ON THE DENSITY OF SUMSETS

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Abstract. Let $\mu^\star : \mathcal{P}(\mathbf{N}) \to \mathbf{R}$ be a subadditive function such that $\mu^\star(X) \le \mu^\star(\mathbf{N}) = 1$ and $\mu^\star(\{kx + h : x \in X\}) = \frac{1}{k}\,\mu^\star(X)$ for all $X \subseteq \mathbf{N}$ and $h, k \in \mathbf{N}^+$. Then, let μ denote the restriction of $\mu^\star$ to the family $\mathrm{dom}(\mu) := \{X \subseteq \mathbf{N} : \mu^\star(X) = 1 - \mu^\star(X^c)\}$. There are many examples of functions μ of this type (called *quasi-densities*), most notably the asymptotic density, Banach density, logarithmic density, analytic density, and Pólya density.

We show that, for all $n \in \mathbf{N}^+$ and $\alpha \in [0, 1]$, there exists a set $A \subseteq \mathbf{N}$ such that $kA \in \mathrm{dom}(\mu)$ and $\mu(kA) = k\alpha/n$ for every quasi-density μ and every $k = 1, \ldots, n$, where $kA := A + \cdots + A$ denotes the k-fold sumset of A. Moreover, for every $\alpha \in [0, 1]$, there is $A \in \mathrm{dom}(\mu)$ such that $\mu(A) = \alpha$ and $2A = \mathbf{N}$ for every quasi-density μ. Lastly, we prove that for each $\alpha \in [0, 1]$ and every non-empty finite $B \subseteq \mathbf{N}$, there exists $A \subseteq \mathbf{N}$ with $A + B \in \mathrm{dom}(\mu)$ and $\mu(A + B) = \alpha$ for every quasi-density μ.

Proofs rely on the properties of a little known density first considered by R. C. Buck and the "structure" of the set of all quasi-densities. In particular, they are rather different from previously known proofs of special cases of the same results. On the way to proving the main results, we also introduce a new type of representation for positive real numbers that might be of independent interest.

1. Introduction

Given $X_1, \ldots, X_n \subseteq \mathbf{Z}$, we denote by $X_1 + \cdots + X_n$ the *sumset* of $X_1, \ldots, X_n$, that is, the set of all sums of the form $x_1 + \cdots + x_n$ with $x_i \in X_i$ for all $1 \le i \le n$. In particular, we write kX for the *k-fold sumset* of a given $X \subseteq \mathbf{Z}$. Sumsets are some of the most fundamental objects studied in additive combinatorics [9, 12], with a great variety of results relating the "largeness" of the summands $X_1, \ldots, X_n$ to that of the sumset $X_1 + \cdots + X_n$. The aim of this work is to provide several relationships between certain sumsets of integers and their corresponding "sizes."

When the summands are finite, the size is usually taken to be the number of elements of the sets in play. However, in the oppositie, many different notions of size have been considered, each corresponding to some [partially defined] function $\mathcal{P}(\mathbf{Z}) \to \mathbf{R}$ that, while retaining some essential and desirable features of a probability, is better suited than a measure to certain applications. On this direction, many surrogates of probability measures have been already proposed in the literature, including e.g. the asymptotic density, Banach density, logarithmic density, analytic density, and Pólya density. In this work, we deal with a class of functions which are defined axiomatically and include all the previous ones; furthermore, the results will be "uniform" in the choice of these measures of largeness.

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Here, we recall the definitions of the asymptotic density d, the lower asymptotic density $\mathsf{d}_\star$, and the Schnirelmann density σ, where for a set $X \subseteq \mathbf{N}$ we take

$$\mathsf{d}(X) := \lim_{n\to\infty} \frac{|X \cap [\![1,n]\!]|}{n}, \quad \mathsf{d}_\star(X) := \liminf_{n\to\infty} \frac{|X \cap [\![1,n]\!]|}{n}, \quad \text{and} \quad \sigma(X) := \inf_{n\ge 1} \frac{|X \cap [\![1,n]\!]|}{n},$$

with the understanding that the limit in the definition of d has to exist. A few classical results that are somehow related with our work are listed below:

- In [15] (see, in particular, the last paragraph of the section "Added in proof"), B. Volkmann proved that, for all $n \ge 2$ and $\alpha_1, \ldots, \alpha_n, \beta \in \,]0,1]$ with $\alpha_1 + \cdots + \alpha_n \le \beta$, there are $A_1, \ldots, A_n \subseteq \mathbf{N}$ such that $\mathsf{d}(A_i) = \alpha_i$ for each $i = 1, \ldots, n$ and $\mathsf{d}(A_1 + \cdots + A_n) = \beta$.
- In [10, Theorem 1], M. B. Nathanson showed that, for $n \ge 2$ and all $\alpha_1, \ldots, \alpha_n, \beta \in [0,1]$ with $\alpha_1 + \cdots + \alpha_n \le \beta$, there exist $X_1, \ldots, X_n \subseteq \mathbf{N}$ such that $\mathsf{d}_\star(X_1) = \sigma(X_1) = \alpha_i$ for each $i = 1, \ldots, n$ and $\mathsf{d}_\star(X_1 + \cdots + X_n) = \sigma(X_1 + \cdots + X_n) = \beta$.

In a similar vein, A. Faisant et al. have more recently proved the following, see [3, Theorem 1.3]:

Theorem 1.1. *Given $n \in \mathbf{N}^+$ and $\alpha \in [0,1]$, there is a set $A \subseteq \mathbf{N}$ such that $\mathsf{d}(kA) = k\alpha/n$ for each $k = 1, \ldots, n$.*

Their proof combines the equidistribution theorem with the elementary property that, for every $\alpha \in \,]0,1]$, the asymptotic density of the set $\{\lfloor \alpha^{-1} n \rfloor : n \in \mathbf{N}\}$ is equal to α. In the same manuscript, one can also find the following result, see [3, Theorem 1.2]:

Theorem 1.2. *Given $\alpha \in [0,1]$ and a non-empty finite $B \subseteq \mathbf{N}$, there exists a set $A \subseteq \mathbf{N}$ such that $\mathsf{d}(A+B) = \alpha$.*

This is a partial generalization of Theorem 1.1 for the special case where $n = 1$. A complete generalization, on the other hand, was obtained by P.-Y. Bienvenu and F. Hennecart, shortly after [3] being posted on arXiv in Sept. 2018: Their proof is based on a "finite version" of Weyl's criterion for equidistribution due to P. Erdős and P. Turán, see [1, Theorem 1.8] for details and [1, Theorems 1.1.a and 1.5] for additional statements along the same lines.

Yet another item in the spirit of Theorem 1.1 is the following result by N. Hegyvári et al., see [4, Proposition 2.1]:

Proposition 1.3. *Given $\alpha \in [0,1]$, there exists $A \subseteq \mathbf{N}$ with $0 \in A$ and $\gcd(A) = 1$ such that $\mathsf{d}(A) = \alpha$ and $2A = \mathbf{N}$.*

In the present paper, we aim to prove that Theorems 1.1 and 1.2 and Proposition 1.3 hold, much more generally, with the asymptotic density d replaced by an arbitrary *quasi-density* μ (see § 2.2 for definitions) and — what is perhaps more interesting — uniformly in the choice of μ (see Theorems 3.1–3.3 for a precise formulation). Most notably, this implies that Theorems 1.1 and 1.2 are true with d replaced by the upper or the lower Banach densities [12, § 5.7] or by the analytic density [13, § III.1.3], which play a rather important role in additive combinatorics and analytic number theory and for which we are not aware of any similar results in the literature.

We emphasize that the proofs of our generalizations of Theorems 1.1 and 1.2 take a completely different route than the ones found in [1, 3]: The latter critically depend on special features of the asymptotic density, whereas our approach relies on the properties of a little known density first considered by R. C. Buck in [2] and the "structure" of the set of all quasi-densities. This is in line with one of our long-term goals, which was also the motivation for first introducing quasi-densities in [8]: To obtain sharper versions of primal results in additive combinatorics by shedding light on the "(minimal) structural properties" they depend on.

2. Preliminaries

In this section, we establish some notations and terminology used throughout the paper and prepare the ground for the proofs of our main theorems in § 3.

2.1. **Generalities.** We let $\mathbf{H}$ denote either the integers $\mathbf{Z}$ or the non-negative integers $\mathbf{N}$. Given $X \subseteq \mathbf{Z}$ and $h, k \in \mathbf{Z}$, we set $k \cdot X + h := \{kx + h : x \in X\}$. An *arithmetic progression* of $\mathbf{H}$ is then a set of the form $k \cdot \mathbf{H} + h$ with $k \in \mathbf{N}^+$ and $h \in \mathbf{H}$, and we write

- $\mathscr{A}$ for the collection of all finite unions of arithmetic progressions of $\mathbf{H}$;
- $\mathscr{A}_\infty$ for the collection of all subsets of $\mathbf{H}$ that can be expressed as the union of a finite set and countably many arithmetic progressions of $\mathbf{H}$;
- $[\![a, b]\!] := \{x \in \mathbf{Z} : a \le x \le b\}$ for the discrete interval between two integers a and b.

Further terminology and notations, if not explained when first introduced, are standard, should be clear from context, or are borrowed from [8].

2.2. **Densities (and quasi-densities).** We say a function $\mu^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R}$ is an *upper density* (on $\mathbf{H}$) provided that, for all $X, Y \in \mathcal{P}(\mathbf{H})$, the following conditions are satisfied:

(F1) $\mu^\star(X) \le \mu^\star(\mathbf{H}) = 1$;
(F2) $\mu^\star$ is *monotone*, i.e., if $X \subseteq Y$ then $\mu^\star(X) \le \mu^\star(Y)$;
(F3) $\mu^\star$ is *subadditive*, i.e., $\mu^\star(X \cup Y) \le \mu^\star(X) + \mu^\star(Y)$;
(F4) $\mu^\star(k \cdot X + h) = \frac{1}{k}\mu^\star(X)$ for every $k \in \mathbf{N}^+$ and $h \in \mathbf{H}$.

In addition, we say $\mu^\star$ is an *upper quasi-density* (on $\mathbf{H}$) if it satisfies (F1), (F3), and (F4).

Every upper density is obviously an upper quasi-density, and the existence of non-monotone upper quasi-densities is guaranteed by [8, Theorem 1]. It is arguable that non-monotone upper quasi-densities are not very interesting from the point of view of applications. However, it seems meaningful to understand if monotocity is "critical" to our conclusions or can be dispensed with: This is basically our motivation for considering upper quasi-densities instead of limiting ourselves to upper densities (although our main interest lies in the latter).

With the above in mind, we let the *conjugate* of an upper quasi-density $\mu^\star$ be the function

$$\mu_\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto 1 - \mu^\star(\mathbf{H} \setminus X).$$

Then we refer to the restriction μ of $\mu^\star$ to the set

$$\mathcal{D} := \{X \subseteq \mathbf{H} : \mu^\star(X) = \mu_\star(X)\}$$

as the *quasi-density* induced by $\mu^\star$, or simply as a *quasi-density* (on $\mathbf{H}$) if explicit reference to $\mu^\star$ is unnecessary. Accordingly, we call $\mathcal{D}$ the *domain* of μ and denote it by $\mathrm{dom}(\mu)$.

Upper densities (and quasi-densities) were first introduced in [8] and further studied in [6, 7]. Notable examples include the upper asymptotic, upper Banach, upper analytic, upper logarithmic, upper Pólya, and upper Buck densities, see [8, § 6 and Examples 4, 5, 6, and 8] for details. In particular, we recall that the upper Buck density (on $\mathbf{H}$) is the function

$$\mathfrak{b}^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \inf_{A\in\mathscr{A},\, X\subseteq A} \mathsf{d}^\star(A), \tag{1}$$

where we recall from § 2.1 that $\mathscr{A}$ is the collection of all finite unions of arithmetic progressions of $\mathbf{H}$, while $\mathsf{d}^\star$ is the *upper asymptotic density* (on $\mathbf{H}$), that is, the function

$$\mathsf{d}^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \limsup_{n\to\infty} \frac{|X \cap [\![1, n]\!]|}{n}.$$

We shall write $\mathfrak{b}_\star$ and $\mathfrak{b}$, respectively, for the conjugate of and the density induced by $\mathfrak{b}^\star$; we call $\mathfrak{b}_\star$ the lower Buck density and $\mathfrak{b}$ the Buck density (on $\mathbf{H}$). By [8, Example 5], one has

$$\mathfrak{b}_\star(X) = \sup_{A\in\mathscr{A},\, A\subseteq X} \mathsf{d}^\star(A), \quad \text{for every } X \subseteq \mathbf{H}. \tag{2}$$

We remark that the asymptotic density d and the lower asymptotic density $\mathsf{d}_\star$ introduced in § 1 are, respectively, the density induced by and the conjugate of $\mathsf{d}^\star$ (say, for $\mathbf{H} = \mathbf{Z}$): One should keep this in mind when comparing our main results (that is, Theorems 3.1–3.3) with Theorems 1.1 and 1.2 and Proposition 1.3.

2.3. **Basic properties.** Our primary goal in this section is to prove an inequality for the upper and the lower Buck density of sumsets of a certain special form (Proposition 2.3). We start with a recollection of basic facts implicit to or already contained in [8].

Proposition 2.1. *Let $\mu^\star$ be an upper quasi-density on $\mathbf{H}$. The following hold:*

- (i) $\mathfrak{b}_\star(X) \le \mu_\star(X) \le \mu^\star(X) \le \mathfrak{b}^\star(X)$ *for every* $X \subseteq \mathbf{H}$.
- (ii) *If* $h \in \mathbf{H}$ *and* $X \subseteq Y \subseteq \mathbf{H}$, *then* $\mathfrak{b}_\star(X + h) = \mathfrak{b}_\star(X) \le \mathfrak{b}_\star(Y)$.
- (iii) $\mathscr{A} \subseteq \mathrm{dom}(\mathfrak{b}) \subseteq \mathrm{dom}(\mu)$ *and* $\mu(X) = \mathfrak{b}(X)$ *for every* $X \in \mathrm{dom}(\mathfrak{b})$.
- (iv) *If* $m \in \mathbf{N}^+$ *and* $\mathcal{H} \subseteq [\![0, m-1]\!]$, *then* $m \cdot \mathbf{H} + \mathcal{H} \in \mathrm{dom}(\mathfrak{b})$ *and* $\mathfrak{b}(m \cdot \mathbf{H} + \mathcal{H}) = \frac{|\mathcal{H}|}{m}$.
- (v) *If* $X \subseteq \mathbf{H}$ *is finite, then* $X \in \mathrm{dom}(\mathfrak{b})$ *and* $\mathfrak{b}(X) = 0$.
- (vi) *If* $X \in \mathrm{dom}(\mathfrak{b})$, $Y \subseteq \mathbf{H}$, *and* $\mathfrak{b}^\star(Y) = 0$, *then* $X \cup Y \in \mathrm{dom}(\mathfrak{b})$ *and* $\mathfrak{b}(X \cup Y) = \mathfrak{b}(X)$.

Proof. We have already mentioned that $\mathfrak{b}^\star$, as defined in (1), is an upper density (and is hence monotone). With this in mind, (i) follows from [8, Proposition 2(vi), Theorem 3, and Corollary 4], where it is established, in particular, that $\mathfrak{b}^\star$ is the pointwise maximum of the set of all upper quasi-densities; (ii) follows from [8, Proposition 2(iv) and Proposition 15]; (iii) and (iv) follow from [8, Corollary 5 and Proposition 7]; and (v) follows from (i) and [8, Proposition 6]. As for (vi), note that, if $X \in \mathrm{dom}(\mathfrak{b})$, $Y \subseteq \mathbf{H}$, and $\mathfrak{b}^\star(Y) = 0$, then we have by (i), (ii), and (F3) that

$$\mathfrak{b}^\star(X) = \mathfrak{b}_\star(X) \le \mathfrak{b}_\star(X \cup Y) \le \mathfrak{b}^\star(X \cup Y) \le \mathfrak{b}^\star(X) + \mathfrak{b}^\star(Y) = \mathfrak{b}^\star(X),$$

with the result that $X \cup Y \in \mathrm{dom}(\mathfrak{b})$ and $\mathfrak{b}(X \cup Y) = \mathfrak{b}(X)$. ■

The next result shows that $\mathfrak{b}^\star$ and $\mathfrak{b}_\star$ are additive under some circumstances.

Proposition 2.2. *Let $X, Y \subseteq \mathbf{H}$ and $A, B \in \mathscr{A}$, and assume $X \subseteq A$, $Y \subseteq B$, and $A \cap B = \varnothing$. Then $\mathfrak{b}^\star(X \cup Y) = \mathfrak{b}^\star(X) + \mathfrak{b}^\star(Y)$ and $\mathfrak{b}_\star(X \cup Y) = \mathfrak{b}_\star(X) + \mathfrak{b}_\star(Y)$.*

Proof. Given $E, F, G \in \mathscr{A}$ such that $X \subseteq E$, $Y \subseteq F$, and $G \subseteq X \cup Y$, our assumptions imply

$$X \subseteq E \cap A \in \mathscr{A}, \quad Y \subseteq F \cap B \in \mathscr{A}, \quad \text{and} \quad (E \cap A) \cap (F \cap B) \subseteq A \cap B = \varnothing, \tag{3}$$

and

$$\begin{cases} \mathscr{A} \ni G \cap A \subseteq X \quad \text{and} \quad \mathscr{A} \ni G \cap B \subseteq Y, \\ G = (G \cap A) \cup (G \cap B) \quad \text{and} \quad (G \cap A) \cap (G \cap B) = \varnothing. \end{cases} \tag{4}$$

On the other hand, we have by parts (iii) and (iv) of Proposition 2.1 that

$$\mathsf{d}^\star(V \cup W) = \mathsf{d}^\star(V) + \mathsf{d}^\star(W), \quad \text{for all } V, W \in \mathscr{A} \text{ with } V \cap W = \varnothing;$$

and it is a basic fact that, for all non-empty subsets S and T of $\mathbf{R}$,

$$\inf S + \inf T = \inf(S + T) \quad \text{and} \quad \sup S + \sup T = \sup(S + T).$$

So, putting it all together, we conclude from (1) and (3) that

$$\begin{aligned} \mathfrak{b}^\star(X) + \mathfrak{b}^\star(Y) &= \inf\{\mathsf{d}^\star(E) + \mathsf{d}^\star(F) : E, F \in \mathscr{A},\ X \subseteq E, \text{ and } Y \subseteq F\} \\ &\le \inf\{\mathsf{d}^\star(E \cap A) + \mathsf{d}^\star(F \cap B) : E, F \in \mathscr{A},\ X \subseteq E, \text{ and } Y \subseteq F\} \\ &\le \inf\{\mathsf{d}^\star((E \cup F) \cap (A \cup B)) : E, F \in \mathscr{A},\ X \subseteq E, \text{ and } Y \subseteq F\} \\ &= \inf\{\mathsf{d}^\star(G) : G \in \mathscr{A} \text{ and } X \cup Y \subseteq G\} \\ &= \mathfrak{b}^\star(X \cup Y); \end{aligned}$$

and from (2) and (4) that

$$\begin{aligned} \mathfrak{b}_\star(X \cup Y) &= \sup\{\mathsf{d}^\star(G) : G \in \mathscr{A} \text{ and } G \subseteq X \cup Y\} \\ &= \sup\{\mathsf{d}^\star(E \cup F) : E, F \in \mathscr{A},\ E \subseteq X, \text{ and } F \subseteq Y\} \\ &= \sup\{\mathsf{d}^\star(E) + \mathsf{d}^\star(F) : E, F \in \mathscr{A},\ E \subseteq X, \text{ and } F \subseteq Y\} \\ &= \mathfrak{b}_\star(X) + \mathfrak{b}_\star(Y). \end{aligned}$$

This is enough to finish the proof, when considering that $\mathfrak{b}^\star$ is subadditive. ■

It is perhaps worth noticing that Proposition 2.2 does not hold with $\mathfrak{b}^\star$ replaced by $\mathsf{d}^\star$. In fact, set $X := E \cap (2 \cdot \mathbf{H})$ and $Y := F \cap (2 \cdot \mathbf{H} + 1)$, where

$$E := \bigcup_{n \ge 1} [\![(4n)!, (4n+1)!]\!] \quad \text{and} \quad F := \bigcup_{n \ge 1} [\![(4n+2)!, (4n+3)!]\!].$$

Then X and Y are both contained in disjoint arithmetic progressions of $\mathbf{H}$, but it is not difficult to see that $\mathsf{d}^\star(X) = \mathsf{d}^\star(Y) = \mathsf{d}^\star(X \cup Y) = \frac{1}{2}$, cf. [8, Lemma 1].

Proposition 2.3. *Fix $n, t, q \in \mathbf{N}^+$ such that $nt < q$, let V be a non-empty subset of $q \cdot \mathbf{H} + t$, and define $X := q \cdot \mathbf{H} + [\![0, t-1]\!]$ and $S := X \cup V$. Then*

$$\frac{kt}{q} \le \mathfrak{d}_\star(kS) \le \mathfrak{d}^\star(kS) = \frac{kt}{q} + \mathfrak{d}^\star(kV) \le \frac{kt+1}{q}, \quad \textit{for every } k \in [\![1, n]\!].$$

Proof. Fix $k \in [\![1, n]\!]$ and set

$$Z := \bigcup_{i=1}^{k} (iX + (k-i)V) \quad \text{and} \quad W := q \cdot \mathbf{H} + [\![0, kt-1]\!].$$

It is clear that

$$kS = k(X \cup V) = \bigcup_{i=0}^{k} (iX + (k-i)V) = kV \cup Z; \tag{5}$$

and since V is a non-empty subset of $q \cdot \mathbf{H} + t$, there exists $x \in \mathbf{H}$ such that

$$(k-i)qx + (k-i)t \in (k-i)V \subseteq q \cdot \mathbf{H} + (k-i)t, \quad \text{for each } i \in [\![0, k]\!]. \tag{6}$$

Considering that $(k-i)t \le kt - (i+1) + 1$ for all $i \in \mathbf{N}^+$, we obtain from (6) that

$$Z \subseteq \bigcup_{i=1}^{k} (q \cdot \mathbf{H} + [\![0, (t-1)i]\!] + (k-i)t) = \bigcup_{i=1}^{k} (q \cdot \mathbf{H} + [\![(k-i)t, kt-i]\!]) = W. \tag{7}$$

In a similar way, we find that

$$Z \supseteq \bigcup_{i=1}^{k} (q \cdot \mathbf{H} + (k-i)qx + [\![(k-i)t, kt-i]\!]) \supseteq kqx + W. \tag{8}$$

It follows from (5), (8), and parts (i) and (ii) of Proposition 2.1 that

$$\mathfrak{d}_\star(W) = \mathfrak{d}_\star(kqx + W) \le \mathfrak{d}_\star(Z) \le \mathfrak{d}_\star(kS) \le \mathfrak{d}^\star(kS). \tag{9}$$

On the other hand, since $kt \le nt < q$ (by hypothesis), we get from (6) and (7) that

$$Z \subseteq W \in \mathscr{A}, \quad kV \subseteq q \cdot \mathbf{H} + kt \in \mathscr{A}, \quad \text{and} \quad Z \cap kV = \varnothing.$$

Therefore, we conclude from (9) and Propositions 2.1(ii) and 2.2 that

$$\mathfrak{d}_\star(W) \le \mathfrak{d}^\star(kS) = \mathfrak{d}^\star(Z) + \mathfrak{d}^\star(kV) \le \mathfrak{d}^\star(W) + \frac{1}{q}.$$

This finishes the proof, because $\mathfrak{d}_\star(W) = \mathfrak{d}^\star(W) = kt/q$ by Proposition 2.1(iv). ∎

2.4. **A positional representation.** We introduce a non-standard positional representation of real numbers (Proposition 2.5) that will be of key importance in the proof of Theorem 3.1; cf. [11, Theorem 1.6] for an "analogous" result attributed by I. Niven to G. Cantor.

Below, for $x \in \mathbf{R}$ we let $\lfloor x \rfloor$ denote the greatest integer $\le x$ and set $\operatorname{frac}(x) := x - \lfloor x \rfloor$.

Lemma 2.4. *Let α be an irrational number in the interval $[0, 1]$, and fix $m, t \in \mathbf{N}^+$. There exist infinitely many $n \in \mathbf{N}^+$ such that $\lfloor (nt+1)\alpha \rfloor \in m \cdot \mathbf{N}^+$.*

Proof. Since $t\alpha$ is irrational, the sequence $(\text{frac}(Nt\alpha))_{N\ge 0}$ is equidistributed in $[0,1[$. This implies that there exists a set $\mathcal{N} \subseteq \mathbf{N}^+$ such that $\mathsf{d}(\mathcal{N}) = \frac{1-\alpha}{m}$ and $\text{frac}(Nt\alpha) \in \left]0, \frac{1-\alpha}{m}\right[$ for all $N \in \mathcal{N}$, see e.g. [5, Exercise 1.15, p. 6]. Since

$$\text{frac}((Ntm+1)\alpha) = m\text{frac}(Nt\alpha) + \alpha \in \left]0,1\right[,$$

it follows that $\lfloor (Ntm+1)\alpha \rfloor = m\lfloor Nt\alpha \rfloor \in m \cdot \mathbf{N}^+$ for all $N \in \mathcal{N}$. ■

Proposition 2.5. *Let α be an irrational number in the interval $[0,1]$, and fix $n \in \mathbf{N}^+$. There exist sequences $(\beta_i)_{i\ge 1}$ and $(q_i)_{i\ge 0}$ of positive integers with $q_0 = 1$ such that*

$$\alpha = \sum_{i\ge 1} \frac{n!\,\beta_i}{q_1 \cdots q_i} \tag{10}$$

and, for every $i \in \mathbf{N}^+$,

$$\gcd(q_i, nq_0 \cdots q_{i-1}) = 1, \quad \alpha_{i-1} \in \left]0,1\right[, \quad \textit{and} \quad \lfloor q_i\alpha_{i-1} \rfloor \in n! \cdot \mathbf{N}^+, \tag{11}$$

where we have defined

$$\alpha_0 := \alpha \quad \textit{and} \quad \alpha_i := q_1 \cdots q_i \left(\alpha - \sum_{j=1}^{i} \frac{n!\,\beta_j}{q_1 \cdots q_j} \right). \tag{12}$$

Proof. For each irrational number $x \in [0,1]$ and $N \in \mathbf{N}^+$, set

$$\mathcal{Q}(x,N) := \left\{ q \in \mathbf{N}^+ : \gcd(q,N) = 1 \text{ and } \lfloor qx \rfloor \in n! \cdot \mathbf{N}^+ \right\}.$$

It follows by Lemma 2.4 that the set $\mathcal{Q}(x,N)$ is infinite.

We define recursively the sequences $(\beta_i)_{i\ge 1}$ and $(q_i)_{i\ge 0}$ as follows. Set $q_0 := 1$ and, for each $i \in \mathbf{N}^+$, pick

$$q_i \in \mathcal{Q}(\alpha_{i-1}, nq_0 \cdots q_{i-1}). \tag{13}$$

Note that this is possible because α_{i-1} is irrational by its definition in (12). Setting

$$\beta_i := \left\lfloor \frac{q_i\alpha_{i-1}}{n!} \right\rfloor,$$

we get by (13) that $\beta_i \in \mathbf{N}^+$ and, in addition,

$$q_i\alpha_{i-1} - 1 < n!\,\beta_i < q_i\alpha_{i-1}. \tag{14}$$

Clearly, $\alpha_0 = \alpha \in \left]0,1\right[$. If, on the other hand, $\alpha_{i-1} \in \left]0,1\right[$ for some $i \in \mathbf{N}^+$, then it follows by (12) and (14) that $\alpha_i = q_i\alpha_{i-1} - n!\,\beta_i \in \left]0,1\right[$. Thus, we see by induction that

$$\alpha_i \in \left]0,1\right[, \quad \text{for all } i \in \mathbf{N}.$$

We may note, thanks to (13), that $q_i > q_i\alpha_{i-1} > n! \ge 1$, hence $q_i \ge 2$ for all $i \in \mathbf{N}^+$. To conclude, identity (10) is obtained by the fact that

$$\left| \alpha - \sum_{j=1}^{i} \frac{n!\,\beta_j}{q_1 \cdots q_j} \right| = \frac{\alpha_i}{q_1 \cdots q_i} < \frac{1}{2^i}, \quad \text{for all } i \in \mathbf{N}^+.$$

■

3. Main results

This section is devoted to the main results of the paper. We start with a generalization of Theorem 1.1. Recall from § 2.1 that $\mathscr{A}_\infty$ denotes the family of all subsets of $\mathbf{H}$ that can be expressed as the union of a finite set and countably many arithmetic progressions of $\mathbf{H}$.

Theorem 3.1. *Given $n \in \mathbf{N}^+$ and $\alpha \in [0,1]$, there exists $A \in \mathscr{A}_\infty$ such that $kA \in \mathrm{dom}(\mu)$ and $\mu(kA) = k\alpha/n$ for each $k \in [\![1,n]\!]$ and every quasi-density μ on $\mathbf{H}$.*

Proof. Thanks to Proposition 2.1(iii), it will be enough to prove that there exists $A \in \mathscr{A}_\infty$ such that $kA \in \mathrm{dom}(\mathfrak{b})$ and $\mathfrak{b}(kA) = \alpha k/n$ for each $k \in [\![1,n]\!]$. To this end, we distinguish two cases.

Case 1: α is rational. Write $\alpha = a/b$, where $a \in \mathbf{N}$ and $b \in \mathbf{N}^+$. Then set

$$A := \{0\} \cup (nb \cdot \mathbf{H} + [\![1,a]\!]) \in \mathscr{A}_\infty.$$

Since $0 \le a \le b$, it is immediate (by induction) that

$$kA = \{0\} \cup (nb \cdot \mathbf{H} + [\![1,ka]\!]), \quad \text{for every } k \in [\![1,n]\!].$$

By Proposition 2.1(iii)–(vi), this implies that

$$kA \in \mathrm{dom}(\mathfrak{b}) \quad \text{and} \quad \mathfrak{b}(kA) = \frac{ka}{nb} = \frac{\alpha k}{n}, \quad \text{for every } k \in [\![1,n]\!].$$

Case 2: α is irrational. By Proposition 2.5, there exist sequences $(\beta_i)_{i\ge1}$ and $(q_i)_{i\ge0}$ of positive integers with $q_0 = 1$ such that $\gcd(q_i, nq_0\cdots q_{i-1}) = 1$ for every $i \in \mathbf{N}^+$ and

$$\alpha = \sum_{i\ge1} \frac{n!\,\beta_i}{q_1\cdots q_i}. \tag{15}$$

Accordingly, we can recursively define sequences $(X_i)_{i\ge1}$ and $(Y_i)_{i\ge0}$ of subsets of $\mathbf{H}$ by taking $Y_0 := \mathbf{H}$ and, for each $i \in \mathbf{N}^+$,

$$X_i := Y_{i-1} \cap (q_i \cdot \mathbf{H} + [\![0,(n-1)!\,\beta_i - 1]\!]) \quad \text{and} \quad Y_i := Y_{i-1} \cap (q_i \cdot \mathbf{H} + (n-1)!\,\beta_i). \tag{16}$$

Because $q_1, q_2, \ldots$ are pairwise coprime integers, it is straightforward from (16) and the Chinese remainder theorem that, for every $i \in \mathbf{N}^+$,

$$Y_i = \bigcap_{j=1}^{i} (q_j \cdot \mathbf{H} + (n-1)!\beta_j) = q_1\cdots q_i \cdot \mathbf{H} + r_i, \quad \text{for some } r_i \in \mathbf{N}.$$

Consequently, we obtain from Proposition 2.1(iv) that

$$kY_i \in \mathrm{dom}(\mathfrak{b}) \quad \text{and} \quad \mathfrak{b}(kY_i) = \frac{1}{q_0\cdots q_i} \le \frac{1}{2^i}, \quad \text{for all } i,k \in \mathbf{N}^+. \tag{17}$$

Note that the sets $X_1, X_2, \ldots$ are pairwise disjoint and, for every $i \in \mathbf{N}^+$,

$$X_i, Y_i \in \mathscr{A} \setminus \{\varnothing\} \quad \text{and} \quad X_i \cup Y_i \subseteq Y_{i-1}. \tag{18}$$

Then, for each $i \in \mathbf{N}$, define $A_i := X_1 \cup \cdots \cup X_i$ and $B_i := A_i \cup Y_i$. We set

$$A := \bigcup_{i\ge1} A_i = \bigcup_{i\ge1} X_i.$$

By construction, it is obvious that $A \in \mathscr{A}_\infty$. So, to finish the proof, it only remains to show that $kA \in \operatorname{dom}(\mathfrak{b})$ and $\mathfrak{b}(kA) = k\alpha/n$ for all $k \in [\![1, n]\!]$.

Fix $k \in [\![1, n]\!]$ and $i \in \mathbf{N}^+$. Since $\mathfrak{b}$ is monotone, it is clear from (17) and (18) that

$$\mathfrak{b}(kX_i) \le \mathfrak{b}(k(X_i \cup Y_i)) \le \frac{1}{2^{i-1}}. \tag{19}$$

On the other hand, it follows from (18) and the above that

$$A_i \subseteq A \subseteq B_i \quad \text{and} \quad A_i, B_i \in \mathscr{A} \setminus \{\varnothing\},$$

implying that

$$kA_i \subseteq kA \subseteq kB_i, \quad kA_i, kB_i \in \operatorname{dom}(\mathfrak{b}), \quad \text{and} \quad \mathfrak{b}(kA_i) \le \mathfrak{b}_\star(kA) \le \mathfrak{b}^\star(kA) \le \mathfrak{b}(kB_i). \tag{20}$$

We claim that

$$\mathfrak{b}(kA_i) = \frac{k}{n} \sum_{j=1}^{i-1} \frac{n!\, \beta_j}{q_1 \cdots q_j} + \mathfrak{b}(kX_i). \tag{21}$$

Indeed, define $Z_j := A_i \setminus A_j = X_{j+1} \cup \cdots \cup X_i$ $(0 \le j < i)$. Then $Z_j \in \mathscr{A} \setminus \{\varnothing\}$ and $Z_j \subseteq Y_j$, and we derive from Proposition 2.3 and (16) that

$$\mathfrak{b}(kZ_j) = \mathfrak{b}(k(X_{j+1} \cup Z_{j+1})) = \frac{k}{n} \cdot \frac{n!\, \beta_{j+1}}{q_1 \cdots q_{j+1}} + \mathfrak{b}(kZ_{j+1}), \quad \text{for each } j \in [\![0, i-2]\!].$$

Thus, recalling that $A_i = Z_0$, we obtain by induction that

$$\mathfrak{b}(kA_i) = \frac{k}{n} \cdot \frac{n!\beta_1}{q_1} + \mathfrak{b}(kZ_1) = \cdots = \frac{k}{n} \sum_{j=1}^{i-1} \frac{n!\, \beta_j}{q_1 \cdots q_j} + \mathfrak{b}(kZ_{i-1}).$$

This suffices to prove the claim (because $X_i = Z_{i-1}$), and in a similar way we find that

$$\mathfrak{b}(kB_i) = \frac{k}{n} \sum_{j=1}^{i-1} \frac{n!\, \beta_j}{q_1 \cdots q_j} + \mathfrak{b}(k(X_i \cup Y_i)). \tag{22}$$

The proof is essentially the same as the proof of (21), with Z_j replaced by $B_i \setminus A_j$ $(0 \le j < i)$; we omit further details. Therefore, it follows by (15), (19), (21), and (22) that

$$\max\left\{ \left| \mathfrak{b}(kA_i) - \frac{k\alpha}{n} \right|, \left| \mathfrak{b}(kB_i) - \frac{k\alpha}{n} \right| \right\} \le \sum_{j \ge i} \frac{n!\, \beta_j}{q_1 \cdots q_j} + \frac{1}{2^{i-1}}.$$

Consequently, we see that

$$\lim_{i \to \infty} \mathfrak{b}(kA_i) = \lim_{i \to \infty} \mathfrak{b}(kB_i) = \frac{k\alpha}{n},$$

and we conclude, by (20), that $kA \in \operatorname{dom}(\mathfrak{b})$ and $\mathfrak{b}(kA) = k\alpha/n$ (as wished). ■

Theorem 3.2. *Given $\alpha \in [0, 1]$ and a non-empty finite set $B \subseteq \mathbf{H}$, there exists $A \in \mathscr{A}_\infty$ such that $A + B \in \operatorname{dom}(\mu)$ and $\mu(A + B) = \alpha$ for every quasi-density μ on $\mathbf{H}$.*

Proof. Similarly as in the proof of Theorem 3.1, it suffices to prove that there exists $A \in \mathscr{A}_\infty$ such that $A+B \in \mathrm{dom}(\mathfrak{b})$ and $\mathfrak{b}(A+B)=\alpha$. To this end, set $x := \min B$ and $y := \max B$.

We may assume without loss of generality that $x=0$, because $A+B=(A+x)+(B-x)$ and both $A+x$ and $B-x$ are subsets of $\mathbf{H}$, with $|B-x|=|B|$. Therefore, B is a subset of $\mathbf{N}$; and we can suppose that $y \neq 0$, or else the conclusion follows by Theorem 3.1.

Now, the statement to be proved is trivial for $\alpha=0$ or $\alpha=1$ (just take $A := \varnothing$ in the former case and $A := \mathbf{H}$ in the latter). Consequently, let $\alpha \in \,]0,1[$ and pick $h,k \in \mathbf{N}^+$ such that

$$\frac{h}{k} < \alpha < \frac{h+1}{k} \quad \text{and} \quad h \geq 2y+1.$$

Then $k\alpha - h \in \,]0,1[$ and $h-y-1 \geq y$, and we derive from Theorem 3.1 that there exists a set $C \in \mathscr{A}_\infty \cap \mathrm{dom}(\mathfrak{b})$ such that $\mathfrak{b}(C)=k\alpha-h$. So, we define

$$A := (k \cdot \mathbf{H} + [\![0, h-y-1]\!]) \cup (k \cdot C + h - y).$$

Then it is straightforward that

$$A \in \mathscr{A}_\infty \quad \text{and} \quad A+B = (k \cdot \mathbf{H} + [\![0, h-1]\!]) \cup (k \cdot C + h),$$

and it follows by Propositions 2.1(iv) and 2.2 that

$$\mathfrak{b}^\star(A+B) = \mathfrak{b}^\star(k \cdot \mathbf{H} + [\![0,h-1]\!]) + \mathfrak{b}^\star(k \cdot C + h) = \frac{h+\mathfrak{b}(C)}{k} = \alpha.$$

Likewise, we calculate that $\mathfrak{b}_\star(A+B)=\alpha$. Thus, $A+B \in \mathrm{dom}(\mathfrak{b})$ and $\mathfrak{b}(A+B)=\alpha$. ∎

Theorem 3.3. *Given $\alpha \in [0,1]$, there exists a set $A \subseteq \mathbf{H}$ with $0 \in A$ and $\gcd(A)=1$ such that $2A=\mathbf{H}$, $A \in \mathrm{dom}(\mu)$, and $\mu(A)=\alpha$ for every quasi-density μ on $\mathbf{H}$.*

Proof. Once again, it suffices to prove that there exists $A \in \mathrm{dom}(\mathfrak{b})$ such that $\mathfrak{b}(A)=\alpha$, cf. the proofs of Theorems 3.1 and 3.2. To this end, set

$$Q := \{x^2+y^2 : x,y \in \mathbf{N}\} \quad \text{and} \quad X := (Q \cup (-Q)) \cap \mathbf{H}.$$

We know from Lagrange's four square theorem that $2Q=\mathbf{N}$, and from [6, Theorem 4.2] that $\mathfrak{b}(Q)=0$. It follows that $2X=\mathbf{H}$. Moreover, it is clear from the definition of $\mathfrak{b}^\star$ that

$$\mathfrak{b}^\star((-Q)\cap \mathbf{H}) = \mathfrak{b}^\star(Q \cap (-\mathbf{H})) \leq \mathfrak{b}^\star(Q) = 0.$$

Therefore, we find that

$$X \in \mathrm{dom}(\mathfrak{b}) \quad \text{and} \quad \mathfrak{b}(X)=0.$$

On the other hand, Theorem 3.1 guarantees that $\mathfrak{b}(Y)=\alpha$ for some $Y \in \mathrm{dom}(\mathfrak{b})$. So, letting $A := X \cup Y$ and putting all pieces together, we get from Proposition 2.1(vi) that

$$2A = \mathbf{H}, \quad A \in \mathrm{dom}(\mathfrak{b}), \quad \text{and} \quad \mathfrak{b}(A)=\alpha.$$

This finishes the proof, when considering that $0 \in Q \subseteq A$ and $1 \leq \gcd(A) \leq \gcd(Q) = 1$. ∎

4. Closing remarks

Looking at the statement of Theorem 3.1, it is natural to ask whether assuming $A \in \operatorname{dom}(\mu)$, for some fixed quasi-density μ on $\mathbf{H}$, is sufficient to guarantee that $2A \in \operatorname{dom}(\mu)$.

By [4, Proposition 2.2], the answer is negative for the asymptotic density d. On the other hand, it follows by [8, Remark 3] that, in the classical framework of Zermelo-Fraenkel set theory with the axiom of choice, there exists a density μ on $\mathbf{H}$ such that $\operatorname{dom}(\mu) = \mathbf{H}$; hence, in this case, the answer is positive.

One can still wonder what happens with the Buck density $\mathfrak{b}$, especially in light of the role played by $\mathfrak{b}$ in the proofs of § 3. Again, the answer turns out to be in the negative. In fact, set

$$V := \{n! + n : n \in \mathbf{N}\} \quad \text{and} \quad A := \{x^2 + y^2 : x, y \in V\}.$$

Since $\mathfrak{b}^\star$ is monotone, it follows from [6, Theorem 4.2], similarly as in the proof of Theorem 3.3, that $A \in \operatorname{dom}(\mathfrak{b})$ and $\mathfrak{b}(A) = 0$. However, we will show that $2A \notin \operatorname{dom}(\mathfrak{b})$. First, note that

$$2A = \left\{x_1^2 + x_2^2 + x_3^2 + x_4^2 : x_1, x_2, x_3, x_4 \in V\right\}.$$

Next, fix $k \in \mathbf{N}^+$ and $h \in \mathbf{N}$. By Lagrange's four square theorem, there exist $y_1, y_2, y_3, y_4 \in \mathbf{N}$ such that $h = \sum_{i=1}^4 y_i^2$. Set, for each $i \in [\![1, 4]\!]$, $n_i := (h+1)k + y_i$ and $x_i := n_i! + n_i$, and note that $x_i \in V$, $x_i \ge h$, and $n_i \ge k$. It is then easily checked that

$$\sum_{i=1}^4 x_i^2 \equiv \sum_{i=1}^4 (n_i!(n_i! + 2n_i) + n_i^2) \equiv \sum_{i=1}^4 n_i^2 \equiv \sum_{i=1}^4 y_i^2 \equiv h \bmod k.$$

Therefore $(k \cdot \mathbf{H} + h) \cap 2A$ is non-empty and, since k and h were arbitrary, we conclude that the only arithmetic progression of $\mathbf{H}$ containing $2A$ is $\mathbf{H}$ itself, with the result that $\mathfrak{b}^\star(2A) = 1$.

Suppose for a contradiction that $\mathfrak{b}_\star(2A) \neq 0$. From (2), this is only possible if $2A$ contains an arithmetic progression of $\mathbf{H}$, implying that there is a constant $C \in \mathbf{R}^+$ such that $|2A \cap [1, m]| \ge Cm$ for all large m. The latter is, however, a contradiction, because it is clear that

$$|2A \cap [\![1, m]\!]| \le |V \cap [\![1, \sqrt{m}\,]\!]|^4 \le \sup\{n^4 : n \in \mathbf{N} \text{ and } n! \le \sqrt{m}\} = o(m), \quad \text{as } m \to \infty.$$

It follows that $\mathfrak{b}_\star(2A) = 0 \neq \mathfrak{b}^\star(2A)$, and hence $2A \notin \operatorname{dom}(\mathfrak{b})$.

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