

ON PRO- p GROUPS WITH QUADRATIC COHOMOLOGY

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ABSTRACT. The main purpose of this article is to study pro- p groups with quadratic \mathbb{F}_p -cohomology algebra, i.e. H^\bullet -quadratic pro- p groups. Prime examples of such groups are the maximal Galois pro- p groups of fields containing a primitive root of unity of order p .

We show that the amalgamated free product and HNN-extension of H^\bullet -quadratic pro- p groups is H^\bullet -quadratic, under certain necessary conditions. Moreover, we introduce and investigate a new family of pro- p groups that yields many new examples of H^\bullet -quadratic groups: p -RAAGs. These examples generalise right angled Artin groups in the category of pro- p groups. Finally, we explore “Tits alternative behaviour” of H^\bullet -quadratic pro- p groups.

1. INTRODUCTION

1.1. Number-theoretic motivation. For a field K , let \bar{K}_s denote its separable closure. The absolute Galois group of K is the profinite group $G_K = \text{Gal}(\bar{K}_s/K)$. One of the main challenges in current Galois theory is to describe absolute Galois groups of fields among profinite groups. Already describing the maximal pro- p quotient $G_K(p)$ of G_K among pro- p groups, for p a prime number, is a remarkable challenge.

The most amazing advancement in Galois theory in the last decades is the proof of the *Bloch-Kato Conjecture* by M. Rost and V. Voevodsky, with the contribution of Ch. Weibel (cf. [44] [49], [17]). One of the most important consequences of this result — now called the *Norm Residue Theorem* — is the following: if K contains a root of unity of order p , then the \mathbb{F}_p -cohomology algebra of G_K — i.e. the graded algebra $\bigoplus_{n \geq 0} H^n(G_K, \mathbb{F}_p)$, endowed with the cup product and with \mathbb{F}_p as a trivial G_K -module — is a *quadratic algebra* over \mathbb{F}_p , namely, all its elements of positive degree are combinations of products of elements of degree 1, and its defining relations are homogeneous relations of degree 2 (see Definition 2.1). This property is inherited by the maximal pro- p quotient $G_K(p)$ (cf. [36, § 2]).

It is therefore of major interest to study pro- p groups whose \mathbb{F}_p -cohomology algebra is quadratic, which we call H^\bullet -*quadratic* (or simply *quadratic*) pro- p groups. This class has been investigated in some recent papers, in order to find obstructions

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in the realization of pro- p groups as the maximal pro- p quotient of absolute Galois groups (cf. [36, 37, 39, 47]). Unfortunately, as it often happens in profinite group theory, there is an astounding lack of examples of H^\bullet -quadratic pro- p groups. We will try to partly remedy this lack of examples in our work.

1.2. Main Results. In the present paper we start a systematic investigation of H^\bullet -quadratic pro- p groups from various points of view.

First of all, we deal with torsion in H^\bullet -quadratic pro- p groups. In the next proposition we will collect some facts that might be already known to experts. For further discussion on finite H^\bullet -quadratic 2-groups see Remark 2.10.

Proposition A. *Let G be a finitely generated pro- p group.*

- (a) *If p is odd and G is H^\bullet -quadratic, then G is torsion-free.*
- (b) *If $p = 2$, G is finite and abelian, then G is H^\bullet -quadratic if and only if G is 2-elementary abelian.*
- (c) *If $p = 2$ and G is finite, then every subgroup of G is quadratic if and only if G is 2-elementary abelian.*

Next, we study the closure of the class of H^\bullet -quadratic pro- p groups under free constructions in the category of pro- p groups, such as *amalgamated free products* and *HNN-extensions* (see Section 3 for the definitions). The corresponding results for free products and direct products are well known to experts. In Section 3 we prove the following theorems.

Theorem B. *Let G be a finitely generated pro- p group which can be written as a proper amalgam $G = G_1 \amalg_H G_2$ with H^\bullet -quadratic pro- p groups G_1, G_2, H . Assume that the restriction maps*

$$(1.1) \quad \text{res}_{G_i, H}^\bullet : H^\bullet(G_i, \mathbb{F}_p) \longrightarrow H^\bullet(H, \mathbb{F}_p),$$

with $i = 1, 2$, satisfy the following conditions:

- (i) $\text{res}_{G_1, H}^1$ and $\text{res}_{G_2, H}^1$ are surjective;
- (ii) $\ker(\text{res}_{G_i, H}^2) = \ker(\text{res}_{G_i, H}^1) \wedge H^1(G_i, \mathbb{F}_p)$ for both $i = 1, 2$.

Then also G is H^\bullet -quadratic.

Theorem C. *Let G_0 be a H^\bullet -quadratic pro- p group, and let $A, B \leq G_0$ two isomorphic H^\bullet -quadratic subgroups, with isomorphism $\phi : A \rightarrow B$. Assume that*

- (i) *the restriction map $\text{res}_{G_0, A}^1$ is surjective,*
- (ii) *$\ker(\text{res}_{G_0, A}^2) = \ker(\text{res}_{G_0, A}^1) \wedge H^1(G_0, \mathbb{F}_p)$ and*
- (iii) *the map $f_{G_0}^1 : H^1(G_0, \mathbb{F}_p) \rightarrow H^1(A, \mathbb{F}_p)$ is the zero map, where*

$$f_{G_0}^1 = \text{res}_{G_0, A}^1 - \phi^* \circ \text{res}_{G_0, B}^1,$$

and $\phi^* : H^1(B, \mathbb{F}_p) \rightarrow H^1(A, \mathbb{F}_p)$ is the map induced by ϕ .

Assume further that $G = \text{HNN}(G_0, A, \phi)$ is proper. Then G is H^\bullet -quadratic.

Moreover, we exhibit several examples to show that each numbered condition in Theorem B and Theorem C is essential for the proofs (see Examples 3.7, 3.8, 3.14, 3.15 and 3.16). In passing we find two new criteria to ensure that an amalgam of pro- p groups is *proper*, see Proposition 5.19 and Proposition 5.22. In particular, in Proposition 5.19 we show that the amalgamated free product of two uniform groups G_1 and G_2 over an isomorphic uniform subgroup H is always proper, provided that the generators of H are part of bases of both G_1 and G_2 .

In the rest of this section p will be a prime number greater or equal to 3.
For $p = 2$ see Section 1.3.

We proceed by collecting some known results on p -adic analytic pro- p groups to characterise H^\bullet -quadratic groups in this class (see Section 4 for the definitions).

Theorem D. *Let p be an odd prime. A p -adic analytic pro- p group G is H^\bullet -quadratic if and only if G is uniform.*

Note that this provides a new characterisation of uniform pro- p groups among p -adic analytic ones. As a direct consequence we have that if a p -adic analytic pro- p group G can be realised as $G_K(p)$ for some K containing a root of unity of order p then every open subgroup of G is uniform, i.e., G is a hereditarily uniform pro- p group. Note that hereditarily uniform pro- p groups have been classified and all those groups can be realised as maximal pro- p Galois groups (cf. [36], [51] and [21]; see also [20]).

We continue by looking at H^\bullet -quadratic groups from a more “combinatorial” point of view. Namely we introduce a new class of pro- p groups, called *generalised Right Angled Artin pro- p Groups* — p -RAAGs for short. Define a p -graph $\Gamma = (\mathcal{G}, f)$ to be an oriented graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ together with labels $f(e) = (f_1(e), f_2(e)) \in p\mathbb{Z}_p \times p\mathbb{Z}_p$ for every edge $e \in \mathcal{E}$. The p -RAAG associated to Γ is the pro- p group defined by the pro- p presentation

$$G_\Gamma = \langle x_1, \dots, x_n \in \mathcal{V} \mid [x_i, x_j] = x_i^{f_1(e)} x_j^{f_2(e)} \text{ for } e = (x_i, x_j) \in \mathcal{E} \rangle$$

(see Definition 5.3 for the precise details). The class of p -RAAGs is extremely interesting. For instance, we can completely determine the \mathbb{F}_p -cohomology algebra of an H^\bullet -quadratic p -RAAG and this only depends on the underlying graph.

Theorem E. *Let p be an odd prime, and let G_Γ be an H^\bullet -quadratic p -RAAG with associated p -graph $\Gamma = (\mathcal{G}, f)$. Then*

$$H^\bullet(G_\Gamma, \mathbb{F}_p) \cong \Lambda_\bullet(\mathcal{G}^{\text{op}}),$$

where $\Lambda_\bullet(\mathcal{G}^{\text{op}})$ denotes the exterior Stanley-Reisner \mathbb{F}_p -algebra associated to the graph \mathcal{G} .

In Theorem 5.32 we show that there is a further condition, besides yielding a H^\bullet -quadratic pro- p RAAG, that a p -graph Γ must satisfy in order to generate a p -RAAG which occurs as a Galois group $G_K(p)$ for some field K .

Next we prove that p -RAAGs provide an abundance of new examples of H^\bullet -quadratic pro- p groups.

Theorem F. *Let p be an odd prime, and let $\Gamma = (\mathcal{G}, f)$ be a p -graph. Then the following are equivalent.*

- (i) \mathcal{G} is triangle-free.
- (ii) The generalised p -RAAG G_Γ is mild.
- (iii) G_Γ has cohomological dimension $\text{cd}(G_\Gamma) = 2$.

Furthermore, if these conditions are satisfied, G_Γ is quadratic.

In fact, by a classical theorem of P. Erdős et al., the number of graphs on n vertices that do not involve a “triangle” is asymptotic to $2^{n^2/4}$ for n that tends to infinity; while the total number of graphs on n vertices is asymptotic to $2^{n^2/2}$ for n

that tends to infinity. The condition of *mildness* for a pro- p group, introduced by J. Labute in [23], is quite technical and we direct the reader to § 2.5 and references therein for the definition.

It is worth underlining that by Theorems E–F, the family of generalised p -RAAGs is a rich source of pro- p groups whose \mathbb{F}_p -cohomology algebras and \mathbb{F}_p -graded group algebras are *Koszul*, see Corollary 5.34 (for an overview on Koszulity in Galois cohomology see [27]).

In light of Theorem F, we start a careful investigation of p -RAAGs associated to triangle p -graphs. It so happens that a classical example of Mennicke, of a *finite* 3-generated 3-related pro- p group, can be written as a triangle p -RAAG (cfr. Example 5.14); so there are p -RAAGs that are not quadratic, because they contain non-trivial torsion. Furthermore, it is in general very hard to decide whether a given presentation of a pro- p group yields a finite group. So we content ourselves to classify the possible H^\bullet -quadratic p -RAAGs that arise from triangle p -RAAGs.

Theorem G. *Let p be an odd prime, and G be a p -RAAG associated to a triangle p -graph, i.e., let*

$$(1.2) \quad G = \langle x, y, z \mid [x, y] = x^{\alpha_1} y^{\alpha_2}, [y, z] = y^{\beta_2} z^{\beta_3}, [z, x] = z^{\gamma_3} x^{\gamma_1} \rangle,$$

where $\alpha_1, \alpha_2, \beta_2, \beta_3, \gamma_1, \gamma_3 \in p\mathbb{Z}_p$. If G is H^\bullet -quadratic, then either:

- (a) G is a metabelian uniform pro- p group; or
- (b) G is isomorphic to a 3-dimensional uniform open subgroup of the p -Sylow subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$.

Additionally, if G is metabelian in part (a) of the previous theorem, it has to belong to one of three different explicitly described families of pro- p groups. We investigate also part (b) in more detail: namely, we explicitly realise $\mathrm{SL}_2^1(\mathbb{Z}_p)$ as a triangle p -RAAG (see Proposition 6.10). Observe that if a pro- p group G as in (1.2) occurs as the maximal pro- p Galois group of a field containing a root of unity of order p , then it is well-known that condition (a) of Theorem G holds true (cf. [51, Thm. 3]). Therefore, Theorem G recovers partially, in a purely group-theoretic way, a well-known result in Galois theory.

To boot, we analyse whether all *torsion-free* p -RAAGs are H^\bullet -quadratic. We fall short of proving this in full generality, but we can use Theorem B to deal with a large number of them (see § 5.6). In particular, we can prove the following theorem for p -RAAGs associated to *chordal* graphs. A graph is chordal if all cycles of four or more vertices have a chord (see Definition 5.27).

Theorem H. *Let p be an odd prime, and let $\Gamma = (\mathcal{G}, f)$ be a p -graph with \mathcal{G} a chordal graph, such that the associated p -RAAG G_Γ is uniform completable. Then G is a quadratic pro- p group.*

Moreover, we can prove that all p -RAAGs arising from p -graphs on 5 vertices are H^\bullet -quadratic, if they are *uniform completable* (cf. Definition 5.15).

Finally, we investigate the presence of free non-abelian closed subgroups in H^\bullet -quadratic pro- p groups. In the arithmetic case one has a ‘‘Tits alternative type’’ result: if the field K contains a roots of unity of order p , then either $G_K(p)$ contains a free non-abelian closed pro- p subgroup, or every finitely generated subgroup is uniform (cf. [36, Thm. B], [51, Thm. 3] and [5, § 3.1]). We prove the following (see § 7).

Theorem I. *Let p be an odd prime, and let G_Γ be a p -RAAG, with associated p -graph $\Gamma = (\mathcal{G}, f)$. Then either G_Γ is a powerful pro- p group, or it contains a free non-abelian closed pro- p subgroup.*

As an immediate corollary of the previous theorem we deduce: a quadratic p -RAAG is either uniform, or it contains a free non-abelian closed pro- p subgroup.

We also investigate the presence of free non-abelian subgroups in mild pro- p groups. For this, in Proposition 7.5 we show that —under certain conditions— several mild pro- p groups are generalised Golod-Shafarevic pro- p groups (see Section 2.6 for the definition). As a corollary, we deduce that all H^\bullet -quadratic groups with at most 3 generators are either uniform or contain a closed free non-abelian pro- p subgroup (see Corollary 7.6). Motivated by the aforementioned results, we formulate the following.

Conjecture J. *Let G be a finitely generated H^\bullet -quadratic pro- p group. Then either G is a uniform pro- p group (virtually uniform pro- p group if $p = 2$), or it contains a closed free non-abelian pro- p subgroup.*

1.3. Case $p = 2$. As it often happens in the theory of pro- p groups, one needs to take extra care in the case $p = 2$. We will assume a technical condition on H^\bullet -quadratic pro-2 groups (see Remark 2.11) that will ensure that most of our proofs also work for the even prime.

We remark that Theorems B and C hold without change for $p = 2$. Theorem D does not hold for $p = 2$, see Examples 4.2 and 4.3. The same examples show that in Conjecture J we have to write ‘virtually uniform’ instead of ‘uniform’ when $p = 2$.

The definition of p -RAAGs needs to be slightly modified for $p = 2$ (see Section 5), namely labels of edges have to be in $4\mathbb{Z}_2$. After this modification, Theorems E, F, G and H hold for $p = 2$. Moreover, if we assume that G_Γ is torsion free, our proof of Theorem I works also for $p = 2$ (see Remark 7.2).

Finally, Proposition 7.5 also holds for $p = 2$ under the additional assumption of Remark 2.11.

1.4. Structure of the article. The main definitions and conventions of this article are laid out in Section 2. We prove Proposition A in Section 2.4. Theorems B and C are proved in Section 3, where we also discuss the properness of amalgamated free products and HNN extensions in the category of pro- p groups. Theorem D is proved in Section 4. We introduce the class of p -RAAGs in Section 5, where we also prove Theorems E and F. Uniform completable p -RAAGs are defined and studied in Section 5.4. Triangle p -RAAGs are analysed in Section 6, which also includes a proof of Theorem G. Finally, in Section 7 we investigate “Tits alternative behaviour” in the classes of p -RAAGs and mild pro- p groups, proving Theorem I.

2. PRELIMINARIES

2.1. Quadratic algebras. An associative unital algebra A_\bullet over the finite field \mathbb{F}_p is graded if it decomposes as the direct sum of \mathbb{F}_p -vector spaces $A_\bullet = \bigoplus_{n \geq 0} A_n$ such that $A_n \cdot A_m \subseteq A_{n+m}$ for all $n, m \geq 0$. The graded algebra A_\bullet is of finite type if every space A_n has finite dimension. Hereinafter we will restrict ourselves to graded \mathbb{F}_p -algebras of finite type, and to finite dimensional vector spaces over \mathbb{F}_p .

For an \mathbb{F}_p -vector space V , let $T_\bullet(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ denote the free graded tensor algebra generated by V . For $\Omega \subseteq V \otimes V$, let (Ω) denote the two-sided ideal generated by Ω in $T_\bullet(V)$.

Definition 2.1. Let V be a vector space, and let Ω be a subset of $V^{\otimes 2}$. We put

$$Q(V, \Omega) = \frac{T_{\bullet}(V)}{(\Omega)}.$$

A graded algebra $A_{\bullet} = \bigoplus_{n \geq 0} A_n$ is called quadratic if the morphism of vector spaces $\mu: A_1^{\otimes 2} \rightarrow A_2$, given by multiplication, induces an epimorphism of graded algebras $T_{\bullet}(A_1) \rightarrow A_{\bullet}$ whose kernel is the two-sided ideal $(\ker(\mu))$, i.e., $Q(A_1, \ker(\mu)) \simeq A_{\bullet}$.

Example 2.2. Let V be a vector space. One has the following quadratic algebras:

- (a) the tensor algebra $T_{\bullet}(V) = Q(V, 0)$;
- (b) the trivial algebra $Q(V, V^{\otimes 2}) = \mathbb{F}_p \oplus V$ — i.e., $Q(V, V^{\otimes 2})$ is concentrated in degree 0 and 1;
- (c) the symmetric algebra

$$S_{\bullet}(V) = \bigoplus_{n \geq 0} S_n(V) = Q(V, \{v \otimes w - w \otimes v \mid v, w \in V\});$$

- (d) the exterior algebra

$$\Lambda_{\bullet}(V) = \bigoplus_{n \geq 0} \Lambda_n(V) = Q(V, \{v \otimes v \mid v \in V\})$$

— observe that for $p = 2$ one has $\Lambda_{\bullet}(V) = A_{\bullet}(V)/(v^2 \mid v \in V)$.

One may combine two quadratic algebras to obtain a new one as follows (cf. [32, § 3.1]).

Example 2.3. Let $A_{\bullet} = \bigoplus_{n \geq 0} A_n$ and $B_{\bullet} = \bigoplus_{n \geq 0} B_n$ be two quadratic algebras, and put $\Omega_A \subseteq A_1^{\otimes 2}$ and $\Omega_B \subseteq B_1^{\otimes 2}$ such that $A_{\bullet} = Q(A_1, \Omega_A)$ and $B_{\bullet} = Q(B_1, \Omega_B)$.

- (a) The direct sum $C_{\bullet} = A_{\bullet} \oplus B_{\bullet}$ is the quadratic algebra

$$C_{\bullet} = Q(A_1 \oplus B_1, \Omega_A \oplus \Omega_B \oplus (A_1 \otimes B_1) \oplus (B_1 \otimes A_1))$$

— namely, $C_n = A_n \oplus B_n$ for every $n \geq 1$, and $ab = ba = 0$ for every $a \in A_1$ and $b \in B_1$.

- (b) The wedge product $C_{\bullet} = A_{\bullet} \wedge B_{\bullet}$ is the quadratic algebra

$$C_{\bullet} = Q(A_1 \oplus B_1, \Omega_A \oplus \Omega_B \oplus \{a \otimes b + b \otimes a \mid a \in A_1, b \in B_1\})$$

— namely, $ba = -ab$ for every $a \in A_1$ and $b \in B_1$, and for every $n \geq 1$ $C_n = A_n \oplus B_n \oplus \bigoplus_{i+j=n} (A_i \wedge B_j)$ (note that in [32, 27] this algebra is called the skew-commutative tensor product $A_{\bullet} \otimes^{-1} B_{\bullet}$).

2.2. Presentations of pro- p groups. Throughout this paper, subgroups of pro- p groups are assumed to be closed (in the pro- p topology) and generators will be intended as topological generators. In particular, given two (closed) subgroups H_1 and H_2 of a pro- p group G , the subgroup $[H_1, H_2]$ is the (closed) subgroup of G generated by the commutators $[g_1, g_2]$, with $g_i \in H_i$ for $i = 1, 2$. Also, for a positive integer n , G^n denotes the (closed) subgroup of G generated by the n -th powers of elements of G . Similarly, all homomorphisms between pro- p groups will be assumed continuous. Finally, we will always consider \mathbb{F}_p as a trivial G -module.

We will now recall some basic facts and definitions for presentations of pro- p groups. The experienced reader might wish to skip ahead to the next section.

For a pro- p group G , set $G_{(2)} = G^p[G, G]$ and

$$(2.1) \quad G_{(3)} = \begin{cases} G^p[[G, G], G] & \text{if } p \geq 3 \\ G^4[[G, G], G] & \text{if } p = 2 \end{cases}.$$

Then $G_{(2)}$ and $G_{(3)}$ coincide with the second and third terms of the p -Zassenhaus filtration of G , respectively (cf. [7, § 11.1]). The pro- p group G is finitely generated if and only if $G/G_{(2)}$ is a vector space of finite dimension over \mathbb{F}_p , and $d(G) := \dim G/G_{(2)}$ is the minimal number of generators of G . Moreover, the equality $H^1(G, \mathbb{F}_p) = \text{Hom}(G, \mathbb{F}_p)$ implies the isomorphism of vector spaces

$$(2.2) \quad H^1(G, \mathbb{F}_p) \cong (G/G_{(2)})^*,$$

where $\cdot^* = \text{Hom}(\cdot, \mathbb{F}_p)$ denotes the dual for vector spaces. Thus, $d(G) = \dim H^1(G, \mathbb{F}_p)$.

A presentation

$$(2.3) \quad 1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

of a finitely presented pro- p group G is said to be *minimal* if one of the following equivalent conditions is satisfied:

- (i) $F \rightarrow G$ induces an isomorphism $F/F_{(2)} \cong G/G_{(2)}$;
- (ii) $R \leq F_{(2)}$.

If (2.3) is minimal, it follows from [29, § 1.6] that

$$H^2(G, \mathbb{F}_p) \cong H^1(R, \mathbb{F}_p)^F \cong (R/R^p[R, F])^*.$$

A minimal subset $\mathcal{R} \subseteq F$ which generates R as normal subgroup is called a set of defining relations of G . Thus, $r(G) := \dim H^2(G, \mathbb{F}_p)$ is the cardinality of a set of defining relations.

For a pro- p group G and a minimal presentation (2.3), set $d = d(G)$ and let $\mathcal{X} = \{x_1, \dots, x_d\}$ be a basis of F . We will identify \mathcal{X} with the induced basis of G via $F \rightarrow G$. Given $r \in R \subseteq F_{(2)}$, one may write

$$(2.4) \quad r = \begin{cases} \prod_{i < j} [x_i, x_j]^{a_{ij}} \cdot r' & \text{if } p \neq 2 \\ \prod_{i=1}^d x_i^{2a_{ii}} \cdot \prod_{i < j} [x_i, x_j]^{a_{ij}} \cdot r' & \text{if } p = 2 \end{cases} \quad r' \in F_{(3)},$$

with $0 \leq a_{ij} < p$, and such numbers are uniquely determined by r (cf. [50, Prop. 1.3.2]).

2.3. Pro- p groups and \mathbb{F}_p -cohomology. The \mathbb{F}_p -cohomology of a pro- p group G comes endowed with the cup-product

$$\smile: H^i(G, \mathbb{F}_p) \times H^j(G, \mathbb{F}_p) \longrightarrow H^{i+j}(G, \mathbb{F}_p)$$

which is bilinear and graded-commutative, i.e. $\beta \smile \alpha = (-1)^{ij} \alpha \smile \beta$ for $\alpha \in H^i(G, \mathbb{F}_p)$ and $\beta \in H^j(G, \mathbb{F}_p)$, so that $H^\bullet(G, \mathbb{F}_p) = \bigoplus_{n \geq 0} H^n(G, \mathbb{F}_p)$ is a graded algebra (cf. [29, § I.4]).

Let G be a finitely presented pro- p group with minimal presentation (2.3), with \mathcal{X} as above, and let $\mathcal{X}^* = \{\alpha_1, \dots, \alpha_d\}$ be the basis of $H^1(G, \mathbb{F}_p)$ dual to \mathcal{X} . Moreover, set $m = r(G)$ and let $\mathcal{R} = \{r_1, \dots, r_m\}$ be a set of defining relations of G . Cup-products of elements of degree 1 and defining relations are connected by the following (cf. [50, Prop. 1.3.2] and [27, Prop. 7.1]).

Proposition 2.4. *Let G , \mathcal{X} , \mathcal{X}^* and \mathcal{R} be as above. Then for every $r_h \in \mathcal{R}$ one has a bilinear pairing*

$$\mathrm{trg}^{-1}(\alpha_i \smile \alpha_j) \cdot r_h = \begin{cases} -a_{ij} & \text{if } i < j \\ a_{ij} & \text{if } j < i \\ -\binom{p}{2} a_{ii} & \text{if } i = j, \end{cases}$$

where a_{ij} are the exponents in the expression of r_h as in (2.4), interpreted as elements of \mathbb{F}_p . In particular, $\alpha_i \smile \alpha_j \neq 0$ if and only if $a_{ij} \neq 0$ for some $r_h \in \mathcal{R}$.

Moreover, one has the following (cf. [27, Thm. 7.3]).

Proposition 2.5. *Let G be a finitely generated pro- p group. Then the following are equivalent.*

- (i) *The cup-product induces an epimorphism $H^1(G, \mathbb{F}_p)^{\otimes 2} \rightarrow H^2(G, \mathbb{F}_p)$.*
- (ii) *One has an equality $R \cap F_{(3)} = R^p[R, F]$ for every i .*

If the above two conditions hold, then $r_i \in R \setminus F_{(3)}$ for any set of defining relations $\mathcal{R} = \{r_1, \dots, r_m\}$ of G .

Hence, if a pro- p group G can have a chance to be quadratic (see Definition 2.7), it has to satisfy one of the equivalent conditions of Proposition 2.5. In the rest of the paper we will mostly restrict ourselves to this situation. In fact, one of the easiest shapes of relations satisfying (ii) above is involved in the definition of p -RAAGs (see Section 5.1).

Remark 2.6 (Gauss reduction of relations). *Let G and \mathcal{X} be as above and set $d = d(G)$, $m = r(G)$. We will consider the lexicographic order $<$ on the set of couples $\{(i, j) \mid 1 \leq i < j \leq m\}$. The quotient $F_{(2)}/F_{(3)}$ is an \mathbb{F}_p -vector space with basis $\{[x_i, x_j]F_{(3)} \mid 1 \leq i < j \leq d\}$. Suppose that G satisfies the conditions of Proposition 2.4 and let $\mathcal{R} = \{r_1, \dots, r_m\}$ be a set of defining relations of G . Then $R/R \cap F_{(3)}$ is a subspace with basis $\bar{\mathcal{R}} = \{r_h F_{(3)} \mid 1 \leq h \leq m\}$ and we will write*

$$r_h F_{(3)} = \sum_{1 \leq i < j \leq d} a(h, i, j) \cdot [x_i, x_j] F_{(3)}$$

with $a(h, i, j) \in \mathbb{F}_p$. Let $A = (a(h, i, j))_{h, (i, j)}$ be the $m \times \binom{d}{2}$ matrix of the coefficients of the elements of \mathcal{R} in $F_{(2)}/F_{(3)}$. Gauss reduction on A yields a lexicographically ordered sequence $(i_1, j_1) < \dots < (i_m, j_m)$ and a set $\mathcal{R}' = \{r'_1, \dots, r'_m\} \subseteq F$ of new relators for G such that:

$$r'_h \equiv \prod_{1 \leq i < j \leq d} [x_i, x_j]^{b(h, i, j)} \pmod{F_{(3)}}, \quad h = 1, \dots, m$$

with exponents $b(h, i, j) \in \mathbb{F}_p$ such that

$$b(h, i, j) = \begin{cases} 1 & \text{if } (i, j) = (i_h, j_h) \\ 0 & \text{if } (i, j) < (i_h, j_h) \end{cases}$$

The commutator $[x_{i_h}, x_{j_h}]$ may be considered the “leading term” of r_h . This will be relevant in Section 7.

2.4. H^\bullet -quadratic pro- p groups. We introduce the main object of our investigations.

Definition 2.7. *A pro- p group is called H^\bullet -quadratic (or simply quadratic) if the \mathbb{F}_p -cohomology algebra $H^\bullet(G, \mathbb{F}_p)$, endowed with the cup-product, is a quadratic algebra.*

In the rest of the paper we will refer to H^\bullet -quadratic pro- p groups simply as *quadratic pro- p groups*.

Let G be a quadratic pro- p group. If $p \neq 2$, then $\alpha \smile \alpha = 0$ for every $\alpha \in H^\bullet(G, \mathbb{F}_p)$ by graded-commutativity. If we further assume that $\alpha \smile \alpha = 0$ for every α also in the case $p = 2$, then one has an epimorphism of quadratic algebras

$$(2.5) \quad \Lambda_\bullet H^1(G, \mathbb{F}_p) \twoheadrightarrow H^\bullet(G, \mathbb{F}_p).$$

Recall that $\text{cd}(G)$ denotes the cohomological dimension of G (cf. [29, § III.3]). In particular, if G is finitely generated then one has the inequalities

$$(2.6) \quad \text{cd}(G) \leq d(G) \quad \text{and} \quad r(G) \leq \binom{d(G)}{2},$$

so that G is finitely presented.

Notation. With a slight abuse of notation, from now on we denote the cup product $\alpha \smile \beta$ of $\alpha \in H^i(G, \mathbb{F}_p)$ and $\beta \in H^j(G, \mathbb{F}_p)$ by $\alpha \wedge \beta \in H^{i+j}(G, \mathbb{F}_p)$. Moreover, when it is clear from context, the cup-product symbol will be omitted.

Example 2.8. *Free pro- p groups are trivially quadratic, as $\text{cd}(G) = 1$ for a free pro- p group G (cf. [29, Prop. 3.5.17]), i.e. $H^n(G, \mathbb{F}_p) = 0$ for $n \geq 2$.*

Given a field K , let $G_K(p)$ denote the maximal pro- p quotient of the absolute Galois group G_K — i.e. $G_K(p)$ is the Galois group of the maximal p -extension of K . Then one has the following consequence of the Norm Residue Theorem (cf. [8, § 24.3]), which is one of the reasons of our interest in quadratic pro- p groups.

Theorem 2.9. *Let K be a field containing a root of 1 of order p . Then the maximal pro- p Galois group $G_K(p)$ of K is quadratic.*

The well-known Artin-Schreier Theorem (and its pro- p version) implies that the only finite group which occurs as maximal pro- p Galois group is C_2 , the cyclic group of order 2. We will now prove Proposition A.

Proof of Proposition A. For (a), assume by contradiction that G is not torsion-free. Then G contains a subgroup H which is cyclic of order p , so that $H^n(H, \mathbb{F}_p) \cong \mathbb{F}_p$ for every $n \geq 0$ and $\text{cd}(H)$ is infinite. Thus, also $\text{cd}(G)$ is infinite by [29, Prop. 3.3.5], contradicting (2.6). This proves claim (a).

Regarding (b), it is well known that the cohomology algebra of a 2-elementary abelian group C_2^d ($d \in \mathbb{N}$) with coefficients in the finite field \mathbb{F}_2 is the symmetric algebra $S_\bullet(H^1(C_2^d, \mathbb{F}_2))$. Thus, C_2^d is quadratic.

Now, let $G \neq \{1\}$ be a finite abelian 2-group. Then $G \cong \mathbb{Z}/2^{n_1} \times \dots \times \mathbb{Z}/2^{n_d}$ for some positive integers d and $1 \leq n_1 \leq \dots \leq n_d$. Let $\mathcal{X} = \{x_1, \dots, x_d\}$ be a generating set of G such that each generator x_i generates a cyclic factor of order 2^{n_i} . Then G has a minimal presentation of the form $\langle \mathcal{X} | R \rangle$ with $R = \{x_k^{2^{n_k}}, [x_i, x_j] \mid 1 \leq k \leq d, 1 \leq i < j \leq d\}$. If $n_d > 1$ then the equivalent conditions in Proposition 2.5 do not hold, and thus G is not quadratic. This yields item (b).

For (c), by item (b), all elements of G must have order 2 and hence G is abelian. \square

We were informed by J. Mináč that a similar result to our Proposition A has been obtained also by him together with S. Chebolu (unpublished).

Remark 2.10 (Proposition A for $p = 2$). *We believe that, if a finite 2-group is quadratic, then it must be elementary abelian. One can check this directly for several finite 2-groups: dihedral groups D_{2^n} (see [1, Chap. IV, Thm. 2.7]), quaternion group and generalised quaternion groups (see [1, Chap. IV, Thm. 2.9] and [1, Chap. IV, Lem. 2.11]) and extraspecial 2-groups (see [1, Rem. 2.13]).*

Unfortunately, we have not been able to verify our conviction in full generality, but we have two additional comments. First, we can show—using a spectral sequence argument—that the Mennicke 2-group M of order 2^{11} given by the presentation

$$M = \langle a, b, c \mid [a, b] = b^{-2}, [b, c] = c^{-2}, [c, a] = a^{-2} \rangle$$

is not quadratic. We have decided to not include a proof, as this is very similar to the cited examples in [1, Chap. IV].

Secondly, the method of proof of item (b) in Proposition A cannot be applied in general, as there are 2-groups that do not satisfy the condition above. These include finite 2-groups G with balanced presentations (i.e., with $d(G) = r(G)$), which among other groups include the Mennicke 2-group M , the quaternion group and the generalised quaternion groups.

Remark 2.11. *Henceforth we will always implicitly assume that α^2 is trivial in $H^2(G, \mathbb{F}_2)$ for every $\alpha \in H^1(G, \mathbb{F}_p)$ in the case $p = 2$. By Proposition 2.4 and [50, Prop. 1.3.2], this is equivalent to assuming that $a_{ii} = 0$ for every $i \in \{1, \dots, d\}$ and for every defining relation $r_h \in \mathcal{R}$ of G . If $G = G_K(p)$ for some field K , then this holds if $\sqrt{-1} \in K$.*

2.5. Mild pro- p groups. A good source of quadratic pro- p groups is the class of *mild* pro- p groups. Such groups were introduced by J. P. Labute in [23] to study the Galois groups of pro- p extensions of number fields with restricted ramification. Since a quadratic pro- p group needs to satisfy the conditions of Proposition 2.5, we give the definition of mild pro- p groups among those satisfying these conditions. For the general definition and properties of mild pro- p groups we refer to [23, 12, 14].

Definition 2.12. *Let G be a finitely presented pro- p group with presentation (2.3) and satisfying Proposition 2.5. Let $\mathcal{X} = \{x_1, \dots, x_d\}$ be a basis of F , and consider the graded algebra $M_\bullet(G) = Q(V, \Omega)$, where $V = F/F_{(2)} = \langle \bar{x}_1, \dots, \bar{x}_d \rangle$ — where \bar{x}_i denotes the coset of x_i in $F/F_{(2)}$ — and*

$$\Omega = \left\{ \sum_{i < j} a_{ij} (\bar{x}_i \otimes \bar{x}_j - \bar{x}_j \otimes \bar{x}_i) \right\}$$

with a_{ij} as in (2.4). The presentation (2.3) is said to be mild if one has an equality of formal power series

$$(2.7) \quad \sum_{n \geq 0} \dim(M_n(G)) \cdot T^n = \frac{1}{1 - d(G)T + r(G)T^2}.$$

A pro- p group satisfying Proposition 2.5 is said to be mild if it has a mild presentation.

The most interesting — and useful — feature of mild pro- p groups is that they have cohomological dimension equal to 2 (cf. [23, Thm. 1.2]). In fact, it is fairly easy to decide if a pro- p group with cohomological dimension 2 is quadratic.

Proposition 2.13. *Let G be a finitely generated pro- p group with $\text{cd}(G) = 2$ which satisfies Proposition 2.5, and let Θ be the kernel of the epimorphism of vector spaces $H^1(G, \mathbb{F}_p)^{\otimes 2} \rightarrow H^2(G, \mathbb{F}_p)$ induced by the cup-product. If*

$$(2.8) \quad \Theta \wedge H^1(G, \mathbb{F}_p) = H^1(G, \mathbb{F}_p)^{\otimes 3},$$

then G is quadratic.

Proof. Since $H^n(G, \mathbb{F}_p)$ is trivial for $n \geq 3$, one just needs to check whether $H^3(G, \mathbb{F}_p) = 0$ follows from the relations Θ in $H^2(G, \mathbb{F}_p)$. But this follows immediately from (2.8). \square

This fact has the following consequence.

Remark 2.14. *Let G be a mild pro- p group. By Proposition 2.5 and Proposition 2.13, if the cup-product induces an epimorphism of vector spaces $H^1(G, \mathbb{F}_p)^{\otimes 2} \rightarrow H^2(G, \mathbb{F}_p)$, then G is quadratic.*

Usually, it is quite difficult to check whether a finitely presented pro- p group is mild using the definition. Rather, one has the following handy criterion to check whether a pro- p group is mild.

Proposition 2.15 ([14, p. 789]). *Let G be a finitely presented pro- p group such that $H^2(G, \mathbb{F}_p) \neq 0$. Assume that $H^1(G, \mathbb{F}_p)$ admits a decomposition $H^1(G, \mathbb{F}_p) = V_1 \oplus V_2$ such that the following holds:*

- (i) *the cup-product $V_1 \otimes V_1 \rightarrow H^2(G, \mathbb{F}_p)$ is trivial;*
- (ii) *the cup-product $V_1 \otimes V_2 \rightarrow H^2(G, \mathbb{F}_p)$ is surjective.*

Then G is mild.

We remark that the condition of mildness of a pro- p group satisfying Proposition 2.5 depends only on the “shape” of its defining relations modulo $F_{(3)}$.

Remark 2.16. *Let G be a mild quadratic pro- p group with minimal presentation (2.3) and defining relations $\mathcal{R} = \{r_1, \dots, r_m\}$. If $\tilde{\mathcal{R}} = \{\tilde{r}_1, \dots, \tilde{r}_m\}$ is a subset of F with $\tilde{r}_h \equiv r_h \pmod{F_{(3)}}$ for every $h = 1, \dots, m$, then also the pro- p group $\tilde{G} = F/\tilde{\mathcal{R}}$ with defining relations $\tilde{\mathcal{R}}$ is mild and quadratic — since $H^2(\tilde{G}, \mathbb{F}_p) \cong H^2(G, \mathbb{F}_p)$ by Proposition 2.3.*

The previous remark will be used in Section 7.

2.6. Generalised Golod-Shafarevic pro- p groups. A generalised Golod-Shafarevich pro- p group is a pro- p group which satisfies a weighted version of the celebrated Golod-Shafarevich condition. We briefly recall the definition and we reference the reader [11, § 4.1] for a deeper treatment.

Let $\mathcal{X} = \{x_1, \dots, x_d\}$ and $U = \{u_1, \dots, u_d\}$ be two finite sets of the same cardinality. The free pro- p group F on \mathcal{X} can be embedded in the free associative algebra $\mathbb{F}_p\langle\langle U \rangle\rangle$ over U via the Magnus embedding $\iota : x_i \mapsto 1 + u_i$. Choose weights $w(u_i) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ for $x_i \in \mathcal{X}$ and extend this to a weight function $w : \mathbb{F}_p\langle\langle U \rangle\rangle \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ by setting

$$w(1) = 0, \quad w(0) = \infty, \quad w(u_{i_1} \dots u_{i_k}) = w(u_{i_1}) + \dots + w(u_{i_k})$$

$$w\left(\sum c_\alpha m_\alpha\right) = \min\{w(m_\alpha) \mid c_\alpha \neq 0\}.$$

For an element $f \in F$, define the *valuation* of f by $D(f) = w(\iota(f) - 1)$ and, for a subset S of F , define $H_{S,D}(T) = \sum_{s \in S} T^{D(s)}$.

A pro- p group G is said to be *generalised Golod-Shafarevic* if there exist:

- (1) a minimal presentation (2.3) for G , with generators $\mathcal{X} = \{x_1, \dots, x_d\}$ and relators \mathcal{R} ;
- (2) a weight function $w : \mathbb{F}_p\langle\langle U \rangle\rangle \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ (with associated valuation D) on $F(\mathcal{X})$ and
- (3) a real number $T_0 \in (0, 1)$

such that

$$1 - H_{\mathcal{X},D}(T_0) + H_{\mathcal{R},D}(T_0) < 0.$$

In Section 7 we will use the following theorem.

Theorem 2.17 ([11, Thm. 7.1]). *A generalised Golod-Shafarevic pro- p groups contains a free non-abelian closed subgroup.*

3. COMBINATORIAL GROUP THEORY FOR QUADRATIC PRO- p GROUPS

In the next section we concern ourselves with finding new ways to construct quadratic groups from old ones.

3.1. Free and direct products. The following results are well-known to experts.

Proposition 3.1. *Let G_1, G_2 be two finitely generated quadratic pro- p groups.*

- (a) *The free pro- p product $G = G_1 * G_2$ is again quadratic; furthermore, we have $H^\bullet(G, \mathbb{F}_p) \cong H^\bullet(G_1, \mathbb{F}_p) \oplus H^\bullet(G_2, \mathbb{F}_p)$.*
- (b) *The direct product $G = G_1 \times G_2$ is again quadratic; furthermore, we have $H^\bullet(G, \mathbb{F}_p) \cong H^\bullet(G_1, \mathbb{F}_p) \wedge H^\bullet(G_2, \mathbb{F}_p)$.*

Proof. Statement (a) follows from [29, § IV.1]. Statement (b) follows from [29, § II.4, Ex. 7]. Note that both statements also follow from the discussion following Example 2.2. \square

Remark 3.2. *Let G_1, G_2 be two quadratic pro- p groups which occur as maximal pro- p Galois groups of fields. By [36, Thm. C], the direct product $G_1 \times G_2$ may occur as a maximal pro- p Galois group only if one of the two groups is abelian.*

As we have seen above, quadratic pro- p groups are closed under taking free and direct products. There are other universal “free product-like” constructions and, in the rest of this section, we will explore to what extent the class of quadratic pro- p groups is closed under these constructions.

3.2. Amalgamated products. Let G_1 and G_2 be pro- p groups and let $\phi_i: H \rightarrow G_i$ (for $i \in \{1, 2\}$) be continuous monomorphisms of pro- p groups. An *amalgamated free pro- p product* (or simply *amalgam*) of G_1 and G_2 with *amalgamated subgroup* H is defined to be the pushout

$$\begin{array}{ccc} H & \xrightarrow{\phi_1} & G_1 \\ \phi_2 \downarrow & & \downarrow \psi_1 \\ G_2 & \xrightarrow{\psi_2} & G \end{array}$$

in the category of pro- p groups, which is unique (cf. [42, § 9.2]). We write $G = G_1 \amalg_H G_2$.

An amalgamated free pro- p product $G = G_1 \amalg_H G_2$ is said to be *proper* if the homomorphisms ψ_i ($i = 1, 2$) are monomorphisms. In that case one identifies G_1 , G_2 and H with their images in G .

Remark 3.3. *For a proper amalgam $G = G_1 \amalg_H G_2$ it follows from [42, Prop. 9.2.13(a)] that*

$$\text{cd}(G) \leq \max\{\text{cd}(G_1), \text{cd}(G_2), \text{cd}(H) + 1\}.$$

Example 3.4. *For $p \neq 2$, let G be a Demushkin group with presentation*

$$G = \langle x_1, \dots, x_d \mid x_1^q [x_1, x_2] [x_3, x_4] \cdots [x_{d-1}, x_d] = 1 \rangle,$$

with $d = d(G) \geq 4$ and q a power of p . Let $G_1, G_2 \leq G$ be the subgroups generated by x_1, x_2 and by x_3, \dots, x_d respectively, and $H \leq G$ the pro-cyclic subgroup generated by $[x_2, x_1] x_1^{-q} = [x_3, x_4] \cdots [x_{d-1}, x_d]$. Then G is (isomorphic to) the proper amalgam $G_1 \amalg_H G_2$ (cf. [42, Ex. 9.2.12]; the same holds also if $p = 2$). Note that $\text{cd}(G_1) = \text{cd}(G_2) = 1$, but $\text{cd}(G) = 2$.

We come to the proof of the first of the two main results of this section: Theorem B.

Remark 3.5. *For a pro- p group G and a subgroup $H \leq G$, the restriction map*

$$\text{res}_{G,H}^\bullet: H^\bullet(G, \mathbb{F}_p) \longrightarrow H^\bullet(H, \mathbb{F}_p),$$

induced by $\text{res}_{G,H}^n$ for every $n \geq 0$, is a morphism of graded algebras (cf. [29, Prop. 1.5.3]).

Proof of Theorem B. By hypothesis (i), the maps $\text{res}_{G_i,H}^1$ are surjective, for $i = 1, 2$. Set $V_i = \ker(\text{res}_{G_i,H}^1)$, and let $W_i \subseteq H^1(G_i, \mathbb{F}_p)$ be a complement for V_i for each i . Then we may identify both W_1 and W_2 with $H^1(H, \mathbb{F}_p)$ via $\text{res}_{G_1,H}^1$ and $\text{res}_{G_2,H}^1$, respectively, and with an abuse of notation we write $W = W_1 = W_2$, so that $H^1(G_i, \mathbb{F}_p) = V_i \oplus W$ for $i = 1, 2$. Since H is quadratic, one has an isomorphism of quadratic algebras

$$H^\bullet(H, \mathbb{F}_p) \simeq \frac{\Lambda_\bullet(W)}{(\Omega_H)},$$

for some subspace $\Omega_H \subseteq \Lambda_2(W)$. Moreover, by hypothesis (ii), there exist subspaces $\Omega_i \subseteq \Lambda_2(V_i) \oplus (V_i \wedge W)$ and isomorphisms of quadratic algebras

$$H^\bullet(G_i, \mathbb{F}_p) \simeq \frac{\Lambda_2(V_i \oplus W)}{(\Omega_i \cup \Omega_H)}$$

for $i = 1, 2$. In particular, in $H^\bullet(G_i, \mathbb{F}_p)$ one has a relation $a + b = 0$ with $a \in H^1(G_i, \mathbb{F}_p) \wedge V_i$ and $b \in \Lambda_2(W)$ if and only if $a \in \Omega_i$ and $b \in \Omega_H$.

Since $G = G_1 \amalg_H G_2$ is a proper amalgam, by [42, Prop. 9.2.13] the monomorphisms $H \rightarrow G_i$ and $G_i \rightarrow G$ for $i = 1, 2$ induce a long exact sequence in cohomology

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(G, \mathbb{F}_p) & \xrightarrow{f_G^1} & \cdots & \xrightarrow{f_H^{n-1}} & H^{n-1}(H, \mathbb{F}_p) \\ & & & & & & \downarrow \\ & & & & & & H^n(G, \mathbb{F}_p) \xrightarrow{f_G^n} H^n(G_1, \mathbb{F}_p) \oplus H^n(G_2, \mathbb{F}_p) \xrightarrow{f_H^n} H^n(H, \mathbb{F}_p) \\ & & & & & & \downarrow \\ & & & & & & H^{n+1}(G, \mathbb{F}_p) \xrightarrow{f_G^{n+1}} H^{n+1}(G_1, \mathbb{F}_p) \oplus H^{n+1}(G_2, \mathbb{F}_p) \xrightarrow{f_H^{n+1}} \cdots \end{array}$$

for $n \geq 1$, where

$$\begin{aligned} f_G^n(\xi) &= (\text{res}_{G, G_1}^n(\xi), \text{res}_{G, G_2}^n(\xi)), \\ f_H^n(\eta, \eta') &= \text{res}_{G_1, H}^n(\eta) - \text{res}_{G_2, H}^n(\eta') \end{aligned}$$

for every $\xi \in H^n(G, \mathbb{F}_p)$ and $\eta \in H^n(G_1, \mathbb{F}_p)$, $\eta' \in H^n(G_2, \mathbb{F}_p)$ (cf. [15, Thm. 1.13]).

Since G_1 , G_2 and H are quadratic, Remark 3.5 implies that the restriction maps $\text{res}_{G_1, H}^\bullet$ and $\text{res}_{G_2, H}^\bullet$ are epimorphisms of quadratic algebras. Therefore also the maps f_H^n are surjective for all $n \geq 1$. Thus,

$$H^\bullet(G, \mathbb{F}_p) \cong \mathbb{F}_p \oplus \left(\bigoplus_{n \geq 1} \ker(f_H^n) \right),$$

where \mathbb{F}_p is the degree 0 part — note that this is a morphism of graded algebras, as the cup-product commutes with the restriction maps (cf. [29, Prop. 1.5.3]).

For every $n \geq 1$, we may decompose the map f_H^n as

$$H^n(G_1, \mathbb{F}_p) \oplus H^n(G_2, \mathbb{F}_p) \xrightarrow{\varphi_n} H^n(H, \mathbb{F}_p) \oplus H^n(H, \mathbb{F}_p) \xrightarrow{\psi_n} H^n(H, \mathbb{F}_p),$$

where $\varphi_n = \text{res}_{G_1, H}^n \oplus \text{res}_{G_2, H}^n$ and $\psi_n = \pi_n - \pi'_n$, with π_n and π'_n the canonical projections onto the first and the second summand respectively. Clearly, $\ker(\psi_n)$ is the diagonal $\Delta(H^n(H, \mathbb{F}_p)) \subseteq H^n(H, \mathbb{F}_p) \oplus H^n(H, \mathbb{F}_p)$. Moreover, by hypothesis (ii) and by Remark 3.5, $\ker(\text{res}_{G_i, H}^\bullet)$ is generated as ideal of $H^\bullet(G_i, \mathbb{F}_p)$ by $\ker(\text{res}_{G_i, H}^1)$ for $i = 1, 2$, so that

$$(3.1) \quad \ker(\text{res}_{G_i, H}^n) = \ker(\text{res}_{G_i, H}^1) \wedge H^{n-1}(G_i, \mathbb{F}_p)$$

for each $n \geq 1$ and $i = 1, 2$. Therefore, one has

$$(3.2) \quad \begin{aligned} \ker(f_H^n) &= \ker(\varphi_n) \oplus \Delta(H^n(H, \mathbb{F}_p)) \\ &\cong (V_1 \wedge H^{n-1}(G_1, \mathbb{F}_p)) \oplus (V_2 \wedge H^{n-1}(G_2, \mathbb{F}_p)) \oplus H^n(H, \mathbb{F}_p) \end{aligned}$$

for all $n \geq 1$ (here we identify $H^n(H, \mathbb{F}_p)$ with $H^n(G_i, \mathbb{F}_p)/\ker(\text{res}_{G_i, H}^n)$).

Now let A_\bullet be the quadratic algebra $\Lambda_\bullet(V_1 \oplus W \oplus V_2)/(\Omega_G)$, with $\Omega_G = \Omega_1 \oplus \Omega_2 \oplus \Omega_H \oplus (V_1 \wedge V_2)$. In particular, one has the isomorphisms of graded algebras $A_\bullet/(V_1) \cong H^\bullet(G_2, \mathbb{F}_p)$, $A_\bullet/(V_2) \cong H^\bullet(G_1, \mathbb{F}_p)$ and

$$(3.3) \quad A_\bullet/(V_1 \oplus V_2) \cong \Lambda_\bullet(W)/(\Omega_H) \cong H^\bullet(H, \mathbb{F}_p).$$

Moreover, for every $n \geq 1$, we have the isomorphisms of vector spaces

$$(3.4) \quad \frac{V_i \wedge (\Lambda_{n-1}(V_i \oplus W))}{(\Omega_i)_n} \cong V_i \wedge H^{n-1}(G_i, \mathbb{F}_p),$$

where $(\Omega_i)_n$ denotes the part of degree n of the ideal (Ω_i) . Let

$$\phi_\bullet: A_\bullet \longrightarrow H^\bullet(G_1, \mathbb{F}_p) \oplus H_1(G_2, \mathbb{F}_p)$$

be the morphism of quadratic algebras given by $\phi_1|_{V_1} = \text{id}_{V_1} \oplus 0$, $\phi_1|_{V_2} = 0 \oplus \text{id}_{V_2}$, and $\phi_1|_W = \text{id}_W \oplus \text{id}_W$ (here 0 denotes the 0-map). Since $\psi_1 \circ \phi_1|_{V_1 \oplus V_2} = 0$, and $f_H^1 \circ \phi_1|_W = 0$, by quadraticity of A_\bullet one has that the image of ϕ_n is contained in $\ker(f_H^n)$ for every $n \geq 1$. Moreover, by (3.1), (3.3) and (3.4), the map $\phi_n: A_n \rightarrow \ker(f_H^n)$ is an isomorphism. Therefore, $\phi_\bullet: A_\bullet \rightarrow \text{im}(\phi_\bullet)$ is an isomorphism of quadratic algebras, and $H^\bullet(G, \mathbb{F}_p)$ is quadratic. \square

Remark 3.6. *By duality (2.2), the restriction maps (1.1), with $k = 1$, are surjective if and only if the monomorphisms $H \rightarrow G_1$ and $H \rightarrow G_2$ induce monomorphisms*

$$\frac{H}{H_{(2)}} \longrightarrow \frac{G_1}{(G_1)_{(2)}}, \quad \frac{H}{H_{(2)}} \longrightarrow \frac{G_1}{(G_1)_{(2)}}.$$

In other words, one may find bases \mathcal{X}_1 and \mathcal{X}_2 for G_1 and G_2 respectively, such that H is generated as a subgroup of G_1 and G_2 by $\mathcal{X}_1 \cap H$ and $\mathcal{X}_2 \cap H$ respectively.

Let \mathcal{X}_1 and \mathcal{X}_2 be as above and, for $i \in \{1, 2\}$, let $x_h, x_j \in \mathcal{X}_i \cap H$. By Proposition 2.4, condition (ii) of the statement of Theorem B holds in the following case: if $a_{hj} \neq 0$ for some relation r of G_i , then there is a relation $[x_h, x_j] = t$, with $t \in \Phi(H)$. In particular, as it will be clear later, Theorem B works for generalised p -RAAGs: a relation of G_i is also a relation of H (see Section 5).

The following two examples of proper amalgams of quadratic pro- p groups show that both conditions of the statement of Theorem B are essential.

Example 3.7 (Condition (i) is essential). *Let $G_1 = \langle x_1, y_1 \rangle$ and $G_2 = \langle x_2, y_2 \rangle$ be two 2-generated free pro- p groups, and set $z_1 = [x_1, [x_1, y_1]]$ and $z_2 = [x_2, [x_2, y_2]]$. Then $G_1, G_2, \langle z_1 \rangle$ and $\langle z_2 \rangle$ are all quadratic pro- p groups. Let G be the proper amalgam $G_1 \amalg_{z_1=z_2} G_2$ (the amalgam is proper by [40, Thm. 3.2]). Then G is not quadratic by [4, Cor. 9.2]. Here the restriction maps $\text{res}_{G_i, \langle z_i \rangle}^1$ are trivial for both $i = 1, 2$: indeed $z_i \in \Phi(G_i)$, so that $\alpha(z_i) = 0$ for any $\alpha \in H^1(G_i, \mathbb{F}_p)$.*

Example 3.8 (Condition (ii) is essential). *For $p \neq 2$ let G_1, G_2 be the pro- p groups with minimal presentations*

$$G_1 = \langle x_1, x_2, x_3 \mid x_1^p[x_2, x_3] = 1 \rangle, \quad G_2 = \langle x_2, x_3, x_4 \mid x_4^p[x_2, x_3] = 1 \rangle.$$

Then G_1, G_2 are isomorphic free-by-Demushkin pro- p groups (cf. [22, Thm. 2]). Also, they are quadratic by [38, Prop. 4.3]. Let $H \leq G_1, G_2$ be the closed subgroup generated by x_2, x_3 . Then H is a free 2-generated pro- p group by the Freiheitsatz (cf. [43]) and thus it is quadratic as well.

Set $G = G_1 \amalg_H G_2$. By [42, Ex. 9.2.6(a)], G is a proper amalgam. Let $\{\alpha_1, \dots, \alpha_4\}$ be a basis of $H^1(G, \mathbb{F}_p)$ dual to $\{x_1, \dots, x_4\}$. Then, since $\text{cd}(H) = 1$, the long exact sequence in cohomology induces an isomorphism in degree 2

$$(3.5) \quad \text{res}_{G, G_1}^2 \oplus \text{res}_{G, G_2}^2: H^2(G, \mathbb{F}_p) \xrightarrow{\sim} H^2(G_1, \mathbb{F}_p) \oplus H^2(G_2, \mathbb{F}_p),$$

with $H^2(G_1, \mathbb{F}_p) = \langle \text{res}_{G, G_1}^2(\alpha_2 \alpha_3) \rangle$ and $H^2(G_2, \mathbb{F}_p) = \langle \text{res}_{G, G_2}^2(\alpha_2 \alpha_3) \rangle$, both isomorphic to \mathbb{F}_p . Moreover, $\text{res}_{G_i, H}^1$ is surjective and

$$\ker(\text{res}_{G_i, H}^2) = H^2(G_i, \mathbb{F}_p) \neq \ker(\text{res}_{G_i, H}^1) \wedge H^1(G_i, \mathbb{F}_p),$$

Since the map ϕ^* commutes with the cup product for every $n \geq 1$ and $\alpha_1, \dots, \alpha_n \in H^1(G_0, \mathbb{F}_p)$, one has

$$\begin{aligned} \text{res}_{G_0, A}^n(\alpha_1 \cdots \alpha_n) &= \text{res}_{G_0, A}^1(\alpha_1) \cdots \text{res}_{G_0, A}^1(\alpha_n) \\ &= \phi^* \text{res}_{G_0, B}^1(\alpha_1) \cdots \phi^* \text{res}_{G_0, B}^1(\alpha_n) \\ &= \phi^* \text{res}_{G_0, B}^n(\alpha_1 \cdots \alpha_n), \end{aligned}$$

and hence by hypothesis (ii) the maps $f_{G_0}^n$ are zero maps for every $n \geq 1$. Therefore, for every $n \geq 2$ one has a short exact sequence of vector spaces

$$(3.6) \quad 0 \longrightarrow H^{n-1}(A, \mathbb{F}_p) \xrightarrow{\delta^{n-1}} H^n(G, \mathbb{F}_p) \xrightarrow{\text{res}_{G, G_0}^n} H^n(G_0, \mathbb{F}_p) \longrightarrow 0.$$

We will identify $H^n(G_0, \mathbb{F}_p)$ and $H^n(A, \mathbb{F}_p)$ as subspaces of $H^n(G, \mathbb{F}_p)$.

Let $\alpha_t \in H^1(G, \mathbb{F}_p)$ be the generator of $\ker(\text{res}_{G, G_0}^1)$ — i.e. α_t is dual to $t \in G$. By Remark 3.11, for every $a \in A$ one has a relation $[a, t] = a'$, with $a' \in A \cap (G_0)_{(2)}$. Hence, by Proposition 2.4, the cup-product $\alpha \alpha_t \in H^2(G, \mathbb{F}_p)$ is not trivial for every $\alpha \in H^1(G_0, \mathbb{F}_p)$ such that $\text{res}_{G_0, A}^1(\alpha) \neq 0$. Therefore $\text{im}(\delta^1) = \ker(\text{res}_{G, G_0}^2) = \alpha_t \wedge H^1(A, \mathbb{F}_p)$. Thus, for every $n \geq 1$, one has an isomorphism of vector spaces

$$(3.7) \quad H^n(G, \mathbb{F}_p) = H^n(G_0, \mathbb{F}_p) \oplus (\alpha_t \wedge H^{n-1}(A, \mathbb{F}_p))$$

and $H^\bullet(G, \mathbb{F}_p)$ is generated in degree 1. Moreover, from (3.7) one deduces

$$H^n(G, \mathbb{F}_p) \cong \frac{H^n(G_0, \mathbb{F}_p) \oplus (\alpha_t \wedge H^{n-1}(G_0, \mathbb{F}_p))}{\alpha_t \wedge \ker(\text{res}_{G_0, A}^{n-1})},$$

for every $n \geq 2$. By hypothesis (i), $\ker(\text{res}_{G_0, A}^{n-1}) = \ker(\text{res}_{G_0, A}^1) \wedge H^{n-2}(G_0, \mathbb{F}_p)$, that is, it is generated in degree 1. Thus $\alpha_t \wedge \ker(\text{res}_{G_0, A}^{n-1})$ is generated in degree 2 for every $n \geq 2$. It follows that $H^\bullet(G, \mathbb{F}_p)$ is a quadratic algebra. \square

The following is an example of a proper HNN-extension satisfying all the hypothesis of Theorem C and it is a new example of quadratic pro- p group obtained in this way.

Example 3.12. Let $G_0 = \langle x, y, z \rangle$ be free abelian pro- p group. Set $A = \langle x, y \rangle \leq G_0$ and let $\phi: A \rightarrow A$ be the isomorphism induced by $x \mapsto xy^p$ and $y \mapsto yz^p$. Consider $G = \text{HNN}(G_0, A, \phi)$. It follows easily using [42, Prop. 9.4.3(2)] that G is a proper HNN-extension. Thus, G has a presentation

$$G = \langle x, y, z, t \mid [x, y] = [y, z] = [z, x] = 1, [x, t] = y^p, [y, t] = z^p \rangle.$$

Let $\{\alpha_x, \alpha_y, \alpha_z\} \subseteq H^1(G_0, \mathbb{F}_p)$ be a basis dual to $\{x, y, z\}$. Then $\ker(\text{res}_{G_0, A}^1) = \langle \alpha_z \rangle$ and

$$\ker(\text{res}_{G_0, A}^2) = \langle \alpha_x \alpha_z, \alpha_y \alpha_z \rangle = \ker(\text{res}_{G_0, A}^1) \wedge H^1(G_0, \mathbb{F}_p).$$

Moreover, $\phi^*: H^1(A, \mathbb{F}_p) \rightarrow H^1(A, \mathbb{F}_p)$ is the identity. Therefore, G is quadratic pro- p group.

On the other hand, Theorem C can also be used to show that certain HNN-extensions are not proper.

Example 3.13. Let G be the pro- p group with presentation

$$\langle x_1, x_2, x_3, x_4 \mid [x_1, x_2][x_3, x_4] = 1, [x_1, x_3] = 1, [x_1, x_4][x_2, x_3] = 1, [x_2, x_4] = 1 \rangle.$$

Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a basis of $H^1(G, \mathbb{F}_p)$ dual to $\{x_1, x_2, x_3, x_4\}$. Then we have $\alpha_1 \alpha_2 = \alpha_3 \alpha_4$, $\alpha_1 \alpha_4 = \alpha_2 \alpha_3$ and therefore $\{\alpha_1 \alpha_2, \alpha_1 \alpha_3, \alpha_1 \alpha_4, \alpha_2 \alpha_4\}$ is a basis of

$H^2(G, \mathbb{F}_p)$ by Proposition 2.3. It is clear that $\alpha_i \alpha_j \alpha_k = 0$ for any triple (i, j, k) ; also it is easy to check that conditions (i)-(iii) of Theorem C are satisfied.

Let $N, H \leq G$ be the subgroups generated by x_2, x_3, x_4 and by x_1 , respectively. Then N is normal in G . The short exact sequence (7.3) implies that $\text{cd}(G) \geq 3$. Thus G is not quadratic.

Notice that we can realise G as $\text{HNN}(G_0, G_0, \phi)$ with

$$G_0 = \langle x_2, x_4 \mid [x_2, x_4] = 1 \rangle * \langle x_3 \rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_p) * \mathbb{Z}_p, \quad t = x_1$$

and $\phi(x_2) = x_2[x_3, x_4]$, $\phi(x_3) = x_3$, $\phi(x_4) = x_4[x_2, x_3]$. Hence, $\text{HNN}(G_0, G_0, \phi)$ is not proper by Theorem C.

The following is a list of examples of proper HNN-extensions which are not quadratic pro- p groups, each of which does not satisfy one of the hypothesis of Theorem C. That the given HNN-extensions are proper one can show easily using [42, Prop. 9.4.3(2)].

Example 3.14 (Condition (i) is essential). Let $G_0 = \langle x, y \rangle$ be a free abelian pro- p group. Set $A = \langle x^p \rangle$ and $B = \langle y^p \rangle$ and let $\phi: A \rightarrow B$ be the isomorphism induced by $x^p \mapsto y^p$. Consider $G = \text{HNN}(G_0, A, \phi)$. Thus G has a presentation

$$G = \langle x, y, t \mid [x, y] = 1, [x^p, t] = (x^{-1}y)^p \rangle,$$

and G is not quadratic, as Proposition 2.5 is not satisfied. Indeed, the map $\text{res}_{G_0, A}^1$ is not surjective.

Example 3.15 (Condition (ii) is essential). Let G_0 be the pro- p group with minimal presentation

$$G_0 = \langle x, y, z \mid x^p[y, z] = 1 \rangle,$$

let $A = B \leq G_0$ be the subgroup generated by y, z (in particular, A is a free 2-generated pro- p group by the *Freiheitsatz*, cf. [43]) and let $\phi: A \rightarrow A$ be the identity. Consider $G = \text{HNN}(G_0, A, \phi)$. Thus, G has a presentation

$$G = \langle x, y, z, t \mid x^p[y, z] = [t, y] = [t, z] = 1 \rangle.$$

Let $\{\alpha_x, \dots, \alpha_t\} \subseteq H^1(G, \mathbb{F}_p)$ and, with a slight abuse of notation, consider $\alpha_x, \alpha_y, \alpha_z$ as elements of $H^1(G_0, \mathbb{F}_p)$. Then $\ker(\text{res}_{G_0, A}^1) = \langle \alpha_x \rangle$ and

$$\ker(\text{res}_{G_0, A}^2) = \langle \alpha_y \alpha_z \rangle \neq \ker(\text{res}_{G_0, A}^1) \wedge H^1(G_0, \mathbb{F}_p).$$

On the other hand, the long exact sequence in cohomology induced by the HNN-extension implies that $H^2(G, \mathbb{F}_p) = \langle \alpha_y \alpha_z, \alpha_t \alpha_y, \alpha_t \alpha_z \rangle$, whereas $H^3(G, \mathbb{F}_p) = 0$ as $\text{cd}(G_0) = 2$ and $\text{cd}(A) = 1$, so that G is not quadratic.

Example 3.16 (Condition (iii) is essential). Let $G_0 = \langle x, y \rangle$ be a free abelian pro- p group. Set $A = \langle x \rangle$, $B = \langle y \rangle$ and let $\phi: A \rightarrow B$ be the isomorphism induced by $x \mapsto y$. Consider $G = \text{HNN}(G_0, A, \phi)$. Thus, G has a presentation

$$G = \langle x, y, t \mid [x, y] = 1, [x, t] = x^{-1}y \rangle = \langle x, t \mid [x, [x, t]] = 1 \rangle$$

and G is not quadratic, as Proposition 2.5 is not satisfied. Indeed, the map $f_{G_0}^1$ is not trivial.

4. ANALYTIC PRO- p GROUPS

A pro- p group G is said to be *powerful* if $p \geq 3$ and $[G, G] \leq G^p$, or $p = 2$ and $[G, G] \leq G^4$. Here, $[G, G]$ and G^p denote the commutator subgroup and the subgroup generated by all p th powers. Recall that the *descending p -central series* of G is defined inductively by $P_1(G) = G$ and $P_{n+1}(G) = P_n(G)^p [P_n(G), G]$ for $n \geq 1$. A pro- p group G is called *uniform* if it is finitely generated, powerful and

$$|P_i(G) : P_{i+1}(G)| = |G : P_2(G)|$$

for all $i \in \mathbb{N}$. It is worth noting that a powerful finitely generated pro- p group is uniform if and only if it is torsion-free (see [7, Thm. 4.5]). Uniform pro- p groups play an important role in the theory of p -adic Lie groups (see [7]).

In light of Lazard's work we have the following (cf. [25], see also [48, Thm. 5.1.5]).

Proposition 4.1. *Let G be a uniform pro- p group. Then*

$$H^\bullet(G, \mathbb{F}_p) \cong \Lambda_\bullet H^1(G, \mathbb{F}_p).$$

We can now prove Theorem D.

Proof of Theorem D. If G is uniform, then the claim follows by Proposition 4.1.

Assume now that G has quadratic \mathbb{F}_p -cohomology. Since G is torsion-free, one has $d(G) \leq \dim(G)$, by [19, Prop. 1.2]. Moreover, G is a Poincaré-Duality pro- p group so that $\dim(G) = \text{cd}(G)$. Now by (2.6),

$$d(G) \leq \dim(G) = \text{cd}(G) \leq d(G),$$

so $\text{cd}(G) = d(G)$ and $H^\bullet(G, \mathbb{F}_p) \cong \Lambda_\bullet(H^1(G, \mathbb{F}_p))$ (cf. [36, Prop. 4.3]). In particular, G is powerful by [48, Thm. 5.1.6], and hence uniform. \square

The following examples show that the above theorem cannot be extended to $p = 2$, even for torsion-free groups.

Example 4.2. *Let $G_d = N_d \rtimes H$ be the pro-2 group with*

$$N_d = \langle x_1, \dots, x_d \mid [x_i, x_j] = 1, 1 \leq i < j \leq d \rangle \cong \mathbb{Z}_2^d, \quad H = \langle y \mid \rangle \cong \mathbb{Z}_2$$

and $y^{-1}x_iy = x_i^{-1}$, for $1 \leq i \leq d$. Then G_d is 2-adic analytic and torsion-free, but it is not uniform. Moreover the group G_d is quadratic, by [39, Thm. 3.16]. Thus, the hypothesis $p \geq 3$ in the above theorem cannot be removed.

There are also infinite quadratic pro-2 groups with torsion.

Example 4.3. *The infinite dihedral pro-2 group $D_\infty = \mathbb{Z}_2 \rtimes C_2$, with action given by inversion, is isomorphic to the free pro-2 product $C_2 * C_2$. By Proposition 3.1, we have*

$$H^\bullet(G, \mathbb{F}_2) \cong H^\bullet(C_2, \mathbb{F}_2) \oplus H^\bullet(C_2, \mathbb{F}_2) \cong \mathbb{F}_2[X_1] \oplus \mathbb{F}_2[X_2]$$

with X_1, X_2 indeterminates. Hence G is 2-adic analytic and quadratic, but it contains torsion elements.

Remark 4.4. *Since there are uncountably many uniform pro- p groups which are not commensurable (cf. [46, Thm. 1.1]), we have that there are uncountably many non-commensurable quadratic pro- p groups. Taking free products, it is then easy to see that there are also uncountably many non-isomorphic quadratic non-uniform pro- p groups.*

5. GENERALISED p -RAAGS

Right angled Artin groups (RAAGs for short) are a combinatorial construction that plays a prominent role in Geometric Group Theory. These can be defined as (abstract) groups given by a presentation where all relators are commutators of generators. One possible way to obtain a pro- p group out of a RAAG is to consider the pro- p completion of the group; this yields pro- p groups whose structure remains quite close to that of a RAAG. In this section we introduce a generalised construction of RAAGs for pro- p groups. For more evidence of the novelty and flexibility of this construction see Section 6.

5.1. p -Graphs and p -RAAGs. We will state some conventions that we will keep for the rest of the article.

An (*oriented*) graph is a couple $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where \mathcal{V} is a finite set, whose elements are called *vertices*, and $\mathcal{E} \subseteq \mathcal{V}^2$, whose elements are called *edges*. For an edge $e = (x_1, x_2) \in \mathcal{E}$, $o(e) := x_1$ and $t(e) := x_2$ are called the *origin* and the *terminus* of e , respectively. The *opposite edge* of $e = (x_1, x_2) \in \mathcal{E}$ is $\bar{e} := (x_2, x_1) \in \mathcal{V}^2$. We denote the set of *opposite edges* of edges in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as $\bar{\mathcal{E}}$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. A *loop* in \mathcal{G} is an edge $e \in \mathcal{E}$ with $|\{o(e), t(e)\}| = 1$. An (*unoriented*) *circuit of length 2* in \mathcal{G} is couple $\{e, \bar{e}\}$ for $e \in \mathcal{E}$. An (*unoriented*) *circuit* in \mathcal{G} is a sequence of distinct edges e_1, \dots, e_n with $n \geq 3$ such that $\{o(e_i), t(e_i)\} \cap \{o(e_{i+1}), t(e_{i+1})\} \neq \emptyset$, for $i = 1, \dots, n-1$ and $\{o(e_n), t(e_n)\} \cap \{o(e_1), t(e_1)\} \neq \emptyset$. A graph \mathcal{G} is said to be *combinatorial* if it has no loops and no circuits of length 2. Note that, in particular, a combinatorial graph has a natural “orientation”, i.e. only one of the pairs $(x_1, x_2), (x_2, x_1)$ with $x_1, x_2 \in \mathcal{V}$ can appear in \mathcal{E} and $\mathcal{E} \cap \bar{\mathcal{E}} = \emptyset$. A p -labelling of a combinatorial graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a function $f : \mathcal{E} \rightarrow p\mathbb{Z}_p \times p\mathbb{Z}_p$ if $p \geq 3$, and $f : \mathcal{E} \rightarrow 4\mathbb{Z}_2 \times 4\mathbb{Z}_2$ if $p = 2$.

Definition 5.1. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a combinatorial graph.

- (a) The graph \mathcal{G} is said to be *complete* if $\bar{\mathcal{E}} = \mathcal{V}^2 \setminus (\mathcal{E} \cup \{(v, v) \mid v \in \mathcal{V}\})$.
- (b) A couple $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ with $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$ is said to be a *subgraph* of \mathcal{G} .
- (c) A subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ of \mathcal{G} is said to be *full* if $\mathcal{E}' = \mathcal{E} \cap (\mathcal{V}')^2$.
- (d) A full subgraph \mathcal{G}' of \mathcal{G} is said to be a *clique* of \mathcal{G} if \mathcal{G}' is complete.

Definition 5.2. A p -graph is a couple $\Gamma = (\mathcal{G}, f)$ where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a combinatorial graph and f is a p -labelling of \mathcal{G} .

Definition 5.3. Let $\Gamma = (\mathcal{V}, \mathcal{E}, f)$ be a p -graph with p -labelling $(f_1, f_2) : \mathcal{E} \rightarrow p\mathbb{Z}_p \times p\mathbb{Z}_p$. The *generalised Right Angled Artin pro- p group* (p -RAAG for short) associated to Γ , denoted by G_Γ , is the pro- p group defined by the following pro- p presentation:

$$(5.1) \quad G_\Gamma = \langle \mathcal{V} \mid [o(e), t(e)] = o(e)^{f_1(e)} t(e)^{f_2(e)} \text{ for } e \in \mathcal{E} \rangle$$

We present a couple of examples to clarify the definition.

Example 5.4. Let \mathcal{G} be a graph, let $c \equiv (0, 0) \in p\mathbb{Z}_p \times p\mathbb{Z}_p$ be the constant p -labelling on \mathcal{G} and set $\Gamma = (\mathcal{G}, c)$; then G_Γ is the pro- p completion of the abstract RAAG associated to \mathcal{G} .

Example 5.5. Let Γ_1 and Γ_2 be the p -graphs

$$\Gamma_1 = \quad x_1 \xrightarrow{(a,b)} x_2 \quad \text{and} \quad \Gamma_2 = \begin{array}{ccc} & & x_2 \\ & \nearrow^{(\lambda,p^2)} & \\ x_1 & \xrightarrow{(0,0)} & x_3 \end{array}$$

with $a, b, \lambda \in p\mathbb{Z}_p$. Then the pro- p groups G_{Γ_1} and G_{Γ_2} are defined by the presentations

$$G_{\Gamma_1} = \langle x_1, x_2 \mid [x_1, x_2] = x_1^a x_2^b \rangle \quad \text{and} \\ G_{\Gamma_2} = \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_1^\lambda x_2^{p^2}, [x_1, x_3] = 1 \rangle,$$

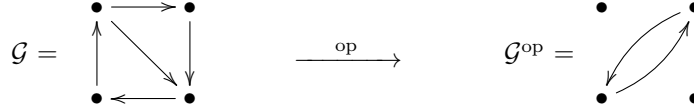
respectively.

We will see in Lemma 6.5, that G_{Γ_1} is a 2-generated Demushkin group, hence a uniform pro- p group.

Remark 5.6. One of our main motivations to introduce generalised p -RAAGs is that G_Γ satisfies the equivalent conditions of Proposition 2.5 and Remark 3.6.

5.2. Cohomology of quadratic p -RAAGs. For a set S , we will denote by $\mathbb{F}_p S$ the vector space with basis S .

Let $\Gamma = (\mathcal{G}, f)$ be a p -graph. In this subsection we will show that the \mathbb{F}_p -cohomology of a quadratic p -RAAG G_Γ is completely determined by its ‘‘underlying graph’’ \mathcal{G} . Recall that, the opposite graph $\mathcal{G}^{\text{op}} = (\mathcal{V}^{\text{op}}, \mathcal{E}^{\text{op}})$ is defined by $\mathcal{V}^{\text{op}} = \mathcal{V}$, and $\mathcal{E}^{\text{op}} = \mathcal{V}^2 \setminus (\mathcal{E} \cup \bar{\mathcal{E}})$. For instance,



Definition 5.7. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. The exterior Stanley-Reisner algebra associated to \mathcal{G} is the quadratic algebra $\Lambda_\bullet(\mathcal{G}^{\text{op}}) = Q(\mathbb{F}_p \mathcal{V}, \Omega)$ with

$$\Omega = \{v \otimes w + w \otimes v \mid (v, w) \notin \mathcal{E} \cup \bar{\mathcal{E}}\} \subseteq \mathbb{F}_p \mathcal{V}^{\otimes 2}$$

(cf. [31, § 1.3 and § 3.2]). Namely, we kill the wedge product of two vertices if they are not connected in \mathcal{G} . In particular, $\Lambda_\bullet(\mathcal{G}^{\text{op}})$ is quadratic and graded-commutative.

Let G_Γ be a generalised p -RAAG with associated p -graph $\Gamma = (\mathcal{G}, f)$ and underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Clearly, by (2.2) one has $H^1(G_\Gamma, \mathbb{F}_p) \cong \Lambda_1(\mathcal{G}^{\text{op}})$. In degree 2 one has the following.

Lemma 5.8. Let G_Γ be a generalised p -RAAG with associated p -graph $\Gamma = (\mathcal{G}, f)$ and underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then $H^2(G_\Gamma, \mathbb{F}_p) \cong \Lambda_2(\mathcal{G}^{\text{op}})$.

Proof. Set $d = d(G_\Gamma)$. Let $\mathcal{X} = \{x_1, \dots, x_d\}$ and \mathcal{R} be the basis and the set of defining relations induced by Γ , respectively. Also let $\mathcal{X}^* = \{\alpha_1, \dots, \alpha_d\}$ be the dual basis of \mathcal{X} in $H^1(G_\Gamma, \mathbb{F}_p)$. Then by Proposition 2.4 one has

$$(5.2) \quad H^2(G_\Gamma, \mathbb{F}_p) = \bigoplus_{\substack{i < j \\ (x_i, x_j) \in \mathcal{E}}} \mathbb{F}_p \cdot \alpha_i \alpha_j,$$

with $\alpha_j \alpha_i = \alpha_i \alpha_j$ for all i, j . Finally, (5.2) coincides with $\Lambda_2(\mathbb{F}_p \mathcal{V})/(\Omega)$, with Ω as in Definition 5.7. \square

Once we know that a p -RAAG is quadratic, the previous lemma completely determines the \mathbb{F}_p -cohomology.

Proof of Theorem E. The proof follows at once from the definition of quadraticity and Lemma 5.8. \square

In the setting of Theorem E, we may describe the cohomology algebra of G_Γ as follows. Let \mathcal{X} be a basis of G_Γ induced by the homomorphism $F \rightarrow G_\Gamma$ in (2.3) and let $\mathcal{X}^* = \{\alpha_1, \dots, \alpha_d\}$ be its dual basis of $H^1(G_\Gamma, \mathbb{F}_p)$. Fix an integer n with $1 \leq n \leq \text{cd}(G_\Gamma)$. Then, by Theorem E, we have

$$\beta = \alpha_{i_1} \cdots \alpha_{i_n} \neq 0, \quad \text{with } 1 \leq i_1 < \dots < i_n \leq n,$$

if, and only if, $(x_{i_j}, x_{i_l}) \in \mathcal{E}$ for every $i_j, i_l \in I = \{i_1, \dots, i_n\}$. Namely, β is not trivial in $H^n(G_\Gamma, \mathbb{F}_p)$ if, and only if, there exists a clique \mathcal{G}_I of \mathcal{G} with $\mathcal{V}(\mathcal{G}_I) = \{x_{i_1}, \dots, x_{i_n}\}$. Let us denote by $\text{Cl}_n(\Gamma)$ the set of n -cliques in Γ . In particular, one has the following.

Corollary 5.9. *Let G_Γ be a quadratic p -RAAG with associated p -graph $\Gamma = (\mathcal{G}, f)$ and underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. For $1 \leq n \leq \text{cd}(G_\Gamma)$ and $1 \leq i_1 < \dots < i_n \leq n$, the assignment*

$$\alpha_{i_1} \cdots \alpha_{i_n} \mapsto \begin{cases} \mathcal{G}_I & \text{if } \alpha_{i_1} \cdots \alpha_{i_n} \neq 0, \\ 0 & \text{if } \alpha_{i_1} \cdots \alpha_{i_n} = 0, \end{cases}$$

with \mathcal{G}_I the n -clique of Γ defined as above, induces an isomorphism of vector spaces

$$H^n(G_\Gamma, \mathbb{F}_p) \xrightarrow{\sim} \mathbb{F}_p \text{Cl}_n(\Gamma).$$

Since we have a clear picture of the cohomology ring of a quadratic p -RAAG, we can describe more precisely how the underlying graph influences the cohomology.

5.3. Triangle-free p -RAAGs. In this section we prove that several graphs always yield quadratic pro- p groups. The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be *triangle-free*, if there are no triples $\{x_i, x_j, x_h\} \subseteq \mathcal{V}$ such that

$$(x_i, x_j), (x_i, x_h), (x_j, x_h) \in \mathcal{E} \cup \bar{\mathcal{E}}.$$

Proof of Theorem F. Assume \mathcal{G} is triangle-free. By Remark 5.6, G_Γ satisfies the conditions of Proposition 2.5. Consider the quadratic algebra $A_\bullet(\mathcal{G}) = Q(\mathbb{F}_p \mathcal{V}, \Omega)$ where

$$\Omega = \{x_i \otimes x_j - x_j \otimes x_i \mid i < j, (x_i, x_j) \in \mathcal{E} \cup \bar{\mathcal{E}}\}.$$

By the definition of a p -RAAG, the coefficients a_{ij} from (2.4) are either zero or one, depending on whether $(x_i, x_j) \in \mathcal{E}$. Hence $A_\bullet(\mathcal{G}) = M_\bullet(G_\Gamma)$ from Definition 2.12. By [52, § 4.2.2], one has an equality of formal power series

$$\sum_{n \geq 0} \dim(A_n(\mathcal{G})) \cdot T^n = \frac{1}{1 - dT + rT^2},$$

where $d = |\mathcal{V}|$ and $r = |\mathcal{E}|$. This yields (ii).

Condition (ii) implies (iii) by [14, Thm. 2.12].

Assume that $\text{cd}(G_\Gamma) = 2$, and suppose that Γ contains a triangle $T = (\mathcal{T}, f|_{\mathcal{T}})$ as a full subgraph, with $\mathcal{V}(\mathcal{T}) = \{x_1, x_2, x_3\}$. Let H be the subgroup of G_Γ generated by x_1, x_2, x_3 . Then $\text{cd}(H) \leq \text{cd}(G_\Gamma)$ by [29, Prop. 3.3.5] and H is powerful. If H is torsion-free, then $\text{cd}(H) = 3$ by Proposition 4.1 — in contradiction with (iii). On

the other hand, if G has non-trivial torsion, then $\text{cd}(H) = \infty$ — again contradicting (iii). Thus, (iii) implies (i).

Finally, if the three conditions hold, Proposition 2.13 and Lemma 5.8 imply that $H^\bullet(G_\Gamma, \mathbb{F}_p)$ is a quadratic algebra. \square

For example, we can show that every “cycle” p -graph yields a quadratic pro- p group.

Example 5.10. *Let G be the pro- p group with minimal presentation*

$$G = \langle x_1, \dots, x_d \mid x_1^p[x_1, x_2] = x_2^p[x_2, x_3] = \dots = x_d^p[x_d, x_1] = 1 \rangle,$$

with $d \geq 4$. By Theorem F, G is quadratic. Note that the abelianization $G/[G, G]$ is a p -elementary abelian group; moreover, G does not occur as a maximal pro- p Galois group by Theorem 5.32.

The above theorem shows that many p -RAAGs are quadratic, since every p -graph with triangle-free underlying graph yields a quadratic group. The precise magnitude of triangle-free graphs was calculated by Erdős, Kleitman and Rothschild; we record their result below.

Theorem 5.11 ([10, Thm. 2]). *The number of triangle-free graphs on n vertices is asymptotic to $2^{n^2/4+o(n^2)}$ for n tending to infinity.*

Proposition 2.13 raises the following questions, one the “dual” of the other.

Question 5.12. *Is every mild pro- p group satisfying Proposition 2.5 quadratic?*

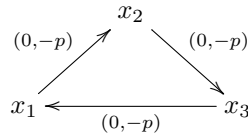
Question 5.13. *Is every finitely presented quadratic pro- p group of cohomological dimension 2 mild?*

By Theorem F, the above questions have a positive answer for p -RAAGs.

5.4. Triangle-free p -RAAGs. In the previous section we saw that a triangle-free p -graph always yields a quadratic pro- p group. In particular, the p -RAAG associated to a triangle-free p -graph is always torsion-free. This is also the case if every edge of Γ is labelled by $(0, 0)$, i.e. G_Γ is the pro- p completion of a RAAG.

On the other hand, it turns out that general p -RAAGs have a surprisingly rich structure. For instance, it is possible for a p -RAAG to be finite.

Example 5.14. *The p -RAAG G_Γ associated to the p -graph*



is given by the presentation

$$G = \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_2^{-p}, [x_2, x_3] = x_3^{-p}, [x_3, x_1] = x_1^{-p} \rangle.$$

This is a finite p -group (see [45, § 4.4, Ex. 2(e)]).

Of course a p -RAAG as in Example 5.14 cannot be quadratic, by Proposition A and Remark 2.10. Hence we need to somehow exclude the possibility that “triangles collapse the group”. One such condition is given by the following definition.

Definition 5.15. *The p -RAAG G_Γ associated to the p -graph $\Gamma = (\mathcal{V}, \mathcal{E}, f)$ is said to be uniform completable if there exist a subset $\tilde{\mathcal{E}} \subseteq \mathcal{V}^2$ and a p -labelling $\tilde{f}: \tilde{\mathcal{E}} \rightarrow p\mathbb{Z}_p \times p\mathbb{Z}_p$ such that*

- (1) $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}})$ is a combinatorial graph,
- (2) $\mathcal{E} \subseteq \tilde{\mathcal{E}}$ and $\tilde{f}|_{\mathcal{E}} = f$,
- (3) $G_{(\tilde{\mathcal{G}}, \tilde{f})}$ is a uniform pro- p group.

The above definition just says that a p -RAAG is uniform completable if its p -graph can be “completed” to the combinatorial p -graph of a uniform pro- p group.

Example 5.16. *We will see later that the p -RAAG G_{Γ_1} from Example 5.5 is a uniform pro- p group. Hence it is uniform completable.*

We can complete the p -graph Γ_2 from Example 5.5 as below; see Section 6 for the details.

$$\Gamma_2 = \begin{array}{ccc} & x_2 & \\ (\lambda, p^2) \nearrow & & \\ x_1 & \xleftarrow{(0,0)} & x_3 \end{array} \quad \mapsto \quad \tilde{\Gamma}_2 = \begin{array}{ccc} & x_2 & \\ (\lambda, p^2) \nearrow & & (0,0) \searrow \\ x_1 & \xleftarrow{(0,0)} & x_3 \end{array}$$

On the other hand, again using the methods from Section 6, one can show that the p -graph Γ_3 below cannot be completed to give a uniform group.

$$\Gamma_3 = \begin{array}{ccc} & x_2 & \xrightarrow{(p,p)} x_3 \\ (p,p) \uparrow & & \downarrow (p,p) \\ x_1 & \xleftarrow{(p,0)} & x_4 \end{array}$$

5.4.1. *Complete p -graphs and p -subgraphs.* We will now exhibit a criterion to check if a p -RAAG associated to a complete p -graph is uniform completable (or equivalently uniform).

Proposition 5.17. (a) *Let $G = \langle x_1, \dots, x_d \rangle$ be a uniform pro- p group with $d = d(G)$ and assume that for all $1 \leq i < j \leq d$ there exist $\alpha_{ij}, \beta_{ij} \in p\mathbb{Z}_p$ such that $[x_i, x_j] = x_i^{\alpha_{ij}} x_j^{\beta_{ij}}$. Then there exists a complete p -graph Γ such that $G = G_\Gamma$.*
 (b) *Let Γ be a complete p -graph and let $G_\Gamma = \langle x_1, \dots, x_d \rangle$ be its associated generalised p -RAAG. Then G_Γ is uniform if and only if every triple of generators x_r, x_s, x_t in G_Γ generates a torsion free pro- p group (which must be uniform of dimension 3). In particular, G_Γ is quadratic if and only if it is torsion free, and therefore uniform.*

Proof. If G is a uniform pro- p group as in (a), then there is a presentation of the required form over a complete p -graph Γ by [7, Proposition 4.32]. This proves (a).

Let Γ be a complete p -graph. Modulo reversing some arrows, the p -RAAG G_Γ has a presentation

$$(5.3) \quad G_\Gamma = \langle x_1, \dots, x_d \mid [x_i, x_j] = x_i^{\alpha_{ij}} x_j^{\beta_{ij}}, 1 \leq i < j \leq d \rangle.$$

Clearly G_Γ is a powerful group. Let $H_{r,s,t}$ be the subgroup of G_Γ generated by a triple of generators x_r, x_s, x_t in G_Γ . If G_Γ is uniform, then it is torsion free and therefore $H_{r,s,t}$ is torsion free as well. Now suppose that $H_{r,s,t}$ is torsion free for

every triple of generators x_r, x_s, x_t in G_Γ . Then $H_{r,s,t}$ is uniform of dimension 3. Hence the \mathbb{Z}_p -Lie algebra $L_{H_{r,s,t}}$ associated to $H_{r,s,t}$ has the presentation

$$(5.4) \quad L_{H_{r,s,t}} = \left\langle X_r, X_s, X_t \left| \begin{array}{l} [X_r, X_s] = \alpha'_{r,s}X_r + \beta'_{r,s}X_s, \\ [X_s, X_t] = \alpha'_{s,t}X_s + \beta'_{s,t}X_t, \\ [X_t, X_r] = \alpha'_{t,r}X_t + \beta'_{t,r}X_r \end{array} \right. \right\rangle$$

for some $\alpha'_{ij}, \beta'_{ij} \in p\mathbb{Z}_p$. In fact, (5.4) follows from (5.3) by direct computation using (6.2) and the hypothesis. Define the powerful \mathbb{Z}_p -Lie algebra L given by the presentation

$$L = \langle X_1, \dots, X_d \mid [X_i, X_j] = \alpha'_{ij}X_i + \beta'_{ij}X_j, \ 1 \leq i < j \leq d \rangle.$$

It is easy to see that the free \mathbb{Z}_p -module L with basis X_1, \dots, X_d has a Lie algebra structure with the given value of the Lie bracket on the generators; it suffices to check the Jacobi identity for triples of generators. Consider the uniform pro- p group G_L associated to L via the Lazard correspondence using the map $\exp : L \rightarrow G_L$. Using the properties of the commutator Campbell-Hausdorff formula $\Psi : L \rightarrow G_L$ ([7, Lem. 7.12(iii)]), one can show that

$$\exp([X_r, X_s]) = [\exp(X_r), \exp(X_s)] = \exp(X_r)^{\alpha_{rs}} \exp(X_s)^{\beta_{rs}}.$$

Thus the map $G_L \rightarrow G$ defined by $\exp(X_r) \mapsto x_r$ yields a surjective homomorphism, which induces the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(G, \mathbb{F}_p) & \xrightarrow{\text{inf}_{G_L, N}^1} & H^1(G_L, \mathbb{F}_p) & \xrightarrow{\text{res}_{G_L, N}^1} & H^1(N, \mathbb{F}_p)^G \\ & & & & & & \uparrow \\ & & & & & & H^2(G, \mathbb{F}_p) \xrightarrow{\text{inf}_{G_L, N}^2} H^2(G_L, \mathbb{F}_p) \end{array}$$

where N denotes the kernel of $G_L \rightarrow G$. The map $\text{inf}_{G_L, N}^1$ is an isomorphism because we may identify the bases of G_L and of G . Moreover, also the map $\text{inf}_{G_L, N}^2$ is an isomorphism, since Proposition 4.1 yields $H^2(G, \mathbb{F}_p) \cong H^2(G_L, \mathbb{F}_p) \cong \Lambda_2 H^1(G, \mathbb{F}_p)$. Therefore, $H^1(N, \mathbb{F}_p)^G$ is trivial, so that $N = \ker(G_L \rightarrow G)$ is trivial too. \square

Proposition 5.17 also gives a handy criterion to check if a complete p -RAAG is uniform completable. In fact, we believe that deciding whether a generalised p -RAAG is not uniform completable boils down to the same question for p -subgraphs that are triangles. This can be a very difficult task. We attempt to give some partial answers in Section 6 where we will study the groups arising from p -graphs of the form

$$\begin{array}{ccc} & x_2 & \\ (\alpha_1, \alpha_2) \nearrow & & \searrow (\beta_2, \beta_3) \\ x_1 & \longleftarrow & x_3 \\ & (\gamma_3, \gamma_1) & \end{array}$$

We will call these groups *triangle p -RAAGs*.

Furthermore, it is hard to decide whether the p -RAAG associated to a p -subgraph $\Gamma_1 \subset \Gamma$ embeds into the p -RAAG associated to Γ . This can be addressed for complete p -subgraphs of uniform completable p -graphs.

Lemma 5.18. *Let $\Gamma = ((\mathcal{V}, \mathcal{E}), f)$ be a p -graph such that the associated p -RAAG G_Γ is uniform completable, and let $\Delta = ((\mathcal{V}_\Delta, \mathcal{E}_\Delta), f_\Delta)$ be a complete p -subgraph of Γ . Then the p -RAAG G_Δ is uniform, and it embeds in G_Γ .*

Proof. Let $\tilde{\Gamma}$ be a complete p -graph which completes Γ . Then Δ is a p -subgraph of $\tilde{\Gamma}$, and one has the morphisms of pro- p groups

$$G_\Delta \xrightarrow{\phi_\Delta} G_\Gamma \xrightarrow{\phi_{\tilde{\Gamma}}} G_{\tilde{\Gamma}},$$

with $G_{\tilde{\Gamma}}$ uniform. Set $\psi = \phi_{\tilde{\Gamma}} \circ \phi_\Delta$. Since Δ is complete, G_Δ is powerful, and therefore also $\text{im}(\psi)$ is powerful. Moreover, $G_{\tilde{\Gamma}}$ is torsion-free, thus also $\text{im}(\psi)$ is torsion-free, and hence uniform. Finally, both G_Δ and $\text{im}(\psi)$ are minimally generated by \mathcal{V}_Δ . Hence, $\psi: G_\Delta \rightarrow \text{im}(\psi)$ is an isomorphism. \square

In the rest of this section we will try to apply Theorem B to show that several families of p -RAAGs yield quadratic pro- p groups. We have seen in Remark 3.6 that p -RAAGs automatically satisfy the cohomological hypotheses of Theorem B. It turns out that the main obstruction to apply the theorem is the fact that we do not have a general criterion to decide whether a certain p -RAAG G_Γ is a *proper* amalgam of two p -RAAGs associated to full subgraphs of Γ . In the next subsection, we will add two novel criteria for an amalgam of pro- p groups to be proper.

5.5. Proper amalgams of p -RAAGs. We will show below that the amalgam of two uniform pro- p groups over a uniform subgroup H is always proper, provided that the generators of H are part of the minimal generating sets of both groups. This adds a new criterion to the known criteria from [42, § 9.2]. We will add yet another new criterion for properness in later in this Section.

Proposition 5.19. *Let $1 \leq d \leq k \leq n$. Let $G_1 = \langle x_1, \dots, x_k \rangle$ and $G_2 = \langle x_d, \dots, x_n \rangle$ be uniform pro- p groups with the isomorphic closed uniform subgroup $H = \langle x_d, \dots, x_k \rangle$. Then the amalgamated free pro- p product $G = G_1 \amalg_H G_2$ is proper.*

Proof. First note that it is sufficient to show that $H^{p^n} = H \cap G_i^{p^n}$ for every n and $i = 1, 2$. In fact, we can then apply [42, Thm. 9.2.4] and we are done.

By symmetry, it is sufficient to show the property for $i = 1$. It is clear that $H^{p^n} \leq H \cap G_1^{p^n}$ for all n . By [7, Thm. 2.7], we have that $G_1^{p^n} = \langle x_1^{p^n}, \dots, x_k^{p^n} \rangle$ and $H^{p^n} = \langle x_d^{p^n}, \dots, x_k^{p^n} \rangle$. Suppose by contradiction that there is $g \in (H \cap G_1^{p^n}) \setminus H^{p^n}$. Then

$$g = x_1^{a_1 p^n} \dots x_{d-1}^{a_{d-1} p^n} x_d^{a_d p^n} \dots x_k^{a_k p^n}$$

for some $a_i \in \mathbb{Z}_p$. Since $x_d^{a_d p^n} \dots x_k^{a_k p^n} \in H$, we also have $x_1^{a_1 p^n} \dots x_{d-1}^{a_{d-1} p^n} \in H$. Thus there exist $c_d, \dots, c_k \in \mathbb{Z}_p$ such that

$$x_1^{a_1 p^n} \dots x_{d-1}^{a_{d-1} p^n} \cdot x_d^{c_d p^n} \dots x_k^{c_k p^n} = 1.$$

Since G_1 is uniform, we can conclude that

$$a_1 = \dots = a_{d-1} = c_d = \dots = c_k = 0,$$

and hence $g \in H^{p^n}$, which yields a contradiction. \square

We now need an auxilliary lemma on free products of pro- p groups and their Frattini subgroups. Recall that the Frattini series of a pro- p group G is defined inductively by $\Phi^1(G) = G$ and $\Phi^{n+1}(G) = \Phi^n(G)^p [\Phi^n(G), \Phi^n(G)]$.

Lemma 5.20. *Let H and K be pro- p groups and let $G = H * K$. Then*

$$\Phi^n(H) = H \cap \Phi^n(G) \quad \text{for all } n.$$

Proof. Clearly $\Phi^n(H) \leq H \cap \Phi^n(G)$. For the other inclusion, we first observe that there exists a retraction $r : G \rightarrow H$, i.e. a continuous homomorphism such that $r|_H = \text{id}_H$, given by $h \mapsto h$ and $k \mapsto 1$ for $h \in H$ and $k \in K$. Hence $r(\Phi^n(G)) \leq \Phi^n(r(G))$ and

$$\begin{aligned} \Phi^n(G) \cap H &= r(\Phi^n(G) \cap H) \leq r(\Phi^n(G)) \cap r(H) \leq \\ &\leq \Phi^n(r(G)) \cap H = \Phi^n(H) \cap H = \Phi^n(H). \end{aligned}$$

□

We are almost ready to state our criterion for an amalgam of p -RAAGs to be proper. First one more proposition which might be of independent interest.

Proposition 5.21. *Let $G = G_\Gamma = \langle x_1, \dots, x_d \rangle$ be a uniform completable p -RAAG. Suppose that the subgroup $H = \langle x_k, \dots, x_d \rangle$ is uniform for some $1 \leq k \leq d$. Then*

$$\Phi^n(H) = H \cap \Phi^n(G)$$

for every $n \in \mathbb{N}$.

Proof. Clearly $\Phi^n(H) \leq H \cap \Phi^n(G)$. Consider the free pro- p group $F = F(x_1, \dots, x_{k-1})$ generated by x_1, \dots, x_{k-1} and the canonical projection $\varphi : F * H \rightarrow G$ given by $x_i \mapsto x_i$ for $i = 1, \dots, d$. Also, denote by \tilde{G} the uniform quotient of G from Definition 5.15 and by $\pi : G \rightarrow \tilde{G}$ the associated projection. Then the map $(\pi \circ \varphi)|_H$ is an isomorphism, since H is uniform by hypothesis. We will use the same symbol for the three different copies of H to make the notation lighter.

Suppose by contradiction that there is $x \in (H \cap \Phi^n(G)) \setminus \Phi^n(H)$. Then, $\pi(x) \in H \cap \Phi^n(\tilde{G}) = \Phi^n(H)$. Moreover, by Lemma 5.20, there exists $y \in \Phi^n(H) = H \cap \Phi^n(F * H)$ such that $\pi(\varphi(y)) = \pi(x)$. Since $y \in H$ and $\pi|_H$ is also an isomorphism, we deduce that $\varphi(y) = x$. Thus $x \in \varphi(\Phi^n(H)) \leq \Phi^n(H)$, which yields a contradiction. □

Let us go back to the task at hand, that is showing the properness of amalgams over uniform subgroups in certain uniform completable cases.

Proposition 5.22. *Let $G_1 = G_{\Gamma_1}$ and $G_2 = G_{\Gamma_2}$ be uniform completable p -RAAGs with underlying p -graphs Γ_1 and Γ_2 , respectively. Let Γ' be a common isomorphic complete p -subgraph of Γ_1 and Γ_2 and let $H = G_{\Gamma'}$. Then the amalgamated product $G_1 \amalg_H G_2$ is proper.*

Proof. By Proposition 5.21, we have that $H \cap \Phi^n(G_i) = \Phi^n(H)$ for $i = 1, 2$ and every $n \in \mathbb{N}$. Now the result follows from [42, Thm. 9.2.4], where we can take $U_{in} = \Phi^n(G_i)$. □

Example 5.23. *Let K_1 and K_2 be uniform completable p -RAAGs and let H be a uniform pro- p group. Consider the groups $G_1 = K_1 \times H$ and $G_2 = H \times K_2$. Then $G = G_1 \amalg_H G_2$ is a proper amalgam by the pro- p version of [42, Thm. 9.2.4] and the fact that any open normal subgroup of H extends to an open normal subgroup of G_i . Note that, if H is p -RAAG, then properness follows also from Proposition 5.22.*

Moreover, if both K_1 and K_2 are quadratic, by Proposition 3.1 (b) and Proposition 4.1 also G_1 and G_2 are quadratic, and for both $i = 1, 2$ one has

$$\begin{aligned} H^1(G_i, \mathbb{F}_p) &= H^1(K_i, \mathbb{F}_p) \oplus H^1(H, \mathbb{F}_p), \\ H^2(G_1, \mathbb{F}_p) &= H^2(K_i, \mathbb{F}_p) \oplus (H^1(K_i, \mathbb{F}_p) \wedge H^1(H, \mathbb{F}_p)) \oplus H^2(H, \mathbb{F}_p), \end{aligned}$$

and the restriction maps $\text{res}_{G_i, H}^1$ and $\text{res}_{G_i, H}^2$ are the projections onto the second summand of $H^1(G_i, \mathbb{F}_p)$ and the third summand of $H^2(G_i, \mathbb{F}_p)$ respectively. Hence, all the hypothesis of Theorem B are satisfied and G is quadratic.

5.6. Some quadratic triangle-ful p -RAAGs. Next we will produce several examples of triangle-ful p -RAAGs that are quadratic. First of all we remark that all “small” uniform completable p -RAAGs are quadratic.

Lemma 5.24. *Let $\Gamma = ((\mathcal{V}, \mathcal{E}), f)$ be a p -graph.*

- (1) *If $|\mathcal{V}| \leq 3$ and $|\mathcal{E}| \leq 2$, then G_Γ is quadratic.*
- (2) *Suppose that $|\mathcal{V}| = 3$ and $|\mathcal{E}| = 3$ and that G_Γ is uniform completable. Then G_Γ is quadratic.*

Proof. In the first case, G_Γ is either a free pro- p group, a Demushkin group or an amalgamated free product of a free pro- p group and a Demushkin group over a pro-cyclic subgroup. The result follows from [40, Thm. 3.2] and Theorem B.

In the second case, G_Γ is a uniform pro- p group by Proposition 5.17. The result follows from Proposition 4.1. \square

Remark 5.25. *We now exhibit some operations that, starting from quadratic p -RAAGs, produce new quadratic p -RAAGs. We point out that, in all the following examples, we can apply Theorem B because of Remark 3.6.*

- (a) **Disjoint union.** *If Γ is the union of two disjoint subgraphs Γ_1 and Γ_2 , then G_Γ is isomorphic to $G_{\Gamma_1} * G_{\Gamma_2}$. Therefore, if Γ_1 and Γ_2 are quadratic p -RAAGs, then G_Γ is a quadratic p -RAAG as well. Moreover, free products of quadratic p -RAAGs are quadratic p -RAAGs.*
- (b) **Mirroring.** *Let G_Γ be a quadratic p -RAAG. Then the amalgamated product $G = G_\Gamma \amalg_{G_{\Gamma'}} G_\Gamma$ of G_Γ with itself over $G_{\Gamma'}$ (identified via the identity map), where Γ' is any full subgraph of Γ , is proper by [42, Ex. 9.2.6(a)]. Hence such a G is quadratic.*
- (c) **Amalgam over uniform subgroups.** *The p -RAAGs obtained from Proposition 5.19 and Proposition 5.22.*
- (d) **RAAGs.** *Pro- p completions of abstract RAAGs can be written as a series of proper HNN-extensions, hence they are quadratic. This fact was already proved by D. Riley and Th. Weigel (unpublished).*

Example 5.26. *Let Γ be a p -graph obtained from a complete p -graph by removing one edge. Using Proposition 5.19, we can show that the p -RAAG G_Γ is quadratic. One can make similar considerations using Proposition 5.22.*

Next we will show that another special class of triangle-ful p -graphs that yields several new quadratic p -RAAGs.

Definition 5.27. *A graph \mathcal{G} is chordal (or triangulated) if it contains no circuits other than triangles as full subgraphs.*

Chordal graphs are characterized by the following property (cf. [6, Prop. 5.5.1]).

Proposition 5.28. *A graph is chordal if and only if it can be constructed recursively by pasting along complete subgraphs, starting from complete graphs.*

Now we will prove Theorem H.

Proof of Theorem H. We proceed by induction on the size of Γ . Clearly, the theorem holds if Γ has only one vertex. Therefore, we can assume that Γ is a graph with more than one vertex. If Γ is complete, then $G = G_\Gamma$ is a uniform pro- p group, and therefore quadratic (cf. Proposition 5.17(b)). Otherwise, by Proposition 5.28, there are proper full subgraphs $\Gamma_1, \Gamma_2, \Delta$ of Γ , with Δ complete such that Γ is obtained by pasting together Γ_1 and Γ_2 , and $\Delta = \Gamma_1 \cap \Gamma_2$. Thus

$$(5.5) \quad G_\Gamma = G_{\Gamma_1} \amalg_{G_\Delta} G_{\Gamma_2},$$

where G_Δ is a uniform pro- p group, since G_Γ is uniform completable. Clearly, also G_{Γ_1} and G_{Γ_2} are uniform completable, and Γ_1, Γ_2 are chordal. Thus, by Proposition 5.22, the amalgam (5.5) is proper. By induction, G_{Γ_1} and G_{Γ_2} are quadratic pro- p groups. Moreover, by Remark 3.6, G_{Γ_1} , G_{Γ_2} and G_Δ satisfy the hypotheses of Theorem B. Hence, by Theorem B, G is a quadratic pro- p group. \square

We believe that all uniform completable p -RAAGs are quadratic, but this might be very hard to prove given the limited knowledge of properness of amalgamated free products in the category of pro- p groups.

As evidence of the power of our methods, we remark that every p -graph on at most 4 vertices always yields a quadratic p -RAAG. Moreover, all p -graphs on 5 vertices but those with underlying graph \mathcal{H} as in Figure 1 can be handled with the same methods.

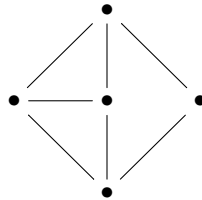


FIGURE 1. The graph \mathcal{H}

If the p -labels on \mathcal{H} are “symmetric along the horizontal axis”, this p -graph can be handled via mirroring (i.e. Remark 5.25(b) above).

On the other hand, there are instances where we cannot decide, in general, whether a uniform completable p -graph always yields a quadratic group.

5.7. Generalised p -RAAGs and Galois groups. Throughout this subsection, K denotes a field containing a root of unity of order p , and also $\sqrt{-1}$ if $p = 2$. The following result shows that for some fields K , the maximal pro- p Galois group $G_K(p)$ is a generalised p -RAAG.

Proposition 5.29. *Let G be a finitely generated solvable pro- p group which occurs as $G_K(p)$ for some field K . Then $G \cong G_\Gamma$ for a complete p -graph Γ .*

Proof. By [36, Cor. 4.9], if G is solvable then it is uniform, and moreover every 2-generated subgroup of G is again uniform — i.e. it is a 2-generated Demushkin group. Thus, given a basis $\mathcal{X} = \{x_1, \dots, x_d\}$ of G , one has $[x_i, x_j] \in \langle x_i, x_j \rangle$ and Proposition 5.17 yields the claim. \square

Clearly, the Bloch-Kato conjecture implies that being quadratic is a necessary condition for a generalised p -RAAG to occur as maximal pro- p Galois group for some field K . It is natural asking whether there are further conditions that a generalised p -RAAG must fulfill in order to occur as maximal pro- p Galois group of a field K . For example, one has the following obstruction.

Example 5.30. *Let $\Gamma = (\mathcal{G}, f)$ be the square p -graph with labels all equal to $(0, 0)$. Then, by [36, Thm. 5.6], the p -RAAG G_Γ can not be realized as $G_K(p)$ for any field K .*

We will see shortly another necessary condition about how the edges of a p -graph $\Gamma = (\mathcal{G}, f)$ and their labels patch together. Let (x_i, x_j) be an edge of \mathcal{G} and let $H_{ij} = \langle x_i, x_j \rangle$ be the subgroup of the p -RAAG G_Γ generated by x_i and x_j . By Example 5.5 and Lemma 6.5, if H_{ij} is uniform then it is a 2-generated Demushkin group and there exist $u_{ij}, w_{ij} \in H_{ij}$ such that

$$(5.6) \quad H_{ij} = \langle u_{ij}, w_{ij} \mid u_{ij}w_{ij}u_{ij}^{-1} = w_{ij}^{1+\lambda_{ij}} \rangle,$$

for some $\lambda_{ij} \in p\mathbb{Z}_p$. In particular, the structure of H_{ij} induces a homomorphism of pro- p groups

$$(5.7) \quad \theta_{ij}: H_{ij} \rightarrow 1 + p\mathbb{Z}_p, \quad \theta_{ij}(u_{ij}) = 1 + \lambda_{ij}, \quad \theta_{ij}(w_{ij}) = 1.$$

Recall that $1 + p\mathbb{Z}_p = \{1 + p\lambda \mid \lambda \in \mathbb{Z}_p\}$, which is isomorphic to \mathbb{Z}_p if $p \neq 2$, and to $C_2 \oplus \mathbb{Z}_2$ if $p = 2$.

Definition 5.31. *A p -graph $\Gamma = (\mathcal{G}, f)$, with underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, is called cyclotomic if:*

- (a) *for every edge $(x_i, x_j) \in \mathcal{E}$, the subgroup H_{ij} is uniform, and*
- (b) *for all edges $(x_i, x_j), (x_j, x_h) \in \mathcal{E}$ we have $\theta_{ij}(x_j) = \theta_{jh}(x_j)$ (cf. (5.7)).*

Namely, in a cyclotomic p -graph $\Gamma = (\mathcal{G}, f)$ the homomorphisms θ_{ij} induced by the edges of \mathcal{G} , which are 2-generated Demushkin groups, agree on common vertices. The following shows that being cyclotomic is a necessary condition for a p -graph in order to give rise to a generalised p -RAAG which is a maximal pro- p Galois group.

Theorem 5.32. *Let K be a field containing a root of unity of order p and let $\Gamma = (\mathcal{G}, f)$ be a p -graph with underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Suppose that $G_\Gamma \cong G_K(p)$ for some field K . Then Γ is cyclotomic.*

Proof. Let

$$\theta: G_\Gamma \longrightarrow 1 + p\mathbb{Z}_p$$

be the p th cyclotomic character induced by the action of $G_K(p)$ on the roots of unity of order a power of p lying in the maximal pro- p extension $K(p)$ of K (cf., e.g., [29, Def. 7.3.6]). For an edge $(x_i, x_j) \in \mathcal{E}$, the subgroup H_{ij} is the maximal pro- p Galois group of the subextension $K(p)/K(p)^{H_{ij}}$.

Since H_{ij} is not free, then H_{ij} is uniform by [36, Thm. 4.6], with $\theta_{ij}: H_{ij} \rightarrow 1 + p\mathbb{Z}_p$ the cyclotomic character of the extension $K(p)/K(p)^{H_{ij}}$, which coincides with the restriction $\theta|_{H_{ij}}$. Therefore, for all edges $(x_i, x_j), (x_j, x_h) \in \mathcal{E}$, one has $\theta_{ij}(x_j) = \theta(x_j) = \theta_{jh}(x_j)$. \square

The following result shows that cyclotomic p -graphs are also a good source of non-degenerate p -RAAGs.

Proposition 5.33. *Let $\Gamma = (\mathcal{G}, f)$ be a cyclotomic p -graph with underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then the p -RAAG G_Γ is uniform completable.*

Proof. By Remark 5.25(a), we may assume that \mathcal{G} is connected, so that every vertex $x_i \in \mathcal{V}$ belongs to some edge. Let $\theta: G_\Gamma \rightarrow 1 + p\mathbb{Z}_p$ be the homomorphism induced by the homomorphisms θ_{ij} for every edge of Γ — since Γ is cyclotomic, θ is well defined. Moreover, let (2.3) be the minimal presentation of G_Γ induced by Γ and let $\hat{\theta}: F \rightarrow 1 + p\mathbb{Z}_p$ be the composition of the projection $F \rightarrow G_\Gamma$ with θ .

Consider the following normal subgroups

$$K(G_\Gamma) = \left\langle y^{-\theta(x)}xyx^{-1} \mid x \in G_\Gamma, y \in \ker(\theta) \right\rangle \triangleleft G_\Gamma,$$

$$K(F) = \left\langle y^{-\hat{\theta}(x)}xyx^{-1} \mid x \in F, y \in \ker(\hat{\theta}) \right\rangle \triangleleft F$$

(cf. [9, § 3]). Note that $K(G_\Gamma) \leq \Phi(G_\Gamma) \cap \ker(\theta)$ and $K(F) \leq \Phi(F) \cap \ker(\hat{\theta})$. By (5.6), R is generated as normal subgroup of F by the relations

$$w_{ij}^{-\hat{\theta}(u_{ij})}u_{ij}w_{ij}u_{ij}^{-1}, \quad \text{for } (x_i, x_j) \in \mathcal{E}(\mathcal{G}),$$

and consequently $R \leq K(F)$. Therefore, [9, Thm. 5.6] implies that the quotient $\bar{G}_\Gamma := G_\Gamma/K(G_\Gamma)$ is torsion-free and it splits as semi-direct product

$$\bar{G}_\Gamma \cong \mathbb{Z}_p^m \rtimes (G_\Gamma/\ker(\theta)), \quad \text{for some } m \geq 0,$$

with action $\bar{x}z\bar{x}^{-1} = z^{\bar{\theta}(\bar{x})}$ for all $z \in \mathbb{Z}_p^m$ and $\bar{x} \in \bar{G}_\Gamma$, where $\bar{\theta}: \bar{G}_\Gamma \rightarrow 1 + p\mathbb{Z}_p$ is the morphism induced by θ (namely, the pro- p group G_Γ endowed with the morphism θ is “Kummerian”, following the language of [9]). In particular, $[x, y] \in \langle x, y \rangle \leq \bar{G}_\Gamma$. Therefore, by Proposition 5.17 (a) there exists a complete p -graph $\tilde{\Gamma}$ such that $\bar{G}_\Gamma \cong G_{\tilde{\Gamma}}$. Since $\mathcal{V}(\tilde{\Gamma}) = \mathcal{V}(\Gamma)$, $\tilde{\Gamma}$ is a completion of Γ . \square

The class of *Koszul graded algebras* is a particular class of quadratic algebras, singled out by S. Priddy in [35] — the definition of Koszul graded algebra is highly technical; we refer to [32, Ch. 2] and to [27, § 2]. Recently, Koszul graded algebras became of great interest in the context of Galois cohomology (see, e.g., [33, 34, 27]). In particular, L. Positselski conjectured in [34] that the cohomology algebra $H^\bullet(G_K(p), \mathbb{F}_p)$ is Koszul, if $G_K(p)$ is finitely generated. Later, Th. Weigel conjectured in [53] that the graded group algebra

$$\text{gr}(\mathbb{F}_p G_K(p)) = \bigoplus_{n \geq 0} I^n / I^{n+1}, \quad I^0 = \mathbb{F}_p G_K(p),$$

where I denotes the augmentation ideal of the group algebra $\mathbb{F}_p G_K(p)$ (cf. [27, § 3.2]), is also a Koszul graded algebra. By [27, Thm. F], a positive answer to Weigel’s conjecture implies a positive answer to Positselski’s conjecture. Usually it is quite hard to check whether a graded algebra is Koszul. Nonetheless, in the setting of generalised p -RAAGs we can easily deduce the following.

Corollary 5.34. *Let $\Gamma = (\mathcal{G}, f)$ be a p -graph and let G_Γ be the associated p -RAAG.*

- (i) *If G_Γ is quadratic, then the cohomology algebra $H^\bullet(G_\Gamma, \mathbb{F}_p)$ is Koszul.*
- (ii) *If Γ is triangle-free, then the graded group algebra $\text{gr}(\mathbb{F}_p G_\Gamma)$ — and thus also the cohomology algebra $H^\bullet(G_\Gamma, \mathbb{F}_p)$ — is Koszul.*

Proof. Statement (i) follows from Theorem E, and from the fact that the exterior Stanley-Reisner algebra associated to a graph is Koszul (cf. [13], see also [31, § 3.2] and [52, § 4.2.2]). Statement (ii) follows from Theorem F and [28, Thm. 1.3]. \square

Thus, generalised p -RAAGs provide a huge source of pro- p groups for which Positselski's and Weigel's conjectures hold. The above result raises the following question.

Question 5.35. *Let G_Γ be a quadratic p -RAAG with associated p -graph Γ . Is the graded algebra $\text{gr}(\mathbb{F}_p G_\Gamma)$ Koszul?*

6. TRIANGLE p -RAAGS

In the following section we will slightly change the focus of our investigation. Until now we were mainly concerned with finding new examples of quadratic pro- p groups in the family of p -RAAGs. Here we will mainly be concerned with the determination of the isomorphism classes of quadratic p -RAAGs arising from triangle p -graphs. These will be called *triangle p -RAAGs*.

6.1. The Lazard correspondence. Given a powerful pro- p group G and $n \in \mathbb{N}$, we have $P_{n+1}(G) = G^{p^n} = \{x^{p^n} \mid x \in G\}$ (see [7, Thm. 3.6]). Moreover, if G is uniform, then the mapping $x \mapsto x^{p^n}$ is a homeomorphism from G onto G^{p^n} (see [7, Lem. 4.10]). This shows that each element $x \in G^{p^n}$ admits a unique p^n th root in G , which we denote by $x^{p^{-n}}$.

As in the case of pro- p groups, a \mathbb{Z}_p -Lie algebra L is called *powerful* if $L \cong \mathbb{Z}_p^d$ for some $d > 0$ as \mathbb{Z}_p -module and $(L, L)_{Lie} \subseteq pL$ if p is odd, or $(L, L)_{Lie} \subseteq 4L$ if $p = 2$.

If G is an analytic pro- p group, then it has a characteristic open subgroup which is uniform. For every open uniform subgroup $H \leq G$, $\mathbb{Q}_p[H]$ can be made into a normed \mathbb{Q}_p -algebra, call it A , and $\log(H)$, considered as a subset of the completion \hat{A} of A , will have the structure of a Lie algebra over \mathbb{Z}_p . There is a different construction of an intrinsic Lie algebra over \mathbb{Z}_p for uniform groups. The uniform group U and its Lie algebra over \mathbb{Z}_p , call it L_U , are identified as sets, and the Lie operations are defined by

$$(6.1) \quad g + h = \lim_{n \rightarrow \infty} (g^{p^n} h^{p^n})^{p^{-n}}$$

and

$$(6.2) \quad (g, h)_{Lie} = \lim_{n \rightarrow \infty} [g^{p^n}, h^{p^n}]^{p^{-2n}} = \lim_{n \rightarrow \infty} (g^{-p^n} h^{-p^n} g^{p^n} h^{p^n})^{p^{-2n}}.$$

It turns out that L_U is a powerful \mathbb{Z}_p -Lie algebra and it is isomorphic to the \mathbb{Z}_p -Lie algebra $\log(U)$ (cf. [7, Cor. 7.14]).

On the other hand, if L is a powerful \mathbb{Z}_p -Lie algebra, then the Campbell-Hausdorff formula induces a group structure on L ; the resulting group is a uniform pro- p group. If this construction is applied to the \mathbb{Z}_p -Lie algebra L_U associated to a uniform group U , one recovers the original group. Indeed, the assignment $U \mapsto L_U$ gives an equivalence between the category of uniform pro- p groups and the category of powerful \mathbb{Z}_p -Lie algebras (see [7, Thm. 9.10]).

In light of Theorem F and Proposition 5.17, the structure of quadratic p -RAAGs associated to triangle p -graphs is of particular interest. So we are interested in

torsion-free pro- p groups defined by presentations of the form

$$(6.3) \quad G_{A'} = \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_1^{\alpha'_1} x_2^{\alpha'_2}, [x_2, x_3] = x_2^{\beta'_2} x_3^{\beta'_3}, [x_3, x_1] = x_3^{\gamma'_3} x_1^{\gamma'_1} \rangle$$

with parameters $A' = (\alpha'_1, \alpha'_2, \beta'_2, \beta'_3, \gamma'_1, \gamma'_3) \in p\mathbb{Z}_p^6$. Since the torsion-free group $G_{A'}$ is uniform, we can associate to it the \mathbb{Z}_p -Lie lattice $L_{G_{A'}}$. It follows from (6.2) that $L_{G_{A'}}$ has a (\mathbb{Z}_p -Lie algebra) presentation of the form

$$(6.4) \quad L_A = \left\langle x_1, x_2, x_3 \mid \begin{array}{l} [x_1, x_2] = \alpha_1 x_1 + \alpha_2 x_2, \quad [x_2, x_3] = \beta_2 x_2 + \beta_3 x_3, \\ [x_3, x_1] = \gamma_1 x_1 + \gamma_3 x_3 \end{array} \right\rangle$$

for some $\alpha_1, \alpha_2, \beta_2, \beta_3, \gamma_1, \gamma_3 \in p\mathbb{Z}_p$. Hence, to understand the structure of triangle p -RAAGs, it is important to determine all 3-dimensional \mathbb{Z}_p -Lie lattices of type (6.4). This will be done in the next subsection.

6.2. Triangle Lie algebras.

Lemma 6.1. *Let L be a free \mathbb{Z}_p -module with basis $\{x_1, x_2, x_3\}$ and let $\alpha_1, \alpha_2, \beta_2, \beta_3, \gamma_1, \gamma_3 \in p\mathbb{Z}_p$. Consider the bilinear function $[\cdot, \cdot] : L \times L \rightarrow L$ defined by $[x_i, x_i] = 0$ for $i = 1, 2, 3$ and*

$$[x_1, x_2] = \alpha_1 x_1 + \alpha_2 x_2, \quad [x_2, x_3] = \beta_2 x_2 + \beta_3 x_3, \quad [x_3, x_1] = \gamma_1 x_1 + \gamma_3 x_3.$$

Then $[\cdot, \cdot]$ defines a bracket on L if and only if the following system is satisfied:

$$(6.5) \quad \begin{cases} \alpha_1 \beta_2 - \gamma_1 \beta_3 = 0 \\ \gamma_1 \alpha_2 - \beta_2 \gamma_3 = 0 \\ \alpha_1 \gamma_3 - \alpha_2 \beta_3 = 0 \end{cases}$$

Proof. We only need to check that (6.5) is satisfied if and only if the function $[\cdot, \cdot]$ satisfies the Jacobi identity. Now, a straight-forward computation yields the claim. \square

Proposition 6.2. *Let $\alpha_1, \alpha_2, \beta_2, \beta_3, \gamma_1, \gamma_3 \in p\mathbb{Z}_p$ and consider the \mathbb{Z}_p -Lie lattice L_A defined in (6.4). Then there exist $\eta, \rho, \mu, \lambda \in p\mathbb{Z}_p$, satisfying $\eta\rho - \mu\lambda = 0$, such that L_A is isomorphic to one of the following:*

- $L_1(\eta, \rho, \mu, \lambda) = \langle x, y, z \mid [x, y] = \eta y, [y, z] = \mu y, [z, x] = \lambda z + \rho x \rangle$.
- $L_2(\eta, \mu) = \langle x, y, z \mid [x, y] = 0, [y, z] = \eta y + \mu z, [z, x] = 0 \rangle$.
- $L_3(\eta, \mu) = \langle x, y, z \mid [x, y] = 0, [y, z] = \eta z, [z, x] = \mu z \rangle$.

• For $\eta\mu\lambda \neq 0$,

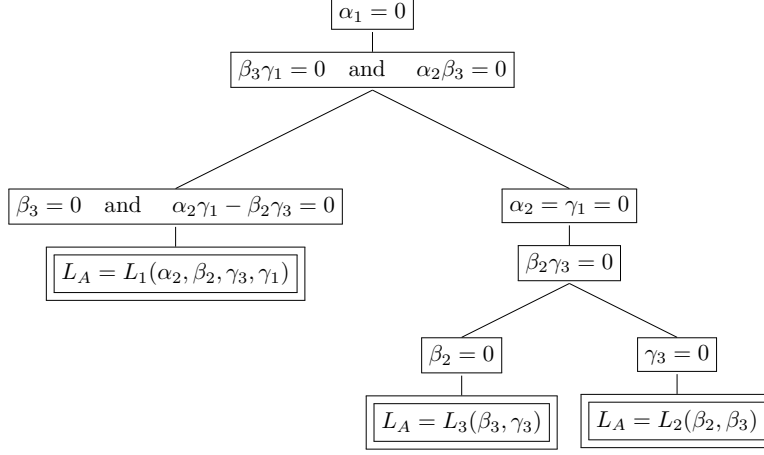
$$L_4(\eta, \mu, \lambda) = \langle x, y, z \mid [x, y] = \eta x + \mu y, [y, z] = \lambda y - \eta z, [z, x] = -\lambda x - \mu z \rangle.$$

• For $\eta\mu\lambda \neq 0$,

$$L_*(\eta, \mu, \lambda) = \langle x, y, z \mid [x, y] = \eta x + \mu y, [y, z] = \lambda y + \eta z, [z, x] = \lambda x + \mu z \rangle.$$

Proof. Since L_A is a Lie lattice, the parameters have to satisfy (6.5), by Lemma 6.1. First suppose that $\alpha_1 = 0$. It is easy to check the statement in this case, see Figure 2 below for a schematic treatment.

If $\alpha_1 \neq 0$ and some other coefficient is zero, we can use a change of basis to go back to the above case.

FIGURE 2. Proof of Proposition 6.2 for $\alpha_1 = 0$.

Suppose now that all coefficients are non-zero. Define the constants

$$q = \frac{\beta_2}{\gamma_1} = \frac{\beta_3}{\alpha_1}, \quad r = \frac{\gamma_1}{\beta_2} = \frac{\gamma_3}{\alpha_2}, \quad s = \frac{\alpha_1}{\beta_3} = \frac{\alpha_2}{\gamma_3}.$$

Then $\alpha_1 \beta_2 = \beta_3 \gamma_1$, together with the definition of r , implies that $\alpha_1 = \beta_3 r$. Since $\alpha_1 = s \beta_1$, we must have $r = s$. Furthermore $\beta_2 \gamma_3 = \alpha_2 \gamma_1$, together with the definition of q , implies that $q \gamma_3 = \alpha_2$. Since $\alpha_2 = \gamma_3 s$, we must have $s = q$. So $q = r = s$.

Finally note that $\beta_2 = q \gamma_1$ and $\gamma_1 = \beta_2 r = \beta_2 q$, therefore $q = \pm 1$. So either $q = r = s = 1$ or $q = r = s = -1$. \square

The unusual numbering in the previous lemma will become clear after Lemma 6.4. Recall that a non-zero element $z \in \mathbb{Z}_p$ can be written as formal power series $z = \sum_{n=N}^{\infty} a_n p^n$, for $a_n \in \{0, \dots, p-1\}$ with $a_N \neq 0$, and we define $v_p(z) = N$.

Lemma 6.3. *Each of the algebras of Proposition 6.2 can be written in the form*

$$L_{\alpha, \beta, \gamma}^{\xi_1, \xi_2, \xi_3} = \langle x, y, z \mid [x, y] = \alpha \xi_1, [y, z] = \beta \xi_2, [z, x] = \gamma \xi_3 \rangle$$

where $\alpha, \beta, \gamma \in p\mathbb{Z}_p$, and $\xi_1 \in \{0, x, y\}$, $\xi_2 \in \{0, y, z\}$ and $\xi_3 \in \{0, z, x, y\}$. Moreover, we have the following isomorphisms of Lie algebras

$$L_1(\eta, \rho, \mu, \lambda) \cong L_{0, \lambda, \eta}^{0, x, y}, \quad L_2(\eta, \mu) \cong L_{0, \mu, \eta}^{0, z, z}, \quad L_3(\eta, \mu) \cong L_{0, \eta, \mu}^{0, z, z}$$

$$L_4(\eta, \mu, \lambda) \cong L_{0, \eta, -\eta}^{0, y, x} \quad \text{and} \quad L_*(\eta, \mu, \lambda) \cong L_{\eta, \eta, -2\rho}^{x, z, y}.$$

Proof. The isomorphisms above can again be obtained by base-change. We will spell out the details in a few cases for the sake of clarity.

Clearly $L_3(\eta, \mu) \cong L_{0, \eta, \mu}^{0, z, z}$. Consider the algebra $L_2(\eta, \mu)$. Without loss of generality we can assume $v_p(\eta) \leq v_p(\mu)$. The change of basis

$$u = x - \frac{\mu}{\eta} z, \quad v = y + \frac{\mu}{\eta} z, \quad t = z$$

yields

$$[u, v] = \mu u, \quad [v, t] = \eta u, \quad [t, u] = 0.$$

and $L_2(\eta, \mu) \cong L_3(\mu, \eta)$. Hence $L_2(\eta, \mu) \cong L_{\mu, \eta, 0}^{y, y, 0} \cong L_{0, \eta, \mu}^{0, z, z} \cong L_3(\mu, \eta)$. The calculations for the other cases are completely analogous and will be omitted. \square

Lemma 6.4. *Let $L = L_{\alpha, \beta, \gamma}^{\xi_1, \xi_2, \xi_3}$ be the Lie algebra defined in Lemma 6.3. Then L is not metabelian if, and only if, $L = L_{\eta, \eta, -2\rho}^{x, z, y}$. Moreover, in this case L is commensurable to the Lie algebra of $\mathrm{SL}_2^1(\mathbb{Z}_p)$.*

Proof. From the proof of Lemma 6.3, we can see that it is sufficient to check that $L_1(\eta, \mu, \lambda, \rho)$ and $L_3(\eta, \mu)$ are metabelian for every allowed choice of coefficients. It is clear that $L_3(\eta, \mu)$ is metabelian.

The derived subalgebra of $L_{0, \lambda, \eta}^{0, y, x}$ is generated by λy and ηx and these elements commute in $L_{0, \lambda, \eta}^{0, y, x}$.

The last claim follows from the fact that $\mathbb{Q}_p \otimes L \cong \mathfrak{sl}_2(\mathbb{Q}_p)$ (see for instance [30, Prop. 2.30]). \square

6.3. Solvable triangle groups. We start by determining the isomorphism classes of p -RAAGs with exactly one edge.

Lemma 6.5. *Let $G_{\alpha, \beta}$ be the pro- p group defined by*

$$\langle x, y \mid [x, y] = x^\alpha y^\beta \rangle$$

with $\alpha, \beta \in p^{1+\varepsilon}\mathbb{Z}$ ($\varepsilon = 0$ if p is odd and $\varepsilon = 1$ if $p = 2$) and set $c = \min\{v_p(\alpha), v_p(\beta)\}$. Then $G_{\alpha, \beta} \cong G_{p^c, 0}$.

Proof. For $\alpha = \beta = 0$, the group $G_{0,0}$ is isomorphic to \mathbb{Z}_p^2 and the statement is clear. We can then suppose that $\alpha \neq 0$ and $v_p(\alpha) \leq v_p(\beta) \leq \infty$.

Note that $G_{\alpha, \beta}$ is a 2-generated powerful pro- p group. In particular, $[G_{\alpha, \beta}, G_{\alpha, \beta}]$ is a finitely generated normal subgroup of $G_{\alpha, \beta}$ and it is non-trivial. In fact, one can easily show that

$$G_{\alpha, \beta} / [G_{\alpha, \beta}, G_{\alpha, \beta}] \cong \mathbb{Z}_p \times \left(\mathbb{Z}_p / p^{v_p(\alpha)} \mathbb{Z}_p \right).$$

Since $G_{\alpha, \beta}$ has positive deficiency, [18, Thm. 4] yields that $G_{\alpha, \beta}$ is a pro- p duality group of dimension 2. Hence, $G_{\alpha, \beta}$ is a p -adic analytic Demushkin group of dimension 2. By [16, Prop. 7.1], $G_{\alpha, \beta}$ is isomorphic to the group

$$H = \langle x, y \mid [x, y] = x^{p^k} \rangle$$

for some positive integer k ($k \geq 2$ for $p = 2$). Comparing the abelianisations of $G_{\alpha, \beta}$ and H , we conclude that $k = v_p(\alpha)$. \square

Lemma 6.6. *Let α, β and $G_{\alpha, \beta}$ be as above. Consider the powerful \mathbb{Z}_p -Lie lattice $L_{\alpha, \beta}$ defined by*

$$\langle x, y \mid [x, y] = \alpha x + \beta y \rangle$$

and the associated pro- p group $G_{L_{\alpha, \beta}}$ under the Lazard correspondence. Then

$$G_{L_{\alpha, \beta}} \cong G_{\alpha, \beta}.$$

Proof. Without loss of generality we may suppose that $v_p(\alpha) \leq v_p(\beta) \leq \infty$. We will first perform a change of basis in $L_{\alpha, \beta}$: set $u = x + (\beta/\alpha)y$ and $v = y$, then $L'_{\alpha, \beta} = \langle u, v \mid [u, v] = \alpha u \rangle_{\mathrm{Lie}}$ is clearly isomorphic to $L_{\alpha, \beta}$.

Let $a = p^{v_p(\alpha)}$. Working directly with the Lie bracket definition of the uniform p -adic analytic pro- p group $H_a = \langle x, y \mid [x, y] = x^a \rangle$, it is easy to see that its associated Lie algebra L_{H_a} is isomorphic to $\langle x, y \mid [x, y] = ax \rangle_{\mathrm{Lie}}$. By a standard

Lie-theoretic computation, it follows that $L'_{\alpha,\beta}$ is isomorphic to L_{H_a} . Therefore $G_{L_{\alpha,\beta}} \cong G_{L_{H_a}}$. By the Lazard correspondence $G_{L_{H_a}} \cong H_a$ and finally $H_a \cong G_{\alpha,\beta}$, by Lemma 6.5. \square

In a similar fashion to the previous section, we are going to define some groups that will turn out to correspond to the above Lie algebras. For $\alpha, \beta, \gamma \in p\mathbb{Z}_p$, and $\xi_1 \in \{0, x, y\}$, $\xi_2 \in \{0, y, z\}$ and $\xi_3 \in \{0, z, x, y\}$, define the pro- p group

$$G_{\alpha,\beta,\gamma}^{\xi_1,\xi_2,\xi_3} = \langle x, y, z \mid [x, y] = \xi_1^\alpha, [y, z] = \xi_2^\beta, [z, x] = \xi_3^\gamma \rangle$$

Also define

$$G_2(\alpha, \beta) = \langle x, y, z \mid [x, y] = 1, [y, z] = y^\alpha z^\beta, [z, x] = 1 \rangle.$$

It turns out that the picture for groups is analogous to that of Lie lattices (cf. Proposition 6.2).

Lemma 6.7. *The groups $G_2(\alpha, \beta)$, $G_{0,\beta,\gamma}^{0,z,z}$ and $G_{0,\beta,\gamma}^{0,y,x}$ are metabelian uniform pro- p groups of dimension 3 for every choice of parameters. Moreover,*

$$L_{G_2(\alpha,\beta)} \cong L_2(\alpha, \beta) \cong L_{0,\beta,\alpha}^{0,z,z}, \quad L_{G_{0,\beta,\gamma}^{0,z,z}} \cong L_{0,\beta,\gamma}^{0,z,z} \quad \text{and} \quad L_{G_{0,\beta,\gamma}^{0,y,x}} \cong L_{0,\beta,\gamma}^{0,y,x}.$$

In particular, $G_{0,\beta,\gamma}^{0,z,z} \cong G_2(\beta, \gamma)$.

Proof. First of all notice that these groups are powerful pro- p groups by definition. We first consider $G_{0,\beta,\gamma}^{0,y,x}$. Define the homomorphisms

$$\varphi_1 : G_{0,\beta,\gamma}^{0,y,x} \rightarrow \langle z, x \mid [z, x] = x^\gamma \rangle \quad \text{and} \quad \varphi_2 : G_{0,\beta,\gamma}^{0,y,x} \rightarrow \langle y, z \mid [y, z] = y^\beta \rangle.$$

By Lemma 6.6, the subgroup generated by x and z is uniform of dimension 2. Moreover, the kernel of φ_1 is generated by y and hence infinite, because y has infinite image via φ_2 . In particular, the dimension of $G_{0,\beta,\gamma}^{0,y,x}$ as a p -adic analytic group must satisfy

$$3 \geq \dim(G_{0,\beta,\gamma}^{0,y,x}) \geq \dim(\text{Im } \varphi_1) + \dim(\text{Ker } \varphi_1) = 2 + 1 = 3.$$

Now, $G_{0,\beta,\gamma}^{0,y,x}$ is a powerful pro- p group with $d(G) = \dim(G)$ and, by [21, Prop. 2.12], it must be uniform. The isomorphism of Lie algebras now follows from the definition of the Lazard Lie bracket via a straight-forward calculation using (6.2).

For the group $G_2(\alpha, \beta)$ the proof is similar to the previous case using the homomorphisms induced by $x \mapsto 1$ and $y \mapsto 1$. Moreover, it is clear that $G_2(\alpha, \beta) = \langle x \rangle \times \langle y, z \rangle$. By Lemma 6.6 applied to the subgroup generated by y and z , we deduce that $L(G_2(\alpha, \beta)) \cong \mathbb{Z}_p \oplus L_{\alpha,\beta} \cong L_2(\alpha, \beta)$.

We need to use a different strategy to prove the claims about $G_{0,\beta,\gamma}^{0,z,z}$. Without loss of generality, we can assume that $v_p(\beta) \geq v_p(\alpha)$. For $\delta \in \mathbb{Z}_p$, since $y^{-1}zy = z^{1-\alpha}$, we deduce that $[z, y^\delta] = z^{(1-\alpha)^\delta - 1}$. Hence,

$$[y^\delta x, z] = [y^\delta, z]^x [x, z] = (z^{1-(1-\alpha)^\delta})^x z^{-\beta} = z^{(1+\beta)(1-(1-\alpha)^\delta)} z^{-\beta}.$$

By setting $[z, y^\delta] = 1$, we obtain the equation $(1+\beta)(1-(1-\alpha)^\delta) = \beta$. Solving for δ we obtain $\delta = \log(1 - \frac{\beta}{1+\beta}) / \log(1-\alpha) \in \mathbb{Z}_p$, since $v_p(\beta) \geq v_p(\alpha)$. Therefore

$$G_{0,\beta,\gamma}^{0,z,z} = \langle y^\delta x, y, z \mid [y^\delta x, y] = 1, [y, z] = z^\alpha, [z, y^\delta x] = 1 \rangle \cong G_2(0, \alpha).$$

In conclusion, the isomorphisms of Lie algebras are obtained using the definition of Lie bracket (6.2) and comparing abelianisations. \square

We are now ready to prove Theorem G.

Proof of Theorem G. Note that the quadratic triangle p -RAAG G_A is powerful. Hence, G_A must be uniform of dimension 3, by Theorem D. Since G_A is uniform, we can consider the associated Lazard Lie algebra. The result now follows from Lemma 6.3, Lemma 6.4 and Lemma 6.7. \square

To the interested reader the last theorem might sound unsatisfactory, as we have a pretty clear picture for the solvable case and not so for the non-solvable one. In the next section, we will content ourselves with the computation of a somewhat special case to outline the general method that could be used to produce many non-solvable triangle groups.

6.4. Unsolvble triangle groups. In this section we will use the methods of [2] to study non-solvable triangle p -RAAGs.

Let $G = G(\eta, \mu, \lambda)$ be the pro- p group defined by the balanced pro- p presentation

$$(6.6) \quad \langle x, y, z \mid [x, y] = x^\eta y^\mu, [y, z] = y^\lambda z^\eta, [z, x] = z^\mu x^\lambda \rangle$$

with $\eta, \mu, \lambda \neq 0$. Suppose additionally that $v_p(\lambda) = v_p(\mu) = v_p(\eta) = 1$. We will show that G is isomorphic to $\mathrm{SL}_2^1(\mathbb{Z}_p)$ by proving that the latter admits a presentation of the form (6.6).

Given a pro- p group G , we can form its *graded Lie algebra*: for $n \geq 1$, define

$$\mathfrak{gr}_n(G) = P_n(G)/P_{n+1}(G) \text{ and } \mathfrak{gr}(G) = \bigoplus_{n \geq 1} \mathfrak{gr}_n(G).$$

Denote by $\phi_n : P_n(G) \rightarrow \mathfrak{gr}_n(G)$ the quotient map. It is straightforward to check that the maps $\pi_n : \mathfrak{gr}_n(G) \rightarrow \mathfrak{gr}_{n+1}(G)$ induced by p -powers are linear and these extend uniquely to the linear map $\pi : \mathfrak{gr}(G) \rightarrow \mathfrak{gr}(G)$. Finally, commutators in the graded components $\mathfrak{gr}_n(G)$ endow $\mathfrak{gr}(G)$ with the structure of a graded Lie $\mathbb{F}_p[\pi]$ -algebra.

We will need the following two lemmas from [2] which can also be checked directly.

Lemma 6.8. *The Lie $\mathbb{F}_p[\pi]$ -algebra $\mathfrak{gr}(G)$ is generated by its homogeneous component $\mathfrak{gr}_1(G)$; moreover $\mathfrak{gr}(G)$ is generated by $\{\phi_1(x) \mid x \in X\}$ for every generating set X of G .*

Lemma 6.9. *The graded Lie algebra $\mathfrak{gr}(\mathrm{SL}_2^1(\mathbb{Z}_p))$ of $\mathrm{SL}_2^1(\mathbb{Z}_p)$ can be presented by*

$$(6.7) \quad \langle \bar{x}_1, \bar{x}_2, \bar{x}_3 \mid [\bar{x}_1, \bar{x}_2] = \pi \bar{x}_1, [\bar{x}_2, \bar{x}_3] = \pi \bar{x}_3, [\bar{x}_3, \bar{x}_1] = \pi \bar{x}_2 \rangle.$$

In particular, it is a free $\mathbb{F}_p[\pi]$ -module of rank 3.

Proposition 6.10. *The group $\mathrm{SL}_2^1(\mathbb{Z}_p)$ has a pro- p presentation $G(\eta, \mu, \lambda)$ of the form (6.6) for every $\eta, \mu, \lambda \in p\mathbb{Z}_p$ and $v_p(\eta) = v_p(\mu) = v_p(\lambda) = 1$.*

Proof. We will write $G = \mathrm{SL}_2^1(\mathbb{Z}_p)$. We proceed by induction. Suppose that, for $n \geq 1$, we defined three generators $x_n, y_n, z_n \in G$ and three coefficients $\eta_n, \mu_n, \lambda_n \in \mathbb{Z}_p$ such that these satisfy the relations \mathcal{R} modulo $P_{n+2}(G)$, that is

$$(6.8) \quad [x_n, y_n] \equiv x_n^{\eta_n} y_n^{\mu_n}, \quad [y_n, z_n] \equiv y_n^{\lambda_n} z_n^{\eta_n}, \quad [z_n, x_n] \equiv z_n^{\mu_n} x_n^{\lambda_n} \pmod{P_{n+2}(G)}$$

$$(6.9) \quad \eta_n \equiv \eta, \quad \mu_n \equiv \mu, \quad \lambda_n \equiv \lambda \pmod{p^{n+1}\mathbb{Z}_p}.$$

By the proof of Lemma 6.3 (or by base-change), the Lie \mathbb{F}_p -algebra $\mathfrak{gr}_1(G(\eta, \mu, \lambda))$ admits a presentation of the form (6.7), hence we have the case $n = 1$.

Let $\delta_1, \delta_2, \delta_3 \in \{0, \dots, p-1\}$ be the integers such that

$$(6.10) \quad \delta_1 \equiv \eta - \eta_n, \quad \delta_2 \equiv \mu - \mu_n, \quad \delta_3 \equiv \lambda - \lambda_n \pmod{p^{n+2}\mathbb{Z}_p}$$

and define the new coefficients $\eta_{n+1} = \eta_n + p^{n+1}\delta_1$, $\mu_{n+1} = \mu_n + p^{n+1}\delta_2$ and $\lambda_{n+1} = \lambda_n + p^{n+1}\delta_3$.

Let $\Delta_1, \Delta_2, \Delta_3 \in P_{n+1}(G)$ and define the new elements $x_{n+1} = x_n\Delta_1$, $y_{n+1} = y_n\Delta_2$ and $z_{n+1} = z_n\Delta_3$. We will prove that we can choose $\Delta_1, \Delta_2, \Delta_3$ so that

$$\begin{aligned} f_1 &= [x_{n+1}, y_{n+1}](x_{n+1}^{\eta_{n+1}} y_{n+1}^{\mu_{n+1}})^{-1} \cdot ([x_n, y_n](x_n^{\eta_n} y_n^{\mu_n})^{-1})^{-1} \\ f_2 &= [y_{n+1}, z_{n+1}](y_{n+1}^{\lambda_{n+1}} z_{n+1}^{\eta_{n+1}})^{-1} \cdot ([y_n, z_n](y_n^{\lambda_n} z_n^{\eta_n})^{-1})^{-1} \\ f_3 &= [z_{n+1}, x_{n+1}](z_{n+1}^{\mu_{n+1}} x_{n+1}^{\lambda_{n+1}})^{-1} \cdot ([z_n, x_n](z_n^{\mu_n} x_n^{\lambda_n})^{-1})^{-1} \end{aligned}$$

are congruent to 1 modulo $P_{n+3}(G)$ for every $\delta_1, \delta_2, \delta_3 \in \mathbb{Z}_p$. This will allow us to define the new generators x_{n+1} , y_{n+1} and z_{n+1} with the required properties. Finally, the elements $x = \lim_{n \rightarrow \infty} x_n$, $y = \lim_{n \rightarrow \infty} y_n$ and $z = \lim_{n \rightarrow \infty} z_n$ clearly deliver a presentation of the form $G(\eta, \mu, \lambda)$.

First we note that $f_i \in P_{n+2}(G)$, $i = 1, 2, 3$. Now, denote by \bar{y} the image of $y \in G$ in $\mathfrak{gr}_1(G)$ and write $\bar{x} = \phi_1(x_n)$, $\bar{y} = \phi_1(y_n)$, $\bar{z} = \phi_1(z_n)$. Thus $\bar{\Delta}_i$ can be written as $\pi^n(\gamma_{i1}\bar{x} + \gamma_{i2}\bar{y} + \gamma_{i3}\bar{z})$ for some $\gamma_{ij} \in \mathbb{F}_p$. Reducing f_i modulo $P_{n+3}(G)$, it is easy to show that

$$\begin{aligned} \bar{f}_1 &= \pi^{n+1}([\bar{x}, \bar{\Delta}_2] + [\bar{\Delta}_1, \bar{y}] - \delta_1\bar{x} - \delta_2\bar{y}) \\ \bar{f}_2 &= \pi^{n+1}([\bar{y}, \bar{\Delta}_3] + [\bar{\Delta}_2, \bar{z}] - \delta_3\bar{y} - \delta_1\bar{z}) \\ \bar{f}_3 &= \pi^{n+1}([\bar{z}, \bar{\Delta}_1] + [\bar{\Delta}_3, \bar{x}] - \delta_2\bar{z} - \delta_3\bar{x}). \end{aligned}$$

Using the relations (6.8) and the expressions of $\bar{\Delta}_i$, we obtain the equations

$$\begin{aligned} (\gamma_{22} + \gamma_{11} - \gamma_{23} - \delta_1)\bar{x} + (\gamma_{22} + \gamma_{11} - \gamma_{13} - \delta_2)\bar{y} + (-\gamma_{23} - \gamma_{13})\bar{z} &= 0 \\ (-\gamma_{31} - \gamma_{21})\bar{x} + (-\gamma_{31} + \gamma_{33} + \gamma_{22} - \delta_3)\bar{y} + (\gamma_{33} - \gamma_{21} + \gamma_{22} - \delta_1)\bar{z} &= 0 \\ (\gamma_{11} - \gamma_{32} + \gamma_{33} - \delta_3)\bar{x} + (-\gamma_{12} - \gamma_{32})\bar{y} + (\gamma_{11} - \gamma_{12} + \gamma_{33} - \delta_2)\bar{z} &= 0. \end{aligned}$$

Therefore we have to solve the 9×9 system of equations obtained by equating all coefficients to 0 with the parameters $\delta_1, \delta_2, \delta_3$. This system can be represented by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{32} \\ \gamma_{33} \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ 0 \\ 0 \\ \delta_3 \\ \delta_1 \\ \delta_3 \\ 0 \\ \delta_2 \end{bmatrix}$$

One can easily check that this system has the solution

$$\frac{1}{2}(\delta_2, -(\delta_2 - \delta_3), \delta_1 - \delta_2, -(\delta_1 - \delta_3), \delta_1, -(\delta_1 - \delta_2), \delta_1 - \delta_3, \delta_2 - \delta_3, \delta_3)$$

□

Remark 6.11. *Note that the 9×9 matrix appearing above has rank 8.*

Finally, in order to get a large family of subgroups of $\mathrm{SL}_2^1(\mathbb{Z}_p)$ as triangle groups, one can consider the subgroups $\langle x^{p^{a_1}}, y^{p^{a_2}}, z^{p^{a_3}} \rangle$ for $a_i \geq 1$ inside $G(\eta, \mu, \lambda)$ with $v_p(\eta) = v_p(\mu) = v_p(\lambda) = 1$. These are powerful pro- p groups and they satisfy relations of the form

$$\begin{aligned} [x^{p^{a_1}}, y^{p^{a_2}}] &= (x^{p^{a_1}})^{\alpha_1} (y^{p^{a_2}})^{\alpha_2}, & [y^{p^{a_2}}, z^{p^{a_3}}] &= (y^{p^{a_2}})^{\beta_2} (z^{p^{a_3}})^{\beta_3}, \\ [z^{p^{a_3}}, x^{p^{a_1}}] &= (z^{p^{a_3}})^{\gamma_3} (x^{p^{a_1}})^{\gamma_1} \end{aligned}$$

for some $\alpha_1, \alpha_2, \beta_2, \beta_3, \gamma_1, \gamma_3 \in p\mathbb{Z}_p$.

It is possible to slightly modify the proof of Proposition 6.10 to show that $\mathrm{SL}_2^k(\mathbb{Z}_p)$ admits a presentation

$$G(\eta, \mu, \lambda) = \langle x, y, z \mid [x, y] = x^\eta y^\mu, [y, z] = y^\lambda z^\eta, [z, x] = z^\mu x^\lambda \rangle$$

with $\eta, \mu, \lambda \in p\mathbb{Z}_p$ and $v_p(\eta) = v_p(\mu) = v_p(\lambda) = k$ for $k \geq 2$.

We believe that all the groups of the form (6.6) are torsion-free and therefore uniform for any choice of parameters. Note that, by [2] and [21, Prop. 2.12], it would be sufficient to show that these groups are infinite.

7. NON-ABELIAN FREE SUBGROUPS

In Galois theory one has the following version of the celebrated *Tits alternative* (cf. [36] and [51, Thm. 3]).

Theorem 7.1. *Let K be a field containing a root of unity of order p . Then either $G_K(p)$ is (metabelian) uniform, or it contains a free non-abelian pro- p subgroup.*

One may ask whether a similar result holds also for quadratic pro- p groups which do not arise as maximal pro- p Galois groups of fields. In the case of p -RAAGs we have Theorem I, which we prove next.

7.1. Non-abelian free subgroups in p -RAAGs.

Proof of Theorem I. Let G_Γ be a p -RAAG with associated p -graph $\Gamma = (\mathcal{G}, f)$ and underlying graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Assume that G_Γ is not powerful. Set $d = d(G)$ and let

$$G_\Gamma = \langle x_1, \dots, x_d \mid R \rangle$$

be the presentation induced by Γ . By Proposition 5.17 (b), \mathcal{G} is not complete, hence there exist two vertices in $\mathcal{V}(\mathcal{G})$ — say x_1 and x_2 — such that $(x_1, x_2) \notin \mathcal{E}(\mathcal{G})$.

Consider the pro- p group $S = \langle a, b \mid a^p = b^p = 1 \rangle$. Note that $S \cong C_p * C_p$. By [41, Theorem 4.6], S has a non-abelian free pro- p subgroup A (for instance, the derived subgroup $[S, S]$ is a non-abelian free pro- p subgroup). Let $\phi: G \rightarrow S$ be the homomorphism defined by $\phi(x_1) = a$, $\phi(x_2) = b$ and $\phi(x_i) = 1$ for $i \neq 1, 2$. Set $H = \langle x_1, x_2 \rangle$. Then $\phi|_H: H \rightarrow S$ is an epimorphism. Let \tilde{A} be the preimage of A under this map. Then $\phi|_{\tilde{A}}: \tilde{A} \rightarrow A$ is an epimorphism, and from the Hopfian property it follows that \tilde{A} is a non-abelian free pro- p subgroup of G . \square

Remark 7.2. *Our proof does not work for $p = 2$, since the infinite dihedral group $C_2 * C_2$ is 2-adic analytic, so does not have a non-abelian free pro-2 subgroup. However, if we assume that G_Γ is torsion free, then Theorem I holds for $p = 2$ as well. Indeed, in the above proof, we just need to take $S = \langle a, b \rangle = \mathbb{Z}_2 * \mathbb{Z}_2$. Nevertheless, we believe that Theorem I holds without any further assumption.*

Remark 7.3. Let $G = G_1 \amalg_H G_2$ be a pro- p group as in Theorem B and suppose that H is not equal to G_1 or G_2 , i.e., the amalgam is non-fictitious. Then the standard graph $S = S(G)$ of G is defined as follows (cf. [41, Sec. 4]):

$$S = G/H \bigcup G/G_1 \bigcup G/G_2, \quad V(S) = G/G_1 \bigcup G/G_2, \quad d_0(gH) = gG_1, \quad d_1(gH) = gG_2.$$

Now let $G = \text{HNN}(G_0, A, \phi) = \langle G_0, t \mid tat^{-1} = \phi(a), a \in A \rangle$ be as in Theorem C. Then the standard graph $S = S(G)$ of G is defined as follows:

$$S = G/A \bigcup G/G_0, \quad V(S) = G/G_0, \quad d_0(gA) = gG_0, \quad d_1(gH) = gtG_0.$$

In both cases S is a pro- p tree (see [41, Thm. 4.1]). Moreover, it is not difficult to see that G acts faithfully and irreducibly on S . By [41, Thm. 3.15], if G is not isomorphic to $C_2 \times \mathbb{Z}_2$ or \mathbb{Z}_p , then G contains a free non-abelian pro- p subgroup.

As we have seen in the previous proof, in a quadratic p -RAAG which is not uniform there must be a ‘‘missing’’ commutator among its relations. It is interesting to remark that the same is true for quadratic *mild* pro- p groups.

Proposition 7.4. Let G be a mild quadratic pro- p group with $r(G) \geq d(G)$. Then there exist linearly independent elements $\alpha, \alpha' \in H^1(G, \mathbb{F}_p)$ such that $\alpha\alpha' = 0$.

Proof. Let (2.3) be a minimal presentation of G . Let $\mathcal{X} = \{x_1, \dots, x_d\}$ be a basis of F and let $\{\alpha_1, \dots, \alpha_d\}$ be a basis of $H^1(G, \mathbb{F}_p)$ dual to \mathcal{X} .

Suppose that $\alpha\alpha' \neq 0$ for every linearly independent couple $\alpha, \alpha' \in H^1(G, \mathbb{F}_p)$. Then, by bilinearity of the cup-product, $\{\alpha_1\alpha_h \mid h = 2, \dots, d\}$ is a set of linearly independent elements of $H^2(G, \mathbb{F}_p)$. This implies that, for every $h = 1, \dots, d-1$, the commutator $[x_1, x_{h+1}]$ appears in some relation. Moreover $r(G) := m \geq d > d-1$. By Remark 2.6 and the above discussion, one may pick a set of defining relations $\mathcal{R} = \{r_1, \dots, r_m\}$ such that

$$(7.1) \quad r_h \equiv [x_1, x_{h+1}] \cdot \prod_{2 \leq i < j \leq d} [x_i, x_j]^{b(h,i,j)} \pmod{F_{(3)}}, \quad \text{for } 1 \leq h \leq d-1$$

and

$$(7.2) \quad r_h \equiv \prod_{2 \leq i < j \leq d} [x_i, x_j]^{b(h,i,j)} \pmod{F_{(3)}}, \quad \text{for } h \geq d$$

for appropriate coefficients $b(h, i, j) \in \mathbb{F}_p$. Let \tilde{G} be the pro- p group with presentation $\langle \mathcal{X} \mid \tilde{\mathcal{R}} \rangle$ where $\tilde{\mathcal{R}} = \{\tilde{r}_1, \dots, \tilde{r}_m\}$ and \tilde{r}_h is the coset representative of r_h modulo $F_{(3)}$ appearing in the right-hand side of (7.1) and (7.2). Since G is mild, by Remark 2.16 \tilde{G} is also mild. Let $N, H \leq \tilde{G}$ be the subgroups generated by $\{x_2, \dots, x_d\}$ and $\{x_1\}$, respectively. Then N is a normal subgroup of \tilde{G} . Thus the short exact sequence of pro- p groups

$$1 \longrightarrow N \longrightarrow \tilde{G} \longrightarrow H \longrightarrow 1,$$

induces the short exact sequences in cohomology

$$(7.3) \quad 0 \longrightarrow H^1(H, H^{n-1}(N, \mathbb{F}_p)) \longrightarrow H^n(\tilde{G}, \mathbb{F}_p) \longrightarrow H^n(N, \mathbb{F}_p)^{\tilde{G}} \longrightarrow 0$$

for every $n \geq 1$ (cf. [29, § II.4, Ex. 4]), since H is infinite pro-cyclic. In particular, from (7.3) for $n = 2$ we can deduce that $H^2(N, \mathbb{F}_p)^{\tilde{G}} \neq 0$. Therefore $H^1(H, H^2(N, \mathbb{F}_p)) \neq 0$. Finally (7.3) for $n = 3$ yields $H^3(\tilde{G}, \mathbb{F}_p) \neq 0$, contradicting $\text{cd}(\tilde{G}) = 2$. \square

7.2. Non-abelian free subgroups in mild pro- p groups. Even with Proposition 7.4 in hand, we were not able to prove an analogous of Theorem I for mild pro- p groups in full generality. One reason for this is that the condition of mildness for a pro- p group G depends only on the shape of the defining relations modulo $G_{(3)}$ (cf. Remark 2.16).

Nevertheless, we can show that many mild pro- p groups contain a free non-abelian subgroup. To do this, we will show that—in several cases— a mild pro- p group is a *generalised Golod-Shafarevic group* (see Section 2.6).

Proposition 7.5. *Let G be a mild quadratic and non-uniform pro- p group and let (2.3) be a minimal presentation of G . Choose a basis $\mathcal{X} = \{x_1, \dots, x_d\}$ of F and a set of defining relations $\mathcal{R} = \{r_1, \dots, r_m\} \subset F$ for G . Suppose that one of the following conditions holds:*

- (a) $r(G) \neq d(G)^2/4$;
- (b) *there are $x_i, x_j \in \mathcal{X}$, $i \neq j$, such that $x_i^3, x_j^3, [x_i, x_j]$ and any other higher commutator involving only x_i and x_j do not appear in any defining relation $r_h \in \mathcal{R}$;*
- (c) *every defining relation consists of a single elementary commutator modulo $F_{(3)}$, i.e., $r_h \equiv [x_{i_h}, x_{j_h}] \pmod{F_{(3)}}$ for some $1 \leq i_h < j_h \leq d$, for all $r_h \in \mathcal{R}$.*

Then G is generalised Golod-Shafarevic. In particular, it contains a free non-abelian pro- p subgroup.

Proof. Note that, by Theorem 2.17, it is sufficient to show that G is a generalised Golod-Shafarevic pro- p group. Set $d = d(G)$ and $m = r(G)$. By [24, Prop. 4],

$$(7.4) \quad m \leq \frac{d^2}{4}.$$

If (7.4) is a strict inequality, then G is a Golod-Shafarevich pro- p group by [54, pg. 224]. Thus it is also generalized Golod-Shafarevich. This settles part (a).

If (7.4) is an equality, then $d = 2n$ for some $n \geq 2$. Without loss of generality, suppose that condition (b) holds with $i = 1, j = 2$. We define a weight function w on $\mathbb{F}_p \langle\langle \mathcal{X} \rangle\rangle$ as follows: choose weights $w(x_i - 1) = 1$, for $i = 1, 2$, and $w(x_j - 1) = A$ for $j \neq 1, 2$, with A to be chosen later. Consider the valuation D associated to w (cf. Section 2.6). It is easy to show that $D(x_i^p) = pD(x_i)$ and $D([x_i, x_j]) = D(x_i) + D(x_j)$, for $1 \leq i < j \leq d$. By (b), if $D(r_h) \leq 5$ for some h , then one has $p = 5$ and $r_h = x_1^{ap} x_2^{bp} r'_h$, with $a, b \in \{0, 1\}$ not both equal to 0, and $D(r'_h) \geq 1 + A$. Thus one may choose \mathcal{R} such that x_1^p and x_2^p possibly appear only in at most two relations, say r_1 and r_2 . Hence $D(r_1), D(r_2) \geq 5$ and $D(r_h) \geq 1 + A$, for $h \geq 3$. It follows that

$$\begin{aligned} 1 - H_{\mathcal{X}, D}(T) + H_{\mathcal{R}, D}(T) &= 1 - (2T + (d-2)T^A) + (T^{D(r_1)} + T^{D(r_2)} + \mathcal{O}(T^A)) \\ &\leq 1 - 2T + 2T^5 + \mathcal{O}(T^{A-1}) \end{aligned}$$

for $T \in (0, 1)$. Since A can be chosen arbitrarily large, there exists $T_0 \in (0, 1)$ such that $1 - H_{\mathcal{X}, D}(T_0) + H_{\mathcal{R}, D}(T_0) < 0$. Hence G is generalized Golod-Shafarevich, and this settles part (b).

Finally, suppose condition (c) holds. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the combinatorial graph with $\mathcal{V} = \mathcal{X}$ and $\mathcal{E} = \{(x_{i_h}, x_{j_h}), h = 1, \dots, m\}$. Since $\text{cd}(G) = 2$ and G is quadratic, the graph \mathcal{G} is triangle-free. By Mantel's Theorem (cf. [26]), \mathcal{G} is a complete bipartite graph on $2n$ vertices and n^2 edges. Thus, after renumbering we

can arrange that the couples (i_h, j_h) are all the couples with $1 \leq i_h, j_h \leq d$, i_h odd and j_h even. We define a weight function w on $\mathbb{F}_p\langle\langle\mathcal{X}\rangle\rangle$ as follows: choose weights $w(x_i - 1) = 1$, for i odd, and $w(x_j - 1) = 2$ for j even. Consider the valuation D associated to w (cf. Section 2.6). It is easy to show that $D([x_i, x_j]) = D(x_i) + D(x_j)$, for $1 \leq i < j \leq d$. Hence $D(r_h) = 3$ for every $h = 1, \dots, m$. It follows that

$$\begin{aligned} 1 - H_{\mathcal{X},D}(T) + H_{\mathcal{R},D}(T) &= 1 - (nT + nT^2) + n^2T^3 \\ &= (1 - nT)(1 - nT^2). \end{aligned}$$

Thus, there exists $T_0 \in (1/n, 1/\sqrt{n})$ such that $1 - H_{\mathcal{X},D}(T_0) + H_{\mathcal{R},D}(T_0) < 0$. Hence G is generalized Golod-Shafarevich. \square

We conclude this section by showing that all “small” quadratic groups are either uniform or contain a free pro- p subgroup.

Corollary 7.6. *Let G be a quadratic pro- p group with $d(G) \leq 3$. Then either G is uniform, or G contains a free non-abelian pro- p subgroup.*

Proof. By Proposition 7.5, it is enough to show that every quadratic pro- p group with $d(G) \leq 3$ which is not analytic is either mild or free. This is clear for $d(G) = 1$. If $d(G) = 2$, by (2.5) the algebra $H^\bullet(G, \mathbb{F}_p)$ is either a trivial quadratic algebra, or the exterior algebra generated by $H^1(G, \mathbb{F}_p)$. In the former case $cd(G) = 1$, and thus G is free; in the latter case G is uniform (cf. Proof of Theorem D). Finally, for $d(G) = 3$ and $r(G) = 1$ it follows from [38, Prop. 4.2 and Prop. 4.3].

If $d(G) = 3$ and $r(G) = 2$ we may use Proposition 2.15. In this case $\dim H^2(G, \mathbb{F}_p) = 2$, so $H^2(G, \mathbb{F}_p)$ is generated by, say, the products $\alpha_1\alpha_2$ and $\alpha_1\alpha_3$. Then the subspaces $V_1 = \langle\alpha_1\rangle$ and $V_2 = \langle\alpha_2, \alpha_3\rangle$ of $H^1(G, \mathbb{F}_p)$ satisfy the hypotheses of Proposition 2.15 and G is mild. \square

In light of the previous results, Conjecture J is very natural. Notice that Conjecture J holds for almost all known examples of quadratic groups: maximal pro- p Galois groups, p -RAAGs, many mild pro- p groups and “small” quadratic pro- p groups.

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