




# Robust bi-objective mean-CVaR portfolio selection: Applications to energy sector<sup>☆</sup>

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## ABSTRACT

A new approach to optimizing or hedging a portfolio of financial positions is presented and tested with applications to energy market. Motivated by uncertainty in the estimation of problem data we consider robust bi-objective optimization problems with mean and conditional value-at-risk objective functions where the underlying probability distribution of portfolio return is only known to belong to a certain set. To tackle the problem of uncertainty we consider two different approaches: in the first one, uncertainty is represented by an elliptic set centered at the sample estimators of mean and covariance matrix; in the second one, uncertainty takes into account experts beliefs. For both approaches, we derive analytical semi-closed-form solutions for the worst case mean-CVaR portfolio; in addition, we provide a characterization of the location of the robust Pareto frontier with respect to the corresponding original Pareto frontier.

## 1. Introduction

Mathematical optimization plays a key role in decision-making, particularly in finance. However, optimization problems often involve conflicting objectives, making it difficult to define a single clear objective function. Additionally, uncertainties in input data arising from measurement errors, imprecise data, or noise add complexity. Recent approaches have addressed these challenges by integrating robust optimization and bi-objective optimization, gaining popularity as highlighted in surveys [1] and comprehensive overviews [2].

Robust bi-objective optimization is crucial in financial decision-making, particularly in portfolio allocation. Since Markowitz's 1952 formulation [3], portfolio allocation theory has developed in various directions. Recognizing that using variance as a proxy for investment risk is suboptimal, due to its equal weighting of gains and losses, scholars have proposed different risk measure, such as, semivariance, expected absolute deviation, Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR) and semivariance-CVaR (see, among others, Rachev et al. [4], Rockafellar and Uryasev [5,6] and Rockafellar et al. [7]). In this paper we focus our attention on CVaR due to its superior properties over VaR, such as monotonicity, convexity, and positive homogeneity [6,8–10]. Additionally, as historical estimators often are characterized by high estimation errors, introducing noise and leading

to disappointing portfolio analyses (see, among others, Hitaj and Zambuno [11] and the reference therein), robust bi-objective decision rules have been proposed.

A common approach to address estimation uncertainty, often used in statistics, involves considering pre-specified ellipsoidal sets or confidence ellipsoids, which regulate the range of the uncertain input parameters (see [12,13]). In this special context, a robust bi-objective model based on ellipsoidal confidence sets, representing possible values of the unknown parameters, emerges as a valuable tool for implementing efficient investment strategies in front of ambiguous parameter estimations. In this direction, while Schöttle and Werner [14] explores closed-form solutions for robust mean–variance portfolios within an elliptical uncertainty set, our paper extends the methodology by focusing on the robust mean-CVaR portfolio optimization problem. An alternative approach, related to uncertainty arising from diverse opinions, investigates portfolios that outperform under the worst-case scenario formed by the recommendations of multiple experts, see [15,16]. The robust investor integrates distinct expert priors derived from various estimates or estimation methods of expected return and risk. Numerous authors have explored both theoretical and practical instances in this context [14,17–20]. Despite the advancements in robust portfolio optimization, the extension of mean-risk analysis to a bi-objective framework remains an area with limited exploration.

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In comparison with existing studies such as Salahi et al. [21], Zhu et al. [22] and Piri et al. [23], which consider robust portfolio optimization techniques with uncertainty only in the mean return, our approach accounts for uncertainty in both the mean and variance of returns, assuming normality. This provides a more comprehensive robust optimization framework. Additionally, unlike [24], which addresses uncertainty in the distribution of returns with a fixed mean return vector and variance/covariance matrix, our paper considers uncertainty in the entire distribution of asset returns without fixing the first two moments. In [19] robust models for VaR and CVaR optimization are developed under joint ambiguity in distribution, mean returns and covariance matrix. Notably, our paper differs from [19] in the following aspects: (i) we exploit the convex structure of the original bi-objective robust optimization problem by using linear scalarization techniques, while Lotfi and Zenios [19] focuses on minimizing VaR or CVaR under a constraint on excess expected return, and (ii) we compare the entire robust efficient frontiers, whereas [19] only considers a single efficient portfolio, minimizing VaR or CVaR under a fixed target return constraint.

To the best of our knowledge, the exploration of mean-risk bi-objective portfolios with ellipsoidal uncertainty sets or the involvement of multiple experts remains limited. Given the growing attention towards energy portfolio optimization over recent decades due to rising energy demands and environmental concerns, our study applies these models to robust portfolios of energy assets, a market characterized by significant uncertainty. It is essential to note that the approaches proposed in this paper are versatile and extend beyond energy portfolios, making them applicable to various financial markets.

The main contributions of this paper are:

- Investigation of the Robust Mean-CVaR Efficient Frontier: Using bi-objective programming methods, we derive the analytical expression for the robust mean-CVaR optimal portfolios under commonly made assumptions on the ambiguity set (e.g., ellipticity). Furthermore, the robust frontier is compared with the mean-CVaR Pareto one (nominal frontier, henceforth).
- Analytical construction of an alternative robust portfolio model: This involves a scenario where the investor faces a finite set of experts' recommendations on input parameters. We derive historical average returns and standard deviations for each expert using non-overlapping sub-samples of the dataset. Importantly, no distributional assumption of the portfolio returns is specified.
- Analyzing the effect of the risk aversion parameter on optimization solutions: This parameter, associated with the coefficient  $\lambda$  in our scalarized problem, illustrates how varying risk preferences influence the composition of optimal portfolios. A comparison of the robust mean-CVaR and robust mean-variance efficient portfolios is performed. Although both approaches depend on the first two moments of the return distribution under the assumption of normality, their efficient frontiers differ.
- Practical application of the proposed robust bi-objective optimization approaches to stock markets: We work first by assuming normality of the asset's returns distribution and an ellipsoidal uncertainty set. Then, we investigate the efficient frontiers under alternative experts' recommendations. The optimal portfolio performances are tested in both the nominal and robust cases. It is important to note that this empirical study serves as an illustration of the proposed methodology rather than a full-scale empirical validation. A more extensive comparison with alternative robust optimization models is left for future research.

The rest of the paper is organized as follows. Section 2 investigates the robust bi-objective mean-CVaR optimization. Section 2.1 addresses the problem of robust mean-CVaR optimization in the special case where the portfolio returns have a multivariate normal distribution but the moments of the returns are only known to belong to a given

elliptical set. Section 2.2 focuses on robust portfolio optimization in the setting of finite rival-scenario uncertainty, assuming there exists a finite number of experts with their own prior distribution. Section 3 presents two practical applications to illustrate the proposed model. While this empirical analysis does not extend to comparisons with other robust models – a task left for future research – it demonstrates the applicability of our method using real market data. By comparing it with the nominal mean-CVaR model, the analysis provides key insights into the potential practical use of our approach in investment settings. To further support the discussion, we include performance measures relevant to practitioners, such as the Sharpe ratio, Omega ratio, and Excess Return on VaR, offering a broader perspective on portfolio performance. A promising direction for future research is to compare our approach with other robust portfolio optimization models, evaluating their performance in different empirical settings and assessing their trade-offs in terms of risk-adjusted returns and computational complexity.

Section 3.1 works under the assumption of normality with parameters varying in an elliptical ambiguity set. In Section 3.2 the robust portfolio optimization with multiple experts is considered for a portfolio comprising members of the S&P500 energy sector. To validate our model, we focus on two important aspects for managers, out-of-sample performance and portfolio diversification. Section 4 provides a summary of our findings and indicates potential future developments.

## 2. Robust mean-CVaR optimization

### 2.1. Bi-objective optimization under elliptic uncertainty set

We begin this subsection with a short presentation of the main notions and techniques in bi-objective optimization. For a more detailed exposition, we refer the reader to Ehrgott [25] and the references therein.

**Definition 1.** Let  $f : \mathcal{C} \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^2$  and let  $\mathcal{X} \subseteq \mathbb{R}^d$ . A point  $\bar{\mathbf{w}} \in \mathcal{C} \cap \mathcal{X}$  is called efficient on the set  $\mathcal{X}$  if and only if whenever there exist  $\mathbf{w} \in \mathcal{C} \cap \mathcal{X}$  and  $i \in 1, 2$  such that  $f_i(\mathbf{w}) < f_i(\bar{\mathbf{w}})$ , then  $f_j(\mathbf{w}) > f_j(\bar{\mathbf{w}}), \forall j \neq i$ . A point  $\bar{\mathbf{w}} \in \mathcal{C} \cap \mathcal{X}$  is called weakly efficient on the set  $\mathcal{X}$  if and only if whenever there exist  $\mathbf{w} \in \mathcal{C} \cap \mathcal{X}$  and  $i \in 1, 2$  such that  $f_i(\mathbf{w}) < f_i(\bar{\mathbf{w}})$ , then  $f_j(\mathbf{w}) \geq f_j(\bar{\mathbf{w}}), \forall j \neq i$ .

Obviously, if a point  $\bar{\mathbf{w}} \in \mathcal{C} \cap \mathcal{X}$  is efficient on the set  $\mathcal{X}$ , then it is also weakly efficient on  $\mathcal{X}$ . The reverse implication holds if  $f$  is strictly convex on the convex set  $\mathcal{X}$ . We denote by  $\text{WEffmin}_{\mathbf{w} \in \mathcal{X}} [f(\mathbf{w})]$  the set of weakly efficient points of  $f$  on  $\mathcal{X}$ .

A usual technique to obtain optimality conditions for a bi-objective optimization problem is to transform it into a family of scalar optimization problems, the so called scalarization of a bi-objective optimization problem. By means of a linear scalarization technique some sufficient optimality conditions can be easily obtained. Let us consider the following scalarized objective function:

$$\Lambda(\mathbf{w}) = \lambda f_1(\mathbf{w}) + (1 - \lambda) f_2(\mathbf{w})$$

Let  $\bar{\mathbf{w}} \in \mathcal{C} \cap \mathcal{X}$ .

If  $\lambda \in [0, 1]$  and  $\Lambda(\bar{\mathbf{w}}) \leq \Lambda(\mathbf{w}), \forall \mathbf{w} \in \mathcal{C} \cap \mathcal{X}$ , then  $\bar{\mathbf{w}} \in \mathcal{C} \cap \mathcal{X}$  is weakly efficient on the set  $\mathcal{X}$ . Moreover, if  $\Lambda(\bar{\mathbf{w}}) < \Lambda(\mathbf{w}), \forall \mathbf{w} \in \mathcal{C} \cap \mathcal{X}$  with  $\mathbf{w} \neq \bar{\mathbf{w}}$ , then  $\bar{\mathbf{w}}$  is efficient on the set  $\mathcal{X}$ . We can also obtain necessary optimality conditions for weak efficiency under suitable convexity assumptions. Let  $f$  be a convex function on the convex set  $\mathcal{C}$  and let  $\mathcal{X} \subseteq \mathbb{R}^d$  be a convex set. If  $\bar{\mathbf{w}} \in \mathcal{C} \cap \mathcal{X}$  is weakly efficient on the set  $\mathcal{X}$ , then there exists  $\lambda \in [0, 1]$  such that  $\Lambda(\bar{\mathbf{w}}) \leq \Lambda(\mathbf{w}), \forall \mathbf{w} \in \mathcal{C} \cap \mathcal{X}$ . We underline that the extreme cases, where  $\lambda = 0, 1$ , consider the unconstrained minimization of only one of the original two objectives, as special cases.

Now we introduce the mean-CVaR framework that will be considered in this work. Suppose that there exist  $d$  risky assets; let  $\mathbf{X} = (X_1, \dots, X_d) \in \mathbb{R}^d$  denote the random vector of assets' returns and let

$\mathbf{w} = (w_1, \dots, w_d) \in \mathcal{X} \subseteq \mathbb{R}^d$  be the proportion of wealth invested in the  $d$  risky assets. Then the portfolio return is defined as  $\mathbf{w}^\top \mathbf{X}$ , and the loss function is

$$\ell(\mathbf{X}, \mathbf{w}) = -\mathbf{w}^\top \mathbf{X}.$$

We consider mean-CVaR optimization with no short-selling and a budget constraint. Thus, we assume that the set of feasible portfolio weights  $\mathcal{W}$  is non-empty, convex and compact and that they add up to one, i.e.,

$$\mathcal{W} = \{\mathbf{w} \in \mathbb{R}^d : \mathbf{w}^\top \mathbf{1} = 1, \mathbf{w} \geq \mathbf{0}\}, \tag{1}$$

where  $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$ . In addition, we assume that  $\mathbf{X}$  is multivariate normally distributed with location parameter  $\mu \in \mathbb{R}^d$  and scale parameter  $\Sigma \in \mathbb{S}_+^d$  (where  $\mathbb{S}_+^d$  denotes the space of positive semidefinite  $d$ -dimensional matrices). Given a confidence level  $\alpha$ , the Value-at-Risk (VaR) associated with the portfolio return  $\mathbf{w}^\top \mathbf{X}$  is defined as:

$$\text{VaR}_\alpha(\ell(\mathbf{X}, \mathbf{w})) = \min \{z \in \mathbb{R} : \mathbb{P}(\ell(\mathbf{X}, \mathbf{w}) \leq z) \geq 1 - \alpha\}.$$

The Conditional Value-at-Risk (CVaR) of the portfolio is defined in [5] as the conditional expectation of the loss of the portfolio exceeding or equal to the VaR, i.e.,

$$\text{CVaR}_\alpha(\ell(\mathbf{X}, \mathbf{w})) = \min_{z \in \mathbb{R}} F_\alpha(\mathbf{w}, \mathbf{X}, z),$$

where

$$F_\alpha(\mathbf{w}, \mathbf{X}, z) = z + \frac{1}{\alpha} \mathbb{E} [(-\mathbf{w}^\top \mathbf{X} - z)^+],$$

where  $(t)^+ = \max\{t, 0\}$ . In the case where the underlying probability distribution of the portfolio returns is a multivariate normal with location and scale parameters given by  $\mathbf{u} = (\mu, \Sigma) \in \mathbb{R}^d \times \mathbb{S}_+^d$  (see, e.g., [16]), CVaR reduces to

$$\text{CVaR}_\alpha(\ell(\mathbf{X}, \mathbf{w})) = -\mathbf{w}^\top \mu + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\|,$$

where

$$\gamma_\alpha = -\frac{\int_{-\infty}^{\Phi^{-1}(\alpha)} t \phi(t) dt}{\alpha}$$

and  $\phi, \Phi$  represent respectively the standard normal density function and cumulative distribution. Notice that, for  $\alpha \leq 0.5$ , we have  $\gamma_\alpha > 0$ .

**Notation 2.** To emphasize the dependence of the Conditional Value-at-Risk on  $\mathbf{w} \in \mathcal{W}$  and  $\mathbf{u} = (\mu, \Sigma)$ , in the following, the risk of the portfolio with weights  $\mathbf{w}$  will be denoted by  $\text{CVaR}_\alpha(\mathbf{w}, \mathbf{u})$ , i.e.,

$$\text{CVaR}_\alpha(\mathbf{w}, \mathbf{u}) = -\mathbf{w}^\top \mu + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\|, \quad \mathbf{u} = (\mu, \Sigma). \tag{2}$$

In addition, we will denote the portfolio expected return as

$$m(\mathbf{w}, \mathbf{u}) = \mathbf{w}^\top \mu.$$

Clearly, for a given  $\mathbf{u} \in \mathbb{R}^d \times \mathbb{S}_+^d$ ,  $m$  depends only on  $\mu$  and not on  $\Sigma$ .

We will consider the following parametric bi-objective optimization problem

$$\text{WEff min}_{\mathbf{w} \in \mathcal{W}} \begin{bmatrix} \text{CVaR}_\alpha(\mathbf{w}, \mathbf{u}) \\ -m(\mathbf{w}, \mathbf{u}) \end{bmatrix}. \tag{P_u}$$

As we have already pointed out, the pair  $\mathbf{u} = (\mu, \Sigma) \in \mathbb{R}^d \times \mathbb{S}_+^d$  is not known exactly but subject to uncertainty. Values for  $\mu$  and  $\Sigma$  are usually estimated from historical data. For this reason, similarly as in [16,18], we suppose that  $\mathbf{u} = (\mu, \Sigma)$  belongs to an ellipsoidal uncertainty set of size  $\delta$  around a nominal point  $(\hat{\mu}, \hat{\Sigma})$

$$\mathcal{U}_\delta(\hat{\mu}, \hat{\Sigma}) := \left\{ \mathbf{u} = (\mu, \Sigma) \in \mathbb{R}^d \times \mathbb{S}_+^d : \|(A^{-\frac{1}{2}})^\top (\mu - \hat{\mu})\|^2 + \|(M^{-\frac{1}{2}})^\top (\Sigma - \hat{\Sigma}) M^{-\frac{1}{2}}\|_{\text{tr}}^2 \leq \delta^2 \right\}. \tag{3}$$

Here,  $A$  and  $M^1$  are two given symmetric and positive semidefinite matrices and  $\|U\|_{\text{tr}}^2 := \text{Tr}(U^\top U) = \sum_{i=1}^d [U^\top U]_{ii}$  denotes the trace

nuclear norm of a matrix  $U$ , the so-called Frobenius norm of  $U$ . We stress that all vectors in  $\mathbb{R}^d$  are considered as columns.

Due to the fact that  $\Sigma \in \mathbb{S}_+^d$ , we know that  $\Sigma$  can be decomposed as

$$\Sigma = (\Sigma^{\frac{1}{2}})^\top \Sigma^{\frac{1}{2}},$$

where  $\Sigma^{\frac{1}{2}}$  is an upper triangular matrix (to be precise,  $(\Sigma^{\frac{1}{2}})^\top$  is the Cholesky matrix corresponding to  $\Sigma$ ). In addition, we observe that for any  $A \in \mathbb{S}_+^d$  we have

$$A^{-1} = [(A^{\frac{1}{2}})^\top A^{\frac{1}{2}}]^{-1} = A^{-\frac{1}{2}} (A^{-\frac{1}{2}})^\top.$$

In what follows, the nominal point  $(\hat{\mu}, \hat{\Sigma})$  will be represented by the mean and covariance estimated from historical data. Any  $(\mu, \Sigma) \in \mathcal{U}_\delta(\hat{\mu}, \hat{\Sigma})$  in the uncertainty set indicates a possible parameter realization. Clearly, more sophisticated uncertainty sets could be considered (see, for instance, Schöttle and Werner [14], where the author consider confidence ellipsoids).

We will consider a robust bi-objective optimization problem where we optimize the portfolio considering the worst outcome within the uncertainty set. This is the worst-case approach defined in [26] and later investigated by Schöttle and Werner [14,29] and Fliege and Werner [18]. More concretely, we consider a robustification of the original problem by substituting the original parametric bi-objective function by its robust counterpart:

$$\text{WEff min}_{\mathbf{w} \in \mathcal{W}} \begin{bmatrix} \max_{\mathbf{u} \in \mathcal{U}_\delta} \text{CVaR}_\alpha(\mathbf{w}, \mathbf{u}) \\ \max_{\mathbf{u} \in \mathcal{U}_\delta} -m(\mathbf{w}, \mathbf{u}) \end{bmatrix}. \tag{P_{\mathcal{U}_\delta}}$$

Notice the negative sign in the second objective function (the expected return of the portfolio), from which one can obtain, for fixed  $\mathbf{w} \in \mathcal{W}$ , the portfolio with worst expectation in the uncertainty set. This follows from the straightforward equality

$$\max_{\mathbf{u} \in \mathcal{U}_\delta} -m(\mathbf{w}, \mathbf{u}) = -\min_{\mathbf{u} \in \mathcal{U}_\delta} m(\mathbf{w}, \mathbf{u}).$$

In the sequel, we calculate the analytical solutions of the robust bi-objective mean-CVaR optimization problem. The calculations given in this section are inspired by Schöttle and Werner [14, Section 3], where the authors concentrate on the mean-variance problem. An alternative formulation of the robust mean-CVaR problem has also been formulated in [19, Appendix A.1], where a constrained scalarization technique is used. However, in that work the robustification is directly performed on the scalarized CVaR optimization problem considered there, where the objective function is the auxiliary function introduced in [5]. However, when joint uncertainties affect the components of the objective function, the commutativity between robustification and scalarization cannot be ensured as proved in Appendix A.1 (a simpler example is also given in [18], Example 3.2).

Differently, following the robust approach introduced in [18], we first consider the robust counterparts of the two components of the objective function, CVaR and mean, and then scalarize the obtained robust bi-objective optimization problem by linear techniques. Finally, as stated in the Introduction, by considering all the possible values of the scalarization parameter  $\lambda$ , we do not concentrate on a single optimal portfolio; instead, we analyze and compare entire efficient frontiers (see Remarks 6 and 7 for details).

<sup>1</sup>  $A$  and  $M$  provide a flexible framework for defining the ellipsoidal uncertainty set. This parameterization ensures that the model can adequately capture both the uncertainty in the mean returns and the covariance matrix of asset returns, with the flexibility to address uncertainties of different magnitudes. We observe that these two matrices have potential connections to the confidence sets associated with the point estimates  $\hat{\mu}$  and  $\hat{\Sigma}$  for  $\mu$  and  $\Sigma$ , as noted in [14], see also [14,26–28] for more information.

**Proposition 3.** Let  $\mathcal{U} = \mathcal{U}_\delta(\hat{\mu}, \hat{\Sigma})$  be as in (3). Then the robust counterpart of the first objective function in  $(P_{\mathcal{U}_\delta})$  is given by

$$\max_{\mathbf{u} \in \mathcal{U}_\delta(\hat{\mu}, \hat{\Sigma})} \text{CVaR}_\alpha(\mathbf{w}, \mathbf{u}) = -\mathbf{w}^\top \hat{\mu} + \max_{\kappa \in [0,1]} \Xi(\kappa, \mathbf{w}), \quad (4)$$

where, for fixed  $\mathbf{w} \in \mathcal{W}$ ,

$$\Xi(\kappa, \mathbf{w}) := \sqrt{\kappa} \delta \|A^{\frac{1}{2}} \mathbf{w}\| + \gamma_\alpha \sqrt{\delta \sqrt{1 - \kappa} \mathbf{w}^\top M \mathbf{w} + \mathbf{w}^\top \hat{\Sigma} \mathbf{w}}$$

is a function from  $[0, 1]$  in  $\mathbb{R}$ .

The proof is reported in A.2.

**Remark 4.** Notice that, for fixed  $\mathbf{w} \in \mathcal{W}$ ,  $\Xi(\kappa, \mathbf{w})$  is a function of the form

$$\Xi(\kappa, \mathbf{w}) = c_1(\mathbf{w})\sqrt{\kappa} + \gamma_\alpha \sqrt{c_2(\mathbf{w})\sqrt{1 - \kappa} + c_3(\mathbf{w})},$$

where  $c_i, i = 1, 2, 3$  are respectively

$$c_1(\mathbf{w}) = \delta \|A^{\frac{1}{2}} \mathbf{w}\|, \quad c_2(\mathbf{w}) = \delta \mathbf{w}^\top M \mathbf{w}, \quad c_3(\mathbf{w}) = \mathbf{w}^\top \hat{\Sigma} \mathbf{w}.$$

We emphasize that  $c_i, i = 1, 2, 3$ , are positive real functions defined on  $\mathcal{W}$ . For fixed  $\mathbf{w} \in \mathcal{W}$ ,  $\Xi$  is a continuous function in  $[0, 1]$ , it is also differentiable in  $(0, 1)$  with the maximum reached at the boundary (i.e., either at  $\kappa^* = 0$  or at  $\kappa^* = 1$ ) or in the interior of  $[0, 1]$  at some point  $\kappa^*$  such that  $\frac{\partial}{\partial \kappa} \Xi(\kappa, \mathbf{w}) \Big|_{\kappa=\kappa^*} = 0$ . The value of  $\kappa^* \in (0, 1)$  can be found numerically and depends on  $\mathbf{w} \in \mathcal{W}$ . For this reason, the notation  $\kappa^*(\mathbf{w})$  is more appropriate.

However, for  $A = M = \hat{\Sigma}$ ,  $\Xi(\kappa, \mathbf{w})$  can be written as

$$\Xi(\kappa, \mathbf{w}) = \sqrt{\mathbf{w}^\top \hat{\Sigma} \mathbf{w}} \max_{\kappa \in [0,1]} \tilde{\Xi}(\kappa)$$

where  $\tilde{\Xi}(\kappa) := \sqrt{\kappa} \delta + \gamma_\alpha \sqrt{\delta \sqrt{1 - \kappa} + 1}$  does not depend on  $\mathbf{w} \in \mathcal{W}$ .<sup>2</sup> Thus the optimal value of problem (4), denoted here by  $\text{CVaR}_\alpha^*(\mathbf{w})$ , is

$$\begin{aligned} \text{CVaR}_\alpha^*(\mathbf{w}) &= -\mathbf{w}^\top \hat{\mu} + \sqrt{\kappa^*} \delta \| \hat{\Sigma}^{\frac{1}{2}} \mathbf{w} \| + \gamma_\alpha \sqrt{\delta \sqrt{1 - \kappa^*} + 1} \| \hat{\Sigma}^{\frac{1}{2}} \mathbf{w} \| \\ &= -\mathbf{w}^\top \hat{\mu} + \tilde{\Xi}(\kappa^*) \| \hat{\Sigma}^{\frac{1}{2}} \mathbf{w} \|, \end{aligned}$$

where  $\kappa^* = \arg\max_{\kappa \in [0,1]} \tilde{\Xi}(\kappa)$  does not depend on  $\mathbf{w} \in \mathcal{W}$ .

Moreover, setting

$$\mu(\kappa^*, \mathbf{w}) := \hat{\mu} + \sqrt{\kappa^*} \delta (\hat{\Sigma}^{\frac{1}{2}})^T z^*(\mathbf{w})$$

$$\Sigma(\kappa^*, \mathbf{w}) := \hat{\Sigma} + \delta \sqrt{1 - \kappa^*} (z^*(\mathbf{w}) z^*(\mathbf{w})^\top)$$

we have that  $\mathbf{u}^*(\mathbf{w}) = (\mu(\kappa^*, \mathbf{w}), \Sigma(\kappa^*, \mathbf{w})) \in \mathcal{U}_{\sqrt{\kappa^*} \delta}(\hat{\mu}) \times \mathcal{U}_{\sqrt{1 - \kappa^*} \delta}(\hat{\Sigma})$  and

$$\text{CVaR}_\alpha^*(\mathbf{w}) = -\mathbf{w}^\top \mu(\kappa^*, \mathbf{w}) + \gamma_\alpha \| \Sigma^{\frac{1}{2}}(\kappa^*, \mathbf{w}) \|.$$

For the second objective function in  $(P_{\mathcal{U}_\delta})$  we have a similar result, as reported below.

**Proposition 5.** Let  $\mathcal{U} = \mathcal{U}_\delta(\hat{\mu}, \hat{\Sigma})$  be as in (3). Then the robust counterpart of the second objective function in  $(P_{\mathcal{U}_\delta})$  is given by

$$\max_{\mathbf{u} \in \mathcal{U}_\delta(\hat{\mu}, \hat{\Sigma})} -m(\mathbf{w}, \mathbf{u}) = -\mathbf{w}^\top \hat{\mu} + \delta \|A^{\frac{1}{2}} \mathbf{w}\|. \quad (5)$$

**Proof.** Since  $m(\mathbf{w}, \mathbf{u})$  depends only on  $\mu$ , we have that

$$\max_{\mathbf{u} \in \mathcal{U}_\delta(\hat{\mu}, \hat{\Sigma})} -m(\mathbf{w}, \mathbf{u}) = \max_{\kappa \in [0,1]} \max_{z \in \mathbb{R}^d: \|z\|^2 \leq 1} \left[ -\mathbf{w}^\top (\hat{\mu} + \sqrt{\kappa} \delta (A^{\frac{1}{2}})^T z) \right]$$

and the inner maximum is realized at  $z^*(\mathbf{w}) = -\frac{A^{\frac{1}{2}} \mathbf{w}}{\|A^{\frac{1}{2}} \mathbf{w}\|}$  so that

$$\max_{\mathbf{u} \in \mathcal{U}_\delta(\hat{\mu}, \hat{\Sigma})} -m(\mathbf{w}, \mathbf{u}) = \max_{\kappa \in [0,1]} -\mathbf{w}^\top \left[ \hat{\mu} + \sqrt{\kappa} \delta (A^{\frac{1}{2}})^T z^*(\mathbf{w}) \right]$$

$$\begin{aligned} &= \max_{\kappa \in [0,1]} \left[ -\mathbf{w}^\top \hat{\mu} + \sqrt{\kappa} \delta \|A^{\frac{1}{2}} \mathbf{w}\| \right] \\ &= -\mathbf{w}^\top \hat{\mu} + \delta \|A^{\frac{1}{2}} \mathbf{w}\|. \quad \blacksquare \end{aligned}$$

We observe that the robust counterparts of both the objective functions of the original problem are convex (see, e.g., [30]).

Propositions 3 and 5 imply that the convex robust mean-CVaR minimization problem  $(P_{\mathcal{U}_\delta})$  is

$$\begin{aligned} \text{WEffmin}_{\mathbf{w} \in \mathcal{W}} &\begin{bmatrix} -\mathbf{w}^\top \hat{\mu} + \max_{\kappa \in [0,1]} \Xi(\kappa, \mathbf{w}) \\ -\mathbf{w}^\top \hat{\mu} + \delta \|A^{\frac{1}{2}} \mathbf{w}\| \end{bmatrix} \\ &= \text{WEffmin}_{\mathbf{w} \in \mathcal{W}} \begin{bmatrix} -\mathbf{w}^\top \hat{\mu} + \Xi(\kappa^*, \mathbf{w}) \\ -\mathbf{w}^\top \hat{\mu} + \delta \|A^{\frac{1}{2}} \mathbf{w}\| \end{bmatrix}, \end{aligned}$$

where  $\kappa^* \in [0, 1]$  is the maximum of  $\Xi(\cdot, \mathbf{w})$ . Taking into account Section 2.1, the whole robust weakly efficient frontier is obtained by considering the following set

$$\begin{aligned} &\bigcup_{\lambda \in [0,1]} \text{argmin}_{\mathbf{w} \in \mathcal{W}} \left\{ \lambda \left( -\mathbf{w}^\top \hat{\mu} + \max_{\kappa \in [0,1]} \Xi(\kappa, \mathbf{w}) \right) \right. \\ &\quad \left. + (1 - \lambda) \left( -\mathbf{w}^\top \hat{\mu} + \delta \|A^{\frac{1}{2}} \mathbf{w}\| \right) \right\} \\ &= \bigcup_{\lambda \in [0,1]} \text{argmin}_{\mathbf{w} \in \mathcal{W}} \left\{ -\mathbf{w}^\top \hat{\mu} + \lambda \max_{\kappa \in [0,1]} \Xi(\kappa, \mathbf{w}) + (1 - \lambda) \delta \|A^{\frac{1}{2}} \mathbf{w}\| \right\}. \end{aligned} \quad (6)$$

**Remark 6.** We emphasize that when  $A = M = \hat{\Sigma}$  (cf. Remark 4), the term  $\Xi(\kappa^*, \mathbf{w})$  reduces to  $\tilde{\Xi}(\kappa^*) \| \hat{\Sigma}^{\frac{1}{2}} \mathbf{w} \|$ , so that

$$\begin{aligned} \text{WEffmin}_{\mathbf{w} \in \mathcal{W}} &\begin{bmatrix} -\mathbf{w}^\top \hat{\mu} + \max_{\kappa \in [0,1]} \Xi(\kappa, \mathbf{w}) \\ -\mathbf{w}^\top \hat{\mu} + \delta \|A^{\frac{1}{2}} \mathbf{w}\| \end{bmatrix} \\ &= \bigcup_{\lambda \in [0,1]} \text{argmin}_{\mathbf{w} \in \mathcal{W}} \left\{ -\mathbf{w}^\top \hat{\mu} + (\lambda \tilde{\Xi}(\kappa^*) + (1 - \lambda) \delta) \| \hat{\Sigma}^{\frac{1}{2}} \mathbf{w} \| \right\}. \end{aligned} \quad (7)$$

The nominal weakly efficient frontier is constructed by considering the set of minimization problems

$$\begin{aligned} &\bigcup_{\lambda \in [0,1]} \text{argmin}_{\mathbf{w} \in \mathcal{W}} \left\{ \lambda \text{CVaR}_\alpha(\mathbf{w}, \hat{\mathbf{u}}) - (1 - \lambda) \mathbf{w}^\top \hat{\mu} \right\} \\ &= \bigcup_{\lambda \in [0,1]} \text{argmin}_{\mathbf{w} \in \mathcal{W}} \left\{ -\mathbf{w}^\top \hat{\mu} + \lambda \gamma_\alpha \| \hat{\Sigma}^{\frac{1}{2}} \mathbf{w} \| \right\}. \end{aligned} \quad (8)$$

In particular, for  $\lambda = 0$ , the scalarized problem for the nominal efficient frontier in (8) is:

$$\min_{\mathbf{w} \in \mathcal{W}} \left\{ -\mathbf{w}^\top \hat{\mu} \right\},$$

which does not rely on the CVaR.

On the contrary, the worst-CVaR weakly efficient frontier depends, for all  $\lambda \in [0, 1]$ , on both  $\hat{\mu}$  and  $\hat{\Sigma}$ . In particular, for  $\lambda = 0$  the mean-CVaR scalarized problem in (7) becomes

$$\min_{\mathbf{w} \in \mathcal{W}} \left\{ -\mathbf{w}^\top \hat{\mu} + \delta \| \hat{\Sigma}^{\frac{1}{2}} \mathbf{w} \| \right\}.$$

**Remark 7.** An alternative bi-objective optimization problem that can be investigated is the following

$$\text{WEffmin}_{\mathbf{w} \in \mathcal{W}} \begin{bmatrix} \text{CVaR}_\alpha^*(\mathbf{w}) \\ -m(\kappa^*, \mathbf{w}) \end{bmatrix}, \quad (9)$$

where  $m(\kappa^*, \mathbf{w}) = -\mathbf{w}^\top \left[ \hat{\mu} + \sqrt{\kappa^*} \delta (A^{\frac{1}{2}})^T z^*(\mathbf{w}) \right]$ . Using Remark 4, we can see that the components of the bi-objective function in (9) correspond, respectively, to the Conditional Value-at-Risk and the expected return of the portfolio with normal distribution  $N(\mu(\kappa^*, \mathbf{w}), \Sigma(\kappa^*, \mathbf{w}))$ , i.e.,

$$\text{CVaR}_\alpha^*(\mathbf{w}) = \text{CVaR}_\alpha(\mathbf{w}, \mathbf{u}(\kappa^*, \mathbf{w}))$$

$$-m(\kappa^*, \mathbf{w}) = -\mathbf{w}^\top \mu(\kappa^*, \mathbf{w}).$$

<sup>2</sup> Fig. 1 reports the plot of  $\tilde{\Xi}$  for fixed  $\gamma_\alpha$  and  $\delta$ .

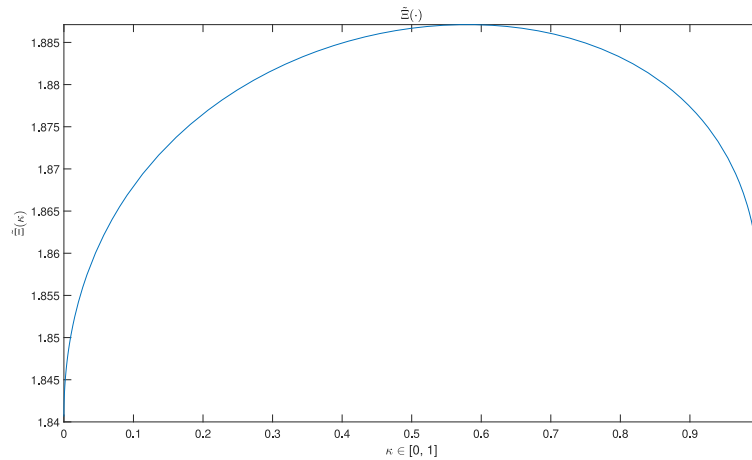


Fig. 1. Representation of  $\tilde{E}(\cdot)$  for  $\gamma_\alpha = 1.755$  and  $\delta = 0.1$ .

The weakly efficient points of the image set

$$\left\{ \begin{bmatrix} \text{CVaR}_\alpha^*(\mathbf{w}) \\ -m(\kappa^*, \mathbf{w}) \end{bmatrix} : \mathbf{w} \in \mathcal{W} \right\} \quad (10)$$

are characterized by the worst CVaR on the uncertainty set  $\mathcal{U}_\delta(\hat{\mu}, \hat{\Sigma})$ , which is realized at  $\mathbf{u}^* = (\mu(\kappa^*, \mathbf{w}), \Sigma(\kappa^*, \mathbf{w}))$ , and the expected return  $-\mathbf{w}^\top \mu(\kappa^*, \mathbf{w}) \in \mathcal{U}_{\delta\sqrt{\kappa^*}}(\hat{\mu})$ . Notice that in this case, differently from problem  $(P_{\mathcal{U}_\delta})$ , function  $m$  depends on  $\kappa^*$ . We will denote the weakly efficient frontier corresponding to (10) by  $\text{WCVaR}(\kappa^*)$ .

In the empirical part we will compare the robust efficient frontier deriving from problem  $(P_{\mathcal{U}_\delta})$  with the nominal efficient frontier (8) but also with the efficient frontier corresponding to (10).

**Remark 8.** With calculations similar to those considered above, we can characterize the solution of the robust bi-objective mean-variance minimization problem:

$$\text{WEff min}_{\mathbf{w} \in \mathcal{W}} \begin{bmatrix} \max_{\mathbf{u} \in \mathcal{U}_\delta} \sigma(\mathbf{w}, \mathbf{u}) \\ \max_{\mathbf{u} \in \mathcal{U}_\delta} -m(\mathbf{w}, \mathbf{u}) \end{bmatrix},$$

where  $\sigma(\mathbf{w}, \mathbf{u}) = \mathbf{w}^\top \Sigma \mathbf{w}$  indicates the variance of the portfolio.

More precisely, proceeding as in Propositions 3 and 5, and assuming  $A = M = \hat{\Sigma}$  the robust weakly efficient frontier is obtained by considering the following set:

$$\bigcup_{\lambda \in [0, 1]} \text{argmin}_{\mathbf{w} \in \mathcal{W}} \left\{ -(1 - \lambda) \mathbf{w}^\top \hat{\mu} + \left( \lambda \tilde{E}_\sigma(\kappa^*) + (1 - \lambda) \delta \right) \|\hat{\Sigma}^{\frac{1}{2}} \mathbf{w}\| \right\} \quad (11)$$

where  $\tilde{E}_\sigma(\kappa^*) = \max_{\kappa \in [0, 1]} \sqrt{\delta \sqrt{1 - \kappa} + 1}$ .

In the empirical part (see Section 3.1), we will argue that the robust mean-variance and robust mean-CVaR frameworks generally produce different optimal portfolio allocations for the same value of  $\lambda$ . Moreover, the two approaches typically result in distinct efficient frontiers, as formally proven in Appendix A.3.

### 2.2. Bi-objective optimization in finite spaces

Let us now consider a finite uncertainty set without any assumption on the probability distribution. For instance, when making a portfolio decision, according to Garlappi et al. [12] and Lutgens and Schotman [15], a robust investor may obtain multiple priors from different experts. Since the investor does not know which prior is the most appropriate, she could try to incorporate all of them simultaneously into the investment decision. In practice, the robust portfolio is the one with minimum mean-CVaR over each of the alternative analyst recommendations.

In this subsection we provide the theoretical background necessary to solve the robust mean-CVaR optimization problem in presence of a finite number of possible alternative distributions of asset's returns and for finite probability spaces. We begin by recalling an alternative expression of CVaR and then proceed to characterize solutions of the worst-case bi-objective problem.

We suppose here that the state space of assets' values is discrete, i.e.,

$$\mathbf{X} \in \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}\},$$

where  $\mathbf{x}^{(\ell)}$ ,  $\ell = 1, \dots, n$  represents the  $\ell$ th element of the state space (or the  $\ell$ th observation) of the portfolio components, i.e.,  $\mathbf{x}^{(\ell)} \in \mathbb{R}^d$ . We also assume that the uncertainty set of probability distributions is given by  $k$  different probabilities  $\pi^{(1)}, \dots, \pi^{(k)}$  (possibly representing the expert priors). We assume that all distributions share a common support space, and scenarios that are not realized in any of these distributions have a probability of zero.

As in the formulation of problem  $(P_{\mathcal{U}_\delta})$ , the robust counterpart of the bi-objective mean-CVaR problem is given by

$$\text{WEff min}_{\mathbf{w} \in \mathcal{W}} \begin{pmatrix} \max_{j=1, \dots, k} \text{CVaR}(\mathbf{w}, \pi^{(j)}) \\ \max_{j=1, \dots, k} \mathbb{E}_{\pi^{(j)}}[-\mathbf{w}^\top \mathbf{X}] \end{pmatrix}.$$

We can rewrite the conditional value at risk for a discrete loss distribution as the solution of a suitable minimization problem. For any  $j = 1, \dots, k$ , let  $F^{(j)} : \mathbb{R}^n \rightarrow \mathbb{R}$  be the functions defined as

$$F_\alpha^{(j)}(\mathbf{w}, z) = z + \frac{1}{\alpha} \sum_{i=1}^n [-\mathbf{x}^{(i)\top} \cdot \mathbf{w} - z]^+ \pi_i^{(j)}, \quad (12)$$

where  $\pi_i^{(j)}$  denotes the probability of  $\mathbf{x}^{(i)}$  under  $\pi^{(j)}$ . We have the following results, see [5, Theorem 1]: as a function of  $z$ ,  $F_\alpha^{(j)}(\mathbf{x}, z)$  is convex and continuously differentiable. The  $\text{CVaR}_\alpha$  of the loss associated with any  $\mathbf{w} \in \mathcal{W}$  can be determined from the formula

$$\varphi_\alpha(\mathbf{w}) = \min_{z \in \mathbb{R}} F_\alpha^{(j)}(\mathbf{w}, z).$$

The set

$$A_\alpha(\mathbf{w}) = \text{arg min}_{z \in \mathbb{R}} F_\alpha^{(j)}(\mathbf{w}, z),$$

is a nonempty closed bounded interval (perhaps reducing to a single point), and the  $\text{VaR}_\alpha$  of the loss is given by

$$z_\alpha(\mathbf{w}) = \min(A_\alpha(\mathbf{w})).$$

The maps  $F_\alpha^{(j)}$  and  $\varphi_\alpha^{(j)}$  ( $j = 1, \dots, n$ ) can be easily proved to satisfy the following

**Proposition 9.** For any  $j = 1, \dots, k$ , the function  $F_\alpha^{(j)}(\mathbf{w}, z)$  is convex with respect to  $(\mathbf{w}, z)$ , and  $\varphi_\alpha^{(j)}(\mathbf{w})$  is convex with respect to  $\mathbf{w}$ .

**Proof.** We observe that  $F_\alpha^{(j)}(\mathbf{w}, z)$  is convex with respect to  $(\mathbf{w}, z)$  since the term  $(\mathbf{w}, z) \mapsto [-(\mathbf{x}^{(j)})^\top \cdot \mathbf{w} - z]^+$  in formula (12) is itself convex with respect to  $(\mathbf{w}, z)$ . Indeed, this term is the composition of the function  $(\mathbf{w}, z) \mapsto f(\mathbf{w}, z) := -(\mathbf{x}^{(j)})^\top \cdot \mathbf{w} - z$  with the nondecreasing convex function  $t \mapsto (t)^+$ .

The convexity of the function  $\varphi_\alpha^{(j)}$  with respect to  $\mathbf{w}$  follows immediately from [31, page. 38–39]. ■

As a consequence of the previous result, the robust CVaR

$$\max_{j=1,\dots,k} \text{CVaR}_\alpha(\mathbf{w}, \pi^{(j)}) = \max_{j=1,\dots,k} \varphi_\alpha^{(j)}(\mathbf{w})$$

being the maximum of a finite number of convex functions on  $\mathbb{R}^d$ , turns out to be convex in  $\mathbf{w}$ .

The set of weakly efficient points of the robust bi-objective mean-CVaR problem can therefore be characterized within the framework described in Section 2, specifically through the family of scalarized problems:

$$\bigcup_{\lambda \in (0,1]} \operatorname{argmin}_{\mathbf{w} \in \mathcal{W}} \left\{ \lambda \max_{j=1,\dots,k} \text{CVaR}_\alpha(\mathbf{w}, \pi^{(j)}) + (1 - \lambda) \max_{j=1,\dots,k} \mathbb{E}_{\pi^{(j)}}[-\mathbf{w}^\top \mathbf{X}] \right\}.$$

### 3. Empirical analysis

In this section we consider two datasets with different characteristics. The first one is composed of monthly observations and the returns of each component are approximately normally distributed according to the Jarque–Bera test. The second one consists of daily observations of assets’ returns<sup>3</sup> that are not necessarily normally distributed. This choice is motivated by the following facts. First, as argued in [32], the underlying distribution of time series behave differently depending on the frequency of observations. In particular, monthly returns series are often observed to follow a normal distribution, while daily returns typically do not. Second, certain investment vehicles, like some hedge funds, report their “net asset value” monthly; consequently, daily returns may not always be available. Both datasets are downloaded from Bloomberg<sup>4</sup> (see Section 3.1 below for more details). Overall the empirical analysis the parameters  $\delta$  and  $\alpha$  are fixed respectively as 0.1 and 5%.

Throughout the empirical analysis, our primary objective is to apply and demonstrate the potential of the proposed model using two real portfolios. The focus is on illustrating its practical application and comparing it with the nominal mean-CVaR approach. To implement the robust approach, we consider hypothetical priors from multiple experts. While our methodology for constructing these priors is inspired by Huang et al. [16], we acknowledge that it differs from the rigorous techniques that practitioners might employ in real-world settings. However, this intentional simplification serves a clear purpose: to provide insight into portfolio behavior under varying assumptions and expert-based priors.

#### 3.1. Robust mean-CVaR portfolio under the assumption of normally distributed asset returns

##### 3.1.1. Efficient portfolios

We work with an energy portfolio with six components (listed in Table 1) selected from the VDE index fund<sup>5</sup> that according to the

<sup>3</sup> With an abuse of notation, in all the empirical part, the term “return” refers to “simple-return”.

<sup>4</sup> [www.bloomberg.com](http://www.bloomberg.com).

<sup>5</sup> VDE: VANGUARD ENERGY ETF is an exchange-traded fund made up of stocks of large, mid-size, and small U.S. companies within the energy sector that are involved with the construction or provision of oil rigs as well as the exploration, production and refining of oil/gas products. The fund seeks to track the performance of the MSCI US Investable Market Energy Index.

**Table 1**

Descriptive statistics for each time series of returns on the period under consideration. Means and standard deviations are reported on annual basis.

Ticker	$\mu$	$\sigma$	Skewness	Kurtosis	$p$ -value	JB-test	Critical value
DINO US	0.178	0.419	0.218	3.169	0.451	1.334	12.208
DRQ US	0.037	0.363	0.140	2.974	0.500	0.478	12.208
DK US	0.213	0.502	0.420	3.406	0.055	5.307	12.208
CTRA US	0.141	0.339	0.456	3.594	0.031	7.202	12.208
CVI US	0.239	0.523	0.276	3.853	0.040	6.291	12.208
BKR US	0.062	0.396	-0.140	3.874	0.059	5.118	12.208

Jarque–Bera test can be assumed normally distributed. The dataset contains 135 monthly observations from January 2010 to March 2022.

Table 1 outlines some descriptive statistics for each time series of returns on the whole period under analysis.<sup>6</sup> For each component of the portfolio, we report the annual mean, the annual standard deviation, skewness and kurtosis of the returns. In addition, the last three columns display the  $p$ -value, the value of the Jarque–Bera test and the corresponding critical value.

We notice that most of the energy equities under analysis are positively skewed with kurtosis around 3. The  $p$ -values show that the Jarque–Bera test does not reject the null hypothesis, for all assets, at the 1% significance level. Therefore, we can assume the portfolio to be normally distributed and use the theory described in Section 2.1.

The nominal CVaR (NCVaR) and the worst CVaR (WCVaR) optimal portfolios are compared in Fig. 2. More precisely, Fig. 2.(a) shows the worst-case versus the nominal CVaR efficient frontiers,<sup>7</sup> together with the frontier  $\text{WCVaR}(\kappa^*)$  (see (10)) introduced in Remark 7. From this figure it is clear that the larger the expected return, the larger the associated risk for both nominal and worst-case models. In addition, we observe that given a fixed value of risk (in our case, the CVaR), the expected return of the nominal optimal portfolio appears to be larger than that of the  $\text{WCVaR}(\kappa^*)$ , while the expected return of the WCVaR optimal portfolio is smaller than that of the  $\text{WCVaR}(\kappa^*)$ . Fig. 2.(b) reports, for selected values of  $\lambda$ , the composition of the worst-case and nominal efficient portfolios, and the corresponding modified Herfindahl index (HI).<sup>8</sup> We point out that, for both the WCVaR and NCVaR, the higher the value of  $\lambda$ , the more diversified the portfolio (HI decreases as  $\lambda$  increases). In addition, we observe that for all values of  $\lambda$  under analysis, the WCVaR portfolio is characterized by a lower HI index, which means that it is more diversified than the NCVaR one. Notice, however, that the difference between the HI indexes of the nominal and robust portfolios decreases as  $\lambda$  increases. In particular, for  $\lambda = 0$  the nominal efficient portfolio is obtained by minimizing

$$-\mathbf{w}^\top \hat{\boldsymbol{\mu}},$$

i.e., ignoring the risk measure. Coherently, all the wealth is invested into the asset with the highest mean (CVI US). For the worst-CVaR efficient frontier the objective function to be minimized (see Remark 6) when  $\lambda = 0$  is

$$-\mathbf{w}^\top \hat{\boldsymbol{\mu}} + \delta \|\hat{\boldsymbol{\Sigma}}^{1/2} \mathbf{w}\|.$$

This function depends not only on the expected value of the portfolio (as it happens instead for the nominal portfolio), but also on

<sup>6</sup> The first column of the table indicates the Bloomberg tickers of the investment universe.

<sup>7</sup> The nominal CVaR efficient frontier is obtained using the historical mean and standard deviations ( $\hat{\boldsymbol{\mu}}$  and  $\hat{\boldsymbol{\Sigma}}$ ) over the period Jan 2010–Dec 2022.

<sup>8</sup> The modified Herfindahl index is defined as

$$\text{HI} = \frac{\sum_{i=1}^n w_i^2 - \frac{1}{n}}{1 - \frac{1}{n}}.$$

By definition,  $\text{HI} \in [0, 1]$ . The equally weighted portfolio, that is deemed as the more diversified one, has a  $\text{HI} = 0$ .

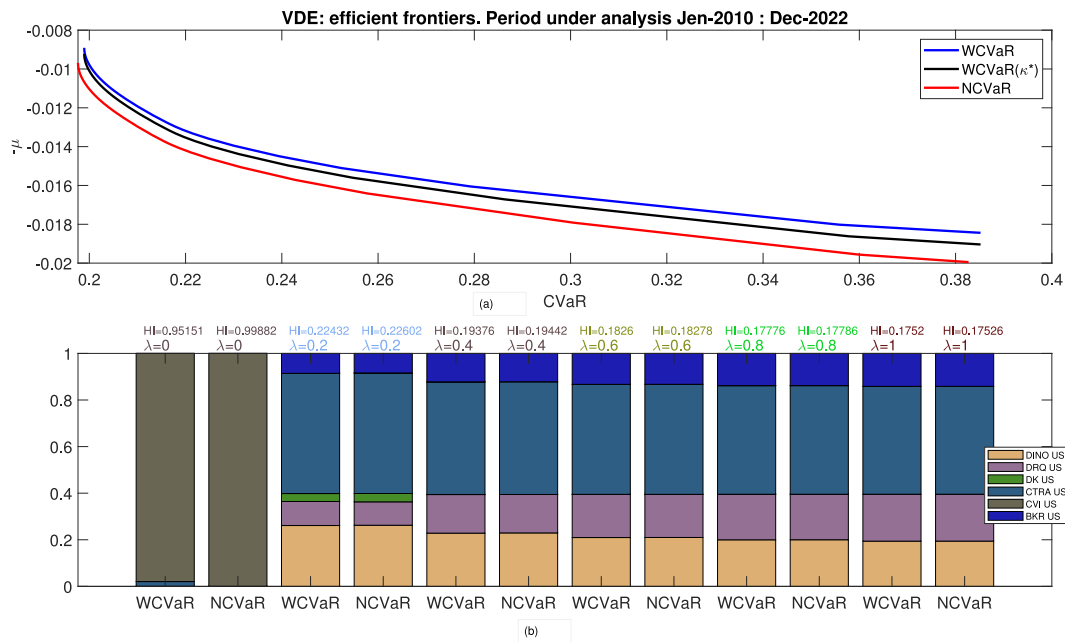


Fig. 2. (a) Robust efficient frontiers for the WCVaR portfolio (blue line), WCVaR( $\kappa^*$ ) (in black), and the nominal frontier (NCVaR, in red). (b) Composition of the WCVaR and NCVaR portfolios and their modified Herfindahl index for selected values of the parameter  $\lambda$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 2  
Efficient points for selected  $\lambda$ . The value of  $\mu$  is reported on annual basis.

$\lambda$	Worst case		$\kappa^*$		Nominal	
	$\mu$	CVaR	$\mu$	CVaR	$\mu$	CVaR
0	0.221	0.38518	0.228	0.38519	0.239	0.38269
0.2	0.126	0.20233	0.130	0.20238	0.136	0.20120
0.4	0.113	0.19935	0.117	0.19936	0.123	0.19807
0.6	0.110	0.19903	0.114	0.19903	0.119	0.19773
0.8	0.108	0.19896	0.112	0.19896	0.117	0.19766
1	0.107	0.19895	0.111	0.19895	0.116	0.19765

its variance. Hence, it is not surprising that the worst-CVaR portfolio concentrates, beside the asset with the highest expectation, on the asset with the lowest volatility (CTRA US).

On the contrary, for  $\lambda = 1$  the nominal and the worst-case are obtained, respectively, through the minimization problems

$$\min_{\mathbf{w} \in W} \left\{ -\mathbf{w}^T \hat{\boldsymbol{\mu}} + \gamma_\alpha \|\hat{\boldsymbol{\Sigma}}^{\frac{1}{2}} \mathbf{w}\| \right\} \quad \text{and} \quad \min_{\mathbf{w} \in W} \left\{ -\mathbf{w}^T \hat{\boldsymbol{\mu}} + \tilde{\Xi}(\kappa^*) \|\hat{\boldsymbol{\Sigma}}^{\frac{1}{2}} \mathbf{w}\| \right\}.$$

The objective functions here involve both the portfolio's expected return and the variance and differ only in the coefficients multiplying  $\|\hat{\boldsymbol{\Sigma}}^{\frac{1}{2}} \mathbf{w}\|$ , i.e.,  $\gamma_\alpha = 1.755$  and  $\tilde{\Xi}(\kappa^*) = \sqrt{\kappa^*} \delta + \gamma_\alpha \sqrt{\delta \sqrt{1 - \kappa^*} + 1} = 1.8871$  (with  $\kappa^* = 0.5803$ ). Notice anyway, that these two values are very close, so that the levels of diversification of the two optimal portfolios are almost the same.

One can also notice, see Table 2 and Fig. 2.(b), that the risk of the portfolio is decreasing with respect to  $\lambda$ . Thus, the more diversified is the portfolio, the less the CVaR is. This is coherent with the financial diversification principle, according to which allocating investments across various financial instruments may reduces risk.

Fig. 3 reports the three efficient frontiers for the baseline parameters. The colored sets represent, for  $\lambda = 0.04, 0.2$  and  $0.6$ , the image set of the bi-objective function

$$\left\{ \left[ \begin{array}{l} \text{CVaR}_\alpha(\mathbf{w}_{\text{nom}}^\lambda, \mathbf{u}) \\ -m(\mathbf{w}_{\text{nom}}^\lambda, \mathbf{u}) \end{array} \right] : \mathbf{u} = (\boldsymbol{\mu}, \boldsymbol{\Sigma}) \in \mathcal{U}_\delta(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}) \right\},$$

where  $\mathbf{w}_{\text{nom}}^\lambda$  denotes the vector of optimal weights for the nominal efficient frontier at  $\lambda$ . The profile of the efficient frontiers and of the

image sets are in line with that obtained by Fliege and Werner [18], where the robust mean–variance bi-objective optimization problem is studied. We highlight that here the focus is on the robust mean-CVaR bi-objective optimization.

However, as can be seen in Fig. 4, the location of the three efficient frontiers depends both on the model specification and on the dataset under analysis (or on expert priors). In this figure, in particular, we present the robust efficient frontier (blue), compared to the original nominal frontier (red) and to WCVaR( $\kappa^*$ ) (black) for two different sets of observations. Some descriptive statistics of the two sets are reported in Table 3. The two diverse sets of observations can be interpreted either as distinct time horizons or as different input data used by two independent experts to construct their priors. The first set of observations covers the period from February 2010 to December 2014 while the second ranges from January 2015 to April 2020. It can be observed from Table 3 that the second set can be considered riskier than the first one, as it is characterized by a lower mean and higher standard deviation. In Fig. 4 the evolution of the efficient frontiers across two experts is analyzed. We observe that the efficient frontiers characterized by higher expected return remain below the others. As explained previously, different locations of the efficient frontiers emerge when the investor deals with the recommendation of multiple experts who may have different priors and, therefore, different sets of estimates of the input parameters.

In addition to the Jarque–Bera test, which examines the normality of marginal distributions, we apply Mardia's test for multivariate normality to enhance the validity of our analysis. This test provides a comprehensive evaluation of the joint distribution. The multivariate normality is tested separately for each period:

- Within the first one, the  $p$ -values for skewness and kurtosis are 0.4668 and 0.4331, respectively. Hence, these results do not provide significant evidence to reject the assumption of multivariate normality.
- Within the second period, the  $p$ -values for skewness and kurtosis are 0.0273 and 0.0486, respectively. These results suggest that even in this case, at a 1% significance level, there is no significant evidence to reject the assumption of multivariate normality.

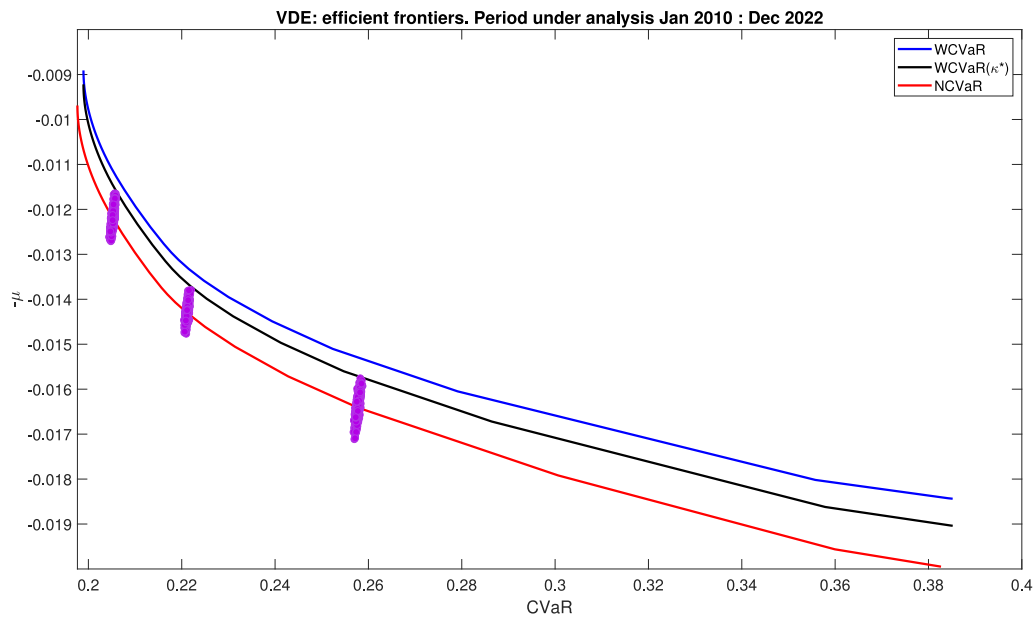


Fig. 3. Robust efficient frontiers (WCVaR, in blue) and WCVaR( $\kappa^*$ ) (in black), for  $\alpha = 5\%$ , compared to the nominal frontier (NCVaR, in red). For selected efficient portfolios, the uncertainty set, is also reported (in purple) for  $\lambda = 0.04, 0.2, 0.6$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

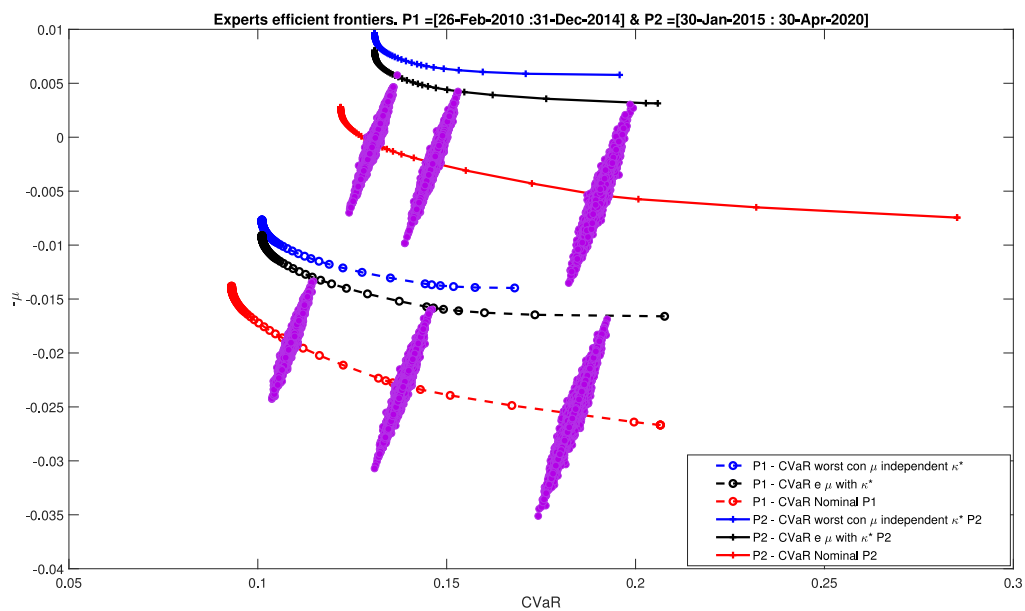


Fig. 4. Robust efficient frontier (blue), compared to the original nominal frontier (red) and the image of the robust efficient portfolios under the nominal parameter (black) based on two different dataset. The first set of observations covers the period from February 2010 to December 2014; the second period ranges from January 2015 to April 2020. For selected efficient portfolios is reported the uncertainty set (in purple) for  $\lambda = 0.04, 0.2, 0.6$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 3

Descriptive statistics for the two periods analyzed. Mean ( $\mu$ ) and standard deviation ( $\sigma$ ) are reported on annual basis.

Bloomberg ticker	Period 1: from 26-Feb-2010 to 31-Dec-2014						Period 2: from 30-Jan-2015 to 30-Apr-2020					
	$\mu$	$\sigma$	skew.	kurt.	$p$ -val	JB test	$\mu$	$\sigma$	skew.	kurt.	$p$ -val	JB test
DINO US	0.28	0.36	-0.34	3.40	0.34	1.53	0.08	0.46	0.23	2.85	0.50	0.61
DRQ US	0.12	0.29	0.14	2.61	0.50	0.55	-0.08	0.38	0.13	2.44	0.50	1.01
DK US	0.38	0.46	0.33	3.19	0.47	1.13	0.10	0.52	0.61	4.00	0.05	2.58
CTRA US	0.29	0.34	0.09	2.56	0.50	0.57	-0.01	0.32	0.25	3.31	0.50	0.91
CVI US	0.43	0.46	-0.09	2.80	0.50	0.18	0.05	0.53	0.42	4.93	0.03	3.77
BKR US	0.11	0.36	-0.47	3.64	0.11	3.21	-0.17	0.42	-0.11	4.40	0.06	2.35

**Table 4**  
Optimal portfolio weights for different levels of  $\lambda$  for the robust mean-CVaR portfolio (WCVaR) and the robust mean-variance portfolio (WV).

	$\lambda = 0.07$		$\lambda = 0.11$	
	WCVaR	WV	WCVaR	WV
DINO US	0.1644	2.12E-05	0.2654	0.0092
DRQ US	6.92E-05	1.95E-06	0.0005	2.84E-06
DK US	0.1462	0.2465	0.0974	0.2183
CTRA US	0.5261	0.2399	0.5675	0.4555
CVI US	0.1632	0.5135	0.0624	0.3170
BKR US	9.22E-05	2.29E-06	0.0068	4.40E-06
	$\lambda = 0.41$		$\lambda = 0.95$	
	WCVaR	WV	WCVaR	WV
DINO US	0.2271	0.2475	0.1953	0.1742
DRQ US	0.1671	0.1329	0.2001	0.2220
DK US	0.0004	0.0157	4.01E-05	3.53E-06
CTRA US	0.4825	0.5007	0.4638	0.4514
CVI US	2.42E-05	9.22E-06	7.90E-06	9.08E-07
BKR US	0.1229	0.1032	0.1408	0.1525

For the sake of completeness, we compute the optimal portfolios for the robust mean-variance optimization problem (WV) introduced in Remark 8. Table 4 presents these solutions, covering the period from January 2010 to December 2022, for various levels of  $\lambda$ , alongside the optimal weights of the robust mean-CVaR (WCVaR) portfolio. We observe that, for all values of  $\lambda$  considered, the two robust models (WCVaR and WV) yield different results, as previously noted in Remark 8.

### 3.1.2. Out-of-sample performance

In the following analysis, we examine the effects of the robust optimization in an out-of-sample setting, using priors from two experts with distinct views. The dataset is divided into three sub-samples: the first two represent the in-sample periods used by each expert to derive a prior on the distribution of asset returns<sup>9</sup> (see also [15,16]) and correspond to two different market conditions.

As explained previously, we assume in a simplified manner that each expert forms their prior by using a different historical data sub-period. Specifically, we assume the first expert derives the empirical mean  $\hat{\mu}$  and covariance matrix  $\hat{\Sigma}$  of the portfolio using monthly observations from January 2010 to December 2014 (sub-period P1). Similarly, the second expert forms his prior using observations from January 2015 to April 2020 (sub-period P2). These two sub-periods capture distinct market conditions<sup>10</sup> (see Fig. 5), which impact the experts' estimated priors. Implicitly, we assume that the two experts hold differing views for the future, which could be interpreted as each having access to different insider insights. This setup allows us to examine how the robust portfolio performs in a market shaped by diverse expert perspectives and priors.

Returns after 2020 identify the third sub-sample and are used to test the out-of-sample performances. This period is characterized by a high volatility, probably as a consequence of the global economic and financial crisis caused by COVID-19 pandemic.

For each expert, the out-of-sample portfolio returns are calculated through a buy-and-hold strategy (see among others [33,34]), which consists of the following steps: first, the optimal weights of the nominal and worst portfolios are determined over the first  $i$  months<sup>11</sup> of the in-sample period (representing the first in-sample window); then,

<sup>9</sup> It is worth noting that these distinct priors are intended to represent different expert views in a simplified form, allowing us to explore how the robust portfolio responds under different hypothetical expert scenarios.

<sup>10</sup> Sub-period P1 is characterized by relatively lower volatility, while P2 shows different return behaviors, possibly reflecting varying economic conditions or shifts in market dynamics.

<sup>11</sup> In this study we consider two different values for  $i$ :  $i = 24$  and  $i = 12$ .

the performances are calculated keeping these weights constant over their first out-of-sample period (composed by  $o$  months<sup>12</sup>); finally, we shift both the in-sample and out-of-sample windows by  $o$  months and roll-over till the end of the time horizon. In this empirical analysis the buy-and-hold procedure is implemented for different levels of  $\lambda$  (0.2, 0.4, 0.6, 0.8) and different lengths of the rolling window strategy (i.  $i = 24$  and  $o = 2$ , ii.  $i = 24$  and  $o = 1$ , iii.  $i = 12$  and  $o = 2$ , iv.  $i = 12$  and  $o = 1$ ).

In order to compare the worst and nominal mean-CVaR energy portfolios for different levels of  $\lambda$  we use the out-of-sample returns and calculate some well known out-of-sample statistics, i.e., the first four moments. We compute the portfolio's out-of-sample Sharpe ratio (SH) and test whether the difference in Sharpe ratios of two portfolios is significant. As alternative risk adjusted performance measure, we also calculate the Excess return on CVaR (ECVaR) and the Omega ratio ( $\Omega$ ), as discussed, e.g., in [33,35]. This performance measures are used to assess and compare the magnitude of potential gains/losses that can be attained by investors. We recall that the out-of-sample Sharpe ratio (SH) is defined as

$$SH = \frac{\mu - \mu_f}{\sigma},$$

where  $\mu$  and  $\sigma$  denote, respectively, the expected return and the standard deviation of the portfolio in the out-of-sample period while  $\mu_f$  indicates the average risk-free rate.<sup>13</sup> The Sharpe ratio quantifies the expected return of a portfolio in excess of the risk-free rate (also called the risk premium) as a fraction of the portfolio total risk, measured by its standard deviation. While Sharpe ratio is the most commonly employed measure to assess fund performance, the common criticism facing its application is when negative excess returns are encountered. During periods of negative excess returns, the appropriateness of the Sharpe ratio as a performance measure has been challenged because with a negative numerator, the ratio no longer provides meaningful interpretation of performance (see among others [38]). Therefore, in the empirical part, in case of negative excess returns we do not calculate the SH ratio. We also tested whether the Sharpe ratios of the two methods under analysis are statistically distinguishable, by employing the circular block bootstrap method proposed in [39]. Excess return on CVaR (ECVaR) is similar to Sharpe ratio where CVaR replaces standard deviation. For the same reasons explained previously, in the empirical part, in case of negative excess returns we do not calculate ECVaR.

The out-of-sample Omega ratio ( $\Omega$ ) is defined as

$$\Omega = \frac{\int_{\tau}^{+\infty} (1 - F(x)) dx}{\int_{-\infty}^{\tau} F(x) dx} = \frac{\mathbb{E}(R - \tau)^+}{\mathbb{E}(\tau - R)^+},$$

where  $R$  indicates the sequence of out-of-sample portfolio returns,  $F$  denotes its cumulative distribution function,  $\tau$  is a specified threshold<sup>14</sup> (see [40] for more details). The reason of calculating  $\Omega$  ratio is due to the fact that this measure does not make any assumptions regarding the return distributions evaluated and incorporates the whole return distribution in the calculation. Returns below (respectively above) the threshold are considered as losses (respectively gains). Values of  $\Omega$  greater than 1 indicate that the portfolio provides more expected gains than expected losses. Investors prefer portfolios with the possible highest  $\Omega$  ratio.

Table 5 reports the out of sample statistics for the nominal and robust mean-CVaR portfolios considering different levels of  $\lambda$  and different lengths for the rolling-window strategy. The results indicate that,

<sup>12</sup> In this study we consider two different values for  $o$ :  $o = 2$  and  $o = 1$ .

<sup>13</sup> As a proxy for the risk-free rate the literature suggests to use the U.S. Treasury Bills with maturity 1 or 3 months (see, e.g., [36,37] and references therein). For simplicity, in this paper we set  $\mu_f = 0$ .

<sup>14</sup> The threshold  $\tau$  can be different from investor to investor. In general, the  $\Omega$  ratio is very sensitive to changes in the value of  $\tau$ . In our analysis  $\tau$  is set equal to 0 (see [34]).

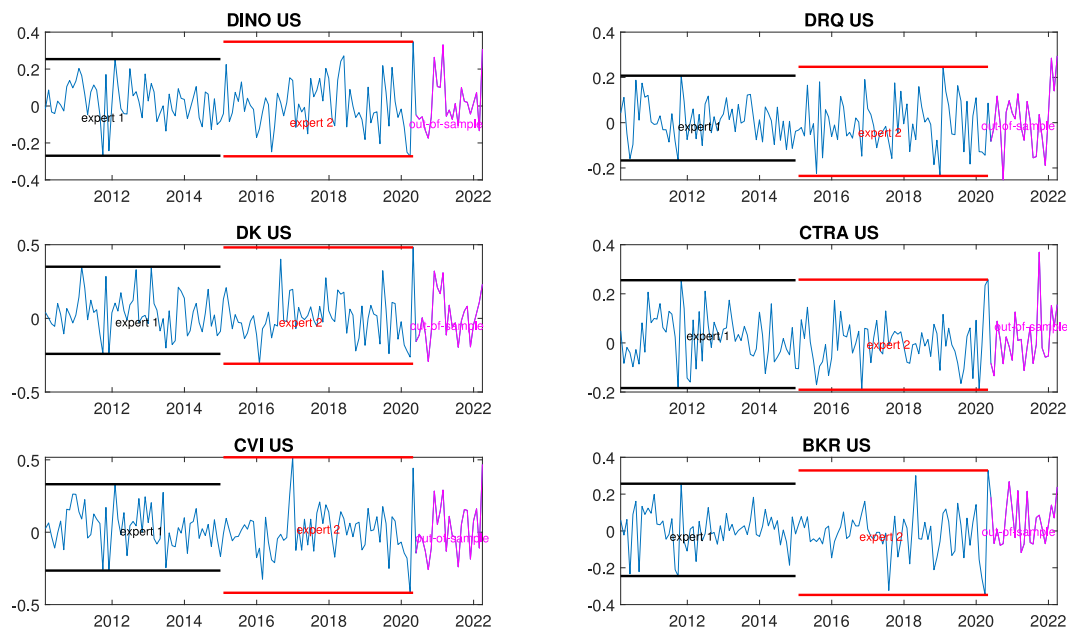


Fig. 5. Asset's returns evolution over time for each component of the portfolio. The time intervals used by Experts 1 and 2 are indicated, respectively, in black and in red; the out-of-sample period corresponds to the pink line. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

for the dataset under analysis, the portfolios obtained from the robust approach consistently outperform those generated by the nominal mean-CVaR model, regardless of the level of  $\lambda$  or the rolling window length. Moreover, the observed differences between the WCVaR and NCVaR Sharpe ratios are, in most cases, statistically significant, highlighting the advantages of incorporating robustness in portfolio selection.

Furthermore, portfolios constructed using as in-sample period P2 seem to be preferable to those obtained from sub-sample P1. This might be explained by observing that the time series of returns in the sub-sample P2 and in the out-of-sample period seem to have similar behavior. Thus, we argue that the differences in the out-of-sample results can be attributed to differences in the two in-sample periods P1 and P2 used for the parameter estimation. In particular, both the nominal and the worst-CVaR portfolio perform better when the market conditions of the in-sample and out-of-sample periods are nearly the same.

### 3.2. Robust mean-CVaR portfolio performance with no distributional assumption

In the previous subsection we addressed worst-CVaR optimization problem in the special case where the portfolio returns are assumed to be normally distributed and their moments are only known to belong to a given set. In the following we analyze the robust mean-CVaR portfolio in absence of any distributional assumption.

The dataset is composed of daily returns from the energy sector of the S&P500 index and spans from January 2, 2001, to April 12, 2022 (for a total of 5551 daily observations)<sup>15</sup> The portfolio is comprised only of the assets included in the S&P500 energy sector at least from 2001, consisting of the 14 assets reported in Table 6.<sup>16</sup> The fundamental statistics of all the components are summarized in Table 6. We observe that all the assets are characterized by skewness different from 0 and kurtosis higher than 3. This suggests that returns cannot be considered

normally distributed, as is also confirmed by the results of the Jarque-Bera test. In fact,  $p$ -values indicate that null hypothesis of normality is rejected at the 5% significance level.

Consequently, we do not impose any distributional assumption and carry out an out-of-sample empirical analysis using the method described in Section 2.2. Index prices and returns dynamics for the period under analysis are illustrated in Fig. 6. Similarly as in [15,16], we consider multiple experts having their own prior on the portfolio distribution.

In particular, we consider four experts, who build their priors using different sub-samples. The priors of all the experts are used to construct the robust portfolio according to the procedure described in Section 2.2. The first expert uses the period that goes from January 2001 to December 2006. As reported in [41], energy commodity prices rose steadily from the late 1990s until the financial crisis in 2007, in part due to the strong demand from China and other emerging markets. Therefore we can consider this as a bull market period.<sup>17</sup>

The second expert uses data from January 2007 to December 2012. This time window includes the Global Financial Crisis, so that it could be considered as a bear market period. It also includes, partly, a period of global economic recovery from the financial crisis, commonly set between 2011 and 2014 and characterized by an increase in the index price (see Fig. 6).

The third expert builds his prior based on daily observations from January 2013 to December, 2018. This period includes the largest oil price drop that took place from 2014 to 2016 and determined, on one hand, by the shale oil revolution, especially in North America, on the other hand, by the reluctance of the Organization of Petroleum Exporting Countries to stabilize the oil market and its decision against cutting production at a 2014 meeting in Vienna.

The fourth expert builds his priors using the period starting from January 2017.

The robust out-of-sample analysis is driven through a generalized buy-and-hold strategy which considers the forecasts of all experts (see below for the details). The out-of-sample period ranges from January 1, 2019-April 12, 2022. We consider four different rolling windows

<sup>15</sup> The dataset has been downloaded from Bloomberg.

<sup>16</sup> The Bloomberg ticker for the energy sector of the S&P500 is S5ENRS Index.

<sup>17</sup> Financial crises are often preceded by a widespread price bubble, and the global financial crisis of 2007–2009 was no exception.

**Table 5**

Out-of-sample statistics, for each expert, for the nominal mean-CVaR portfolio (NCVaR) and the robust mean-CVaR portfolio (WCVaR). Mean, standard deviation and Sharpe ratio are calculated on annual basis. For each expert, each  $\lambda$  and rolling window strategy we indicate in bold the best result and in case of Sharpe ratio when the difference between the two portfolios is statistically significant we indicate it with  $\star$ . With *NA* are indicated the portfolios where the SH and ECVaR are not calculated because the portfolio excess return on the risk free is negative.

Rolling window: 24 months in-sample & 2 months out-of-sample																
	Expert 1								Expert 2							
	$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$		$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$	
	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR
$\mu$	-0.106	<b>-0.075</b>	-0.013	<b>0.009</b>	0.032	<b>0.036</b>	0.039	<b>0.041</b>	0.258	<b>0.276</b>	0.305	<b>0.310</b>	0.315	<b>0.317</b>	0.321	<b>0.322</b>
$\sigma$	0.399	<b>0.376</b>	0.352	<b>0.347</b>	0.343	<b>0.339</b>	0.333	<b>0.332</b>	0.308	<b>0.307</b>	0.327	<b>0.330</b>	<b>0.334</b>	0.336	<b>0.340</b>	0.341
Skew	0.226	<b>0.228</b>	<b>0.061</b>	0.012	<b>-0.039</b>	-0.079	<b>-0.231</b>	-0.263	<b>0.002</b>	-0.003	0.402	<b>0.486</b>	0.575	<b>0.615</b>	0.695	<b>0.713</b>
Kurt	<b>2.624</b>	2.629	2.795	<b>2.794</b>	<b>2.781</b>	2.795	<b>3.011</b>	3.061	2.240	<b>2.044</b>	<b>2.451</b>	2.605	<b>2.784</b>	2.868	<b>3.045</b>	3.086
SH	NA	NA	NA	<b>0.026*</b>	0.092	<b>0.105*</b>	0.119	<b>0.125*</b>	0.839	<b>0.898*</b>	0.931	<b>0.939</b>	0.944	<b>0.945</b>	<b>0.946</b>	0.945
ECVaR	NA	NA	NA	<b>0.004</b>	0.012	<b>0.014</b>	0.015	<b>0.017</b>	0.147	<b>0.190</b>	0.173	<b>0.214</b>	0.179	<b>0.219</b>	0.183	<b>0.222</b>
$\Omega$	0.815	<b>0.860</b>	0.971	<b>1.020</b>	1.073	<b>1.083</b>	1.096	<b>1.102</b>	1.797	<b>1.859</b>	2.000	<b>2.035</b>	2.054	<b>2.061</b>	2.073	<b>2.074</b>
Rolling window: 24 months in-sample & 1 month out-of-sample																
	Expert 1								Expert 2							
	$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$		$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$	
	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR
$\mu$	-0.078	<b>-0.052</b>	-0.001	<b>0.015</b>	0.034	<b>0.037</b>	<b>0.026</b>	0.026	0.173	<b>0.220</b>	0.278	<b>0.287</b>	0.296	<b>0.299</b>	0.307	<b>0.309</b>
$\sigma$	0.420	<b>0.398</b>	0.375	<b>0.370</b>	0.366	<b>0.361</b>	0.356	<b>0.356</b>	<b>0.316</b>	0.318	<b>0.338</b>	0.340	<b>0.343</b>	0.344	<b>0.347</b>	0.348
Skew	<b>0.017</b>	0.050	<b>-0.060</b>	-0.085	<b>-0.114</b>	-0.154	<b>-0.256</b>	-0.269	<b>0.220</b>	0.145	0.441	<b>0.510</b>	0.588	<b>0.623</b>	0.694	<b>0.710</b>
Kurt	<b>2.151</b>	2.186	<b>2.402</b>	2.429	<b>2.445</b>	2.459	<b>2.597</b>	2.631	2.225	<b>1.941</b>	<b>2.344</b>	2.505	<b>2.698</b>	2.790	2.976	3.018
SH	NA	NA	NA	<b>0.042*</b>	0.092	<b>0.102*</b>	0.073	<b>0.073</b>	0.549	<b>0.690*</b>	0.824	<b>0.845*</b>	0.863	<b>0.870*</b>	0.884	<b>0.887</b>
ECVaR	NA	NA	0.000	<b>0.006</b>	0.012	<b>0.015</b>	0.010	<b>0.010</b>	0.099	<b>0.151</b>	0.158	<b>0.198</b>	0.168	<b>0.206</b>	0.175	<b>0.213</b>
$\Omega$	0.874	<b>0.910</b>	0.998	<b>1.032</b>	1.071	<b>1.078</b>	1.055	<b>1.053</b>	1.492	<b>1.632</b>	1.852	<b>1.895</b>	1.924	<b>1.935</b>	1.959	<b>1.963</b>
Rolling window: 12 months in-sample & 2 months out-of-sample																
	Expert 1								Expert 2							
	$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$		$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$	
	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR
$\mu$	<b>0.085</b>	0.039	<b>-0.019</b>	-0.037	<b>-0.060</b>	-0.071	<b>-0.109</b>	-0.120	-0.034	<b>0.022</b>	0.069	<b>0.077</b>	0.087	<b>0.092</b>	0.104	<b>0.107</b>
$\sigma$	0.466	<b>0.434</b>	0.401	<b>0.394</b>	0.388	<b>0.386</b>	0.381	<b>0.380</b>	0.335	<b>0.327</b>	0.325	<b>0.325</b>	<b>0.326</b>	0.327	<b>0.329</b>	0.330
Skew	<b>0.414</b>	0.353	<b>0.127</b>	0.027	<b>-0.021</b>	-0.045	<b>-0.078</b>	-0.081	-0.183	<b>-0.004</b>	0.151	<b>0.182</b>	0.217	<b>0.231</b>	0.255	<b>0.260</b>
Kurt	<b>2.544</b>	2.559	2.452	<b>2.309</b>	<b>2.282</b>	2.291	<b>2.351</b>	2.366	2.262	<b>2.189</b>	<b>2.270</b>	2.293	<b>2.315</b>	2.319	2.313	<b>2.310</b>
SH	<b>0.182*</b>	0.090	NA	NA	NA	NA	NA	NA	NA	<b>0.068*</b>	0.213	<b>0.238*</b>	0.266	<b>0.282*</b>	0.316	<b>0.325*</b>
ECVaR	<b>0.030</b>	0.015	NA	NA	NA	NA	NA	NA	NA	<b>0.011</b>	0.031	<b>0.037</b>	0.038	<b>0.044</b>	0.046	<b>0.051</b>
$\Omega$	<b>1.143</b>	1.067	0.966	<b>0.968</b>	0.893	<b>0.872</b>	0.810	<b>0.831</b>	0.931	<b>1.050</b>	1.165	<b>1.187</b>	1.212	<b>1.227</b>	1.258	<b>1.266</b>
Rolling window: 12 months in-sample & 1 month out-of-sample																
	Expert 1								Expert 2							
	$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$		$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$	
	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR
$\mu$	-0.072	-0.053	-0.065	-0.070	-0.087	-0.095	-0.119	-0.123	-0.073	-0.024	0.032	0.044	0.062	0.072	0.094	0.099
$\sigma$	0.430	0.415	0.392	0.384	0.378	0.376	0.371	0.368	0.334	0.326	0.325	0.326	0.327	0.328	0.331	0.331
Skew	0.537	0.498	0.242	0.134	0.078	0.049	-0.008	-0.022	-0.175	0.022	0.184	0.209	0.228	0.233	0.237	0.238
Kurt	3.130	3.030	2.681	2.514	2.478	2.491	2.590	2.635	2.186	2.147	2.258	2.284	2.301	2.297	2.274	2.268
SH	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.097	<b>0.135*</b>	0.189	<b>0.220*</b>	0.283	<b>0.298*</b>
ECVaR	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.014	<b>0.021</b>	0.027	<b>0.034</b>	0.042	<b>0.047</b>
$\Omega$	0.882	<b>0.908</b>	0.880	<b>0.882</b>	0.837	<b>0.842</b>	0.781	<b>0.783</b>	0.860	<b>0.951</b>	1.070	<b>1.099</b>	1.142	<b>1.167</b>	1.221	<b>1.234</b>

**Table 6**

Descriptive statistics for each component of the S5ENRS Index. Mean and standard deviation are reported on annual basis.

Ticker	$\mu$	$\sigma$	Skewness	Kurtosis	p-value	JB-test
OXY UN	0.158	0.405	-0.78	50.97	0.001	532704
OKE UN	0.150	0.354	-0.20	46.93	0.001	446418
CVX UN	0.100	0.272	0.11	25.29	0.001	114974
COP UN	0.122	0.327	-0.01	16.89	0.001	44608
XOM UN	0.063	0.256	0.24	14.02	0.001	28159
PXD UN	0.207	0.427	-0.34	15.11	0.001	34020
VLO UN	0.195	0.407	0.11	12.84	0.001	22404
SLB UN	0.072	0.373	-0.22	12.53	0.001	21036
HES UN	0.155	0.411	-0.34	14.78	0.001	32196
MRO UN	0.148	0.433	-0.41	25.28	0.001	115004
WMB UN	0.160	0.549	1.86	110.67	0.001	2684386
CTRA UN	0.191	0.408	0.27	8.99	0.001	8379
EOG UN	0.177	0.391	-0.14	13.11	0.001	23665
HAL UN	0.139	0.449	-0.71	25.84	0.001	121119

strategies with in-sample length of 24 and 36 months and out-of-sample length of 2 and 4 months.

In particular, the out-of-sample portfolios are determined through the usual buy-and-old strategy (cf. Section 3.1). The nominal mean-CVaR model is implemented only through the fourth experts' forecast who represents the standard approach in a buy-and-hold procedure.

We now describe the generalized buy-and-hold strategy implemented to obtain the robust portfolio performance. To fix ideas, we consider a rolling window with 24 months in-sample and 4 months out-of-sample; in Fig. 7 we represent for each expert the first five shifts of the rolling windows (blue, red, yellow and violet lines), and, in addition, the shifts of the out-of-sample period (green line), which is common to all experts.

For instance, the first rolling window of the in-sample periods are:

- January 2, 2001-December 31, 2002 for Expert 1,
- January 2, 2007-December 31, 2008 for Expert 2,
- January 2, 2013-December 31, 2014 for Expert 3,
- January 2, 2017-December 31, 2018 for Expert 4.

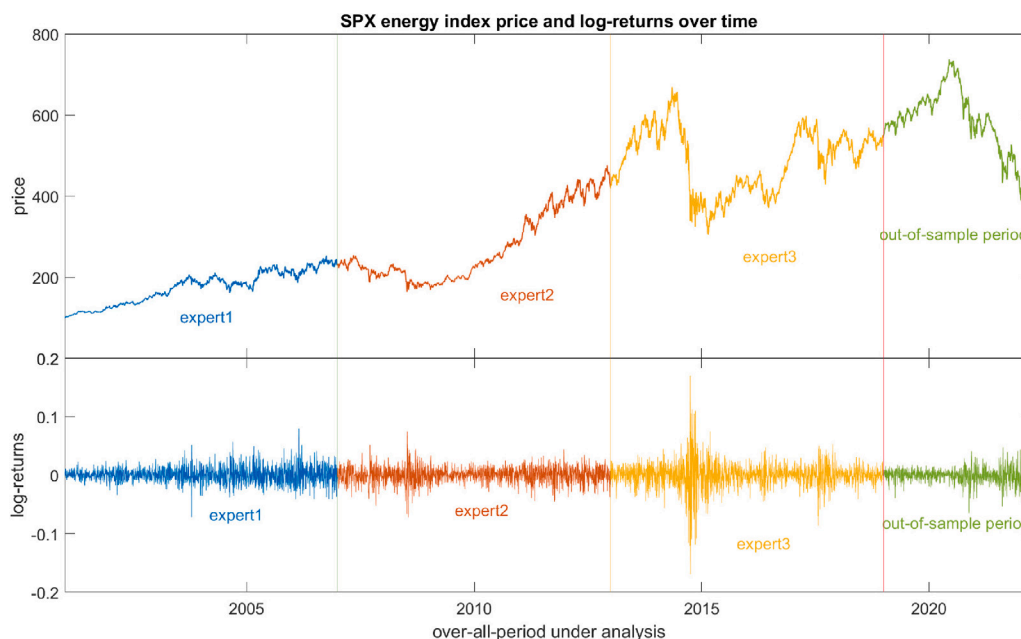


Fig. 6. Daily prices and returns of the S5ENRS index within the period January 1, 2001 - April 12, 2022. The sample is divided into four periods: the first three correspond to the sub-samples used by the first three experts to construct their prior, while the last one indicates the out-of-sample period. The sub-sample used by the fourth expert is not indicated here.

The returns within these time periods are used to obtain multiple estimations of the sample mean and covariance matrix, and, later, to compute the worst-case optimal weights through the method described in Section 2.2. Next, these weights are kept constant from January 1, 2019 to April 30, 2019 to calculate the portfolio's performance in the first out-of-sample window. Successively, the priors of each expert are updated by shifting the samples of four months to determine the optimal weights in the second rolling window. Subsequently, the performances of the worst-CVaR portfolio for the second out-of-sample window, which ranges from May 1, 2019 to August 30, 2019 are calculated keeping the optimal weights constants, as done previously. The procedure is repeated rolling over until the end of the time horizon. In this way, we obtain the out-of-sample portfolio returns and performances covering the period that ranges from January 1, 2019 to April 12, 2022.

Expert 4 obtains the portfolio's out-of-sample performances by implementing the standard buy-and-hold strategy,<sup>18</sup> see Fig. 7. For more details about this procedure see, among others, Cesarone et al. [42] and Elsayed et al. [43].

Fig. 8 illustrates the realized out-of-sample portfolio performances for the robust investor, denoted by WCVaR, and those corresponding to the nominal (historical) investor,<sup>19</sup> indicated as NCVaR. Similarly to the analysis driven in Section 3.1, we examine different levels of scalarization for the mean-CVaR objective function ( $\lambda = 0.2, 0.4, 0.6, 0.8$ ). Fig. 8 indicates that both the NCVaR and WCVaR energy portfolio performances fluctuate considerably especially around February–March 2020, resulting in a sort of instability of the optimal energy portfolios performance. This is probably due to the negative impact of the health crisis on major stock and commodity markets (see [44] for a discussion about the changes in market efficiency during the epidemic). Observing the out-of-sample performances not always is clear which strategy performs better.

<sup>18</sup> We highlight that, for a specified expert and rolling window, the in-sample and the out-of-sample periods do not overlap.

<sup>19</sup> We recall that in this case the investor considers only the prior of Expert 4.

Table 7 reports the main out-of-sample statistics, i.e., the first four moments, Sharpe ratio, Excess return on CVaR and Omega ratio, of the worst and nominal mean-CVaR energy portfolios for different levels of  $\lambda$  and for the four rolling-window strategies considered. As explained in Section 3.1.2 these measures are used to assess and compare the magnitude of potential gains/losses that can be attained by investors adopting the models considered in this paper.

For each statistic reported in Table 7, we highlight the best result in bold. Our analysis reveals significant differences between the worst and nominal portfolios, regardless of the chosen performance measure. When considering the first four moments, the comparison between WCVaR and NCVaR is not always straightforward. In some cases, the WCVaR portfolio exhibits a higher mean return but also higher standard deviation, skewness, and kurtosis than the NCVaR portfolio. Consequently, the preferred portfolio may depend on an investor's individual risk appetite. However, when focusing on the out-of-sample risk-adjusted performance measures (Sharpe ratio, Expected CVaR, and Omega ratio), we find that the robust mean-CVaR portfolio consistently outperforms the nominal approach in most cases, demonstrating its potential advantages in portfolio selection. It is important to note that these conclusions are drawn based on the dataset under analysis.

We also tested whether the Sharpe ratios of the two methods under analysis are statistically distinguishable, by employing the circular block bootstrap method proposed in [39] with a block size of 6. The star symbol is added to values where the difference between the worst-Sharpe ratio and the nominal one is significant at a confidence level of 5%. In these cases, it turns out that the difference between the two SH is statistically significant. This holds true for almost all the rolling-windows strategies and almost all the levels of  $\lambda$  considered.

To summarize, from the out-of-sample results obtained we infer that the robust optimization can represent a promising solution during periods of crises.

#### 4. Conclusions

The main contributions of this article are threefold. First, we have provided an analytical representation of the robust mean-CVaR optimal

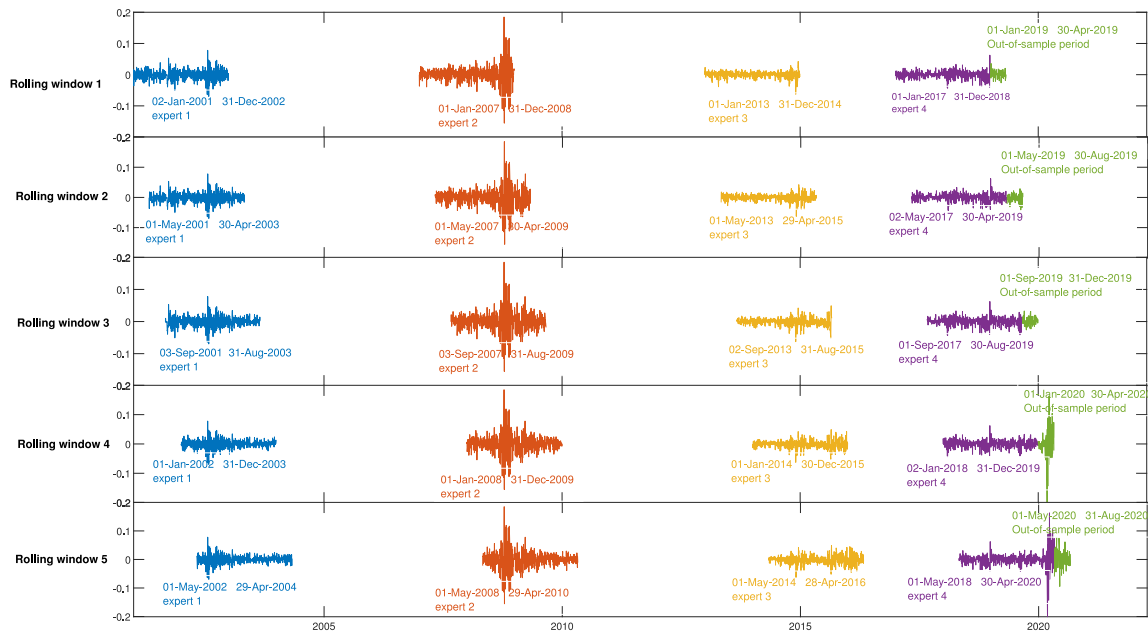


Fig. 7. Experts' prior subsamples over the first five rolling windows (blue, red, yellow, violet lines) and corresponding out-of-sample rolling window in green. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



Fig. 8. Out of sample performances for  $\lambda = 0.2, 0.4, 0.6, 0.8$ , in case of rolling window strategy of 24 months in-sample and 4 months out of sample.

portfolios under normality assumptions of the return distribution. Second, we have constructed an alternative robust portfolio model based on the recommendations of a group of expert advisors. Third, we have illustrated the applicability of these models using two energy portfolios. The first assumes normality in asset returns, with uncertainty related to parameter estimation. The second considers uncertainty in the return distribution, modeled through different expert priors, with the robust portfolio constructed by aggregating these priors appropriately.

The empirical results for the datasets under analysis suggest that the WCVaR approach can provide improvements over the nominal CVaR model. However, the comparison in this study is limited to the nominal model and does not extend to other robust portfolio optimization frameworks. Investigating these alternative approaches remains

an important direction for future research, as a systematic empirical evaluation against other robust methods would offer deeper insights into the advantages and trade-offs of our approach.

Future research could also explore additional bi-objective robust optimization approaches, such as best-case or regret-based CVaR models [16,45]. Moreover, broader mean-risk models could be examined by incorporating alternative risk measures, such as entropic risk or entropic VaR [46] and references therein. Expanding the empirical analysis to compare these methodologies across different datasets and market conditions would provide deeper insights into their practical implications for portfolio optimization.

**Table 7**

Out-of-sample statistics for the two methods under analysis, for different levels of  $\lambda$ , and for different rolling-windows strategy. Mean, standard deviation and Sharpe ratio are reported on annual basis. For each statistic the best result is indicated in bold. When the difference between the Sharpe ratios of the NCVaR and WCVaR portfolios is statistically significant is indicated with  $\star$ .

Rolling window: in-sample = 24 & out_of_sample = 2									
	$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$		
	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	
$\mu$	0.088	<b>0.099</b>	0.084	<b>0.111</b>	0.089	<b>0.103</b>	0.107	<b>0.108</b>	
$\sigma$	0.403	<b>0.362</b>	0.384	<b>0.378</b>	<b>0.367</b>	0.381	0.352	<b>0.351</b>	
Skew	-1.434	<b>-0.538</b>	-0.998	<b>-0.660</b>	<b>-0.635</b>	-0.787	<b>-0.276</b>	-0.809	
Kurt	25.59	<b>11.69</b>	19.29	<b>13.62</b>	15.28	<b>14.62</b>	<b>11.79</b>	15.12	
SH	0.217	<b>0.272*</b>	0.220	<b>0.294*</b>	0.243	<b>0.270*</b>	0.305	<b>0.307</b>	
ECVaR	0.006	<b>0.007</b>	0.006	<b>0.008</b>	0.007	<b>0.007</b>	0.009	<b>0.009</b>	
$\Omega$	1.046	<b>1.053</b>	1.044	<b>1.058</b>	1.048	<b>1.054</b>	1.055	<b>1.057</b>	
Rolling window: 24 months in-sample & 4 months out_of_sample = 4									
	$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$		
	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	
$\mu$	0.077	<b>0.134</b>	0.102	<b>0.120</b>	0.107	<b>0.128</b>	0.105	<b>0.129</b>	
$\sigma$	0.383	<b>0.363</b>	<b>0.350</b>	0.385	<b>0.349</b>	0.387	<b>0.348</b>	0.386	
Skew	-1.143	<b>-0.229</b>	<b>-0.266</b>	-0.611	<b>-0.173</b>	-0.605	<b>-0.155</b>	-0.591	
Kurt	19.87	<b>10.55</b>	<b>11.42</b>	14.06	<b>10.96</b>	14.17	<b>10.74</b>	14.20	
SH	0.202	<b>0.368*</b>	0.291	<b>0.310*</b>	0.306	<b>0.329*</b>	0.301	<b>0.335*</b>	
ECVaR	0.006	<b>0.010</b>	0.008	<b>0.009</b>	0.009	<b>0.009</b>	0.009	<b>0.009</b>	
$\Omega$	1.041	<b>1.072</b>	1.055	<b>1.062</b>	1.058	<b>1.066</b>	1.057	<b>1.068</b>	
Rolling window: 36 months in-sample & 2 months out-of-sample									
	$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$		
	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	
$\mu$	0.072	<b>0.072</b>	<b>0.075</b>	0.075	0.069	<b>0.083</b>	0.080	<b>0.098</b>	
$\sigma$	<b>0.359</b>	0.360	0.356	<b>0.355</b>	<b>0.354</b>	0.379	<b>0.354</b>	0.383	
Skew	<b>-0.861</b>	-1.022	<b>-0.769</b>	-0.877	<b>-0.725</b>	-0.793	<b>-0.670</b>	-0.686	
Kurt	<b>16.58</b>	18.42	<b>16.36</b>	16.40	15.86	<b>15.74</b>	<b>15.58</b>	15.70	
SH	0.201	<b>0.201</b>	0.212	<b>0.212</b>	0.195	<b>0.220</b>	0.225	<b>0.256</b>	
ECVaR	0.006	<b>0.006</b>	0.005	<b>0.005</b>	0.005	<b>0.006</b>	0.006	<b>0.007</b>	
$\Omega$	1.040	<b>1.042</b>	1.042	<b>1.044</b>	1.038	<b>1.045</b>	1.044	<b>1.052</b>	
Rolling window: 36 months in-sample & 4 months out_of_sample									
	$\lambda = 0.2$		$\lambda = 0.4$		$\lambda = 0.6$		$\lambda = 0.8$		
	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	NCVaR	WCVaR	
$\mu$	0.116	<b>0.125</b>	0.110	<b>0.130</b>	0.108	<b>0.132</b>	0.110	<b>0.133</b>	
$\sigma$	<b>0.350</b>	0.365	<b>0.346</b>	0.383	<b>0.346</b>	0.384	<b>0.346</b>	0.384	
Skew	<b>-0.607</b>	-0.783	<b>-0.527</b>	-0.707	<b>-0.488</b>	-0.692	<b>-0.453</b>	-0.684	
Kurt	<b>13.36</b>	16.51	<b>12.22</b>	15.41	<b>11.98</b>	15.30	<b>11.93</b>	15.18	
SH	0.332	<b>0.341*</b>	0.319	<b>0.339*</b>	0.312	<b>0.343*</b>	0.318	<b>0.348*</b>	
ECVaR	0.009	<b>0.010</b>	0.009	<b>0.009</b>	0.009	<b>0.009</b>	0.009	<b>0.010</b>	
$\Omega$	1.065	<b>1.067</b>	1.062	<b>1.069</b>	1.060	<b>1.070</b>	1.061	<b>1.071</b>	

**CRedit authorship contribution statement**

**Asmerilda Hitaj:** Formal analysis, Project administration, Writing – review & editing, Conceptualization, Resources, Methodology, Software, Writing – original draft, Investigation. **Elisa Mastrogiacomo:** Conceptualization, Writing – review & editing, Investigation, Writing – original draft, Methodology, Formal analysis. **Elena Molho:** Writing – original draft, Resources, Writing – review & editing, Supervision, Methodology, Conceptualization, Formal analysis.

**Declaration of competing interest**

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers’ bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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**Appendix**

*A.1. On the commutativity between robustification and scalarization*

In the following example we show that robustification and scalarization do not commute in general. Let us assume that the portfolio  $X$  consists of two positions only: a stock and a risk free asset. The (uncertain) return of the stock will be denoted by  $X$ , while the deterministic return of the risk-free asset will be denoted by  $r$ . To simplify the exposition we set the risk free return  $r$  equal to 0 so that

$$\mu_X = \mathbb{E}[X] = (\hat{\mu}, 0) \quad \text{and} \quad \Sigma_X = \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & 0 \end{pmatrix},$$

where  $\hat{\mu} = \mathbb{E}[X]$  and  $\hat{\sigma} = \sqrt{\mathbb{E}[(X - \hat{\mu})^2]}$ . We consider the feasible set

$$\mathcal{W} = \{(w_1, w_2) \in \mathbb{R}^2 : w_1, w_2 \geq 0, w_1 + w_2 = 1\}$$

and uncertainty on  $\mu_X, \Sigma_X \mathcal{U}$  of the following form

$$\mathcal{U} = \mathcal{U}_\delta(\hat{\mu}, \hat{\sigma}) = \{(\mu, \sigma) \in \mathbb{R} \times [0, +\infty) : (\mu - \hat{\mu})^2 + (\sigma - \hat{\sigma})^2 \leq \delta^2\}$$

for some positive  $\delta$ .

We note that the uncertainty set is an ellipsoidal around the center  $\hat{\mu} \in \mathbb{R}, \hat{\sigma} > 0$ , i.w., we leave out from the uncertainty set the risk-free component. In addition, we observe that the feasible set can be rewritten in the following way:

$$\mathcal{W} = \{(w, 1 - w) \in \mathbb{R}^2 : w \in [0, 1]\},$$

so that, for fixed  $w \in \mathcal{W}$ , the portfolio return at maturity is given by  $wX + (1 - w)r$  with

$$\mathbb{E}[wX + (1 - w)r] = w\mathbb{E}[X] = w\hat{\mu}$$

$$\sigma(wX + (1 - w)r) = w\hat{\sigma}.$$

We introduce the following linear bi-objective problem with

$$\text{WEff min}_{w \in [0,1]} \begin{pmatrix} \text{CVaR}_\alpha(w, \hat{\mu}, \hat{\sigma}) \\ -m(w, \hat{\mu}, \hat{\sigma}) \end{pmatrix} = \text{WEff min}_{w \in [0,1]} \begin{pmatrix} -\hat{\mu}w + \hat{\sigma}\gamma_\alpha w \\ -\hat{\mu}w \end{pmatrix}.$$

We are going to show that the two approaches mentioned above (i.e., i. first scalarization and then robustification and ii. scalarization of the robustified problem) do not coincide in general. The first approach leads to consider, for  $\lambda \in [0, 1]$ , the family of problems

$$\begin{aligned} & \min_{w \in [0,1]} \max_{\mathcal{U}_\delta(\hat{\mu}, \hat{\sigma})} \lambda \text{CVaR}_\alpha(w, \mu, \sigma) + (1 - \lambda)(-m(w, \mu, \sigma)) \\ & \min_{w \in [0,1]} \max_{\mathcal{U}_\delta(\hat{\mu}, \hat{\sigma})} \lambda(-\mu w + \sigma\gamma_\alpha w) + (1 - \lambda)(-\mu w) \quad (\text{RS}) \\ & \min_{w \in [0,1]} \max_{\mathcal{U}_\delta(\hat{\mu}, \hat{\sigma})} -\mu w + \lambda\sigma\gamma_\alpha w. \end{aligned}$$

Using standard methods for constrained optimization problems we can see that the inner maximization problem has the following solution

$$\mu^* = -\frac{\delta}{\sqrt{1 + \lambda^2\gamma_\alpha^2}} + \hat{\mu} \quad \sigma^* = \frac{\delta\lambda\gamma_\alpha}{\sqrt{1 + \lambda^2\gamma_\alpha^2}} + \hat{\sigma}$$

with maximum value

$$\begin{aligned} -\mu^*w + \lambda\sigma^*\gamma_\alpha w &= -\left(-\frac{\delta}{\sqrt{1 + \lambda^2\gamma_\alpha^2}} + \hat{\mu}\right)w + \lambda\left(\frac{\delta\lambda\gamma_\alpha}{\sqrt{1 + \lambda^2\gamma_\alpha^2}} + \hat{\sigma}\right)\gamma_\alpha w \\ &= \left(\delta\sqrt{1 + \lambda^2\gamma_\alpha^2} + \lambda\gamma_\alpha\hat{\sigma} - \hat{\mu}\right)w. \end{aligned}$$

The outer minimization problem now becomes

$$\min_{w \in [0,1]} \left(\delta\sqrt{1 + \lambda^2\gamma_\alpha^2} + \lambda\gamma_\alpha\hat{\sigma} - \hat{\mu}\right)w;$$

since the objective function above is linear, the solution of this problem is

$$w^{*,rs} = \begin{cases} 0, & \text{if } \delta\sqrt{1 + \lambda^2\gamma_\alpha^2} + \lambda\gamma_\alpha\hat{\sigma} - \hat{\mu} \geq 0 \\ 1, & \text{otherwise.} \end{cases} \quad (13)$$

The second approach consists in solving, for  $\lambda \in [0, 1]$ , the family of problems

$$\min_{w \in [0,1]} \left[ \lambda \max_{\mathcal{U}_\delta(\hat{\mu}, \hat{\sigma})} \text{CVaR}_\alpha(w, \mu, \sigma) + (1 - \lambda) \max_{\mathcal{U}_\delta(\hat{\mu}, \hat{\sigma})} (-m(w, \mu, \sigma)) \right]. \quad (\text{SR})$$

We analyze separately the two maximization problems

$$\max_{\mathcal{U}_\delta(\hat{\mu}, \hat{\sigma})} \text{CVaR}_\alpha(w, \mu, \sigma) \quad \text{and} \quad \max_{\mathcal{U}_\delta(\hat{\mu}, \hat{\sigma})} (-m(w, \mu, \sigma)).$$

For the first one, using again standard methods of constrained optimization, we find that

$$\mu^{*,1} = -\frac{\delta}{\sqrt{1 + \gamma_\alpha^2}} + \hat{\mu} \quad \sigma^{*,1} = \frac{\delta\gamma_\alpha}{\sqrt{1 + \gamma_\alpha^2}} + \hat{\sigma}$$

and the maximum value is

$$\text{CVaR}_\alpha(w, \mu^{*,1}, \sigma^{*,1}) = w \left[ \delta\sqrt{1 + \gamma_\alpha^2} + \hat{\sigma}\gamma_\alpha - \hat{\mu} \right].$$

Considering the robustification of the portfolio mean, we obtain the following solution:

$$\mu^{*,2} = \hat{\mu} - \delta \quad \sigma^{*,2} = \hat{\sigma}$$

and the maximum value is

$$-m(w, \mu^{*,2}, \sigma^{*,2}) = -(\hat{\mu} - \delta)w.$$

The outer minimization in (SR) becomes

$$\begin{aligned} & \min_{w \in [0,1]} \lambda \text{CVaR}_\alpha(w, \mu^{*,1}, \sigma^{*,1}) + (1 - \lambda)(-m(w, \mu^{*,2}, \sigma^{*,2})) \\ &= \min_{w \in [0,1]} w \left[ \delta + \lambda\delta\sqrt{1 + \gamma_\alpha^2} - \delta\lambda + \lambda\gamma_\alpha\hat{\sigma} - \hat{\mu} \right]. \end{aligned}$$

The solution is

$$w^{*,sr} = \begin{cases} 0, & \text{if } \delta + \delta\lambda\sqrt{1 + \gamma_\alpha^2} - \delta\lambda + \lambda\gamma_\alpha\hat{\sigma} - \hat{\mu} \geq 0 \\ 1, & \text{otherwise.} \end{cases} \quad (14)$$

By comparing the solutions in (13) and (14), we can see that  $w^{*,rs} = 1$  and  $w^{*,sr} = 0$  whenever

$$\delta\sqrt{1 + \lambda^2\gamma_\alpha^2} + \lambda\gamma_\alpha\hat{\sigma} < \hat{\mu} < \delta \left( 1 + \lambda\sqrt{1 + \gamma_\alpha^2} - \lambda \right) + \lambda\gamma_\alpha\hat{\sigma},$$

or, equivalently,

$$\sqrt{1 + \lambda^2\gamma_\alpha^2} < \frac{\hat{\mu} - \lambda\gamma_\alpha\hat{\sigma}}{\delta} < 1 + \lambda\sqrt{1 + \gamma_\alpha^2} - \lambda.$$

We observe that inequality

$$\sqrt{1 + \lambda^2\gamma_\alpha^2} < 1 + \lambda\sqrt{1 + \gamma_\alpha^2} - \lambda$$

is satisfied for any  $\lambda \in (0, 1)$ ,<sup>20</sup> since the functions

$$K_1(\lambda) := 1 + \lambda\sqrt{1 + \gamma_\alpha^2} - \lambda$$

$$K_2(\lambda) := \sqrt{1 + \lambda^2\gamma_\alpha^2}$$

are strictly increasing with  $K_1(0) = K_2(0) = 1$  and  $K_2'(\lambda) < K_1'(\lambda)$  for all  $\lambda \in (0, 1)$ . In fact, for every  $\lambda \in (0, 1)$

$$K_2'(\lambda) = \frac{\lambda\gamma_\alpha^2}{\sqrt{1 + \lambda^2\gamma_\alpha^2}}$$

$$K_1'(\lambda) = \sqrt{1 + \lambda^2\gamma_\alpha^2} - 1$$

and

$$\frac{\lambda\gamma_\alpha^2}{\sqrt{1 + \lambda^2\gamma_\alpha^2}} < \sqrt{1 + \lambda^2\gamma_\alpha^2} - 1 \quad \Leftrightarrow \quad 0 < \gamma^2(1 - \lambda)^2,$$

which is always satisfied for  $\lambda \in (0, 1)$ . This shows that for many possible choices of parameters  $\hat{\mu}, \hat{\sigma}, \delta$  and  $\gamma_\alpha$  scalarization and robustification do not commute.

### A.2. Proof of proposition 3

We start rewriting the uncertainty set in the following way:

$$\mathcal{U}_\delta(\hat{\mu}, \hat{\Sigma}) = \bigcup_{\kappa \in [0,1]} \mathcal{U}_{\sqrt{\kappa}\delta}(\hat{\mu}) \times \mathcal{U}_{\sqrt{1-\kappa}\delta}(\hat{\Sigma})$$

where

$$\mathcal{U}_{\sqrt{\kappa}\delta}(\hat{\mu}) := \left\{ \mu \in \mathbb{R}^d : \|(A^{-\frac{1}{2}})^T(\mu - \hat{\mu})\|^2 \leq \kappa\delta^2 \right\}$$

<sup>20</sup> Note that for  $\lambda = 0$  or  $\lambda = 1$  problems (RS) and (SR) coincide.

$$\mathcal{U}_{\sqrt{1-\kappa\delta}}(\hat{\Sigma}) := \left\{ \Sigma \in \mathbb{S}^d : \|(M^{-\frac{1}{2}})^\top(\Sigma - \hat{\Sigma})M^{-\frac{1}{2}}\|_{\text{tr}}^2 \leq (1-\kappa)\delta^2 \right\}.$$

For fixed  $\mathbf{w} \in \mathcal{W}$ , the worst CVaR is obtained through the (multiple) maximization

$$\begin{aligned} & \max_{\mathbf{u} \in \mathcal{U}_\delta(\hat{\mu}, \hat{\Sigma})} \text{CVaR}_\alpha(\mathbf{w}, \mathbf{u}) \\ &= \max_{\kappa \in [0,1]} \max_{\Sigma \in \mathcal{U}_{\sqrt{1-\kappa\delta}}(\hat{\Sigma})} \max_{\mu \in \mathcal{U}_{\sqrt{\kappa\delta}}(\hat{\mu})} \left[ -\mathbf{w}^\top \mu + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\| \right]. \end{aligned}$$

We solve the inner maximization problem. To this purpose we observe that, by using the variable change  $z = (A^{-\frac{1}{2}})^\top(\mu - \hat{\mu})$ , the set  $\mathcal{U}_\delta(\hat{\mu})$  can be written as

$$\begin{aligned} \mathcal{U}_{\sqrt{\kappa\delta}}(\hat{\mu}) &= \left\{ \mu \in \mathbb{R}^d : \|(A^{-\frac{1}{2}})^\top(\mu - \hat{\mu})\|^2 \leq \kappa\delta^2 \right\} \\ &= \left\{ \mu \in \mathbb{R}^d : \mu = \delta\sqrt{\kappa}(A^{\frac{1}{2}})^\top z + \hat{\mu} \text{ for } z \in \mathbb{R}^d \text{ s.t. } \|z\|^2 \leq 1 \right\}. \end{aligned}$$

Indeed, if we set  $\mu = \delta\sqrt{\kappa}(A^{\frac{1}{2}})^\top z + \hat{\mu}$  for some  $z \in \mathbb{R}^d$  with  $\|z\|^2 \leq 1$ , we have

$$\begin{aligned} \|(A^{-\frac{1}{2}})^\top(\mu - \hat{\mu})\|^2 &= (\mu - \hat{\mu})^\top A^{-\frac{1}{2}}(A^{-\frac{1}{2}})^\top(\mu - \hat{\mu}) \\ &= \delta^2 \kappa z^\top z \leq \delta^2 \kappa, \end{aligned}$$

i.e.,  $\mu \in \mathcal{U}_{\sqrt{\kappa\delta}}(\hat{\mu})$ . Viceversa, if  $\mu \in \mathcal{U}_{\sqrt{\kappa\delta}}(\hat{\mu})$ , then  $z := \frac{1}{\delta\sqrt{\kappa}}(A^{-\frac{1}{2}})^\top(\mu - \hat{\mu})$ , it is easy to see that  $\|z\|^2 \leq 1$  and that  $\mu = \delta\sqrt{\kappa}(A^{\frac{1}{2}})^\top z + \hat{\mu}$ . Hence

$$\begin{aligned} & \max_{\mu \in \mathcal{U}_{\sqrt{\kappa\delta}}(\hat{\mu})} -\mathbf{w}^\top \mu + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\| \\ &= \max_{z \in \mathbb{R}^d, \|z\|^2 \leq 1} -\mathbf{w}^\top (\hat{\mu} + \sqrt{\kappa\delta}(A^{\frac{1}{2}})^\top z) + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\|. \end{aligned}$$

The above function is maximized at  $z^*(\mathbf{w}) = -\frac{A^{\frac{1}{2}} \mathbf{w}}{\|A^{\frac{1}{2}} \mathbf{w}\|}$  and the maximum is

$$\begin{aligned} & -\mathbf{w}^\top \left( \hat{\mu} + \sqrt{\kappa\delta}(A^{\frac{1}{2}})^\top z^*(\mathbf{w}) \right) + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\| \\ &= -\mathbf{w}^\top \left( \hat{\mu} - \sqrt{\kappa\delta}(A^{\frac{1}{2}})^\top \frac{A^{\frac{1}{2}} \mathbf{w}}{\|A^{\frac{1}{2}} \mathbf{w}\|} \right) + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\| \\ &= -\mathbf{w}^\top \hat{\mu} + \sqrt{\kappa\delta} \mathbf{w}^\top (A^{\frac{1}{2}})^\top \frac{A^{\frac{1}{2}} \mathbf{w}}{\|A^{\frac{1}{2}} \mathbf{w}\|} + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\| \\ &= -\mathbf{w}^\top \hat{\mu} + \sqrt{\kappa\delta} \frac{\|A^{\frac{1}{2}} \mathbf{w}\|^2}{\|A^{\frac{1}{2}} \mathbf{w}\|} + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\| \\ &= -\mathbf{w}^\top \hat{\mu} + \sqrt{\kappa\delta} \|A^{\frac{1}{2}} \mathbf{w}\| + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\|. \end{aligned} \tag{15}$$

Now let us consider the maximization with respect to  $\Sigma$ . We need to maximize the last expression in (15), i.e.,

$$\max_{\Sigma \in \mathcal{U}_{\sqrt{1-\kappa\delta}}(\hat{\Sigma})} -\mathbf{w}^\top \hat{\mu} + \sqrt{\kappa\delta} \|A^{\frac{1}{2}} \mathbf{w}\| + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\|.$$

We stress that, introducing the variable change  $C = (M^{-\frac{1}{2}})^\top(\Sigma - \hat{\Sigma})M^{-\frac{1}{2}}$ , we have

$$\Sigma \in \mathcal{U}_{\sqrt{1-\kappa\delta}}(\hat{\Sigma}) \iff \|C\|_{\text{tr}}^2 \leq (1-\kappa)\delta^2$$

and

$$\Sigma = (M^{\frac{1}{2}})^\top C M^{\frac{1}{2}} + \hat{\Sigma}.$$

Recalling that

$$\begin{aligned} \|\Sigma^{\frac{1}{2}} \mathbf{w}\| &= \sqrt{\langle \Sigma^{\frac{1}{2}} \mathbf{w}, \Sigma^{\frac{1}{2}} \mathbf{w} \rangle} = \sqrt{(\Sigma^{\frac{1}{2}} \mathbf{w})^\top (\Sigma^{\frac{1}{2}} \mathbf{w})} \\ &= \sqrt{\mathbf{w}^\top (\Sigma^{\frac{1}{2}})^\top \cdot (\Sigma^{\frac{1}{2}} \mathbf{w})} = \sqrt{\mathbf{w}^\top \Sigma \mathbf{w}} \end{aligned}$$

we obtain

$$\|\Sigma^{\frac{1}{2}} \mathbf{w}\| = \sqrt{\mathbf{w}^\top \Sigma \mathbf{w}} = \sqrt{\mathbf{w}^\top ((M^{\frac{1}{2}})^\top C M^{\frac{1}{2}} + \hat{\Sigma}) \mathbf{w}},$$

and, consequently,

$$\begin{aligned} & \max_{\Sigma \in \mathcal{U}_{\sqrt{1-\kappa\delta}}(\hat{\Sigma})} -\mathbf{w}^\top \hat{\mu} + \sqrt{\kappa\delta} \|A^{\frac{1}{2}} \mathbf{w}\| + \gamma_\alpha \|\Sigma^{\frac{1}{2}} \mathbf{w}\| \\ &= -\mathbf{w}^\top \hat{\mu} + \sqrt{\kappa\delta} \|A^{\frac{1}{2}} \mathbf{w}\| + \max_{C \in \mathbb{S}_+^d : \|C\|_{\text{tr}}^2 \leq (1-\kappa)\delta^2} \gamma_\alpha \sqrt{\mathbf{w}^\top ((M^{\frac{1}{2}})^\top C M^{\frac{1}{2}} + \hat{\Sigma}) \mathbf{w}}. \end{aligned}$$

Since  $\gamma_\alpha > 0$ , the function above is maximized at

$$C^* = \delta\sqrt{1-\kappa} \frac{M^{\frac{1}{2}} \mathbf{w}}{\|M^{\frac{1}{2}} \mathbf{w}\|} \cdot \frac{\mathbf{w}^\top (M^{\frac{1}{2}})^\top}{\|M^{\frac{1}{2}} \mathbf{w}\|}$$

and the maximum value is

$$\begin{aligned} & -\mathbf{w}^\top \hat{\mu} + \sqrt{\kappa\delta} \|A^{\frac{1}{2}} \mathbf{w}\| \\ &+ \gamma_\alpha \sqrt{\delta\sqrt{1-\kappa} \mathbf{w}^\top (M^{\frac{1}{2}})^\top \frac{M^{\frac{1}{2}} \mathbf{w}}{\|M^{\frac{1}{2}} \mathbf{w}\|} \cdot \frac{\mathbf{w}^\top (M^{\frac{1}{2}})^\top}{\|M^{\frac{1}{2}} \mathbf{w}\|} M^{\frac{1}{2}} \mathbf{w} + \mathbf{w}^\top \hat{\Sigma} \mathbf{w}} \\ &= -\mathbf{w}^\top \hat{\mu} + \sqrt{\kappa\delta} \|A^{\frac{1}{2}} \mathbf{w}\| + \gamma_\alpha \sqrt{\delta\sqrt{1-\kappa} \mathbf{w}^\top M \mathbf{w} + \mathbf{w}^\top \hat{\Sigma} \mathbf{w}}. \end{aligned}$$

In conclusion, for the first objective function in  $(P_{\mathcal{U}_\delta})$ , we are left with the last optimization problem

$$\begin{aligned} & \max_{\kappa \in [0,1]} \left\{ -\mathbf{w}^\top \hat{\mu} + \sqrt{\kappa\delta} \|A^{\frac{1}{2}} \mathbf{w}\| + \gamma_\alpha \sqrt{\delta\sqrt{1-\kappa} \mathbf{w}^\top M \mathbf{w} + \mathbf{w}^\top \hat{\Sigma} \mathbf{w}} \right\} \\ &= -\mathbf{w}^\top \hat{\mu} + \max_{\kappa \in [0,1]} \left\{ \sqrt{\kappa\delta} \|A^{\frac{1}{2}} \mathbf{w}\| + \gamma_\alpha \sqrt{\delta\sqrt{1-\kappa} \mathbf{w}^\top M \mathbf{w} + \mathbf{w}^\top \hat{\Sigma} \mathbf{w}} \right\} \tag{16} \\ &= -\mathbf{w}^\top \hat{\mu} + \max_{\kappa \in [0,1]} \Xi(\kappa, \mathbf{w}), \end{aligned}$$

where, for fixed  $\mathbf{w} \in \mathcal{W}$ ,

$$\Xi(\kappa, \mathbf{w}) := \sqrt{\kappa\delta} \|A^{\frac{1}{2}} \mathbf{w}\| + \gamma_\alpha \sqrt{\delta\sqrt{1-\kappa} \mathbf{w}^\top M \mathbf{w} + \mathbf{w}^\top \hat{\Sigma} \mathbf{w}}$$

is a function from  $[0, 1]$  in  $\mathbb{R}$ .

Here is the content of the first appendix.

### A.3. Robust mean–variance and robust mean-CVaR efficient frontiers

Here we show that the mean–variance and the mean-CVaR robust efficient frontiers do not coincide. As indicated in Remark 8 we need to compare for any  $\lambda \in [0, 1]$  the scalarized problems

$$\min_{\mathbf{w} \in \mathcal{W}} -(1-\lambda)\mathbf{w}^\top \mu + (\lambda\sqrt{\delta+1} + (1-\lambda)\delta) \|\hat{\Sigma}^{1/2} \mathbf{w}\| \tag{17}$$

$$\min_{\mathbf{w} \in \mathcal{W}} -\mathbf{w}^\top \mu + (\lambda\tilde{\Xi}(\kappa^*) + (1-\lambda)\delta) \|\hat{\Sigma}^{1/2} \mathbf{w}\|. \tag{18}$$

For  $\lambda = 1$  the two problems reduces to

$$\begin{aligned} & \min_{\mathbf{w} \in \mathcal{W}} \sqrt{\delta+1} \|\hat{\Sigma}^{1/2} \mathbf{w}\| \\ & \min_{\mathbf{w} \in \mathcal{W}} -\mathbf{w}^\top \mu + \tilde{\Xi}(\kappa^*) \|\hat{\Sigma}^{1/2} \mathbf{w}\| \end{aligned}$$

and they give different optimal weights, since the solution of the first one corresponds to the global minimum variance portfolio, while the second problem also incorporates the mean in the objective function.

For  $\lambda \in [0, 1)$  problem (17) can be rewritten as

$$\begin{aligned} & \min_{\mathbf{w} \in \mathcal{W}} -(1-\lambda)\mathbf{w}^\top \mu + (\lambda\sqrt{\delta+1} + (1-\lambda)\delta) \|\hat{\Sigma}^{1/2} \mathbf{w}\| \\ &= -(1-\lambda) \min_{\mathbf{w} \in \mathcal{W}} -\mathbf{w}^\top \mu + \frac{\lambda\sqrt{\delta+1} + (1-\lambda)\delta}{1-\lambda} \|\hat{\Sigma}^{1/2} \mathbf{w}\| \\ &= -(1-\lambda) \min_{\mathbf{w} \in \mathcal{W}} -\mathbf{w}^\top \mu + \left( \frac{\lambda}{1-\lambda} \sqrt{\delta+1} + \delta \right) \|\hat{\Sigma}^{1/2} \mathbf{w}\| \\ &= -(1-\lambda) \min_{\mathbf{w} \in \mathcal{W}} -\mathbf{w}^\top \mu + \left( \left( -1 + \frac{1}{1-\lambda} \right) \sqrt{\delta+1} + \delta \right) \|\hat{\Sigma}^{1/2} \mathbf{w}\|. \end{aligned}$$

We notice that problems (17) and (18) differ for the coefficients multiplying  $\|\hat{\Sigma}^{1/2} \mathbf{w}\|$ . Let  $A_1, A_2 : [0, 1] \rightarrow \mathbb{R}$  be defined by

$$A_1(\lambda) = \left( -1 + \frac{1}{1-\lambda} \right) \sqrt{\delta+1} + \delta \quad A_2(\lambda) = \lambda\tilde{\Xi}(\kappa^*) + (1-\lambda)\delta.$$

We investigate below the image set of these two functions.

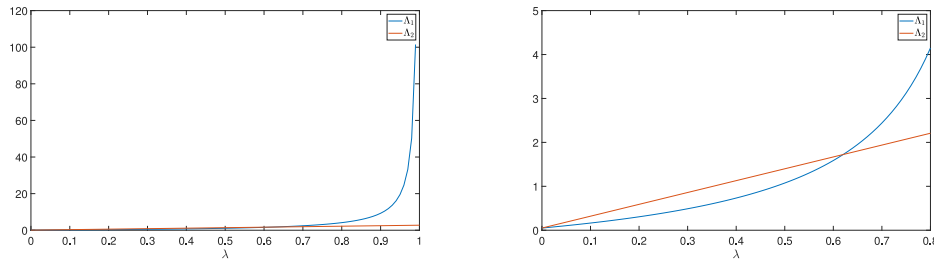


Fig. 9. Plots of  $A_1(\lambda)$  and  $A_2(\lambda)$  for  $\delta = 0.05$ ,  $\alpha = 0.01$  and  $\gamma_\alpha \approx 2.6652$ .

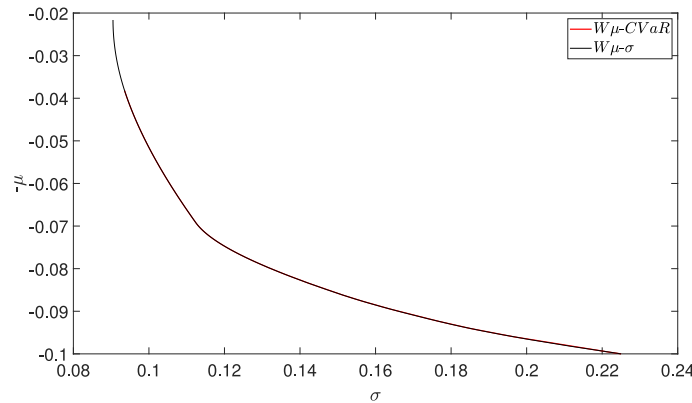


Fig. 10. The figure represents the robust mean-standard deviation (black) and robust mean-CVaR (red) efficient frontiers in the  $(\sigma, \mu)$ -plane.

Concerning  $A_1$ : this is a hyperbolic function, strictly increasing for  $\lambda \in [0, 1)$ , and such that  $A_1(0) = \delta$ .  $A_1$  has vertical asymptote at  $\lambda = 1$  and intersection with the horizontal axis at  $\lambda_1 = -\frac{\delta}{\sqrt{1+\delta}-\delta}$  if  $\delta \neq \frac{1+\sqrt{5}}{2}$ ; otherwise, if  $\delta = \frac{1+\sqrt{5}}{2}$ , there is no intersection with the horizontal axis as the horizontal asymptote is  $\lambda = 0$ . Thus  $\text{Im}_{\lambda \in [0,1)}(A_1) = [\delta, +\infty)$ , where  $\text{Im}(f)$  stands for the image set of a function  $f$ .

Concerning  $A_2$ : this is a linear function with slope  $m = \tilde{\Xi}(\kappa^*) - \delta$  and value  $A(0) = \delta$ . Since

$$\tilde{\Xi}(\kappa^*) = \max_{\kappa \in [0,1]} \left( \sqrt{\kappa} \delta + \gamma_\alpha \sqrt{\delta \sqrt{1-\kappa^*} + 1} \right) \geq \tilde{\Xi}(1) = \delta + \gamma_\alpha > \delta,$$

we can derive that  $m > 0$  for every  $0 \leq \delta$ . In addition,  $A_2 \equiv 0$  for  $\lambda_2 = -\frac{\delta}{\tilde{\Xi}(\kappa^*)-\delta} < 0$ . Thus  $\text{Im}_{\lambda \in [0,1)}(A_2) = [\delta, \tilde{\Xi}(\kappa^*)]$ .

The two functions are represented in Fig. 9 for given values of  $\alpha$  and  $\delta$ .

In conclusion,  $\text{Im}_{\lambda \in [0,1)}(A_2) = [\delta, \tilde{\Xi}(\kappa^*)] \subset \text{Im}_{\lambda \in [0,1)}(A_1) = [\delta, +\infty)$ . This implies that the solutions of the scalarized problems in (18) are “contained” within the set of solutions of (17), but they never coincide since we have a strict inclusion of the image sets associated with the coefficient multiplying  $\|\hat{\Sigma}^{1/2} \mathbf{w}\|$ . This fact is shown in Fig. 10, where we represent the efficient frontiers associated with the two problems for a portfolio of 6 assets with nominal parameters

$$\hat{\mu} = \begin{pmatrix} -0.005 \\ 0.06 \\ 0.07 \\ 0.08 \\ -0.009 \\ 0.10 \end{pmatrix}, \quad \hat{\sigma} = \begin{pmatrix} 0.100 \\ 0.125 \\ 0.150 \\ 0.175 \\ 0.200 \\ 0.225 \end{pmatrix}.$$

Pairwise return correlations is assumed equal to  $\hat{\rho} = 0.40$ ,  $\alpha = 0.01$  and  $\delta = 0.05$ .

**Data availability**

The authors do not have permission to share data.

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