1}(W)$ open in X. Now, suppose that the sections in $\mathcal{S}$ are jointly epic: then for every $w \in W$ there exists $(U, s) \in \mathcal{S}$ and $x \in U$ such that $s(x) = w$. Since $x \in s(s^{-1}(W)) \subseteq W$ we have that W is covered by opens of the form $s(s^{-1}(W))$, which are basic opens in $\mathcal{B}_{\mathcal{S}}$, we can conclude that $W \in \sigma_\pi^{\mathcal{S}}$. □

The previous considerations imply the following:

Corollary C.8. *Consider a map $\pi : E \to X$, with X a topological space, and $\mathcal{S} \subseteq Sec(\pi)$ jointly epic and satisfying* [†]*: then $\mathcal{B}_{\mathcal{S}}$ is a subbase for the topology $\tau_\pi^{\mathcal{S}}$.*

Proof. If $\mathcal{S}$ satisfies [†] then $\sigma_\pi^{\mathcal{S}} \subseteq \tau_\pi^{\mathcal{S}}$, while if the sections in $\mathcal{S}$ are jointly epic the opposite inclusion holds. □

Notice that if $\mathcal{S}$ is jointly surjective and satisfies [†] there is exactly one possible topology on E making π into a local homeomorphism such that the sections in $\mathcal{S}$ are continuous, since any other such topology must be squeezed between $\tau_\pi^{\mathcal{S}}$ and $\sigma_\pi^{\mathcal{S}}$ (which coincide). The equality of the two topologies $\tau_\pi^{\mathcal{S}}$ and $\sigma_\pi^{\mathcal{S}}$ alone is in general not enough to deduce that π is a local homeomorphism; it does however imply that its restriction to the joint image of $\mathcal{S}$ is:

Proposition C.9. *If $\sigma_\pi^{\mathcal{S}} = \tau_\pi^{\mathcal{S}}$, then the composite*

$$\bar{\pi} : \bar{E} = \bigcup_{(U,s) \in \mathcal{S}} s(U) \hookrightarrow E \xrightarrow{\pi} X$$

is a local homeomorphism when $\bar{E}$ is endowed with the topology $\sigma_\pi^{\mathcal{S}}$.

Proof. We can consider $\mathcal{S}$ as a family of sections whose domain is $\bar{E}$ instead of E: of course then the sections in $\mathcal{S}$ are jointly surjective over $\bar{E}$. On the other hand, the inclusion $\sigma_\pi^{\mathcal{S}} \subseteq \tau_\pi^{\mathcal{S}}$ implies that $\mathcal{S}$ satisfies [†], and thus $\bar{\pi}$ is a local homeomorphism. □

Let us conclude this section with a couple of further considerations. The first is that there are some canonical bases for the topology $\tau_\pi^{\mathcal{S}}$, which include $\mathcal{B}_{\mathcal{S}}$ whenever $\mathcal{S}$ is closed under subsections:

Proposition C.10. *Consider $\pi : E \to X$ and $\mathcal{S} \subseteq Sec(\pi)$ as above, and suppose that $\mathcal{S}$ is jointly surjective and satisfies* [†]*:*

(i) If $\mathcal{A}$ is a base for X, then

$$\mathcal{B}_{\mathcal{A}}^{\mathcal{S}} := \{s(V) \mid (U,s) \in \mathcal{S},\ V \in \mathcal{A},\ V \subseteq U\}$$

is a basis for $\tau_\pi^{\mathcal{S}}$;

(ii) The collection

$$\{s(V) \mid (U,s) \in \mathcal{S}, V \subseteq U,\ V \text{ open}\}$$

is a basis for $\tau_\pi^{\mathcal{S}}$;

(iii) If $\mathcal{S}$ is closed under subsections, $\mathcal{B}_{\mathcal{S}}$ is a basis for $\tau_\pi^{\mathcal{S}}$.

Proof. (i) First of all, let us show that $\mathcal{B}_{\mathcal{A}}^{\mathcal{S}}$ is indeed a basis. If $(U,s) \in \mathcal{S}$ and $w \in s(U)$ then $w = s(x)$ for some $x \in U$. Since U is open in X, there must exist some open $A \in \mathcal{A}$ such that $x \in A \subseteq U$, and thus $w \in s(A)$. This implies immediately that the opens in $\mathcal{B}_{\mathcal{A}}^{\mathcal{S}}$ cover E whenever $\mathcal{S}$ is jointly surjective. Consider now two sections (U,s) and (V,t), two basic opens U', $V' \in \mathcal{A}$ such that $U' \subseteq U$ and $V' \subseteq V$ and an element $w \in s(U') \cap t(V')$: there must exist some $x \in t^{-1}(s(U)) = \{x \in U \cap V \mid s(x) = t(x)\}$, which is open in X by the condition [†], such that $w = s(x) = t(x)$. Moreover $x \in U' \cap V'$, and hence we can take a small enough open $A \in \mathcal{A}$ such that $x \in A \subseteq U' \cap V' \cap t^{-1}(s(U))$. This immediately implies that $s_{|A} = t_{|A}$ and $w \in s(A) \subseteq s(U') \cap t(V')$, and thus $\mathcal{B}_{\mathcal{A}}^{\mathcal{S}}$ is a basis.

Every open $\mathcal{B}_{\mathcal{A}}^{\mathcal{S}}$ is open in $\tau_\pi^{\mathcal{S}}$: indeed, if we take $(U,s) \in \mathcal{S}$ and $U' \in \mathcal{A}$ such that $U' \subseteq U$, then for every other element $(V,t) \in \mathcal{S}$ it holds that $t^{-1}(s(U')) = t^{-1}(s(U)) \cap U'$, which is open in X. Therefore, $\mathcal{B}_{\mathcal{A}}^{\mathcal{S}} \subseteq \tau_\pi^{\mathcal{S}}$. Finally, let us show that every open in $\tau_\pi^{\mathcal{S}}$ can be covered by opens in the basis $\mathcal{B}_{\mathcal{A}}^{\mathcal{S}}$. If we consider $w \in W \in \tau_\pi^{\mathcal{S}}$, since $\mathcal{S}$ is jointly surjective there must exist $(U,s) \in \mathcal{S}$ such that $w \in s(U)$, i.e. $w = s(x)$ for some $x \in U$. By definition of $\tau_\pi^{\mathcal{S}}$, the set $s^{-1}(W) \subseteq U$ is open in X, and thus there must exist a basic open $A \in \mathcal{A}$ such that $x \in A \subseteq s^{-1}(W)$, which implies $w \in s(A) \subseteq s(U)$.

(ii) It follows from the previous item, when we take $\mathcal{A} = \mathcal{O}(X)$.

(iii) If $\mathcal{S}$ is closed under subsections it is immediate to see that $\mathcal{B}_{\mathcal{S}} = \mathcal{B}^{\mathcal{S}}_{\mathcal{O}(X)}$. □

Finally, we show that the request for the existence of a family of sections $\mathcal{S}$ can be formulated in terms of a 'local bijection' between the two sets E and X:

Proposition C.11. *Consider a map $p : E \to X$, with X a topological space. The following are equivalent:*

(i) There exists a topology on E such that π is a local homeomorphism;

(ii) There exist a family of sets $\{W_i \subseteq E \mid i \in I\}$ such that

(a) $\bigcup_i W_i = E$;

(b) for every i, $\pi_{|W_i} : W_i \to \pi(W_i)$ is bijective and $\pi(W_i) \subseteq X$ is open;

(c) for every $i, j \in I$, $\pi(W_i \cap W_j) \subseteq X$ is open.

Proof. (i)⇒(ii) is obvious: if π is a local homeomorphism then E is endowed with $\mathcal{O}(E) = \tau_\pi^{Cont(\pi,\mathcal{O}(E))}$. Every $w \in E$ belongs to $s(U)$ for some continuous section $(U, s) \in Cont(\pi, \mathcal{O}(E))$, and $\pi_{|s(U)} : s(U) \to U$ is a homeomorphism. Moreover, one easily verifies that

$$\pi(s(U) \cap t(V)) = t^{-1}(s(U)),$$

which is open in X.

(ii)⇒(i). We can set $U_i := \pi(W_i) \subseteq X$ and

$$s_i : U_i \xrightarrow{\pi^{-1}_{|W_i}} W_i \hookrightarrow E :$$

then one immediately sees that $\mathcal{S} = \{(U_i, s_i) \mid i \in I\}$ is a family of sections for π which is jointly surjective. Finally, a quick calculation shows that

$$\pi(W_i \cap W_j) = s_i^{-1}(s_j(U_j)),$$

which is open by hypothesis: this means that $\mathcal{S}$ also satisfies [†], and thus the topology $\tau_\pi^{\mathcal{S}}$ over E makes π into a local homeomorphism. □

Algebraic spectra and their structural sheaves

As an application of the previous results, we consider two famous cases of spectra of algebraic theories, and show that the usual topologies on their structural sheaves can be recovered using canonical classes of local sections.

Consider a commutative ring R. We recall that the spectrum $\mathrm{Spec}(R)$ is defined as the set of its prime ideals. It can be endowed with the Zariski topology, whose base is that of opens of the kind

$$D(a) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid a \notin \mathfrak{p}\}$$

for every $a \in R$. Now, consider the set

$$E := \coprod_{\mathfrak{p} \in \mathrm{Spec}(R)} R_{\mathfrak{p}} \xrightarrow{\pi} \mathrm{Spec}(R),$$

where π is the canonical map sending each component $R_{\mathfrak{p}}$ to $\mathfrak{p}$. It is well known (see for instance [16, pag. 70]) that the set E can be topologized so that the projection π is a local homeomorphism: the corresponding sheaf $\mathrm{Spec}(R)^{op} \to \mathbf{Set}$ is the *structural sheaf* of the ring R. The topology $\mathcal{O}(E)$ on E is induced by the base

$$\mathcal{B} = \big\{B(a,g) := \{[g]_{\mathfrak{p}} \in R_{\mathfrak{p}} \mid \mathfrak{p} \in D(a)\} \bigm| a \in R,\ g \in R[a^{-1}]\big\},$$

where $[g]_{\mathfrak{p}}$ denotes the class of g seen as an element of the localization $R_{\mathfrak{p}}$. What we can remark now is that $\mathcal{O}(E)$ can now be understood as induced by the canonical class of sections

$$\mathcal{S} = \{s_g : D(a) \to E \mid a \in R,\ g \in R[a^{-1}],\ s_g(\mathfrak{p}) := [g]_{\mathfrak{p}}\} \subseteq Sec(\pi).$$

We notice immediately that $B(a,g) := s_g(D(a))$, and therefore $\mathcal{B} = \mathcal{B}_{\mathcal{S}}$. The sections in $\mathcal{S}$ are obviously jointly epic: every $h \in R_{\mathfrak{p}}$ is a fraction of the kind x/y where $y \notin \mathfrak{p}$, and thus $h = [x/y]_{\mathfrak{p}} = s_{x/y}(\mathfrak{p})$ for $s_{x/y} : D(y) \to E$. We can also show that $\mathcal{S}$ satisfies [†]. To do this consider any two sections $s_{x/a^n} : D(a) \to E$ and $s_{z/b^m} : D(b) \to E$, and take

$$\mathfrak{p} \in s^{-1}_{z/b^m}(s_{x/a^n}(D(a))) = \{\mathfrak{p} \in D(a) \cap D(b) \mid [x/a^n]_{\mathfrak{p}} = [z/b^m]_{\mathfrak{p}}\} :$$

since $[x/a^n]_{\mathfrak{p}} = [z/b^m]_{\mathfrak{p}}$ in $R_{\mathfrak{p}}$, there exists $w \notin \mathfrak{p}$ such that $wxb^m = wza^n$. Now, notice that $\mathfrak{p}$ also belongs to the open $D(wab) = D(w) \cap D(a) \cap D(b)$; conversely, if $\mathfrak{q} \in D(wab)$ then

$$[x/a^n]_{\mathfrak{q}} = [wxb^m/wa^nb^m]_{\mathfrak{q}} = [wza^n/wa^nb^m]_{\mathfrak{q}} = [z/b^m]_{\mathfrak{q}},$$

which means that $D(wab) \subseteq s^{-1}_{z/b^m}(s_{x/a^n}(D(a)))$. Therefore, we have that the subset $s^{-1}_{z/b^m}(s_{x/a^n}(D(a)))$ is open in $\mathrm{Spec}(R)$. Applying Proposition C.5, we conclude that $\tau^{\mathcal{S}}_{\pi}$ makes π into an étale space over $\mathrm{Spec}(R)$; on the other hand, since $\mathcal{S}$ is obviously closed under subsections we can apply Proposition C.10 to conclude that $\mathcal{B}$, as defined above, is a basis for E. In this way, taking the sheaf of cross-sections of this bundle, we recover the structure sheaf of A, as described in [16].

We now turn our attention to spectra of MV-algebras. We recall that an *MV-algebra* (see for instance [10]) is an abelian monoid $(A, \oplus, 0)$ endowed with a further 1-ary operation $\neg$ satisfying the axioms $\neg\neg x = x$, $x \oplus (\neg 0) = \neg 0$ and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. In particular we set $1 := \neg 0$. Any MV-algebras admits an order relation $\leq$ defined by $x \leq y$ if and only if there exists $z \in A$ such that $x \oplus z = y$. We will also make use of the derived operation $x \ominus y := \neg(\neg x \oplus y)$: in particular, $x \leq y$ if and only if $x \ominus y = 0$. Finally, there is a binary *distance operation* d, defined by $d(x, y) := (x \ominus y) \oplus (y \ominus x)$.

An *ideal* of A is a submonoid $I \subseteq A$ such that if $y \in I$ and $x \leq y$ then $x \in I$. The ideal I is said to be *prime* if it is strictly contained in A, and moreover for every x and y in A either $x \ominus y \in I$ or $y \ominus x \in I$. Every ideal I of A provides a congruence $\sim_I$ on the elements of A, defined by $x \sim_I y$ if and only if $d(x, y) \in I$. The quotient by said congruence is denoted by A/I.

We define the *spectrum of* A to be the set $\mathrm{Spec}(A)$ of prime ideals of A. For each $a \in A$, we define $W(a) := \{I \in \mathrm{Spec}(A) \mid a \in I\}$. One immediately checks that $W(0) = \mathrm{Spec}(A)$, $W(1) = \emptyset$ and $W(a) \cap W(b) = W(a \oplus b)$, and thus the sets $W(a)$ are a basis for a topology over $\mathrm{Spec}(A)$. We now introduce the set

$$E := \coprod_{I \in Spec(A)} A/I \xrightarrow{\pi} \mathrm{Spec}(A).$$

It can be endowed with a topology making it into an étale space over $\mathrm{Spec}(A)$, induced by the class of global sections

$$\mathcal{S} = \{\bar{a} : \mathrm{Spec}(A) \to E \mid a \in A,\ \bar{a}(I) := [a]_I\}.$$

Said sections are of course jointly epic over E. As per the condition [†], one can check that for a pair of sections $\bar{a}$ and $\bar{b}$ one has

$$\bar{b}^{-1}(\bar{a}(\mathrm{Spec}(A))) = W(d(a, b)),$$

which is open in $\mathrm{Spec}(A)$: thus the topology $\tau_\pi^{\mathcal{S}}$ makes π into a local homeomorphism. Applying Proposition C.10, we have that the topology $\tau_\pi^{\mathcal{S}}$ is generated by the basis

$$\mathcal{B}^{\mathcal{S}}_{\{W(a)\ \mid\ a \in A\}} = \{\bar{b}(W(a)) | a \in A,\ b \in A\}.$$

We thus recover, by taking the sheaf of cross-sections of this bundle, the structure sheaf of A, as constructed in [10].

Appendix D

Technical lemmas in category theory

Adjunctions between functor categories

Lemma D.1 [34, Proposition 4.4.6]. *Consider a pair of adjoint functors $F \dashv G : \mathcal{A} \to \mathcal{B}$. For any category $\mathcal{C}$ there is an adjunction*

$$[\mathcal{B}, \mathcal{C}] \underset{(-\circ G)}{\overset{(-\circ F)}{\rightleftarrows}} [\mathcal{A}, \mathcal{C}] \quad (\top) \ .$$

Corollary D.2. *Consider a reflective subcategory* $\mathrm{a} \dashv i : \mathcal{A} \hookrightarrow \mathcal{C}$ *and another category* $\mathscr{F}$. *For any pair of functors* $F, G : \mathcal{A} \rightrightarrows \mathscr{F}$, *there is a natural bijection*

$$[\mathcal{A}, \mathscr{F}](F, G) \cong [\mathcal{C}, \mathscr{F}](F\mathrm{a}, G\mathrm{a})$$

Proof. We recall that a right adjoint is full and faithful if and only if the counit of the adjunction is a natural isomorphism (see for instance [34, Lemma 4.5.13]): thus the counit ε of $\mathrm{a} \dashv i$ is a natural isomorphism. Therefore we have the chain of natural bijections

$$[\mathcal{A}, \mathscr{F}](U, V) \cong [\mathcal{A}, \mathscr{F}](U\mathrm{a}i, V) \cong [\mathcal{C}, \mathscr{F}](U\mathrm{a}, V\mathrm{a})$$

where the bijection on the left is just composition with $U\varepsilon$, while the bijection on the right comes from the previous lemma. □

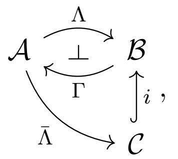
Joint conservativity and preservation of co-/limits

Lemma D.3. *Consider a functor $F : \mathcal{A} \to \mathcal{B}$ and a family of jointly conservative functors $C_i : \mathcal{B} \to \mathcal{C}_i$. Suppose that $\mathcal{A}$, $\mathcal{B}$ and all the $\mathcal{C}_i$ have (co)limits of shape $\mathcal{I}$, and that all the functors C_i and C_iF preserve said (co)limits: then F also preserves said (co)limits.*

Proof. We prove this for colimits, but the argument is the same for limits. Consider a diagram $D : \mathcal{I} \to \mathcal{A}$ and consider in $\mathcal{B}$ the unique arrow $m : \operatorname{colim}(FD) \to F(\operatorname{colim} D)$. If we apply C_i we obtain $C_i(m) : C_i(\operatorname{colim}(FD)) \to C_iF(\operatorname{colim} D)$: but both C_i and C_iF commute with limits of shape $\mathcal{I}$, thus $C_i(m)$ is equivalent up to canonical isomorphisms to the identity of $\operatorname{colim}(C_iFD)$. Since the functors C_i are jointly conservative this immediately implies that m is an isomorphism, and hence F preserves colimits of shape $\mathcal{I}$. □

Restriction of adjunctions to full subcategories

Lemma D.4. *Consider the diagram of 2-categories*



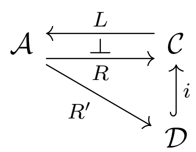
where i is 2-fully faithful. If $i\bar{\Lambda} \simeq \Lambda$, then $\bar{\Gamma} := \Gamma i$ is a right adjoint to $\bar{\Lambda}$, and $\Gamma\Lambda \simeq \bar{\Gamma}\bar{\Lambda}$. The same holds for 1-categories and i fully faithful.

Proof. It is immediate by the following chain of pseudonatural equivalences (resp. natural isomorphisms):

$$\mathcal{A}(X, \Gamma i(Y)) \simeq \mathcal{B}(\Lambda(X), i(Y)) \simeq \mathcal{B}(i\bar{\Lambda}(X), i(Y)) \simeq \mathcal{C}(\bar{\Lambda}(X), Y).$$

□

Lemma D.5. *Consider a diagram of 2-functors*



and suppose $R \cong i \circ R'$: then $L \circ i \dashv R'$.

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Index

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