A Chern-Simons transgression formula for supersymmetric path integrals on spin manifolds

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July 1, 2022

Abstract

Earlier results show that the $N = 1/2$ supersymmetric path integral $\mathfrak{I}^g$ on a closed even dimensional Riemannian spin manifold (X,g) can be constructed in a mathematically rigorous way via Chen differential forms and techniques from non-commutative geometry, if one considers $\mathfrak{I}^g$ as a current on the loop space LX, that is, as a linear form on differential forms on LX. This construction admits a Duistermaat-Heckman localization formula. In this note, fixing a topologic spin structure on X, we prove that any smooth family $g_\bullet = (g_t)_{t\in[0,1]}$ of Riemannian metrics on X canonically induces a Chern-Simons current $\mathfrak{C}^{g_\bullet}$ which fits into a transgression formula for the supersymmetric path integral. In particular, this result entails that the supersymmetric path integral induces a differential topologic invariant on X, which essentially stems from the $\widehat{A}$-genus of X.

1 Motivation

Let X be a compact even dimensional topological spin manifold[1]. The fixed topological spin structure induces an orientation (cf. Corollary E in [17]) on the Fréchet manifold LX of smooth loops $\gamma : \mathbb{T} \to X$, whose tangent space $T_\gamma LX$ at a fixed loop $\gamma \in LX$ is given by the space of vector fields on X along γ, that is, smooth maps $A : \mathbb{T} \to TX$ with $\dot\gamma(s) \in T_{\gamma(t)}X$ for all $s \in \mathbb{T}$. Given a Riemannian metric g on X let $E^g \in C^\infty(LX)$ and $\omega^g \in \Omega^2(LX)$ denote the energy functional and, respectively, the presymplectic form

$$E^g_\gamma := (1/2)\int_{\mathbb{T}} g(\dot\gamma,\dot\gamma), \quad \omega^g_\gamma(A,B) := \int_{\mathbb{T}} g(\nabla_{\dot\gamma}A,B),$$

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[1]We work exclusively in the category of smooth manifolds without boundary.

where we will occasionally identify $\mathbb{T} = [0,1]/\sim$. The following $N = 1/2$ supersymmetric path integral plays a crucial role in the context of Duistermaat-Heckman localization on LX: with

$$\widehat{\Omega}(LX) := \prod_{j=0}^{\infty} \Omega^j(LX)$$

the space of smooth differential forms on LX, one formally sets

$$\mathfrak{I}^g : \widehat{\Omega}(LX) \longrightarrow \mathbb{C}, \quad \mathfrak{I}^g[\sigma] := \int_{LX} e^{-E^g - \omega^g} \wedge \sigma. \tag{1.1}$$

Note that even though LX is oriented, as it stands, the definition of $\mathfrak{I}^g$ does not make sense for (at least) the following reasons:

- there exists no infinite dimensional Lebesgue measure;
- the integral of an inhomogeneous differential form (which are the ones of interest) should by definition be the integral of its top degree part, however, LX is infinite dimensional;
- LX is noncompact, so even if one finds a natural way to integrate differential forms on LX, some care has to be taken concerning the question of finding a class of 'integrable' (smooth) differential forms.

As we are going to explain in a moment, the mathematical solution of these problems is tied together and manifests itself in a construction of $\mathfrak{I}^g$ via Chen integrals and the differential graded Chern character on (X,g). However, in order to motivate our main results, let us continue with our heuristic observations for the moment.
With ι the contraction by the vector field K on LX given by $\gamma \mapsto \dot{\gamma}$, which generates the natural $\mathbb{T}$-action on LX given by rotating loops, and

$$\widehat{\Omega}_{\mathbb{T}}(LX) := \{\sigma \in \widehat{\Omega}(LX) : \mathcal{L}_K \sigma = 0\}$$

the space of $\mathbb{T}$-invariant differential forms, there is a supercomplex

$$\cdots \xrightarrow{d-\iota} \widehat{\Omega}^+_{\mathbb{T}}(LX) \xrightarrow{d-\iota} \widehat{\Omega}^-_{\mathbb{T}}(LX) \xrightarrow{d-\iota} \widehat{\Omega}^+_{\mathbb{T}}(LX) \xrightarrow{d-\iota} \cdots, \tag{1.2}$$

and (with a slight abuse of notation) the dual supercomplex

$$\cdots \xrightarrow{d-\iota} \widehat{\Omega}_+^{\mathbb{T}}(LX) \xrightarrow{d-\iota} \widehat{\Omega}_-^{\mathbb{T}}(LX) \xrightarrow{d-\iota} \widehat{\Omega}_+^{\mathbb{T}}(LX) \xrightarrow{d-\iota} \cdots. \tag{1.3}$$

Note that these complexes are actually well-defined within the differential calculus of Fréchet manifolds. Now, supersymmetry takes the form $(d-\iota)\mathfrak{I}^g = 0$. Moreover, $\mathfrak{I}^g$ is an even current, as LX is formally even-dimensional, so that $\mathfrak{I}^g$ determines an even homology class in the homology of (1.3). Finally, one can derive the following infinite dimensional analogue of the Duistermaat-Heckman localization formula,

$$\mathfrak{I}^g[\sigma] = \int_X \widehat{A}(X,g) \wedge \sigma|_X \quad \text{for all } \sigma \in \widehat{\Omega}(LX) \text{ with } (d-\iota)\sigma = 0,$$

which leads to a simple and differential geometric 'proof' of the Atiyah-Singer index theorem [3, 2, 1], and which was in fact, the main motivation that lead to the discovery of $\mathfrak{I}^g$.

The aim of this paper is to examine the dependence of $\mathfrak{I}^g$ *on* g. To this end, let $g_\bullet = (g_t)_{t\in[0,1]}$ be a smooth family of Riemannian metrics on X and define for every fixed $t \in [0,1]$ a differential form

$$\beta_t^{g_\bullet} \in \Omega^1(LX), \quad \beta_{t,\gamma}^{g_\bullet}(A) := \frac{1}{2}\int_{\mathbb{T}} (dg_t/dt)(\dot{\gamma},A),$$

and the induced odd current

$$\mathfrak{C}_t^{g_\bullet} : \widehat{\Omega}(LX) \longrightarrow \mathbb{C}, \quad \mathfrak{C}_t^{g_\bullet}(\sigma) := \mathfrak{I}^{g_t}(\beta_t^{g_\bullet} \wedge \sigma).$$

In the appendix, we are going to derive the formula

$$(d/dt)\mathfrak{I}^{g_t} = (d-\iota)\mathfrak{C}_t^{g_\bullet} \quad \text{for all } t \in [0,1]. \tag{1.4}$$

This equality has an important consequence: defining the (odd) Chern-Simons current $\mathfrak{C}^{g_\bullet}$ by

$$\mathfrak{C}^{g_\bullet} := \int_0^1 \mathfrak{C}_t^{g_\bullet} dt : \widehat{\Omega}(LX) \longrightarrow \mathbb{C},$$

one gets the transgression formula

$$\mathfrak{I}^{g_1} - \mathfrak{I}^{g_0} = (d-\iota)\mathfrak{C}^{g_\bullet}.$$

These heuristic observations dictate that any mathematical rigorous definition of $\mathfrak{I}^g$ should admit a Chern-Simons type transgression formula, and that the homology class induced by $\mathfrak{I}^g$ in the homology of (1.3) should not depend on a particular choice of a Riemannian metric g on X. Let us denote this homology class with $\mathfrak{I}$. Using Stokes formula it is easy to check that the current

$$\underline{\widehat{A}}(X,g) : \Omega(LX) \longrightarrow \mathbb{C}, \quad \sigma \longmapsto \int_X \widehat{A}(X,g) \wedge \sigma|_X,$$

satisfies $(d-\iota)\underline{\widehat{A}}(X,g) = 0$, and by a standard transgression argument one finds that the induced homology class does not depend on g. In fact, the Duistermaat-Heckman formula dictates that this homology class $\underline{\widehat{A}}(X)$ should be equal to $\mathfrak{I}$.

2 Main results

Let us explain now how these heuristic considerations can be verified in a mathematically rigorous way. To this end, we first explain the natural class of (smooth) integrable differential forms on LX: we turn $\widehat{\Omega}(LX)$ into a complete locally convex Hausdorff space by equipping $\Omega^j(LX)$ with the family of seminorms $\nu_f(\sigma) := \nu(f^*\sigma)$, where f is a smooth

map from a finite dimensional manifold Y to LX, and ν is a continuous seminorm on the Fréchet space $\Omega^j(Y)$, and by equipping $\widehat{\Omega}(LX)$ with the product topology. Given $\sigma \in \Omega(X)$ and $t \in \mathbb{T}$ one defines $\sigma(t) \in \Omega(LX)$ to be the pullback of σ with respect to the evaluation $\gamma \mapsto \gamma(t)$.
Consider the Fréchet space of $\mathbb{T}$-invariant differential forms $\Omega_{\mathbb{T}}(X \times \mathbb{T})$ on $X \times \mathbb{T}$, with $\mathbb{T}$ acting on the second slot. With $\vartheta_{\mathbb{T}} \in \Omega(\mathbb{T})$ the volume form, any $\theta \in \Omega_{\mathbb{T}}(X \times \mathbb{T})$ can be uniquely written in the form $\theta = \theta' + \vartheta_{\mathbb{T}} \wedge \theta''$ with θ',$\theta'' \in \Omega(X)$.
Associated to this construction, there is the space of *entire chains* $\mathsf{C}^{\epsilon}_{\mathbb{T}}(X)$ which is defined as the completion of

$$\mathsf{C}_{\mathbb{T}}(X) := \bigoplus_{N=0}^{\infty} \Omega_{\mathbb{T}}(X \times \mathbb{T}) \otimes \underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})^{\otimes N},$$

with

$$\underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})^{\otimes N} := \Omega_{\mathbb{T}}(X \times \mathbb{T})^{\otimes N}/(\mathbb{C} \cdot 1)$$

and where $\mathsf{C}_{\mathbb{T}}(X)$ is equipped with the following family of seminorms: given any continuous seminorm ν on $\Omega_{\mathbb{T}}(X \times \mathbb{T})$, one gets the induced projective tensor norm

$$\pi_{\nu,N} \quad \text{on} \quad \Omega_{\mathbb{T}}(X \times \mathbb{T}) \otimes \underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})^{\otimes N},$$

and then a seminorm ϵ_ν on $\mathsf{C}_{\mathbb{T}}(X)$ by setting

$$\epsilon_\nu(c) := \sum_{N=0}^{\infty} \frac{\pi_{\nu,N}(c_N)}{\lfloor N/2 \rfloor !}, \tag{2.1}$$

if

$$c = \sum_{N=0}^{\infty} c_N \in \mathsf{C}_{\mathbb{T}}(X), \quad \text{with } c_N \in \Omega_{\mathbb{T}}(X \times \mathbb{T}) \otimes \underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})^{\otimes N} \text{ for all } N.$$

The required family of seminorms is now given by ϵ_ν, where ν is a continuous seminorm on $\Omega_{\mathbb{T}}(X \times \mathbb{T})$.
There exists a uniquely determined continuous map [6], the equivariant *Chen iterated integral map*,

$$\mathrm{Chen}_{\mathbb{T}} : \mathsf{C}^{\epsilon}_{\mathbb{T}}(X) \longrightarrow \widehat{\Omega}(LX).$$

such that for all $N \in \mathbb{N}_{\geq 0}$, $\theta_0, \dots, \theta_N \in \theta \in \Omega_{\mathbb{T}}(X \times \mathbb{T})$, one has

$$\mathrm{Chen}_{\mathbb{T}}(\theta_0 \otimes \cdots \otimes \theta_N) \tag{2.2}$$

$$= \int_{\{0 \leq t_1 \leq \cdots \leq t_N \leq 1\}} \theta_0(0) \wedge (\iota\theta_1'(t_1) + \theta_1''(t_1)) \wedge \cdots \wedge (\iota\theta_N'(t_N) + \theta_N''(t_N))\, dt_1 \cdots dt_N. \tag{2.3}$$

Definition 2.1. The space of *integrable Chen forms* $\widetilde{\Omega}(LX) \subset \widehat{\Omega}(LX)$ is defined as the image of $\mathrm{Chen}_{\mathbb{T}}$.

Set

$$\widetilde{\Omega}_{\mathbb{T}}(LX) := \widetilde{\Omega}(LX) \cap \widehat{\Omega}_{\mathbb{T}}(LX).$$

The following result follows essentially from calculations made in [6]. A detailed proof will be given in Section 3.

Proposition 2.2. *There is a well-defined supercomplex*

$$\cdots \xrightarrow{d-\iota} \widetilde{\Omega}^{+}_{\mathbb{T}}(LX) \xrightarrow{d-\iota} \widetilde{\Omega}^{-}_{\mathbb{T}}(LX) \xrightarrow{d-\iota} \widetilde{\Omega}^{+}_{\mathbb{T}}(LX) \xrightarrow{d-\iota} \cdots . \tag{2.4}$$

The associated dual supercomplex will be denoted with

$$\cdots \xrightarrow{d-\iota} \widetilde{\Omega}_{+}^{\mathbb{T}}(LX) \xrightarrow{d-\iota} \widetilde{\Omega}_{-}^{\mathbb{T}}(LX) \xrightarrow{d-\iota} \widetilde{\Omega}_{+}^{\mathbb{T}}(LX) \xrightarrow{d-\iota} \cdots . \tag{2.5}$$

Let us now give the formula for $\mathfrak{J}^g$. Recall that we have fixed a topologic spin structure on X. Consider the (super) spinor bundle $\Sigma_g \to X$ induced by g, with its (essentially self-adjoint) Dirac operator D_g on the super Hilbert space of L^2-spinors $\Gamma_{L^2}(X,\Sigma_g)$, and the (natural extension to differential forms of all degrees of the) Clifford multiplication

$$c_g : \Omega(X) \longrightarrow \Gamma_{C^\infty}(X,\mathrm{End}(\Sigma_g)).$$

Let $\Psi(X,\Sigma_g)$ denote the super algebra of pseudodifferential operators in $\Sigma_g \to X$. With $H_g := D_g^2$, we define a linear map

$$\begin{aligned}
&F_g : \mathsf{B}_{\mathbb{T}}(X) := \bigoplus_{N=0}^{\infty} \underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})^{\otimes N} \longrightarrow \Psi(X,\Sigma_g),\\
&F_g^{(0)} := H_g,\\
&F_g^{(1)}(\theta) = c_g(d\theta') - [D_g, c_g(\theta')] - c_g(\theta''),\\
&F_g^{(2)}(\theta_1,\theta_2) = (-1)^{|\theta_1'|}\big(c_g(\theta_1'\theta_2') - c_g(\theta_1')c_g(\theta_2')\big),\\
&F_g^{(N)}(\theta_1,\ldots,\theta_N) = 0, \quad \text{if } N \geq 3,
\end{aligned}$$

where here and in the sequel all commutators are super-commutators.
For $M \leq N$ denote with $P_{M,N}$ all tuples $I = (I_1,\ldots,I_M)$ of subsets of $\{1\ldots,N\}$ with $I_1 \cup \cdots \cup I_M = \{1\ldots,N\}$ and with each element of I_a smaller than each element of I_b whenever $a < b$. Given

$$\theta_1,\ldots,\theta_N \in \Omega_{\mathbb{T}}(X \times \mathbb{T}), \quad I = (I_1,\ldots,I_M) \in P_{M,N}, \quad 1 \leq a \leq M,$$

set

$$\theta_{I_a} := (\theta_{i+1},\ldots,\theta_{i+m}), \quad \text{if } I_a = \{j \mid i < j \leq i+m\} \text{ for some } i, m.$$

We finally define a linear map

$$\begin{aligned}
&\Phi^g : \mathsf{B}_{\mathbb{T}}(X) \longrightarrow \Psi(X,\Sigma_g),\\
&\Phi^g(\theta_1,\ldots,\theta_N) = \sum_{M=1}^{N} (-1)^M \sum_{I \in P_{M,N}} \int_{\{0 \leq t_1 \leq \cdots \leq t_M \leq 1\}} e^{-t_1 H_g} F_g(\theta_{I_1}) e^{-(t_2-t_1)H_g} F_g(\theta_{I_2}) \cdots\\
&\qquad \cdots e^{-(t_M - t_{M-1})H_g} F_g(\theta_{I_M}) e^{-(1-t_M)H_g}\, dt_1 \cdots dt_M.
\end{aligned}$$

The linear map

$$\alpha : \mathsf{C}_{\mathbb{T}}(X) \longrightarrow \mathsf{B}_{\mathbb{T}}(X),$$
$$\alpha(\theta_0 \otimes \cdots \otimes \theta_N) := \sum_{k=1}^{N} (-1)^{n_k(n_N - n_k)} (\theta_{k+1} \otimes \cdots \otimes \theta_N \otimes \cdots \otimes \theta_k),$$

where $n_j := |\theta_1| + \cdots |\theta_j| - j$, induces a linear map

$$\alpha_g : \mathrm{Hom}(\mathsf{B}_{\mathbb{T}}(X), \Psi(X,\Sigma_g)) \longrightarrow \mathrm{Hom}(\mathsf{C}_{\mathbb{T}}(X), \Psi(X,\Sigma_g)),$$

given explicitly by

$$[\alpha_g l](\theta_0, \ldots ,\theta_N) = \sum_{k=0}^{N+1} (-1)^{n_k(n_N - n_k)} l(\theta_k, \ldots ,\theta_N, \vartheta_{\mathbb{T}} \wedge \theta_0, \theta_1, \ldots ,\theta_{k-1}).$$

With Str_g the supertrace in $\Gamma_{L^2}(X,\Sigma_g)$, the following is the main result of [7]:

Theorem 2.3. *There exists a uniquely determined current* $\mathfrak{J}^g : \widetilde{\Omega}(LX) \to \mathbb{C}$ *such that for all* $N \in \mathbb{N}_{\geq 0}$, $\theta_0, \ldots ,\theta_N \in \Omega_{\mathbb{T}}(X \times \mathbb{T})$ *one has*

$$\mathfrak{J}^g \left[\int_{\{0 \leq t_1 \leq \cdots \leq t_N \leq 1\}} \theta_0'(0) \wedge (\iota\theta_1'(t_1) + \theta_1''(t_1)) \wedge \cdots \wedge (\iota\theta_N'(t_N) + \theta_N''(t_N)) \, dt_1 \cdots dt_N \right] \tag{2.6}$$
$$= \mathrm{Str}_g \left([\alpha_g \Phi^g](\theta_0, \ldots ,\theta_N) \right).$$

Moreover, $\mathfrak{J}^g$ *is even and* $(d - \iota)\mathfrak{J}^g = 0$*, so that* $\mathfrak{J}^g$ *defines an even homology class in the homology of (2.5), and one has the localization formula*

$$\mathfrak{J}^g[\sigma] = \int_X \widehat{A}(X,g) \wedge \sigma|_X \quad \textit{for all } \sigma \in \widetilde{\Omega}(LX) \textit{ with } (d - \iota)\sigma = 0.$$

That this definition of $\mathfrak{J}^g$ is natural, in the sense that it really serves as an *implementation* of the right hand side of (1.1), has been indicated in [9] using the Pfaffian line bundle. A probabilistic representation of $\mathfrak{J}^g$ has been derived in [10], generalizing the earlier result from [5] for $N = 1$ to all orders.

Assume $g_\bullet = (g_t)_{t\in[0,1]}$ is a smooth family of Riemannian metrics on X. We briefly recall the Bourguignon-Gauduchon machinery for metric changes of the Dirac operator [4]. For any $t \in [0,1]$, define a section $\mathcal{A}_t^{g_\bullet}$ of $\mathrm{End}(TX)$ by

$$g_0(u,v) = g_t(\mathcal{A}_t^{g_\bullet} u,v) \qquad \text{for all} \qquad x \in X, u,v \in T_xX\,.$$

Then $\mathcal{A}_t^{g_\bullet}$ is strictly positive w.r.t. g_t and g_0 and $(\mathcal{A}_t^{g_\bullet})^{-1/2}$ is a pointwise isometry $(TX,g_t) \to (TX,g_0)$. It therefore lifts canonically to an $\mathrm{SO}(n)$-equivariant bundle map

$$\mathcal{A}_t^{g_\bullet,\mathrm{SO}} : \mathrm{SO}(X,g_t) \longrightarrow \mathrm{SO}(X,g_0)\,,$$

where $\mathrm{SO}(X,g_t)$ denotes the bundle of oriented orthonormal frames of X w.r.t. the Riemannian metric g_t.
Now recall that we have fixed a topological spin structure. This implies that every Riemannian metric g_t canonically induces a Riemannian spin structure on X, i.e., a $\mathrm{Spin}(n)$-principal fibre bundle P_{g_t} over X together with a ξ-equivariant map $\pi_{g_t} : P_t \to \mathrm{SO}(X,g_t)$ such that (P_t,π_{g_t}) is a ξ-reduction of $\mathrm{SO}(X,g_t)$. Here, $\xi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the canonically given double cover. Furthermore, (P_{g_t}, π_{g_t}) being associated with a fixed topological spin structure, the map $\mathcal{A}_t^{g_\bullet,\mathrm{SO}}$ lifts to an equivariant bundle map $\mathcal{A}_t^{g_\bullet,P} : P_{g_t} \to P_{g_0}$ and through the associated vector bundle construction, we obtain a fibrewise isometric vector bundle isomorphism

$$\mathcal{A}_t^{g_\bullet,\Sigma} : \Sigma_{g_t} \longrightarrow \Sigma_{g_0} \, ,$$

which moreover satisfies

$$\mathcal{A}_t^{g_\bullet,\Sigma}(c_{g_t}(\theta)(\varphi)) = c_{g_0}(\sqrt{(\mathcal{A}_t^{g_\bullet})'}(\theta))(\mathcal{A}_t^{g_\bullet,\Sigma}(\varphi)) \quad \text{for all} \quad x \in X, \theta \in T_x^*X, \varphi \in (\Sigma_{g_t})_x \, ,$$

where $(\mathcal{A}_t^{g_\bullet})' \in \Gamma_{C^\infty}(X,\mathrm{End}(TX^*))$ denotes the section fibrewise dual to $\mathcal{A}_t^{g_\bullet}$.
With

$$0 < \rho_t^{g_\bullet} = d\mu_{g_0}/d\mu_{g_t} \in C^\infty(X)$$

the Radon-Nikodym density of μ_{g_0} w.r.t. μ_{g_t}, we obtain the unitary operator

$$\begin{aligned} U_t^{g_\bullet} &: \Gamma_{L^2}(X,\Sigma_{g_t}) \longrightarrow \Gamma_{L^2}(X,\Sigma_{g_0}) \\ U_t^{g_\bullet}\varphi(x) &= (\rho_t^{g_\bullet})^{-1/2}\mathcal{A}_t^{g_\bullet,\Sigma}(\varphi(x)) \, , \end{aligned}$$

which we use to define a family $\mathcal{M}^{g_\bullet}$ of ϑ-summable Fredholm modules over $\Omega(X)$ in the sense of Definition 2.1 in [7], by

$$\mathcal{M}_t^{g_\bullet} := \big(\Gamma_{L^2}(X,\Sigma_{g_0}), c_t^{g_\bullet}, Q_t^{g_\bullet}\big) := \big(\Gamma_{L^2}(X,\Sigma_{g_0}), U_t^{g_\bullet} c_{g_t} U_t^{g_\bullet,*}, U_t^{g_\bullet} D_{g_t} U_t^{g_\bullet,*}\big) \, .$$

Next, define

$$\Xi_{g_\bullet,t} : \mathsf{B}_{\mathbb{T}}(X) \longrightarrow \Psi(X,\Sigma_{g_0})$$

by

$$\Xi^{(0)}_{g_\bullet,t} := Q_t^{g_\bullet}, \quad \Xi^{(1)}_{g_\bullet,t}(\theta) = c_t^{g_\bullet}(\theta'), \quad \Xi^{(N)}_{g_\bullet,t}(\theta_1,\dots,\theta_N) = 0, \quad \text{if } N \geq 2,$$

and

$$\Phi_{t,r}^{g_\bullet} : \mathsf{B}_{\mathbb{T}}(X) \longrightarrow \Psi(X,\Sigma_{g_0})$$

with $H_t^{g_\bullet} := (Q_t^{g_\bullet})^2$ for $0 \leq r \leq 1$,

$$\begin{aligned} &\Phi_{t,r}^{g_\bullet} : \mathsf{B}_{\mathbb{T}}(X) \longrightarrow \Psi(X,\Sigma_{g_0}), \\ &\Phi_{t,r}^{g_\bullet}(\theta_1,\dots,\theta_N) = \sum_{M=1}^{N}(-1)^M \sum_{I\in P_{M,N}} \int_{\{0\leq s_1\leq\cdots\leq s_M\leq r\}} e^{-s_1 H_t^{g_\bullet}} F_{g_\bullet,t}(\theta_{I_1}) e^{-(s_2-s_1)H_t^{g_\bullet}} F_{g_\bullet,t}(\theta_{I_2}) \cdots \\ &\qquad\qquad \cdots e^{-(s_M-t_{M-1})H_t^{g_\bullet}} F_{g_\bullet,t}(\theta_{I_M}) e^{-(1-s_M)H_t^{g_\bullet}} \, dt_1 \cdots dt_M. \end{aligned}$$

and

$$
\begin{aligned}
&F_{g_\bullet,t} : \mathsf{B}_{\mathbb{T}}(X) :\longrightarrow \Psi(X,\Sigma_{g_0}),\\
&F^{(0)}_{g_\bullet,t} := H^{g_\bullet}_t,\\
&F^{(1)}_{g_\bullet,t}(\theta) = c^{g_\bullet}_t(d\theta') - [Q^{g_\bullet}_t, c^{g_\bullet}_t(\theta')] - c^{g_\bullet}_t(\theta''),\\
&F^{(2)}_{g_\bullet,t}(\theta_1,\theta_2) = (-1)^{|\theta'_1|}\big(c^{g_\bullet}_t(\theta'_1\theta'_2) - c^{g_\bullet}_t(\theta'_1)c^{g_\bullet}_t(\theta'_2)\big),\\
&F^{(N)}_{g_\bullet,t}(\theta_1,\dots,\theta_N) = 0, \quad \text{if } N\geq 3.
\end{aligned}
$$

The space $\mathrm{Hom}(\mathsf{B}_{\mathbb{T}}(X),\Psi(X,\Sigma_g))$ is turned into a super algebra by means of the product

$$[l_1 l_2](\theta_1,\dots,\theta_N) = \sum_{k=0}^{N} (-1)^{|l_2|(|\theta_1|+\cdots+|\theta_k|-k)} l_1(\theta_1,\dots,\theta_k) l_2(\theta_{k+1},\dots,\theta_N).$$

The following Chern-Simons type transgression formula is the main result of this paper:

Theorem 2.4. *Assume $g_\bullet = (g_t)_{t\in[0,1]}$ is a smooth family of Riemannian metrics on X. Then there exists a uniquely given odd current $\mathfrak{C}^{g_\bullet} : \widetilde{\Omega}(LX) \to \mathbb{C}$ such that for all $N \in \mathbb{N}_{\geq 0}$, $\theta_0,\dots,\theta_N \in \Omega_{\mathbb{T}}(X\times\mathbb{T})$ one has*

$$
\begin{aligned}
&\mathfrak{C}^{g_\bullet}\left[\int_{\{0\leq t_1\leq\cdots\leq t_N\leq 1\}} \theta'_0(0)\wedge(\iota\theta'_1(t_1)+\theta''_1(t_1))\wedge\cdots\wedge(\iota\theta'_N(t_N)+\theta''_N(t_N))\, dt_1\cdots dt_N\right]\\
&= \mathrm{Str}_{g_0}\Big(\Big[\alpha_{g_0}\int_0^1\int_0^1 \Phi^{g_\bullet}_{s,r}(d\Xi_{g_\bullet,t}/dt)\Phi^{g_\bullet}_{s,1-r}\, dr ds\Big](\theta_0,\cdots,\theta_N)\Big).
\end{aligned}
$$

One has $\mathfrak{J}^{g_1} - \mathfrak{J}^{g_0} = (d-\iota)\mathfrak{C}^{g_\bullet}$; in particular, the homology class induced by $\mathfrak{J}^g$ in the homology of (2.5) does not depend on a particular choice of a Riemannian metric g on X.

Remark 2.5. The formula for $\mathfrak{C}^{g_\bullet}$ can be further evaluated by noting that

$$
\begin{aligned}
&Q^{g_\bullet}_t = \frac{1}{2}(\rho^{g_\bullet}_t)^{-1}c_{g_0}((\mathcal{A}^{g_\bullet}_t)^{-1/2}\mathrm{grad}\rho^{g_\bullet}_t) + \mathcal{A}^{g_\bullet,\Sigma}_t D_{g_t}\left(\mathcal{A}^{g_\bullet,\Sigma}_t\right)^{-1},\\
&c^{g_\bullet}_t(\theta) = c_{g_0}(\sqrt{(\mathcal{A}^{g_\bullet}_t)'}(\theta)),\\
&dc^{g_\bullet}_t/dt(\theta) = c_{g_0}((d\sqrt{(\mathcal{A}^{g_\bullet}_t)'}/dt)(\theta)).
\end{aligned}
$$

A local formula for the elliptic first-order differential operator $\mathcal{A}^{g_\bullet,\Sigma}_t D_{g_t}\mathcal{A}^{g_\bullet,\Sigma,-1}_t$ can be found in [4, Théorème 20]. From the above expression for $Q^{g_\bullet}_t$, one can derive an expression for the, in general nonelliptic, first-order differential operator $(d/dt)Q^{g_\bullet}_t$. The needed t-derivative of $\mathcal{A}^{g_\bullet,\Sigma}_t D_{g_t}\mathcal{A}^{g_\bullet,\Sigma,-1}_t$ is recorded in [4, Théorème 21].

As a consequence we get:

Corollary 2.6. *Let X and Y be compact even-dimensional, oriented spin manifolds with fixed topological spin-structures. Assume there exists a diffeomorphism $f : X \to Y$ preserving orientations and topological spin-structures. Then, for any choice of Riemannian metrics g and h on X resp. on Y, the homology class induced by $\mathfrak{J}_X^g$ in the homology of (2.5) equals the homology class of $f^*\mathfrak{J}_Y^h$.*

Proof. Setting $g_1 := f^*h$, the diffeomorphism f becomes an orientation and metric spin-structure preserving isometry $f : (X,g_1) \to (Y,h)$ furnishing unitary equivalences between Clifford multiplications and Dirac operators on (X,g_1) and (Y,h). Formula (2.6) shows that $\mathfrak{J}_X^{g_1}$ and $f^*\mathfrak{J}_Y^h$ are equal, and Theorem 2.4 establishes the claim. □

We denote the homology class of $\mathfrak{J}^g$ for some/any Riemannian metric g on X by $\mathfrak{J}$, which by the previous corollary is a differential topologic invariant of X. Let us identify this invariant: for every Riemannian metric g on X, using Stokes formula, it is easily seen that the current

$$\underline{\widehat{A}}(X,g) : \widetilde{\Omega}(LX) \longrightarrow \mathbb{C}, \quad \sigma \longmapsto \int_X \widehat{A}(X,g) \wedge \sigma|_X$$

satisfies $(d - \iota)\underline{\widehat{A}}(X,g) = 0$, and by Theorem E in combination with Lemma 9.3 from [7], the homology class of $\underline{\widehat{A}}(X,g)$ in (2.5) equals that of $\mathfrak{J}^g$. Moreover, by a standard transgression argument, the homology class of $\underline{\widehat{A}}(X,g)$ does not depend on g. Putting everything together, it follows that this class $\underline{\widehat{A}}(X)$ equals $\mathfrak{J}$.

3 Proof of Proposition 2.2

We have to show that $d - \iota$ maps

$$\widetilde{\Omega}_{\mathbb{T}}(LX) = \widetilde{\Omega}(LX) \cap \widehat{\Omega}_{\mathbb{T}}(LX)$$

to itself. We give $\Omega_{\mathbb{T}}(X \times \mathbb{T})$ the $\mathbb{Z}$-grading

$$\theta' + \vartheta_{\mathbb{T}} \wedge \theta'' \in \Omega_{\mathbb{T}}(X \times \mathbb{T})^j \Leftrightarrow \theta' \in \Omega^j(X), \theta'' \in \Omega^{j+1}(X)$$

and turn it into a locally convex DGA using the differential $d - \iota_{\partial_{\mathbb{T}}}$ with $\partial_{\mathbb{T}}$ the canonic vector field on $\mathbb{T}$. Then $\mathsf{C}_{\mathbb{T}}(X)$ inherits the $\mathbb{Z}$-grading induced by

$$\mathsf{C}_{\mathbb{T}}(X) = \bigoplus_{N=0}^{\infty} \Omega_{\mathbb{T}}(X \times \mathbb{T}) \otimes \underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})[1]^{\otimes N},$$

where $\underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})[1]$ denotes $\underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})$ as a set with the shifted grading

$$\underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})[1]^j := \underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})^{j+1}.$$

With b the Hochschild differential and B the Connes differential in the $\mathbb{Z}$-graded category, the space $\mathsf{C}_{\mathbb{T}}(X)$ becomes a supercomplex with the differential $d+\iota_{\partial_{\mathbb{T}}}+b+B$. By continuity,

the same holds true for $\mathsf{C}^{\epsilon}_{\mathbb{T}}(X)$.
Let

$$\mathbb{A}:\widehat{\Omega}(LX)\longrightarrow\widehat{\Omega}(LX),\quad \sigma\longmapsto\int_{\mathbb{T}}\varphi_{\bullet}^{*}\sigma$$

be the idempotent linear operator obtained by averaging the $\mathbb{T}$-action on LX, where

$$\varphi_s : LX\longrightarrow LX,\quad \gamma\longmapsto\gamma(\bullet+s),\quad s\in\mathbb{T}.$$

Note that it is implicitly used here that $\mathbb{A}$ preserves the image of $\mathrm{Chen}_{\mathbb{T}}$, which follows from a simple calculation. Then, as shown in [6], one has the formulae

$$\mathbb{A}\mathrm{Chen}_{\mathbb{T}}(d-\iota_{\partial_{\mathbb{T}}}+b+B)=(d-\iota\mathbb{A})\mathbb{A}\mathrm{Chen}_{\mathbb{T}}\,,$$

noting that $\iota\mathbb{A}=\mathbb{A}\iota$.
Assume that $\sigma\in\widetilde{\Omega}(LX)$ is $\mathbb{T}$-invariant. This means that $\sigma=\mathrm{Chen}_{\mathbb{T}}(\theta)$ for some $\theta\in\mathsf{C}^{\epsilon}_{\mathbb{T}}(X)$ and that $\mathbb{A}\mathrm{Chen}_{\mathbb{T}}(\theta)=\mathrm{Chen}_{\mathbb{T}}(\theta)$. Then we have

$$(d-\iota)\sigma=d\mathbb{A}\mathrm{Chen}_{\mathbb{T}}(\theta)-\iota\mathbb{A}^2\mathrm{Chen}_{\mathbb{T}}(\theta)=(d-\iota\mathbb{A})\mathbb{A}\mathrm{Chen}_{\mathbb{T}}(\theta)=\mathbb{A}\mathrm{Chen}_{\mathbb{T}}((d-\iota_{\partial_{\mathbb{T}}}+b+B)\theta),$$

which shows that $(d-\iota)\sigma$ is $\mathbb{T}$-invariant and also that $(d-\iota)\sigma$ is a Chen form because $\mathbb{A}$ preserves $\widetilde{\Omega}(LX)$. This completes the proof.

4 Proof of Theorem 2.4

We are going to omit $g_{\bullet}$ everywhere in the notation. Consider the Chern character

$$\mathrm{Ch}_{g_t}:\mathsf{C}^{\epsilon}_{\mathbb{T}}(X)\longrightarrow\mathbb{C},$$

whose value at

$$\theta_0\otimes\cdots\otimes\theta_N\in\mathsf{C}^{\epsilon}_{\mathbb{T}}(X)$$

is given by the RHS of (2.6) for $g=g_t$. Then Ch_{g_t} vanishes on the kernel of $\mathrm{Chen}_{\mathbb{T}}$ and this defines $\mathfrak{J}^{g_t}$. If we can show that $\mathcal{M}^{g_\bullet}$ satisfies the axioms of Definition 6.1 in [7], then (using that Chern characters are invariant under unitary transformations) it follows that the (odd) Chern-Simons form

$$\mathrm{CS}(\mathcal{M}^{g_\bullet}_{\mathbb{T}}):\mathsf{C}^{\epsilon}_{\mathbb{T}}(X)\longrightarrow\mathbb{C}$$

constructed on page 31 in [7] satisfies

$$\mathrm{Ch}_{g_1}-\mathrm{Ch}_{g_0}=(d-\iota_{\partial_{\mathbb{T}}}+b+B)\mathrm{CS}(\mathcal{M}^{g_\bullet}_{\mathbb{T}})$$

and vanishes on the kernel of $\mathrm{Chen}_{\mathbb{T}}$, too. It follows that

$$\mathfrak{C}^{g_\bullet}(\mathrm{Chen}_{\mathbb{T}}(\theta)):=\mathrm{CS}(\mathcal{M}^{g_\bullet}_{\mathbb{T}})(\theta),\quad\theta\in\mathsf{C}^{\epsilon}_{\mathbb{T}}(X),$$

is well-defined and, being invariant under $\mathbb{A}$ (which follows from its very construction), has the desired properties, in view of

$$\mathbb{A}\mathrm{Chen}_{\mathbb{T}}(d - \iota_{\partial_{\mathbb{T}}} + b + B) = (d - \iota)\mathbb{A}\mathrm{Chen}_{\mathbb{T}}.$$

It remains to show (H1) and (H2) from Definition 6.1 in [7], where (H1) is the condition

$$\sup_{t\in[0,1]} \mathrm{tr}\left(e^{-Q_t^2}\right) < \infty,$$

and (H2) is the condition

$$\sup_{t\in[0,1]} \left\|\dot{Q}_t(Q_t^2+1)^{-1/2}\right\| + \sup_{t\in[0,1]} \left\|(Q_t^2+1)^{-1/2}\dot{Q}_t\right\| < \infty.$$

Here, (H1) can be seen as follows: one can appeal to the Lichnerowicz formula for D_t^2 and semigroup domination (cf. Theorem 3.1 in [11]) to get

$$\mathrm{tr}\left(e^{-Q_t^2}\right) \leq \mathrm{rank}(\Sigma_0)e^{-\min_{x\in X}(1/4)\mathrm{scal}_{g_t}(x)}\mathrm{tr}\left(e^{-\Delta_{g_t}}\right),$$

which entails (H1), as $t \mapsto \min_{x\in X}(1/4)\mathrm{scal}_{g_t}(x)$ is clearly continuous, and $t \mapsto \mathrm{tr}\left(e^{-\Delta_{g_t}}\right)$ is smooth by Proposition 6.1 from [14].
To see (H2) note that from elliptic regularity, each $Q_t := U_{g_t}D_{g_t}U_{g_t}^*$ has the same domain of definition $W^{1,2}(X)$. Furthermore, $\dot{Q}_t := (d/dt)Q_t$ is a first order differential operator, which we consider as acting on smooth spinors. The proof of (H2) is based on the following lemma, which is a modification of Lemma 4.17 in [8]:

Lemma 4.1. *Let S be a densely defined, closed linear operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$, and let T be a self-adjoint bounded linear operator in $\mathcal{H}_1$ with $T \geq -\lambda$ for some $\lambda \geq 0$. Assume that $S^*S + T \geq 0$. Then one has*

$$\|S(S^*S+T+1)^{-1/2}\| \leq \sqrt{\lambda+1}\,.$$

Proof. By assumption we have

$$S^*S + 1 \leq S^*S + T + \lambda + 1\,,$$

which means

$$\|(S^*S+1)^{1/2}f\| \leq \|(S^*S+T+\lambda+1)^{1/2}f\| \quad \text{for all} \quad f \in \mathrm{dom}(S^*S)^{1/2}\,.$$

From this we obtain

$$\|(S^*S+1)^{1/2}(S^*S+T+1)^{-1/2}h\| \leq \|(S^*S+T+\lambda+1)^{1/2}(S^*S+T+1)^{-1/2}h\|$$

for all $h \in \mathcal{H}_1$. Using the functional calculus associated with the operator $S^*S + T$, we calculate the norm of the operator appearing on the right hand side to be

$$\|(S^*S + T + \lambda + 1)^{1/2}(S^*S + T + 1)^{-1/2}\| \leq \sup_{t\geq 0} \sqrt{\frac{t+\lambda+1}{t+1}} = \sqrt{\lambda + 1}\,,$$

which implies

$$\|(S^*S + 1)^{1/2}(S^*S + T + 1)^{-1/2}\| \leq \sqrt{\lambda + 1}\,.$$

Now we can estimate

$$\begin{aligned}
\|S(S^*S + T + 1)^{-1/2}\| &= \|S(S^*S + 1)^{-1/2}(S^*S + 1)^{1/2}(S^*S + T + 1)^{-1/2}\| \\
&\leq \sqrt{\lambda + 1}\|S(S^*S + 1)^{-1/2}\| \\
&\leq \sqrt{\lambda + 1}\|(S^*S)^{1/2}(S^*S + 1)^{-1/2}\| \\
&\leq \sqrt{\lambda + 1} \sup_{t\geq 0} \sqrt{\frac{t}{t+1}} \\
&\leq \sqrt{\lambda + 1}\,,
\end{aligned}$$

where we have used the polar decomposition $S = U(S^*S)^{1/2}$ with a partial isometry U on the third line and the functional calculus associated with the operator S^*S on the fourth line. □

Using this lemma, we are going to prove that one has (H2): first of all, note that Q_t acting on $\Gamma_{C^\infty}(X,\Sigma_{g_0})$ is a first order differential operator whose coefficients depend smoothly on $t \in [0,1]$. Since X is compact, it follows that

$$\left\langle \dot{Q}_t\varphi,\psi \right\rangle = (d/dt)\left\langle Q_t\varphi,\psi\right\rangle = (d/dt)\left\langle \varphi,Q_t\psi\right\rangle = \left\langle \varphi,\dot{Q}_t\psi \right\rangle$$

for all $\varphi,\psi \in \Gamma_{C^\infty}(X,\Sigma_{g_0})$, i.e., $\dot{Q}_t$ is symmetric.
Secondly, the operator $Q_t^2 + 1$ being elliptic, it follows from a classical result of Seeley [15] that $(Q_t^2 + 1)^{-1/2}$ is a pseudo-differential operator. In particular, it maps $\Gamma_{C^\infty}(X,\Sigma_{g_0})$ to itself.
Turning to operator norms, note that $\dot{Q}_t(Q_t^2 + 1)^{-1/2}$ is bounded if and only if

$$\sup \left\{ \left| \left\langle \dot{Q}_t(Q_t^2 + 1)^{-1/2}\varphi,\varphi \right\rangle \right| \,:\, \varphi \in \Gamma_{C^\infty}(X,\Sigma_{g_0}) \right\} < \infty\,.$$

The operators $\dot{Q}_t$ and $(Q_t^2 + 1)^{-1/2}$ being symmetric this, in turn, is equivalent to $(Q_t^2 + 1)^{-1/2}\dot{Q}_t$ being bounded. Hence, it suffices to show that

$$\sup_{t\in[0,1]} \left\| \dot{Q}_t(Q_t^2 + 1)^{-1/2} \right\| < \infty\,. \tag{4.1}$$

To this end, we first use the unitary invariance of the functional calculus to compute

$$\left\|\dot{Q}_t(Q_t^2+1)^{-1/2}\right\| = \left\|\dot{Q}_t((U_tD_{g_t}U_t^*)^2+1)^{-1/2}\right\| = \left\|\dot{Q}_tU_t(D_{g_t}^2+1)^{-1/2}U_t^*\right\|$$
$$= \left\|U_t^*\dot{Q}_tU_t(D_{g_t}^2+1)^{-1/2}\right\| .$$

Next, we decompose

$$U_t^*\dot{Q}_tU_t = a_t\circ\nabla_t+\tau_t,$$

with ∇_t the spinor connection of Σ_{g_t}, and

$$a_t\in\Gamma_{C^\infty}(X,\mathrm{Hom}(T^*X\otimes\Sigma_{g_t},\Sigma_{g_t})),\quad \tau_t\in\Gamma_{C^\infty}(X,\mathrm{End}(\Sigma_{g_t})),$$

so that by the Lichnerowicz formula we have

$$U_t^*\dot{Q}_tU_t(D_{g_t}^2+1)^{-1/2} = a_t\nabla\left(\nabla^*\nabla+\tfrac{1}{4}\mathrm{scal}_{g_t}+1\right)^{-1/2}+\tau_t\left(D_{g_t}^2+1\right)^{-1/2} . \tag{4.2}$$

Because $\|(D_{g_t}^2+1)^{-1/2}\|\leq 1$, the operator norm of the second term on the right hand side is bounded by $\|\tau_t\|$, which is continuous in t. Hence,

$$\sup_{t\in[0,1]}\|\tau_t\left(D_{g_t}^2+1\right)^{-1/2}\|<\infty\,.$$

Regarding the first term on the right hand side of (4.2), we appeal to the above lemma with

$$S=\nabla,\quad T=(1/4)\mathrm{scal}_{g_t},\quad \lambda_t:=(1/4)\max_{x\in X}|\mathrm{scal}_{g_t}(x)|,$$

to see that

$$\|a_t\nabla\left(\nabla^*\nabla+\tfrac{1}{4}\mathrm{scal}_{g_t}+1\right)^{-1/2}\|\leq\|a_t\|\sqrt{\lambda_t+1}\,,$$

which is also continuous in t, thereby completing the proof of (4.1) and hence also of Theorem 2.4.

Appendix: formal proof of formula (1.4)

We start by calculating the derivative of $\mathfrak{I}^{g_t}$ w.r.t. t,

$$(d/dt)\mathfrak{I}^{g_t}[\sigma]=\int_{LX}(d/dt)e^{-E^{g_t}-\omega^{g_t}}\wedge\sigma=\int_{LX}e^{-E^{g_t}-\omega^{g_t}}\wedge(d/dt)\left(-E^{g_t}-\omega^{g_t}\right)\wedge\sigma\,.$$

Let $\nabla(t)$ denote the Levi-Civita connection for g_t, and let $\gamma\in LX$, $Y,Z\in T_\gamma LX$. The t-derivative appearing in the integrand on the right-hand side is

$$(d/dt)\left(-E_\gamma^{g_t}-\omega_\gamma^{g_t}\right)(Y,Z)=-\frac{1}{2}\int_{\mathbb{T}}g_t'(\dot\gamma,\dot\gamma)-\int_{\mathbb{T}}g_t'\left(Y,\tfrac{\nabla(t)}{ds}Z\right)-\int_{\mathbb{T}}g_t\left(Y,\tfrac{\nabla(t)'}{ds}Z\right)\,, \tag{4.3}$$

where we have used primes to denote derivatives w.r.t. t and dots to denote derivatives w.r.t. the loop parameter s.

Using that the covariant derivative commutes with every contraction, the second integral in (4.3) is equal to

$$\frac{1}{2}\int_{\mathbb{T}} g_t'\left(Y,\tfrac{\nabla(t)}{ds}Z\right)+\frac{1}{2}\int_{\mathbb{T}}\left\{\dot\gamma g_t'(Y,Z)-\tfrac{\nabla(t)}{ds}(g_t'(Y,\cdot\,))(Z)\right\}$$
$$=\frac{1}{2}\int_{\mathbb{T}} g_t'\left(Y,\tfrac{\nabla(t)}{ds}Z\right)-\frac{1}{2}\int_{\mathbb{T}}\tfrac{\nabla(t)}{ds}(g_t'(Y,\cdot\,))(Z)$$
$$=\frac{1}{2}\int_{\mathbb{T}}\left\{g_t'\left(Y,\tfrac{\nabla(t)}{ds}Z\right)-g_t'\left(Z,\tfrac{\nabla(t)}{ds}Y\right)\right\}-\frac{1}{2}\int_{\mathbb{T}}(\nabla(t)_{\dot\gamma}g_t')(Y,Z)\,.$$

For the third term on the right-hand side of (4.3), we use the well-known formula (see, e.g., [16, Proposition 2.3.1]) for the time derivative of the Levi-Civita connection,

$$\int_{\mathbb{T}} g_t\left(Y,\tfrac{\nabla(t)'}{ds}Z\right)=\frac{1}{2}\int_{\mathbb{T}}\{(\nabla(t)_Z g'(t))(Y,\dot\gamma)+(\nabla(t)_{\dot\gamma}g_t')(Y,Z)-(\nabla(t)_Y g_t')(Z,\dot\gamma)\}\ .$$

Putting the above together, we obtain

$$(d/dt)\left(-E_\gamma^{g_t}-\omega_\gamma^{g_t}\right)(Y,Z)=-\frac{1}{2}\int_{\mathbb{T}} g_t'(\dot\gamma,\dot\gamma)-\frac{1}{2}\int_{\mathbb{T}}\left\{g_t'\left(Y,\tfrac{\nabla(t)}{ds}Z\right)-g_t'\left(Z,\tfrac{\nabla(t)}{ds}Y\right)\right\}$$
$$+\frac{1}{2}\int_{\mathbb{T}}\{(\nabla(t)_Y g_t')(\dot\gamma,Z)-(\nabla(t)_Z g_t')(\dot\gamma,Y)\}\ . \tag{4.4}$$

On the other hand, defining the 1-form $\beta_t^{g_\bullet}$ on LX by

$$(\beta_t^{g_\bullet})_\gamma(Y)=\frac{1}{2}\int_{\mathbb{T}} g_t'(\dot\gamma,Y)\,,$$

its exterior derivative $d\beta_t^{g_\bullet}$ is defined by the Cartan formula [12, 33.12],

$$d(\beta_t^{g_\bullet})_\gamma(Y,Z)=Y\beta_t^{g_\bullet}(\widetilde Z)-Z\beta_t^{g_\bullet}(\widetilde Y)-\beta_t^{g_\bullet}([\widetilde Y,\widetilde Z])\,, \tag{4.5}$$

where $\widetilde Y$ and $\widetilde Z$ are local extensions of Y,Z, i.e., vector fields defined on a neighborhood of $\gamma\in LX$ with $\widetilde Y_\gamma=Y$ and $\widetilde Z_\gamma=Z$ (this definition is independent of the extensions $\widetilde Y,\widetilde Z$), and where we have used the usual identification of tangent vectors with the derivations they induce on the algebra of smooth functions on LX.
To compute the right hand side of (4.5), fix $\gamma\in LX$ and let $\eta,\xi:(-\varepsilon,\varepsilon)\to LX$ be smooth with $\eta(0)=\xi(0)=\gamma$ and $\dot\eta(0)=Y$, $\dot\xi(0)=Z$. Then

$$Y\beta_t^{g_\bullet}(\widetilde Z)=\frac{1}{2}\frac{d}{d\tau}_{|\tau=0}\int_{\mathbb{T}}(g_t')_{\eta(\tau)(s)}\left(\tfrac{\partial}{\partial s}\eta(\tau)(s),\widetilde Z_{\eta(\tau)}(s)\right)ds$$
$$=\frac{1}{2}\int_{\mathbb{T}}\left\{(\nabla(t)_{Y(s)}g_t')(\dot\gamma(s),Z(s))+g_t'\left(\tfrac{\nabla(t)}{\partial\tau}\tfrac{\partial}{\partial s}\eta(\tau)(s),Z(s)\right)+g_t'\left(\dot\gamma(s),\tfrac{\nabla(t)}{d\tau}\widetilde Z_{\eta(\tau)}(s)\right)\right\}_{|\tau=0}ds$$
$$=\frac{1}{2}\int_{\mathbb{T}}\left\{(\nabla(t)_{Y(s)}g_t')(\dot\gamma(s),Z(s))+g_t'\left(\tfrac{\nabla(t)}{ds}Y(s),Z(s)\right)+g_t'\left(\dot\gamma(s),\tfrac{\nabla(t)}{d\tau}\widetilde Z_{\eta(\tau)}(s)\right)\right\}_{|\tau=0}ds\,,$$

where the last equality comes from the well-known identity

$$\frac{\nabla(t)}{\partial\tau}\frac{\partial}{\partial s}\eta(\tau)(s) = \frac{\nabla(t)}{\partial s}\frac{\partial}{\partial\tau}\eta(\tau)(s)\,.$$

Analogously, we have

$$\begin{aligned}&Z\beta^{g_\bullet}\beta_t(\widetilde{Y}) = \\ &= \frac{1}{2}\int_{\mathbb{T}}\left\{(\nabla(t)_{Z(s)}g_t')(\dot\gamma(s),Y(s)) + g_t'\left(\tfrac{\nabla(t)}{ds}Z(s),Y(s)\right) + g_t'\left(\dot\gamma(s),\tfrac{\nabla(t)}{d\tau}\widetilde{Y}_{\xi(\tau)}(s)\right)\right\}_{|\tau=0}ds\,.\end{aligned}$$

To calculate $[\widetilde{Y},\widetilde{Z}]_\gamma(s)$, we use that the space of smooth vector fields on LX forms a Lie subalgebra of the space of bounded variations [12, Theorem 32.8]. To this end, fix $s\in\mathbb{T}$, let $f\in C^\infty(X)$, denote by $\mathrm{ev}_s : LX\to X$ the smooth evaluation map $\gamma\mapsto\gamma(s)$, and define $\widetilde{f} := f\circ\mathrm{ev}_s\in C^\infty(LX)$. Then

$$\widetilde{Z}_\gamma\widetilde{f} = d\widetilde{f}_\gamma\widetilde{Z}_\gamma = df_{\gamma(s)}d(\mathrm{ev}_s)_\gamma\widetilde{Z}_\gamma = df_{\gamma(s)}\widetilde{Z}_\gamma(s)\,,$$

so that

$$\widetilde{Y}_\gamma(\widetilde{Z}\widetilde{f}) = \frac{d}{d\tau}_{|\tau=0} df_{\eta(\tau)(s)}\widetilde{Z}_{\eta(\tau)}(s) = (\nabla(t)_{Y(s)}df)(Z(s)) + df\tfrac{\nabla(t)}{d\tau}\widetilde{Z}_{\eta(\tau)}(s)\,,$$

showing

$$\begin{aligned}[\widetilde{Y},\widetilde{Z}]_\gamma(s)f = [\widetilde{Y},\widetilde{Z}]_\gamma\widetilde{f} &= \mathrm{Hess}\, f(Y(s),Z(s)) + df\tfrac{\nabla(t)}{d\tau}\widetilde{Z}_{\eta(\tau)}(s)_{|\tau=0}\\ &\quad - \mathrm{Hess}\, f(Z(s),Y(s)) - df\tfrac{\nabla(t)}{d\tau}\widetilde{Y}_{\xi(\tau)}(s)_{|\tau=0}\\ &= \left(\tfrac{\nabla(t)}{d\tau}\widetilde{Z}_{\eta(\tau)}(s)_{|\tau=0} - \tfrac{\nabla(t)}{d\tau}\widetilde{Y}_{\xi(\tau)}(s)_{|\tau=0}\right)f\,.\end{aligned}$$

We have proved

$$d(\beta_t^{g_\bullet})_\gamma(Y,Z) = (d/dt)\left(-E_\gamma^{g_t} - \omega_\gamma^{g_t}\right)(Y,Z) + \iota\beta_t^{g_\bullet}\,.$$

Hence, for any differential form σ on LX we have

$$(d/dt)\mathfrak{I}^{g_t}[\sigma] = \int_{LX} e^{-E^{g_t}-\omega^{g_t}}\wedge(d-\iota)\beta_t^{g_\bullet}\wedge\sigma = \int_{LX} e^{-E^{g_t}-\omega^{g_t}}\wedge\beta_t^{g_\bullet}\wedge(d-\iota)\sigma\,,$$

where the last equality follows from the fact that by definition one has

$$(d-\iota)\mathfrak{I}^{g_t}[\sigma] = \mathfrak{I}^{g_t}[(d-\iota)\sigma] = 0.$$

Defining

$$\mathfrak{C}_t^{g_\bullet}(\sigma) := \int_{LX} e^{-E^{g_t}-\omega^{g_t}}\wedge\beta_t^{g_\bullet}\wedge\sigma\,,$$

we end up with

$$(d/dt)\mathfrak{I}^{g_t} = (d-\iota)\mathfrak{C}_t^{g_\bullet},$$

formally proving (1.4).

Acknowledgements: The authors would like to thank Matthias Ludewig and Konrad Waldorf for very helpful discussions!

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