Linear Algebra and its Applications 697 (2024) 468–486

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

GLT sequences and automatic computation of the symbol

N.S. Sarathkumar [a], S. Serra-Capizzano [b,c,*]

[a] *Department of Mathematics, Cochin University of Science and Technology, Kerala, India*
[b] *Department of Science and High Technology, University of Insubria, Italy*
[c] *Department of Information Technology, University of Uppsala, Sweden*

ARTICLE INFO

Article history:
Received 27 April 2023
Received in revised form 16 March 2024
Accepted 21 March 2024
Available online 27 March 2024
Submitted by T. Ehrhardt

Dedicated to Prof. Albrecht Böttcher on the occasion of his 70th birthday, thanking him for inspiring different generations of mathematicians with the beauty and the rigour of his results

MSC:
15A18
15B05
65N06
65E99

Keywords:
Singular value and eigenvalue asymptotics
Generalized Locally Toeplitz sequences

ABSTRACT

Spectral and singular value symbols are valuable tools to analyse the eigenvalue or singular value distributions of matrix-sequences in the Weyl sense. More recently, Generalized Locally Toeplitz (GLT) sequences of matrices have been introduced for the spectral/singular value study of the numerical approximations of differential operators in several contexts. As an example, such matrix-sequences stem from the large linear systems approximating Partial Differential Equations (PDEs), Fractional Differential Equations (FDEs), Integro Differential Equations (IDEs), using any discretization on reasonable grids via local methods, such as Finite Differences, Finite Elements, Finite Volumes, Isogeometric Analysis, Discontinuous Galerkin etc. Studying the asymptotic spectral behaviour of GLT sequences is useful in analysing classical techniques for the solution of the corresponding PDEs/FDEs/IDEs and in designing novel fast and efficient methods for the corresponding large linear systems or related large eigenvalue problems. The theory of GLT sequences, in combination with the results concerning the asymptotic spectral distribution of perturbed sequences of matrices, is one of the most powerful and successful tools for computing the spectral symbol f. In this regard, it would be beneficial to design an automatic procedure to compute the spectral symbols of such matrix-sequences and Ahmed Ratnani partially pursued it. Here, in

* Corresponding author at: Department of Science and High Technology, University of Insubria, Italy.
E-mail addresses: sarathkumarns05@gmail.com (N.S. Sarathkumar), s.serracapizzano@uninsubria.it (S. Serra-Capizzano).

https://doi.org/10.1016/j.laa.2024.03.024

Spectral symbol

the case of one-dimensional and two-dimensional differential problems, we continue in this direction by proposing an automatic procedure for computing the symbol of the underlying sequences of matrices, assuming that it is a GLT sequence satisfying mild conditions.

1. Introduction and preliminaries

Differential equations (DE) are widely used in physics, engineering, and applied science to model real-world problems. Closed forms of analytical solutions for such DEs are generally not available and when available, they come in analytic forms which are either implicit or computationally not tractable. Therefore, it is important to approximate the solution u of the given DEs by a suitable numerical method. When discretizing linear partial differential equations using linear numerical methods, the actual calculation of numerical solutions is reduced to solving a linear system $A_n u_n = b_n$. The size d_n of this linear system increases, that is $d_k < d_m$ if $k < m, k, m \in \mathbb{N}$, and hence d_n tends to infinity, as the fineness parameter $h = h(n)$ tends to zero. Therefore, we have not just a single linear system but a sequence of linear systems of increasing sizes and often it is observed in practice that the sequence of discretized matrices $\{A_n\}_n$ belongs to the $*$-algebra of Generalized Locally Toeplitz (GLT) matrix-sequences. As a consequence, such a GLT matrix-sequence $\{A_n\}_n$ shows a GLT symbol representing the singular value distribution. In the Hermitian setting the GLT symbol also represents the asymptotic spectral distribution, which in turn is somewhat similar to the spectrum of the differential operator associated with the PDE/FDE/IDE under consideration. Notice that in concrete application we need also the asymptotic notion of quasi-Hermitian matrix-sequence and this matter is briefly discussed in Remark 2.

Definition 1.1. Let $\{A_n\}_n$ be a matrix-sequence and $f : D \subset \mathbb{R}^k \to \mathbb{C}$ be a measurable function. We assume that $\mu_k(D)$ belongs to $(0, \infty)$, μ_k being the Lebesgue measure in $\mathbb{R}^k$, and that A_n is of size d_n, with d_n strictly increasing sequence of natural positive numbers, $n \in \mathbb{N}^+$ or more generally n belonging to an infinite subset of $\mathbb{N}^+$.

- We say that $\{A_n\}_n$ has an asymptotic singular value distribution with symbol f, and write $\{A_n\}_n \sim_\sigma f$, if, for all $F \in C_c(\mathbb{R})$, the space of complex-valued continuous functions defined on $\mathbb{R}$ and with bounded support,

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\sigma_i(A_n)) = \frac{1}{\mu_k(D)} \int_D F(|f(x)|)dx,$$

  where $\sigma_i(A_n)$, $i = 1, \cdots, d_n$, are the singular values of A_n.

- We say that $\{A_n\}_n$ has an asymptotic eigenvalue distribution with symbol f, and write $\{A_n\}_n \sim_\lambda f$, if, for all $F \in C_c(\mathbb{C})$, the space of complex-valued continuous functions defined on $\mathbb{C}$ and with bounded support,

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\lambda_i(A_n)) = \frac{1}{\mu_k(D)} \int_D F(f(x))dx,$$

  where $\lambda_i(A_n)$, $i = 1, \cdots, d_n$, are the eigenvalues of A_n.

Solving large linear systems in an efficient way is fundamental to compute accurate solutions in a reasonable time. In this direction, it is well known that the convergence properties of mainstream iterative solvers, such as multigrid and preconditioned Krylov methods, depend heavily on the spectral characteristics of the matrices to which they are applied. Therefore, the knowledge of the asymptotic spectral distribution of the sequence $\{A_n\}_n$ is a useful tool that we can use to select or design the best solver and discretization method. In support of this claim, we recall the notable estimates of the superlinear convergence of the conjugate gradient method obtained by Beckermann and Kuijlaars in [9], which assume positive definiteness, the existence of an asymptotic spectral distribution of the considered matrices, and proper assumptions on the convergence speed to zero of the minimal eigenvalues. In the same direction, we recall the seminal work by Axelsson and Lindskög [1] and the review paper by Kuijlaars [21], where again the concept of spectral distribution is employed in the analysis.
We choose to work with special spaces of matrix-sequences, that is the GLT $*$-algebras introduced in [26,27] (see also the seminal work by Tilli [28]) or a proper GLT subalgebra, which makes it easier to find spectral symbols from sequences of several discretized matrices for practical applications [2,4,7,6,17,18]. It has often been observed that the sequences $\{A_n\}_n$ coming from the discretization of linear DEs belong to one of the GLT spaces, so it has been possible to analyse the spectral behaviour of such sequences, thus justifying the interest in these spaces. Sequences $\{A_n\}_n$ with a spectral behaviour that can be analysed by resorting to GLT spaces have been encountered in the context of Finite Differences, Finite Elements, Finite Volumes, Isogeometric Analysis, and so on [10,14–16,24,26,27]. In essence the whole class of GLT sequences can be seen as a wide generalization of Toeplitz matrix-sequences [13], taking into account the variation of the coefficients along the diagonals [12].

The GLT spaces are built so that the association between sequences and symbols is an isomorphism of algebras and an isometry of metric spaces. Hence, we can compute the symbol of a matrix-sequence $\{A_n\}_n$ from sums, products, and limits of simpler sequences for which we already know the spectral symbol. In fact, it would be beneficial to have an automatic procedure to compute the spectral symbols of such matrix-sequences. This objective was already proposed in item 3 of [17, Chapter 11], and Ahmed Ratnani partially pursued it. Here we give a partial answer to this by introducing an effortless and automatic procedure for computing the symbol in the form of Fourier series.

In the following, we need a few terminologies and results taken mainly from [23].

Throughout this article, we use the multi-index notation (see [18]). For a given multi-index $\boldsymbol{i} \in \mathbb{Z}^d$ its components are denoted by $i_1, \ldots, i_d$, i.e., $\boldsymbol{i} = (i_1, \ldots, i_d)$. Furthermore, we use the notations below.

- $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ are vectors of all zeroes, ones, twos,....
- $\boldsymbol{h} \leq \boldsymbol{k}$ means that $h_r \leq k_r$ for all $r = 1, \ldots, d$.
- The multi-index interval $[\boldsymbol{h}, \boldsymbol{k}]$ is the set $\{\boldsymbol{j} \in \mathbb{Z}^d : \boldsymbol{h} \leq \boldsymbol{j} \leq \boldsymbol{k}\}$. We assume the elements in the interval $[\boldsymbol{h}, \boldsymbol{k}]$ to be ordered in the standard lexicographic manner:

$$\Big[\cdots [[(j_1, \ldots, j_d)]_{j_d = h_d \ldots, k_d}]_{j_{d-1} = h_{d-1}, \ldots, k_{d-1}} \cdots \Big]_{j_1 = h_1, \ldots, k_1}.$$

- $\boldsymbol{j} = \boldsymbol{h}, \ldots, \boldsymbol{k}$ means that $\boldsymbol{j}$ varies from $\boldsymbol{h}$ to $\boldsymbol{k}$ according to the lexicographic ordering.
- $\boldsymbol{s} = \boldsymbol{h} * \boldsymbol{k}$ with $* \in \{\pm, \cdot, /\}$ means $\boldsymbol{s} = (s_1, \ldots, s_d)$ with $s_r = h_r * k_r$ for all $r = 1, \ldots, d$ (i.e. componentwise addition, subtraction, multiplication division).
- $N(\boldsymbol{m}) = \prod_{j=1}^{d} m_j$ and $\boldsymbol{m} \to \infty$ means that $\min(\boldsymbol{m}) = \min_{j=1,\ldots,d} m_j \to \infty$.

In all this work, by a sequence of matrices (or matrix-sequence) we mean a sequence of the form $\{A_{\boldsymbol{n}}\}_n$, where:

- n varies in some infinite subset of $\mathbb{N}$;
- $\boldsymbol{n} = \boldsymbol{n}(n)$ is a multi-index in $\mathbb{N}^d$ which depends on n, and $\min_{j=1,\ldots,d} n_j(n) \to \infty$ when $n \to \infty$ and this is typical of many approximations of DEs in d dimensions, $d \geq 2$;
- $A_{\boldsymbol{n}}$ is a square matrix depending on a multi-index $\boldsymbol{n} = \boldsymbol{n}(n)$ and of size $d_n = N(\boldsymbol{n}(n)) = \prod_{j=1}^{d} n_j(n)$;
- In the setting of d-level GLT matrix-sequences we always have a matrix-size of the form $N(\boldsymbol{n}) = \prod_{j=1}^{d} n_j$ [26,18]. Furthermore, when discretizing differential operators in a d-dimensional domain we find $\boldsymbol{n} = \boldsymbol{n}(n)$, n natural number associated to the fineness parameter, so that with reference to Definition 1.1 we find $d_n = N(\boldsymbol{n}(n)) = \prod_{j=1}^{d} n_j(n)$. Finally, regarding the current work, in the numerical tests in 2D i.e. $d = 2$, for the sake of simplicity, we use a specific dependency on the parameter n that is $d_n = N(\boldsymbol{n}(n)) = n^2$, $\boldsymbol{n} = (n, n)$, $n \geq 1$.

Remark 1. With respect to the last notational item and Definition 1.1, it should be noted that in the pure d-level GLT matrix-sequences, the limits are made directly on the multi-index $\boldsymbol{n}$ which does not necessarily depend on a common parameter n. The need of introducing n comes typically from PDE applications in which one naturally finds a common fineness parameter. Furthermore, in that setting, the size is not necessarily $d_n = N(\boldsymbol{n}(n)) = \prod_{j=1}^{d} n_j(n)$, when the domain is not a d-dimensional rectangle or when the approximation nodes do not form a Cartesian grid, independently of the used discretization scheme (Finite Differences, Finite Elements, Finite Volumes, Isogeometric Analysis, Discontinuous Galerkin, etc). The practical need of treating the resulting general setting is transformed in a more advanced GLT theory, the theory of reduced GLT matrix-sequences, which was started in [26][pp. 398-399], [27][Section 3.1.4], and culminated in the systematic work by Barbarino [3]: in that setting d_n can be an increasing sequence of positive integers dictated by the shape of the domain and by the approximation rule for locating the nodes in the domain.

Definition 1.2. Let $\mathcal{E} = \{\{A_{\boldsymbol{n}}\}_n\colon A_{\boldsymbol{n}} \in \mathcal{M}_{N(\boldsymbol{n})}(\mathbb{C})\}$, $\mathcal{M}_m(\mathbb{C})$ being the set of all complex square matrices of size m, and define the functions $q_p : \mathcal{E} \to \mathbb{R}, 1 \le p \le \infty$ as

$$q_p(\{A_{\boldsymbol{n}}\}_n) = \inf\left\{\limsup_{n\to\infty} \frac{\|N_{\boldsymbol{n}}\|_p}{N(\boldsymbol{n})^{1/p}} :\ R_{\boldsymbol{n}} + N_{\boldsymbol{n}} = A_{\boldsymbol{n}},\ \operatorname{rank}(R_{\boldsymbol{n}}) = o(N(\boldsymbol{n}))\right\},\ 1 \le p < \infty.$$

$$q_\infty(\{A_{\boldsymbol{n}}\}_n) = \inf\left\{\limsup_{n\to\infty} \|N_{\boldsymbol{n}}\| :\ R_{\boldsymbol{n}} + N_{\boldsymbol{n}} = A_{\boldsymbol{n}},\ \operatorname{rank}(R_n) = o(N(\boldsymbol{n}))\right\}.$$

Here the infimum is taken over all such decompositions of $A_{\boldsymbol{n}}$, $\|\cdot\|$ is operator norm that is the maximal singular value of its argument and for $p \in [1,\infty)$ the notation $\|\cdot\|_p$ indicates the Schatten p norm that is the l^p vector norm applied to the vector of the singular values of its argument [11]. We notice that the Schatten 2 norm of a matrix X of size $m_1 \times m_2$ coincides with its Frobenius norm $\|X\|_F = \left(\sum_{i,j=1}^{m_1,m_2} |X_{ij}|^2\right)^{\frac{1}{2}}$, while the operator norm $\|\cdot\|$ is the Schatten p norm with $p = \infty$.

Definition 1.3. A matrix-sequence $\{A_{\boldsymbol{n}}\}_n$ is said to be a norm bounded matrix-sequence if there exists a nonnegative number M independent of n such that, for all n, $\|A_{\boldsymbol{n}}\| \le M$.

We now state two lemmas that are helpful for the following section.

Lemma 1.1. [23] *Suppose $\{A_{\boldsymbol{n}}\}_n$ is a norm bounded matrix-sequence. Then $q_p(\{A_{\boldsymbol{n}}\}_n) = 0$ if and only if $\|A_{\boldsymbol{n}}\|_p^p = o(N(\boldsymbol{n}))$ i.e. $\|A_{\boldsymbol{n}}\|_p = o([N(\boldsymbol{n})]^{1/p})$. For $p = \infty$ we simply have $q_\infty(\{A_{\boldsymbol{n}}\}_n) = 0$ if and only if $\|A_{\boldsymbol{n}}\| = o(1)$.*

Lemma 1.2. [23,25,29] *Let $\{A_{\boldsymbol{n}}\}_n$ and $\{B_{\boldsymbol{n}}\}_n$ be two matrix-sequences and let $p \in [1,\infty)$. If $\|A_{\boldsymbol{n}} - B_{\boldsymbol{n}}\|_p^p = o(N(\boldsymbol{n}))$, then $q_p(\{A_{\boldsymbol{n}} - B_{\boldsymbol{n}}\}_n) = 0$. For $p = \infty$ we simply have $q_\infty(\{A_{\boldsymbol{n}} - B_{\boldsymbol{n}}\}_n) = 0$ if and only if $\|A_{\boldsymbol{n}} - B_{\boldsymbol{n}}\| = o(1)$.*

Now we recall the definition of $\tilde{\mathcal{G}}^p$ introduced in [23]. For any $p \in [1,\infty]$, $\tilde{\mathcal{G}}^p$ is a proper subspace of the space of all GLT sequences. Even though all of the results in [23] is for the one-dimensional situation, they still hold for any d-dimensional scenario. Hence, without offering any proof, we provide the result for d dimensional setting.
Let

$$Z = \{\{A_{\mathbf{n}}\}_n \in \mathcal{E} : q_p(\{A_{\mathbf{n}}\}_n) = 0\},$$
$$\mathcal{A}_p = \{\{A_{\mathbf{n}}\}_n \in \mathcal{E} \ : \ q_p(\{A_{\mathbf{n}}\}_n) < \infty\},$$
$$\mathcal{G}^p = \{\{A_{\mathbf{n}}\}_n \in \mathcal{A}_p : \{A_{\mathbf{n}}\}_n \sim_{GLT} f\},$$

and $\tilde{\mathcal{G}}^p = \mathcal{G}^p/Z$ be the quotient space of $\mathcal{G}^p$.

Theorem 1.1. [23] *$\tilde{\mathcal{G}}_p, 1 \leq p \leq \infty$, are Banach spaces with respect to the norms induced by the seminorms q_p. In particular, $\tilde{\mathcal{G}}^2$ is a Hilbert space and $\tilde{\mathcal{G}}_\infty$ forms a C^*-algebra with *-structure $\{A_{\mathbf{n}}\}_n^* = \{A_{\mathbf{n}}^*\}_n$.*

Theorem 1.2. [23] *The Banach spaces $\tilde{\mathcal{G}}^p$ and $L^p(D)$ are isometrically isomorphic for every $1 \leq p \leq \infty$. In particular, $\tilde{\mathcal{G}}^\infty$ and $L^\infty(D)$ are isomorphic as C^*-algebras. The function that makes $\tilde{\mathcal{G}}^p$ isometrically isomorphic to $L^p(D)$, whenever $\{A_{\mathbf{n}}\}_n$ is a GLT sequence with symbol f, is*

$$\phi_p(\{A_{\mathbf{n}}\}_n) = \begin{cases} f & \text{if } p = \infty, \\ (2\pi)^{-\frac{d}{p}} f & \text{if } p \neq \infty, \end{cases}$$

where $D = [0,1]^d \times [-\pi,\pi]^d$.

Remark 2. For quasi-Hermitian in the current context of matrix-sequences $\{A_n\}_n$, A_n square of size d_n, d_n increasing sequence of positive integers, we mean that the skew-Hermitian part is negligible in the sense that the spectral norm of A_n is uniformly bounded and the Schatten 1 norm of the skew-Hermitian part divided by d_n tends to zero as n tends to infinity. In such a setting if the Hermitian part of $\{A_n\}_n$ admits a spectral symbol which is necessarily real-valued a.e., then the whole quasi-Hermitian matrix-sequences $\{A_n\}_n$ admits the same real-valued spectral symbol even when the matrices A_n are not Hermitian. Such type of results are important when dealing with discretization of diffusion-convection-reaction DEs in which the convection part is responsible usually for the lack of real symmetry [8] or when dealing with compact perturbation of Jacobi matrix-sequences in the context of zeros of orthogonal polynomials [20].

Remark 3. In the previous discussion we used the expression “GLT spaces” and now we clarify this point. In reality there exist infinitely many GLT $*$-algebras and even more GLT spaces. Regarding the GLT $*$-algebras, if one take the dimensionality parameter d positive integer and r the matrix-order of the GLT symbol, r positive integer, then for every $d, r \geq 1$ there exist r-block d-level GLT $*$-algebras. For $r = d = 1$ the material can be found in [17], the multilevel case is studied in [18], while the r block cases $r \geq 1$ are studied in [7,6], respectively for $d = 1$ and $d \geq 1$. These $*$ algebras have been proved to isometrically equivalent to the spaces of all possible GLT symbols (see [2] and the PhD Thesis of Giovanni Barbarino at Normal Superior School in Pisa), which are represented as $r \times r$ complex matrix-valued measurable function defined on $[0,1]^d \times [-\pi,\pi]^d$. Hence, given this correspondence between GLT symbol and classes of GLT matrix-sequences, e.g. in [23] subspaces of the given GLT $*$-algebra have been considered by putting a restriction on the vector space of possible GLT symbols. We mention that all this generality is suggested by the applications, e.g. by numerical methods for PDEs and FDEs, by the analysis of the zeros of orthogonal polynomials etc; see e.g. [9,14,15,19,21,22,20]. Finally, other GLT spaces (even richer!) and for dealing e.g. with multigrid methods [27,5], vector PDEs [5], and domains of non Cartesian types are treated via the notion of reduced GLT $*$-algebras in [26][pp. 398-399], [27][Section 3.1.4], and [3].

The structure of the current article is as follows. We introduce a technique to compute the spectral symbol of GLT sequences in Section 2. In Section 3 and Section 4, we demonstrate the method for one-dimensional and two-dimensional settings, respectively. Section 5 contains concluding remarks and open problems.

2. A method for computing the spectral symbol

We now provide a characterization for q_p that will be helpful. Let $A_{\boldsymbol{n}} = U_{\boldsymbol{n}}\text{diag}(\sigma_1(A_{\boldsymbol{n}}), \sigma_2(A_{\boldsymbol{n}}), \dots, \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}}))V_{\boldsymbol{n}}^*$ be a singular value decomposition (SVD) of $A_{\boldsymbol{n}}$, where $\sigma_1(A_{\boldsymbol{n}}) \geq \sigma_2(A_{\boldsymbol{n}}) \geq \dots \geq \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}})$. For $j = 0, 1, \dots, N(\boldsymbol{n})$, let

$$R_{\boldsymbol{n},j} = U_{\boldsymbol{n}}\text{diag}(\sigma_1(A_{\boldsymbol{n}}), \dots, \sigma_j(A_{\boldsymbol{n}}), 0, \dots, 0)V_{\boldsymbol{n}}^*, \quad R_{\boldsymbol{n},0} = O_{\boldsymbol{n}},$$
$$N_{\boldsymbol{n},j} = U_{\boldsymbol{n}}\text{diag}(0, \dots, 0, \sigma_{j+1}(A_{\boldsymbol{n}}), \dots, \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}}))V_{\boldsymbol{n}}^*,$$

and $D_{A_{\boldsymbol{n}}} = \{(R_{\boldsymbol{n},j}, N_{\boldsymbol{n},j}) : j = 0, 1, \dots N(\boldsymbol{n})\}$.

Lemma 2.1. *Let $\{A_{\boldsymbol{n}}\}_n$ be a matrix-sequence. Then*

$$q_p(\{A_{\boldsymbol{n}}\}_n) = \inf\left\{\limsup_{n\to\infty} \frac{\|N_{\boldsymbol{n}}\|_p}{N(\boldsymbol{n})^{1/p}} : (R_{\boldsymbol{n}}, N_{\boldsymbol{n}}) \in D_{A_{\boldsymbol{n}}},\ \text{rank}(R_{\boldsymbol{n}}) = o(N(\boldsymbol{n}))\right\}.$$

Proof. It is straightforward to prove that

$$q_p(\{A_{\boldsymbol{n}}\}_n) \leq \inf\left\{\limsup_{n\to\infty} \frac{\|N_{\boldsymbol{n}}\|_p}{N(\boldsymbol{n})^{1/p}} : (R_{\boldsymbol{n}}, N_{\boldsymbol{n}}) \in D_{A_{\boldsymbol{n}}},\ \text{rank}(R_{\boldsymbol{n}}) = o(N(\boldsymbol{n}))\right\}.$$

For the other inequality the reasoning is as follows. Consider a decomposition $A_{\boldsymbol{n}} = R_{\boldsymbol{n}} + N_{\boldsymbol{n}}$ with $\text{rank}(R_{\boldsymbol{n}}) = o(N(\boldsymbol{n}))$. Let $\sigma_1(A_{\boldsymbol{n}}) \geq \sigma_2(A_{\boldsymbol{n}}) \geq \dots \geq \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}})$ be the singular values of $A_{\boldsymbol{n}}$ arranged in non-increasing order. Set $r_{\boldsymbol{n}}$ =rank $R_{\boldsymbol{n}}$ and

$$\tilde{R}_{\boldsymbol{n}} = U_{\boldsymbol{n}}\,\text{diag}(\sigma_1(A_{\boldsymbol{n}}), \dots, \sigma_{r_{\boldsymbol{n}}}(A_{\boldsymbol{n}}), 0, \dots, 0)V_{\boldsymbol{n}}^*,$$
$$\tilde{N}_{\boldsymbol{n}} = U_{\boldsymbol{n}}\,\text{diag}(0, \dots, 0, \sigma_{r_{\boldsymbol{n}}+1}(A_{\boldsymbol{n}}), \dots, \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}}))V_{\boldsymbol{n}}^*.$$

We know that from [11], for all $i + j \leq N(\boldsymbol{n}) + 1$,

$$\sigma_{i+j-1}(A_{\boldsymbol{n}}) \leq \sigma_i(R_{\boldsymbol{n}}) + \sigma_j(N_{\boldsymbol{n}}).$$

From this, we deduce

$$\sum_{i=r_{\boldsymbol{n}}+1}^{N(\boldsymbol{n})} \frac{\sigma_i^p(A_{\boldsymbol{n}})}{N(\boldsymbol{n})} \leq \frac{\|N_{\boldsymbol{n}}\|_p^p}{N(\boldsymbol{n})}.$$

Hence,

$$\limsup_{n\to\infty} \frac{\|\tilde{N}_{\boldsymbol{n}}\|_p}{N(\boldsymbol{n})^{1/p}} \leq \limsup_{n\to\infty} \frac{\|N_{\boldsymbol{n}}\|_p}{N(\boldsymbol{n})^{1/p}}, \quad \operatorname{rank}(\tilde{R}_{\boldsymbol{n}}) = o(N(\boldsymbol{n})), \quad (\tilde{R}_{\boldsymbol{n}}, \tilde{N}_{\boldsymbol{n}}) \in D_{A_{\boldsymbol{n}}}.$$

Thus,

$$q_p(\{A_{\boldsymbol{n}}\}_n) \geq \inf\left\{\limsup_{n\to\infty} \frac{\|N_n\|_p}{N(\boldsymbol{n})^{1/p}} : (R_{\boldsymbol{n}}, N_{\boldsymbol{n}}) \in D_{A_{\boldsymbol{n}}},\ \operatorname{rank}(R_{\boldsymbol{n}}) = o(N(\boldsymbol{n}))\right\}. \quad \square$$

From Theorem 1.2, it can be seen that $\tilde{\mathcal{G}}^2$ is a Hilbert space and is isomorphic to $L^2(D)$. Let $\{A_{\boldsymbol{n}}\}_n, \{B_{\boldsymbol{n}}\}_n \in \tilde{\mathcal{G}}^2$, then

$$Re\langle\{A_{\boldsymbol{n}}\}_n, \{B_{\boldsymbol{n}}\}_n\rangle = \frac{1}{4}(q_2^2(\{A_{\boldsymbol{n}}\}_n + \{B_{\boldsymbol{n}}\}_n) - q_2^2(\{A_{\boldsymbol{n}}\}_n - \{B_{\boldsymbol{n}}\}_n)), \tag{1}$$

$$Im\langle\{A_{\boldsymbol{n}}\}_n, \{B_{\boldsymbol{n}}\}_n\rangle = \frac{1}{4}(q_2^2(\{A_{\boldsymbol{n}}\}_n + \{iB_{\boldsymbol{n}}\}_n) - q_2^2(\{A_{\boldsymbol{n}}\}_n - \{iB_{\boldsymbol{n}}\}_n)). \tag{2}$$

Theorem 2.1. *Let $\{A_{\boldsymbol{n}}\}_n$, $\{B_{\boldsymbol{n}}\}_n \in \tilde{\mathcal{G}}^2$ be two norm bounded matrix-sequences. Then*

$$\langle\{A_{\boldsymbol{n}}\}_n, \{B_{\boldsymbol{n}}\}_n\rangle = \lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})}\langle B_{\boldsymbol{n}}, A_{\boldsymbol{n}}\rangle_F,$$

where $\langle B_{\boldsymbol{n}}, A_{\boldsymbol{n}}\rangle_F$ is the Frobenius innerproduct, $\langle B_{\boldsymbol{n}}, A_{\boldsymbol{n}}\rangle_F = trace(B_{\boldsymbol{n}}^* A_{\boldsymbol{n}})$.

Proof.

$$q_2^2(\{A_{\boldsymbol{n}}\}_n) = \inf\left\{\limsup_{n\to\infty} \frac{\|N_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} :\ (R_{\boldsymbol{n}}, N_{\boldsymbol{n}}) \in D_{A_{\boldsymbol{n}}},\ \operatorname{rank}(R_{\boldsymbol{n}}) = o(N(\boldsymbol{n}))\right\}.$$

Since $\{R_{\boldsymbol{n}}\}_n$ is norm bounded, by Lemma 1.1, we deduce $\|R_{\boldsymbol{n}}\|_2^2 = o(N(\boldsymbol{n}))$. Thus,

$$\begin{aligned} q_2^2(\{A_{\boldsymbol{n}}\}_n) &= \inf\left\{\limsup_{n\to\infty} \frac{\|N_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} + \limsup_{n\to\infty} \frac{\|R_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})}\right\} \\ &= \limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})}. \end{aligned}$$

Let $\|A_{\boldsymbol{n}}\| \leq m$, for all n. Consider $F \in C_c(\mathbb{R})$ such that $x^2\chi_{[0,m]} \leq F(x) \leq x^2\chi_{[0,m+\epsilon]}$. Now Definition 1.1 implies that $\lim_{n\to\infty} \sum_{i=1}^{N(\boldsymbol{n})} \frac{F(\sigma_i(A_{\boldsymbol{n}}))}{N(\boldsymbol{n})}$ exists. Hence,

$$\lim_{n\to\infty} \sum_{i=1}^{N(\boldsymbol{n})} \frac{F(\sigma_i(A_{\boldsymbol{n}}))}{N(\boldsymbol{n})} = \limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} = \lim_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})}.$$

By using equations (1) and (2), we obtain

$$\langle \{A_{\boldsymbol{n}}\}_n, \{B_{\boldsymbol{n}}\}_n \rangle = \lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})} \langle B_{\boldsymbol{n}}, A_{\boldsymbol{n}} \rangle_F. \quad \square$$

Now we define two sets

$$D_{m,\boldsymbol{n}} = \{i : \sigma_i(A_{\boldsymbol{n}}) > m\}, \qquad C_{m,\boldsymbol{n}} = \{i : \sigma_i(A_{\boldsymbol{n}}) \le m\},$$

where $\sigma_i(A_{\boldsymbol{n}})$ is the i-th singular value of $A_{\boldsymbol{n}}$ and $\sigma_1(A_{\boldsymbol{n}}) \ge \sigma_2(A_{\boldsymbol{n}}) \ge \ldots \ge \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}})$.

Lemma 2.2. *Let $\{A_{\boldsymbol{n}}\}_n$ be a matrix-sequence such that* $\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})} = 0$. *Then $q_2(\{A_{\boldsymbol{n}}\}_n) = 0$ if and only if $\|A_{\boldsymbol{n}}\|_2^2 = o(N(\boldsymbol{n}))$.*

Proof. If $\|A_{\boldsymbol{n}}\|_2^2 = o(N(\boldsymbol{n}))$, then by Lemma 1.2, $q_2(\{A_{\boldsymbol{n}}\}_n) = 0$.
Conversely suppose that $q_2(\{A_{\boldsymbol{n}}\}_n) = 0$. Let $A_{\boldsymbol{n}} = U_{\boldsymbol{n}}\mathrm{diag}(\sigma_1(A_{\boldsymbol{n}}), \sigma_2(A_{\boldsymbol{n}}), \ldots, \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}}))V_n^*$ be an SVD of $A_{\boldsymbol{n}}$, where $\sigma_1(A_{\boldsymbol{n}}) \ge \sigma_2(A_{\boldsymbol{n}}) \ge \ldots \ge \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}})$. For a nonnegative real number m, let

$$A_{\boldsymbol{n},m} = U_{\boldsymbol{n}}\mathrm{diag}(0, \ldots, 0, \sigma_{j_m+1}(A_{\boldsymbol{n}}), \ldots, \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}}))V_{\boldsymbol{n}}^*,$$
$$N_{\boldsymbol{n},m} = U_{\boldsymbol{n}}\mathrm{diag}(\sigma_1(A_{\boldsymbol{n}}), \ldots, \sigma_{j_m}(A_{\boldsymbol{n}}), 0, \ldots, 0)V_{\boldsymbol{n}}^*,$$

where $\sigma_{j_m}(A_{\boldsymbol{n}}) > m$ and $\sigma_{j_m+1}(A_{\boldsymbol{n}}) \le m$. Now $A_{\boldsymbol{n}} = A_{\boldsymbol{n},m} + N_{\boldsymbol{n},m}$, and $q_2(\{A_{\boldsymbol{n}}\}_n) = 0$ implies $q_2(\{A_{\boldsymbol{n},m}\}_n) = 0$. By Lemma 1.1, $\|A_{\boldsymbol{n},m}\|_2^2 = o(N(\boldsymbol{n}))$. Then

$$\limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} = \limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n},m}\|_2^2 + \|N_{\boldsymbol{n},m}\|_2^2}{N(\boldsymbol{n})} = \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})}.$$

Since the left hand side is independent of m, we find

$$\limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} = \lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})} = 0.$$

Then, $\limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} = \lim_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} = 0. \quad \square$

Lemma 2.3. *Let $\{A_{\boldsymbol{n}}\}_n$ be a matrix-sequence that possesses asymptotic singular value distribution and satisfies* $\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})} = 0$. *Then,* $q_2^2(\{A_{\boldsymbol{n}}\}_n) = \lim_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})}$.

Proof. Consider $F \in C_c(\mathbb{R}), k \in \mathbb{N}$ such that $x^2\chi_{[0,m]} \leq F(x) \leq x^2\chi_{[0,m+\frac{1}{k}]}$. Then from Definition 1.1,

$$\limsup_{n\to\infty} \sum_{i\in C_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})} \leq \lim_{n\to\infty} \sum_{i=1}^{N(\boldsymbol{n})} \frac{F(\sigma_i(A_{\boldsymbol{n}}))}{N(\boldsymbol{n})} \leq \liminf_{n\to\infty} \sum_{i\in C_{m+\frac{1}{k},\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})}.$$

Let $A_{\boldsymbol{n}} = U_{\boldsymbol{n}}\text{diag}(\sigma_1(A_{\boldsymbol{n}}), \sigma_2(A_{\boldsymbol{n}}), \ldots, \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}}))V_{\boldsymbol{n}}^*$ be an SVD of $A_{\boldsymbol{n}}$, where $\sigma_1(A_{\boldsymbol{n}}) \geq \sigma_2(A_{\boldsymbol{n}}) \geq \ldots \geq \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}})$. For a nonnegative real number m, let

$$A_{\boldsymbol{n},m} = U_{\boldsymbol{n}}\text{diag}(0, \ldots, 0, \sigma_{j_m+1}(A_{\boldsymbol{n}}), \ldots, \sigma_{N(\boldsymbol{n})}(A_{\boldsymbol{n}}))V_{\boldsymbol{n}}^*,$$

where $\sigma_{j_m}(A_{\boldsymbol{n}}) > m$ and $\sigma_{j_m+1}(A_{\boldsymbol{n}}) \leq m$. Then,

$$\limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}} - A_{\boldsymbol{n},m}\|_2^2}{N(\boldsymbol{n})} = \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})},$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}} - A_{\boldsymbol{n},m}\|_2^2}{N(\boldsymbol{n})} = \lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})} = 0.$$

Now

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} - \liminf_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} &\leq \limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} - \limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n},m}\|_2^2}{N(\boldsymbol{n})} \\
&\quad + \liminf_{n\to\infty} \frac{\|A_{\boldsymbol{n},m+\frac{1}{k}}\|_2^2}{N(\boldsymbol{n})} - \liminf_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} \\
&= \limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} + \liminf_{n\to\infty} \frac{-\|A_{\boldsymbol{n},m}\|_2^2}{N(\boldsymbol{n})} \\
&\quad - \left(\limsup_{n\to\infty} \frac{-\|A_{\boldsymbol{n},m+\frac{1}{k}}\|_2^2}{N(\boldsymbol{n})} + \liminf_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} \right) \\
&\leq \limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}} - A_{\boldsymbol{n},m}\|_2^2}{N(\boldsymbol{n})} - \liminf_{n\to\infty} \frac{\|A_{\boldsymbol{n}} - A_{\boldsymbol{n},m+\frac{1}{k}}\|_2^2}{N(\boldsymbol{n})} \\
&\leq \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})}.
\end{aligned}$$

Since $\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})} = 0$, we have

$$\limsup_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} = \liminf_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})}.$$

Now $q_2^2(\{A_{\boldsymbol{n}}\}_n) \leq \lim_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})}$ is implied directly by the definition of q_2.

Suppose that $q_2^2(\{A_{\boldsymbol{n}}\}_n) < \lim_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})}$. Then there exist two matrix sequences $\{R_{\boldsymbol{n}}\}_n$ and $\{N_{\boldsymbol{n}}\}_n$, such that $(R_{\boldsymbol{n}}, N_{\boldsymbol{n}}) \in D_{A_{\boldsymbol{n}}}$, $\mathrm{rank}(R_{\boldsymbol{n}}) = o(N(\boldsymbol{n}))$, and

$$\limsup_{n\to\infty} \frac{\|N_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} < \lim_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})}.$$

Now there exists $\epsilon > 0$, such that

$$\begin{aligned}
\epsilon &< \limsup_{n\to\infty} \frac{\|R_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})} \\
&= \limsup_{n\to\infty} \sum_{i\in C_{m,\boldsymbol{n}}} \frac{\sigma_i^2(R_{\boldsymbol{n}})}{N(\boldsymbol{n})} + \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(R_{\boldsymbol{n}})}{N(\boldsymbol{n})} \\
&< \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})},
\end{aligned}$$

since $\limsup_{n\to\infty} \sum_{i\in C_{m,\boldsymbol{n}}} \frac{\sigma_i^2(R_{\boldsymbol{n}})}{N(\boldsymbol{n})} = 0$. However $\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})} > \epsilon$, which is a contradiction. Hence $q_2^2(\{A_{\boldsymbol{n}}\}_n) = \lim_{n\to\infty} \frac{\|A_{\boldsymbol{n}}\|_2^2}{N(\boldsymbol{n})}$. □

Lemma 2.4. *If both* $\{A_{\boldsymbol{n}}\}_n$ *and* $\{B_{\boldsymbol{n}}\}_n$ *belong to* $\tilde{\mathcal{G}}^2$ *and suppose that* $\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})} = 0$ *and* $\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(B_n)}{N(\boldsymbol{n})} = 0$. *Then*

$$\langle \{A_{\boldsymbol{n}}\}_n, \{B_{\boldsymbol{n}}\}_n \rangle = \lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})} \langle B_{\boldsymbol{n}}, A_{\boldsymbol{n}} \rangle_F.$$

Proof. The result follows from equations (1), (2), and Lemma 2.3. □

In the following we introduce an automatic method for calculating the Fourier coefficients of the spectral symbol of GLT sequences. Let $E_{\boldsymbol{n}}^{\boldsymbol{rs}} = D_{\boldsymbol{n}}(e^{2\pi i \boldsymbol{r}\cdot\mathbf{x}}) \cdot T_{\boldsymbol{n}}(e^{i\boldsymbol{s}\cdot\mathbf{t}}), \mathbf{x} \in [0,1]^d, \mathbf{t} \in [-\pi,\pi]^d$, where $D_{\boldsymbol{n}}(a) = \underset{\boldsymbol{j}=\boldsymbol{1},\ldots,\boldsymbol{n}}{\mathrm{diag}}\, a(\frac{\boldsymbol{j}}{\boldsymbol{n}})$ and $T_{\boldsymbol{n}}(f)$ is the $\boldsymbol{n}$-th multilevel Toeplitz matrix associated with f (see [18]) and we recall that $\boldsymbol{j}$ varies from $\mathbf{1}$ to $\boldsymbol{n}$ according to the lexicographic ordering. Consider the matrix-sequence $\{E_{\boldsymbol{n}}^{\boldsymbol{rs}}\}_n$, where $\boldsymbol{n} \to \infty$ as $n \to \infty$. Then $\{E_{\boldsymbol{n}}^{\boldsymbol{rs}}\}_n \sim_{GLT} e^{2\pi i \boldsymbol{r}\cdot\mathbf{x}} e^{i\boldsymbol{s}\cdot\mathbf{t}}$. By Theorem 1.2, $\{\{E_{\boldsymbol{n}}^{\boldsymbol{rs}}\}_n + Z : \boldsymbol{r}, \boldsymbol{s} \in \mathbb{Z}^d\}$ is an orthonormal basis for $\tilde{\mathcal{G}}^2$.

The following theorem shows how to determine the Fourier series representation of the spectral symbol of the GLT sequences.

Theorem 2.2. *Let* $\{A_{\boldsymbol{n}}\}_n \in \tilde{\mathcal{G}}^2$ *with symbol* $f \in L^2([0,1]^d \times [-\pi,\pi]^d)$, *let* $\sum_{\mathbf{r},\mathbf{s}=-\infty}^{\infty} a_{\mathbf{rs}} e^{i2\pi\mathbf{r}\cdot\mathbf{x}} e^{i\mathbf{s}\cdot\mathbf{t}}$ *be the Fourier series representation of* f, *and assume that* $\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i\in D_{m,\boldsymbol{n}}} \frac{\sigma_i^2(A_{\boldsymbol{n}})}{N(\boldsymbol{n})} = 0$. *Then*

$$a_{\mathbf{rs}} = \lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})} \langle E_{\boldsymbol{n}}^{\boldsymbol{rs}}, A_{\boldsymbol{n}} \rangle_F. \tag{3}$$

Proof. The result follows from Theorem 1.2 and Lemma 2.4,

$$\begin{aligned} a_{\mathbf{rs}} &= \frac{1}{(2\pi)^d} \int_D f \cdot e^{-i2\pi\mathbf{r}\cdot\mathbf{x}} e^{-i\mathbf{s}\cdot\mathbf{t}} d\mathbf{x} d\mathbf{t} \\ &= \langle \{A_{\boldsymbol{n}}\}_n, \{E_{\boldsymbol{n}}^{\boldsymbol{rs}}\}_n \rangle \\ &= \lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})} \langle E_{\boldsymbol{n}}^{\boldsymbol{rs}}, A_{\boldsymbol{n}} \rangle_F. \quad \square \end{aligned} \tag{4}$$

Corollary 2.1. *Let* $\{A_{\boldsymbol{n}}\}_n$ *be a matrix-sequence,* $a_{\mathbf{rs}} = \lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})} \langle E_{\boldsymbol{n}}^{\boldsymbol{rs}}, A_{\boldsymbol{n}} \rangle_F$ *and* $\sum_{\mathbf{r},\mathbf{s}=-\infty}^{\infty} |a_{\mathbf{rs}}|_2^2 < \infty$. *If* $\lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})} \|A_{\boldsymbol{n}} - \sum_{\mathbf{r},\mathbf{s}=-\infty}^{\infty} a_{\mathbf{rs}} E_{\boldsymbol{n}}^{\boldsymbol{rs}}\|_2^2 = 0$, *then* $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \sum_{\mathbf{r},\mathbf{s}=-\infty}^{\infty} a_{\mathbf{rs}} e^{i2\pi\mathbf{r}\cdot\mathbf{x}} e^{i\mathbf{s}\cdot\mathbf{t}}$.

3. Computation of the symbol: the 1-dimensional case

Consider the second-order differential problem:

$$\begin{cases} -(a(x)u'(x))' = f(x), & x \in (0,1), \\ \quad u(0) = \alpha, \quad u(1) = \beta, \end{cases} \tag{5}$$

where $a : [0,1] \to \mathbb{R}$ is continuous. We consider the discretization of this differential equation by using the classical second-order central FD scheme. We choose a discretization parameter $n \in \mathbb{N}$ and $x_j = jh$ for all $j \in [0, n+1]$, where $h = \frac{1}{n+1}$. Note that, for $j = 1, 2, \ldots, n$, we can approximate $-(a(x)u'(x))'|_{x=x_j}$ by the following FD formula:

$$\begin{aligned} -(a(x)u'(x))' &\approx -\frac{a(x_{j+\frac{1}{2}})u'(x_{j+\frac{1}{2}}) - a(x_{j-\frac{1}{2}})u'(x_{j-\frac{1}{2}})}{h} \\ &\approx -\frac{1}{h}\left(a(x_{j+\frac{1}{2}}) \frac{u(x_{j+1}) - u(x_j)}{h} - a(x_{j-\frac{1}{2}}) \frac{u(x_j) - u(x_{j-1})}{h} \right) \\ &= \frac{1}{h^2}\Big(-a(x_{j+\frac{1}{2}})u(x_{j+1}) + (a(x_{j+\frac{1}{2}}) + a(x_{j-\frac{1}{2}}))u(x_j) - a(x_{j-\frac{1}{2}})u(x_{j-1}) \Big). \end{aligned} \tag{6}$$

Table 1
Computation of $\|\gamma_n\|_2$ for increasing values of m.

m	100	500	1000	5000
$\|\gamma_n\|_2$	0.0300	0.0057	0.0028	0.0005

Then, we approximate the solution of (5) by the piecewise linear, globally continuous function that takes the value u_j in x_j for $j = 0, \dots, n+1$, where $u_0 = \alpha, u_{n+1} = \beta$, and $u = (u_1, \dots, u_n)^T$ is the solution of the linear system

$$-a(x_{j+\frac{1}{2}})u_{j+1} + (a(x_{j+\frac{1}{2}}) + a(x_{j-\frac{1}{2}}))u_j - a(x_{j-\frac{1}{2}})u_{j-1} = h^2 f(x_j), \quad j = 1, \dots, n. \tag{7}$$

The matrix of the linear system (7) is the tridiagonal symmetric matrix given by

$$A_n = \begin{pmatrix} a(x_{\frac{1}{2}}) + a(x_{\frac{3}{2}}) & -a(x_{\frac{3}{2}}) & & & & \\ -a(x_{\frac{3}{2}}) & a(x_{\frac{3}{2}}) + a(x_{\frac{5}{2}}) & -a(x_{\frac{5}{2}}) & & & \\ & -a(x_{\frac{5}{2}}) & a(x_{\frac{5}{2}}) + a(x_{\frac{7}{2}}) & -a(x_{\frac{7}{2}}) & & \\ & & -a(x_{\frac{7}{2}}) & \ddots & \ddots & \\ & & & \ddots & \ddots & -a(x_{n-\frac{1}{2}}) \\ & & & & -a(x_{n-\frac{1}{2}}) & a(x_{n-\frac{1}{2}}) + a(n + x_{\frac{1}{2}}) \end{pmatrix}.$$

In this example, we will see that Theorem 2.2 allows us to compute the singular value and eigenvalue distribution of the sequence of discretization matrices $\{A_n\}_n$. Using (3), we can compute the function $f_l = \sum_{r,s=-l}^{l} a_{rs} e^{i2\pi rx} e^{is\theta}$. Then $\lim_{l\to\infty} f_l = f$ is the symbol of $\{A_n\}_n$. Thus $\{A_n\}_n \sim_{GLT} f$ and since A_n is Hermitian, f must be a real-valued function and $\{A_n\}_n \sim_\lambda f$.

Let us take $a(x) = 3 + \cos(2x)$. A simple computation shows that $a_{rs} = \lim_{m\to\infty} \frac{1}{m} \langle E_m^{rs}, A_m \rangle_F = 0$, for every $s \in \mathbb{Z} \setminus \{-1, 0, 1\}$.
In Fig. 1, we plotted the graph of $Re(f_{50})$ and $Im(f_{50})$. Now, for $l = 3$ and $m = 1000$,

$$f_3 = \begin{bmatrix} e^{-6i\pi x} \\ e^{-4i\pi x} \\ e^{-2i\pi x} \\ 1 \\ e^{2i\pi x} \\ e^{4i\pi x} \\ e^{6i\pi x} \end{bmatrix}^T \begin{bmatrix} 0.0085 - i0.0761 & -0.0118 + i0.1520 & 0.0099 - i0.0759 \\ 0.1510 - i0.1158 & -0.0250 + i0.2312 & 0.0166 - i0.1156 \\ 0.0545 - i0.2512 & -0.1039 + i0.5016 & 0.0561 - i0.2509 \\ -3.4541 & 6.9079 & -3.4541 \\ 0.0545 + i0.2512 & -0.1039 - i0.5016 & 0.0561 + i0.2509 \\ 0.0151 + i0.1158 & -0.0250 - i0.2312 & 0.0166 + i0.1156 \\ 0.0085 + i0.0761 & -0.0118 - i0.1520 & 0.0099 + i0.0759 \end{bmatrix} \begin{bmatrix} e^{-i\theta} \\ 1 \\ e^{i\theta} \end{bmatrix}. \tag{8}$$

For $l = 3$ and $n = 400$, in Table 1, we computed the norm of γ_n for increasing values of m, where γ_n is the vector of the samples $Re(f_l(\frac{j}{\sqrt{n}}, \frac{2\pi k}{\sqrt{n}})), j, k = 0, 1, \dots, \sqrt{n} - 1$. Indeed,

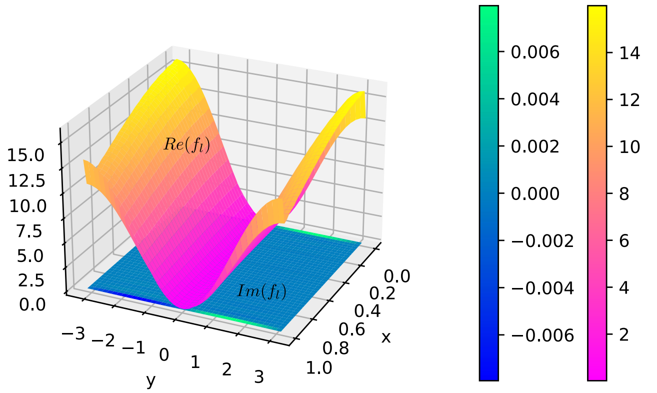

Fig. 1. Real part of f_{50} and imaginary part of f_{50}. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

Table 2
Computation of $\|\zeta_n - \eta_n\|_2$ for increasing values of n and l.

l \ n	20^2	40^2	60^2	80^2	100^2
3	0.4274	0.2165	0.1540	0.1251	0.1090
10	0.4085	0.1983	0.1321	0.1008	0.0825
15	0.4095	0.1942	0.1294	0.1008	0.0825

Table 3
Computation of $\|f - f_l\|_2$ for increasing values of m and l.

l \ m	100	400	600	800	1000
3	0.44278	0.41892	0.41802	0.41770	0.41755
10	0.36266	0.29050	0.28750	0.28645	0.28596
15	0.35961	0.24996	0.24994	0.24323	0.24240

since $Im(f_l)$ is negligible, it is enough to consider the real part of f_l. In Table 2, for increasing values of n and l, we computed the 2-norm of the difference $\gamma_n = \zeta_n - \eta_n$, where

- ζ_n is the vector of the eigenvalues of A_n
- η_n is the vector of the samples $Re(f_l(\frac{j}{\sqrt{n}}, \frac{2\pi k}{\sqrt{n}})), j, k = 0, 1, \ldots, \sqrt{n} - 1$.

Both ζ_n and η_n are arranged in non-increasing order.

We know that from [17] that the symbol of $\{A_n\}_n$ is $f(x, \theta) = a(x)(2 - 2\cos(\theta))$. In Table 3, we computed, for increasing values of m and l, the 2-norm of the function $f - f_l$. In Fig. 2, we plotted the spectrum of A_n together with values $Re(f_l(\frac{j}{\sqrt{n}}, \frac{2\pi k}{\sqrt{n}}))$ and $f(\frac{j}{\sqrt{n}}, \frac{2\pi k}{\sqrt{n}}), j, k = 0, 1, \ldots, \sqrt{n} - 1$, for $n = 3600$ and $l = 10$. The eigenvalues of A_n and the sampling of $Re(f_l)$ and f are depicted in non-increasing order.

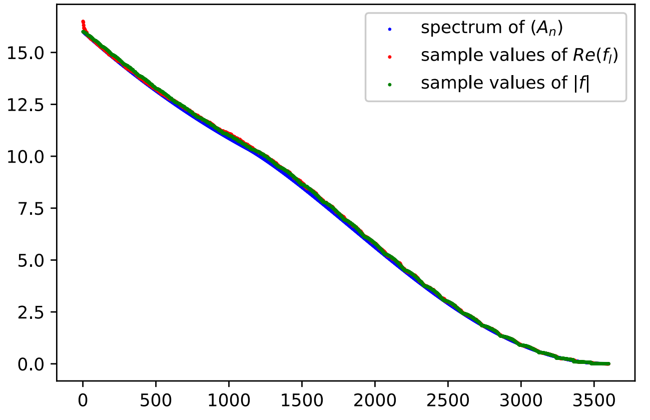

Fig. 2. Spectrum of A_n and samplings $Re(f_l(\frac{j}{\sqrt{n}}, \frac{2\pi k}{\sqrt{n}})), f(\frac{j}{\sqrt{n}}, \frac{2\pi k}{\sqrt{n}}), j, k = 0, 1, \ldots, \sqrt{n} - 1$, for $n = 60^2$ and $l = 10$.

4. Computation of the symbol: the 2-dimensional case

Consider the differential problem

$$\begin{cases} -\sum_{h,k=1}^{2} \frac{\partial}{\partial x_h}(a_{h,k}\frac{\partial u}{\partial x_k}) + \sum_{k=1}^{2} b_k \frac{\partial}{\partial x_k} + cu = f, & \text{in } (0,1)^2, \\ u = 0 & \text{on } \partial((0,1)^2), \end{cases}$$

where $a_{hk} \in C^1((0,1)^2) \cup C([0,1]^2)$ whose partial derivatives $\frac{\partial a_{hk}}{\partial x_l}$, are bounded over $(0,1)^2, b_k \in C([0,1]^2)$, and $c, f \in C([0,1]^2)$.

FD discretization: The problem can be reformulated as follows:

$$\begin{cases} -\sum_{h,k=1}^{2} a_{h,k} \frac{\partial^2 u}{\partial x_h \partial x_k} + \sum_{k=1}^{2} s_k \frac{\partial}{\partial x_k} + cu = f, & \text{in } (0,1)^2, \\ u = 0 & \text{on } \partial((0,1)^2), \end{cases} \tag{9}$$

where

$$s_k = b_k - \sum_{h=1}^{2} \frac{\partial a_{hk}}{\partial x_h}, \quad k = 1, 2.$$

We examine the classical discretization of (9) using the second-order central FD. We choose $\boldsymbol{n} = (n,n) \in \mathbb{N}^2$ and we set $\boldsymbol{h} = \frac{\mathbf{1}}{\boldsymbol{n}+\mathbf{1}} = (\frac{1}{n+1}, \frac{1}{n+1}) = (h_1, h_2)$ and $\mathbf{x}_{\boldsymbol{j}} = \boldsymbol{jh} = (j_1 h_1, j_2 h_2)$ for $\boldsymbol{j} = \mathbf{0}, \ldots, \boldsymbol{n} + \mathbf{1}$, $\mathbf{1}$ being $(1,1)$ according to the notation in the introduction. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$. Then, for $\boldsymbol{j} = \mathbf{0}, \ldots, \boldsymbol{n} + \mathbf{1}$, we can approximate the terms appearing in (9) as follows:

$$a_{kk}\frac{\partial^2 u}{\partial x_k^2}\bigg|_{\mathbf{x}=\mathbf{x_j}} \approx a_{kk}(\mathbf{x_j})\frac{1}{h_k^2}(u(\mathbf{x}_{\boldsymbol{j}+e_k}) - 2u(\mathbf{x_j}) + u(\mathbf{x}_{\boldsymbol{j}-e_k})), \qquad k=1,2,$$

$$a_{hk}\frac{\partial^2 u}{\partial x_h \partial x_k}\bigg|_{\mathbf{x}=\mathbf{x_j}} \approx a_{hk}(\mathbf{x_j})\frac{u(\mathbf{x}_{\boldsymbol{j}+e_k+e_h}) - u(\mathbf{x}_{\boldsymbol{j}+e_k-e_h}) - u(\mathbf{x}_{\boldsymbol{j}-e_k+e_h}) + u(\mathbf{x}_{\boldsymbol{j}-e_k-e_h})}{4h_h h_k},$$

$$h,k=1,2, \quad h \neq k,$$

$$s_k\frac{\partial u}{\partial x_k}\bigg|_{\mathbf{x}=\mathbf{x_j}} \approx s_k(\mathbf{x_j})\frac{1}{2h_k}(u(\mathbf{x}_{\boldsymbol{j}+e_k}) - u(\mathbf{x}_{\boldsymbol{j}-e_k})), \qquad k=1,2,$$

$$cu|_{\mathbf{x}=\mathbf{x_j}} = c(\mathbf{x_j})u(\mathbf{x_j}).$$

The evaluations $u(\mathbf{x_j})$ of the solutions of (9) at the grid points $\mathbf{x_j}$ are approximated by the values $u_{\boldsymbol{j}}$, where $u_{\boldsymbol{j}} = 0$ for $\boldsymbol{j} \in [\mathbf{0}, \boldsymbol{n}+\mathbf{1}] \setminus [\mathbf{1}, \boldsymbol{n}]$, and the vector $\boldsymbol{u} = (u_{\mathbf{1}}, \ldots, u_{\boldsymbol{n}})^T$ is the solution of the linear system

$$-\sum_{k=1}^{2} a_{kk}(\mathbf{x_j})\frac{1}{h_k^2}(u_{\boldsymbol{j}+e_k} - 2u_{\boldsymbol{j}} + u_{\boldsymbol{j}-e_k}) + \sum_{k=1}^{2} s_k(\mathbf{x_j})\frac{1}{2h_k}(u_{\boldsymbol{j}+e_k} - u_{\boldsymbol{j}-e_k}) + c(\mathbf{x_j})u_{\boldsymbol{j}}$$
$$-\sum_{h,k=1,h\neq k}^{2} a_{hk}(\mathbf{x_j})\frac{u_{\boldsymbol{j}+e_k+e_h} - u_{\boldsymbol{j}+e_k-e_h} - u_{\boldsymbol{j}-e_k+e_h} + u_{\boldsymbol{j}-e_k-e_h}}{4h_h h_k} = f(\mathbf{x_j}), \quad \boldsymbol{j} = \mathbf{1}, \ldots, \boldsymbol{n} \tag{10}$$

We now want to understand the structure of the matrix $A_{\boldsymbol{n}}$ associated with the linear system (10). Note that $A_{\boldsymbol{n}}$ admits the following natural decomposition:

$$A_{\boldsymbol{n}} = \sum_{h,k=1}^{2} \frac{1}{h_h h_k}\left(\operatorname*{diag}_{\boldsymbol{j}=\mathbf{1},\ldots,\boldsymbol{n}} a_{hk}(\mathbf{x_j})\right) K_{\boldsymbol{n},hk} + \sum_{k=1}^{2}\frac{1}{h_k}\left(\operatorname*{diag}_{\boldsymbol{j}=\mathbf{1},\ldots,\boldsymbol{n}} s_k(\mathbf{x_j})\right) H_{\boldsymbol{n},k} + \left(\operatorname*{diag}_{\boldsymbol{j}=\mathbf{1},\ldots,\boldsymbol{n}} c(\mathbf{x_j})\right),$$

where,

$$K_{\boldsymbol{n},11} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \otimes I_n, \qquad K_{\boldsymbol{n},22} = I_n \otimes \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

$$H_{\boldsymbol{n},1} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \otimes I_n, \qquad H_{\boldsymbol{n},2} = I_n \otimes \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

$$K_{\boldsymbol{n},12} = K_{\boldsymbol{n},21} = -\frac{1}{4}\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

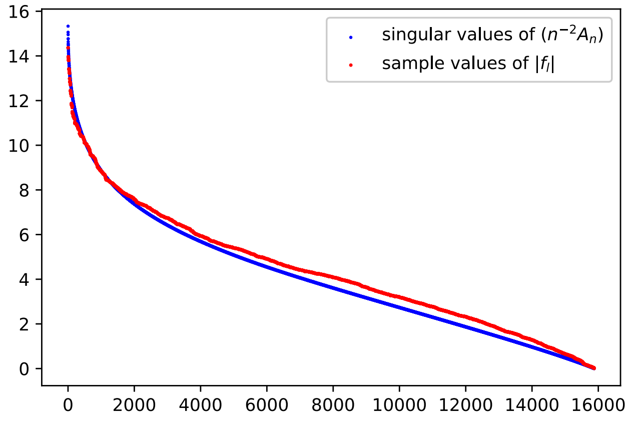

Fig. 3. Singular values of A_n and samplings $|f_{\boldsymbol{l}}(\boldsymbol{j}, \mathbf{k})|$, for $d_n = n^2 = 126^2$ and $\boldsymbol{l} = (-3, 3)$.

Table 4
Computation of $\|\zeta_n - \eta_n\|_2$ for increasing values of n for $\boldsymbol{l} = (-2, 2)$, $(-3, 3)$.

$\boldsymbol{l}$ \ $d_n = n^2$	60^2	77^2	96^2	117^2	126^2
(-2,2)	0.5137	0.4493	0.4048	0.3708	0.3506
(-3,3)	0.5641	0.4667	0.4044	0.3662	0.3433

Now, using equation (3), we can compute the Fourier coefficients of the symbol of the matrix-sequence $\{n^{-2}A_{\boldsymbol{n}}\}_n$ and we can find the partial sum of the Fourier series representation of the symbol f, $f_{\boldsymbol{l}} = \sum_{\mathbf{r},\mathbf{s}=-\boldsymbol{l}}^{\boldsymbol{l}} a_{\mathbf{rs}} e^{i2\pi\mathbf{r}\cdot\mathbf{x}} e^{i\mathbf{s}\cdot\theta}$.
Let us take $a_{11}(x, y) = x + y, a_{12}(x, y) = xy, a_{21}(x, y) = x^2y, a_{22}(x, y) = xy^2, b_1(x, y) = b_2(x, y) = x + y, c(x, y) = xy$.

In Fig. 3, we plotted the singular values of $n^{-2}A_{\boldsymbol{n}}$ together with sample values of $|f_{\boldsymbol{l}}|$, for $d_n = N(\boldsymbol{n}) = n^2 = 126^2$, $\boldsymbol{n} = (n, n)$ and $\boldsymbol{l} = (-3, 3)$. Both the singular values of $n^{-2}A_{\boldsymbol{n}}$ and the sampling of $|(f_{\boldsymbol{l}})|$ are depicted in non-increasing order. In Table 4, we computed, for increasing values of n and $\boldsymbol{l}$, the 2-norm of the difference $\zeta_n - \eta_n$, where

- ζ_n is the vector of the singular values of $n^{-2}A_{\boldsymbol{n}}$
- η_n is the vector of the samples $|f_{\boldsymbol{l}}|$

Both ζ_n and η_n are arranged in non-increasing order.

5. Conclusions

The major achievement of the present work is a step toward computing automatically the spectral symbol of a given matrix-sequence under the assumption that it belongs to the GLT class. We remind once again that the approximation via local methods of

PDE/FDE/IDE leads to GLT matrix-sequences under mild assumptions on the gridding. Then the spectral information provided by the spectral symbol f can be effectively used for designing fast iterative solvers and/or analyzing their convergence properties.

As a future work we plan to design more general procedures for handling more complicate structures and especially those of dense type as it occurs generally when dealing with matrix-sequences stemming from either FDEs or IDEs since the underlying operators are of intrinsic nonlocal nature.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Acknowledgements

N.S. Sarathkumar is thankful to CSIR for financial support through CSIR-SRF. The research of S. Serra-Capizzano was partly supported by the Italian INdAM-GNCS agency, Project CUP-E53C22001930001. Furthermore, the work of S. Serra-Capizzano was funded from the European High-Performance Computing Joint Undertaking (JU) under grant agreement No 955701. The JU receives support from the European Union's Horizon 2020 research and innovation programme and Belgium, France, Germany, Switzerland. Finally Stefano Serra-Capizzano is grateful for the support of the Laboratory of Theory, Economics and Systems – Department of Computer Science at Athens University of Economics and Business. The authors are also grateful to V. B. Kiran Kumar for valuable comments and suggestions and to the anonymous Reviewer for the excellent work that allowed us to improve the presentation, the mathematical rigour, and the whole content of the paper.

References

[1] Owe Axelsson, Gunhild Lindskog, On the rate of convergence of the preconditioned conjugate gradient method, Numer. Math. 48 (5) (1986) 499–523.
[2] Giovanni Barbarino, Equivalence between GLT sequences and measurable functions, Linear Algebra Appl. 529 (2017) 397–412.
[3] Giovanni Barbarino, A systematic approach to reduced GLT, BIT Numer. Math. 62 (3) (2022) 681–743.
[4] Giovanni Barbarino, Carlo Garoni, From convergence in measure to convergence of matrix-sequences through concave functions and singular values, Electron. J. Linear Algebra 32 (2017) 500–513.
[5] Giovanni Barbarino, Carlo Garoni, Mariarosa Mazza, Stefano Serra-Capizzano, Rectangular GLT sequences, Electron. Trans. Numer. Anal. 55 (2022) 585–617.

[6] Giovanni Barbarino, Carlo Garoni, Stefano Serra-Capizzano, Block generalized locally Toeplitz sequences: theory and applications in the multidimensional case, Electron. Trans. Numer. Anal. 53 (2020) 113–216.
[7] Giovanni Barbarino, Carlo Garoni, Stefano Serra-Capizzano, Block generalized locally Toeplitz sequences: theory and applications in the unidimensional case, Electron. Trans. Numer. Anal. 53 (2020) 28–112.
[8] Giovanni Barbarino, Stefano Serra-Capizzano, Non-Hermitian perturbations of Hermitian matrix-sequences and applications to the spectral analysis of the numerical approximation of partial differential equations, Numer. Linear Algebra Appl. 27 (3) (2020) e2286.
[9] Bernhard Beckermann, Arno B.J. Kuijlaars, Superlinear convergence of conjugate gradients, SIAM J. Numer. Anal. 39 (1) (2001) 300–329.
[10] Bernhard Beckermann, Stefano Serra-Capizzano, On the asymptotic spectrum of finite element matrix sequences, SIAM J. Numer. Anal. 45 (2) (2007) 746–769.
[11] Rajendra Bhatia, Matrix Analysis, vol. 169, Springer Science & Business Media, 2013.
[12] Albrecht Böttcher, Sergei Grudsky, Variable-coefficient Toeplitz matrices with symbols beyond the Wiener algebra, in: Numerical Methods for Structured Matrices and Applications, in: Oper. Theory Adv. Appl., vol. 199, Birkhäuser Verlag, Basel, 2010, pp. 191–202.
[13] Albrecht Böttcher, Bernd Silbermann, Introduction to Large Truncated Toeplitz Matrices, Universitext, Springer-Verlag, New York, 1999.
[14] Marco Donatelli, Carlo Garoni, Carla Manni, Stefano Serra-Capizzano, Hendrik Speleers, Spectral analysis and spectral symbol of matrices in isogeometric collocation methods, Math. Comput. 85 (300) (2016) 1639–1680.
[15] Marco Donatelli, Mariarosa Mazza, Stefano Serra-Capizzano, Spectral analysis and structure preserving preconditioners for fractional diffusion equations, J. Comput. Phys. 307 (2016) 262–279.
[16] Carlo Garoni, Spectral distribution of PDE discretization matrices from isogeometric analysis: the case of L^1 coefficients and non-regular geometry, J. Spectr. Theory 8 (1) (2018) 297–313.
[17] Carlo Garoni, Stefano Serra-Capizzano, Generalized Locally Toeplitz Sequences: Theory and Applications, vol. I, Springer Monogr. Math., 2017.
[18] Carlo Garoni, Stefano Serra-Capizzano, Generalized Locally Toeplitz Sequences: Theory and Applications, vol. II, Springer Monogr. Math., 2018.
[19] Carlo Garoni, Hendrik Speleers, Sven-Erik Ekström, Alessandro Reali, Stefano Serra-Capizzano, Thomas J.R. Hughes, Symbol-based analysis of finite element and isogeometric b-spline discretizations of eigenvalue problems: exposition and review, Arch. Comput. Methods Eng. 26 (5) (2019) 1639–1690.
[20] Leonid Golinskii, Stefano Serra-Capizzano, The asymptotic properties of the spectrum of nonsymmetrically perturbed Jacobi matrix sequences, J. Approx. Theory 144 (1) (2007) 84–102.
[21] Arno B.J. Kuijlaars, Convergence analysis of Krylov subspace iterations with methods from potential theory, SIAM Rev. 48 (1) (2006) 3–40.
[22] Arno B.J. Kuijlaars, Stefano Serra-Capizzano, Asymptotic zero distribution of orthogonal polynomials with discontinuously varying recurrence coefficients, J. Approx. Theory 113 (1) (2001) 142–155.
[23] V.B. Kiran Kumar, Rahul Rajan, N.S. Sarathkumar, Banach spaces of GLT sequences and function spaces, Electron. J. Linear Algebra 38 (2022) 295–316.
[24] Mariarosa Mazza, Ahmed Ratnani, Stefano Serra-Capizzano, Spectral analysis and spectral symbol for the 2D curl-curl (stabilized) operator with applications to the related iterative solutions, Math. Comput. 88 (317) (2019) 1155–1188.
[25] Stefano Serra-Capizzano, Spectral behavior of matrix sequences and discretized boundary value problems, Linear Algebra Appl. 337 (2001) 37–78.
[26] Stefano Serra-Capizzano, Generalized locally Toeplitz sequences: spectral analysis and applications to discretized partial differential equations, in: Special Issue on Structured Matrices: Analysis, Algorithms and Applications, Cortona, 2000, Linear Algebra Appl. 366 (2003) 371–402.
[27] Stefano Serra-Capizzano, The GLT class as a generalized Fourier analysis and applications, Linear Algebra Appl. 419 (1) (2006) 180–233.
[28] Paolo Tilli, Locally Toeplitz sequences: spectral properties and applications, Linear Algebra Appl. 278 (1–3) (1998) 91–120.
[29] Evgenij E. Tyrtyshnikov, A unifying approach to some old and new theorems on distribution and clustering, Linear Algebra Appl. 232 (1996) 1–43.