Perturbative approach to analog Hawking radiation in dielectric media in the subcritical regime

S. Trevisan[1,2] F. Belgiorno[3,4] and S. L. Cacciatori[1,2]

[1]*Department of Science and High Technology, Università dell'Insubria, Via Valleggio 11, IT-22100 Como, Italy*
[2]*INFN sezione di Milano, via Celoria 16, IT-20133 Milano, Italy*
[3]*Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo 32, IT-20133 Milano, Italy*
[4]*INdAM-GNFM, Piazzale Aldo Moro, 5 Roma, Italy*

We take into account the subcritical case for dielectric media by exploiting an approximation allowing us to perform perturbative analytical calculations and still not implying low dispersive effects. We show that in the background of a specific solitonlike solution, pair-creation occurs and can display a thermal behavior governed by an effective temperature. The robustness of the approach is also corroborated by the analysis of the $\phi\psi$-model related to the standard Hopfield model, for which analogous results are obtained.

I. INTRODUCTION

We study the problem of the so-called subcritical case for the Hawking effect in analog gravity, which is characterized by the absence of a real horizon but still a process of pair-creation takes place. The phenomenon has been extensively studied in the physical literature, because of the Vancouver experiment with water [1,2], where measurements of a thermal spectrum appear, despite the actual absence of a horizon, as shown in subsequent analysis [3–5]. As to analytical calculations, the first one, again devoted to the case of scattering in water, appeared in [6], and it was provided by means of the so-called Bremmer approximation [7,8], adapted to an ordinary differential equation of the nth order. Subsequent analytical calculations, involving also the subcritical case for water and Bose–Einstein condensates (BEC), again in the limit of low dispersive effects, are found in [9,10], and exploit a Korteweg–De Vries (KdV) equation emerging in the approximation of no copropagating modes [9]. It is worth mentioning also the first experiment with water [11], and we stress that both the subcritical and the transcritical cases for water were studied also in [12–14].

For the case of the analog Hawking effect in dielectric media, horizonless situations were numerically taken into account in [15,16], but for the subcritical case, a fully analytical calculation is still missing.

We have shown in previous papers [17,18] that an equation of the Orr-Sommerfeld type inherited by Nishimoto's works (see e.g. [19,20]) enables us to treat in a quite unified way the case with a horizon, also called the transcritical case, which amounts to the presence of a real turning point (real horizon) in the limit of weak dispersive effects, where the weakness of dispersion is indicated by a suitable small parameter ϵ.

In this paper, we focus on the subcritical case for dielectric media; i.e. we take into account configurations where a real turning point is missing. We still obtain a fourth-order differential equation governing the phenomenon, but we adopt a different attitude and a different expansion parameter with respect to the transcritical case mentioned above. Indeed, we consider a linearization of the equation around a specific solitonlike background solution, and we exploit an expansion in terms of a parameter η which represents the weakness of the soliton amplitude. In the comoving frame of the background solution, we obtain a static situation that mimics that of the Hawking effect in the transcritical case, but with no horizon. Even if it could seem that weak dispersion is implied by our picture, actually this is not the case: Indeed, the dispersion parameter ϵ is *a priori* not restricted to be small, and this is the main difference with respect to the standard picture described in [17], where ϵ is small and a so-called singular perturbation theory is to be allowed. In the framework we discuss, one is allowed to adopt a regular perturbative expansion in the parameter η, which represents a strong advantage.

We show that for the particular profile which is taken into account, in some limit thermality is simulated, with an effective temperature which is one-third of the one of the corresponding transcritical case.

Explicit calculations are first carried out for a modified $\phi\psi$-model whose aim is to simplify as possible the dispersion relation associated with the model, and to allow more straightforward analytical calculations and a clearer exposition of the basic idea. Then we corroborate the robustness of the approach by applying it to the $\phi\psi$-model introduced in [21], then discussed elsewhere, and again taken into account in [17,22]. The aforementioned $\phi\psi$-model will be mentioned as the "original" model and represents a simplification of the Hopfield model [23] which is a standard way to discuss the electromagnetic field in dielectric media.

II. THE MODIFIED $\phi\psi$-MODEL AND ITS SOLITONIC SOLUTIONS

The model we will consider is defined by a Lagrangian function involving two real scalar fields ϕ and ψ, with the aim of simulating some features of the behavior of the electromagnetic field in dielectric media. As is well known, a way to obtain this goal consists in the Hopfield model [23], where the interaction between the electromagnetic field and the atoms/molecules of the dielectric medium is taken into account by replacing the aforementioned microscopic objects with a mesoscopic polarization field. One still obtains the correct Sellmeier equation for the dispersion relation. In our case, intending to simplify as much as possible both the analytical calculations and the dispersion relation, we replace the polarization field with the field ψ and the electromagnetic field with ϕ, with a setup aimed at reproducing the Cauchy dispersion relation most straightforwardly. We stress that our picture below will be corroborated also in the trickier case of the $\phi\psi$-model discussed in [21], where the Hopfield model is reduced in the most direct way to a model reproducing exactly the Sellmeier dispersion relation using a couple of scalar fields ϕ, ψ.

In our present model, the Lagrangian, expressed in the lab frame, with respect to spacetime variables t_l, x_l, is

$$\mathcal{L} = \frac{1}{2}(\partial_{t_l}\phi)^2 + \frac{1}{2}((\partial_{t_l}\psi)^2 + \mu^2\psi^2) + g\phi\partial_{x_l}\psi - \frac{\lambda}{4!}\psi^4. \quad (1)$$

We can write it in a covariant form, which will be useful to pass to the frame comoving with the pulse:

$$\mathcal{L} = \frac{1}{2}(v^\nu\partial_\nu\phi)^2 + \frac{1}{2}((v^\nu\partial_\nu\psi)^2 + \mu^2\psi^2) + g\phi n^\nu\partial_\nu\psi - \frac{\lambda}{4!}\psi^4, \quad (2)$$

where in the lab frame we have $v^\nu_{\text{lab}} = (1, 0)$ and $n^\nu_{\text{lab}} = (0, 1)$, while in a boosted frame we get $v^\nu = (\gamma, -V\gamma)$ and $n^\nu = (-V\gamma, \gamma)$ ($c = 1$ everywhere).

The equations of motion that follow from (2) are

$$(v^\nu\partial_\nu)^2\phi - gn^\nu\partial_\nu\psi = 0, \quad (3)$$

$$(v^\nu\partial_\nu)^2\psi + gn^\nu\partial_\nu\phi - \mu^2\phi + \frac{\lambda}{3!}\psi^3 = 0. \quad (4)$$

The free-field solutions (for $\lambda = 0$) are plane waves $e^{i\omega_{\text{lab}}t_l - ik_{\text{lab}}x_l}$ which satisfy

$$n_0^2(\omega_{\text{lab}}) := \frac{k_{\text{lab}}^2}{\omega_{\text{lab}}^2} = \frac{\mu^2}{g^2} + \frac{\omega_{\text{lab}}^2}{g^2} =: A + B\omega_{\text{lab}}^2, \quad (5)$$

where $\omega_{\text{lab}} = v^\mu k_\mu$ and $k_{\text{lab}} = -n^\mu k_\mu$. The quantity n_0 is, by definition, the refractive index of the medium. Equation (5), in the limit as $\omega_{\text{lab}} \to 0$, gives rise to the Cauchy dispersion relation, which is frequently used to describe dielectric media at low frequencies. Indeed, as $\omega_{\text{lab}} \to 0$ we get

$$n_0(\omega_{\text{lab}}) \simeq \sqrt{A} + \frac{B}{2\sqrt{A}}\omega_{\text{lab}}^2. \quad (6)$$

To describe ordinary dielectrics, we require $A > 1$, i.e. $\mu > g$.

The Lagrangian (2) admits a conserved current, which is related by the Noether theorem to the invariance (in its complexified version) under phase shifting of the fields $\phi \mapsto e^{i\alpha}\phi$, $\psi \mapsto e^{i\alpha}\psi$:

$$J^\nu := \frac{i}{2}[(v^\nu\phi^*(v^\alpha\partial_\alpha\phi) + v^\nu\psi^*(v^\alpha\partial_\alpha\psi) + gn^\nu\psi^*\phi - \text{c.c.}]. \quad (7)$$

The zeroth component gives rise to the conserved charge

$$\int_{\Sigma_t} dxJ^0 =: Q, \quad (8)$$

where Σ_t is a spacelike hypersurface. The associated (conserved) scalar product is

$$\left(\binom{\phi}{\psi}, \binom{\tilde{\phi}}{\tilde{\psi}}\right) = \frac{i}{2}\int_{\Sigma_t}\left[v^0\phi^*v^\alpha\partial_\alpha\tilde{\phi} - v^0\tilde{\phi}v^\alpha\partial_\alpha\phi^*\right.$$
$$+ v^0\psi^*v^\alpha\partial_\alpha\tilde{\psi} - v^0\tilde{\psi}v^\alpha\partial_\alpha\psi^*$$
$$\left. + gn^0(\psi^*\tilde{\phi} - \tilde{\psi}\phi^*)\right]. \quad (9)$$

It is straightforward now to provide a canonical quantization of the free-Lagrangian, compute the propagators, and perform standard perturbative Quantum Field Theory (QFT) computations with the vertex $i\lambda$: we will not deal with this in this paper. The free particles of the theory are *polaritons* which satisfy the Cauchy dispersion relation, and the interaction term is a modelization of the nonlinear

response that can happen in dielectrics, better known as the Kerr effect. The norm of the free particle states $\Phi_k = \binom{\phi_k}{\psi_k} \sim e^{-ik_\mu x^\mu}$ has a simple expression in the laboratory frame:

$$\|\Phi_k\|^2 = Q(\phi_k, \psi_k) \propto 2\omega_{\text{lab}} \left(1 + \frac{\omega_{\text{lab}}^2}{\mu^2 + \omega_{\text{lab}}^2}\right). \quad (10)$$

From Eq. (10) we can clearly distinguish the positive-norm modes ($\omega_{\text{lab}} > 0$) from the negative-norm modes ($\omega_{\text{lab}} < 0$). In Appendix A we give a more detailed analysis of the current two-vector J^μ, and we prove that the norm is proportional to $\text{sign}(\omega_{\text{lab}})$ in *any* inertial frame.

The equations of motion (3) also admit a solitonic solution that propagates rigidly at a fixed velocity V with respect to the laboratory:

$$\psi_s(x_l - Vt_l) = \frac{\alpha}{\cosh(\beta(x_l - Vt_l))}, \quad (11)$$

$$\alpha^2 = \frac{12V^2\beta^2}{\lambda}, \quad (12)$$

$$\beta^2 = \frac{1}{V^4}(\mu^2 V^2 - g^2). \quad (13)$$

III. THE LINEARIZED EOM

In this section we consider the linearization of the equations of motions (EOMs) (3) around a solitonlike background:

$$\psi_B(x_l - Vt_l) = \sqrt{\frac{2|\eta|}{|\lambda|}} \frac{1}{\cosh(\beta(x_l - Vt_l))}. \quad (14)$$

We will treat η and β as independent parameters to allow valid results not only for the soliton (11) but also for other backgrounds with the same shape. Indeed, it is known that solitons of this type can emerge also as *approximated* solutions of electrodynamics inside nonlinear dielectrics: specifically, they are solutions of the so-called nonlinear Schrödinger equation, which derives from some approximations made on the Maxwell equations inside those media. We shall comment further on our choice in the following section.

We want to study the scattering of the asymptotic normal modes against the perturbation given by the soliton traveling across the medium: if the scattering involves both positive and negative-norm modes, this can be interpreted as a sign of instability of the system, which decays by emitting particles, as in the pioneering computation by Hawking for black hole evaporation.

The linearized EOMs are

$$(v^\nu \partial_\nu)^2 \phi - gn^\nu \partial_\nu \psi = 0, \quad (15)$$

$$(v^\nu \partial_\nu)^2 \psi + gn^\nu \partial_\nu \psi - \mu^2 \phi + \frac{\lambda}{2} \psi_B^2 \psi = 0. \quad (16)$$

Notice that, in the linearized equation, the effect of the perturbation is a shift of the parameter μ^2:

$$\mu^2 \mapsto \mu^2 - \frac{\lambda}{2!} \psi_B^2(x_l - Vt_l), \quad (17)$$

which induces a shift in the refractive index [see (5)]

$$n^2(\omega_{\text{lab}}, x_l, t_l) = n_0^2(\omega_{\text{lab}}) + \delta n^2(x_l - Vt_l), \quad (18)$$

$$\delta n^2(x_l - Vt_l) = -\frac{\eta}{g^2} \frac{1}{\cosh^2(\beta(x_l - Vt_l))}. \quad (19)$$
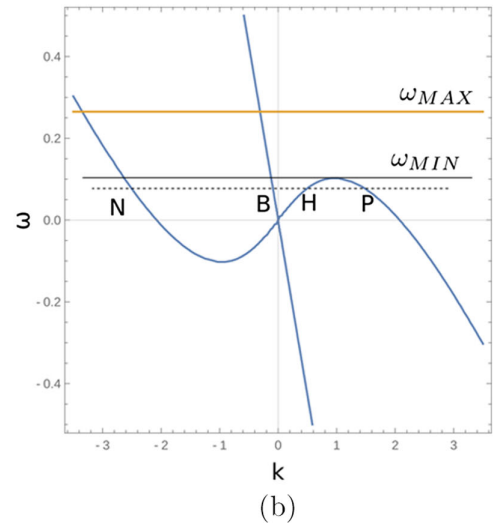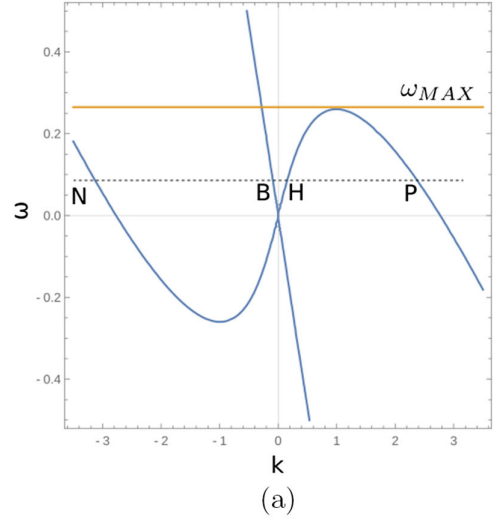

(a)

(b)

FIG. 1. (a) The dispersion relation (5) represented in the comoving frame with the background, with $g = 1$, $\mu = 1.2$; for $0 < \omega < \omega_{\text{MAX}}$ there are four real solutions. (b) The dispersion relation (18) represented at the peak of the perturbation, i.e. $x_l = Vt_l$, for $\eta = -1$. The modes $0 < \omega < \omega_{\text{MIN}}$ do not experience an event horizon (*subcritical regime*).

To include both cases $\delta n > 0$ and $\delta n < 0$, we will consider η as a real quantity, with

$$\text{sign}(\eta) := \text{sign}(\lambda).$$

The dispersion relations (5) and (18) are represented in Fig. 1, as seen in the comoving frame with the background.

We decouple Eqs. (15) and (16) by applying the operator $(v^\nu \partial_\nu)^2$ to the second equation; then, we eliminate ϕ using $(v^\nu \partial_\nu)^2 \phi = g n^\nu \partial_\nu \psi$. In this way, we obtain the following decoupled equation only for the field ψ:

$$(v^\nu \partial_\nu)^4 \psi + g^2 (n^\nu \partial_\nu)^2 \psi - \mu^2 (v^\nu \partial_\nu)^2 \psi$$
$$+ \frac{\lambda}{2} (v^\nu \partial_\nu)^2 (\psi_B^2 \psi) = 0. \tag{20}$$

It is convenient to write the equations in the comoving coordinates $t = \gamma(t_l - V x_l)$, $x = \gamma(x_l - V t_l)$. Since the potential term is independent of the comoving time, we seek a solution in the form $\psi = e^{-i\omega t} f(x)$, where the constant ω is a conserved quantity (cf. e.g. [21]). In this way we end up with a fourth-order equation for $f(x)$ only:

$$0 = V^4 \gamma^4 f^{(4)}(x) + 4iV^3 \omega \gamma^4 f^{(3)}(x) + \gamma^2 f''(x)[\eta V^2 \text{sech}^2(\beta x/\gamma) + g^2 - \mu^2 V^2 - 6V^2 \omega^2 \gamma^2]$$
$$+ 2iV\gamma f'(x)[\eta \text{sech}^2(\beta x/\gamma)(2i\beta V \tanh(\beta x/\gamma) + \omega \gamma) + \omega \gamma(g^2 - \mu^2 - 2\omega^2 \gamma^2)] + f(x)[-2\beta^2 \eta V^2 \text{sech}^4(\beta x/\gamma)$$
$$- \eta \text{sech}^2(\beta x/\gamma)(\omega \gamma + 2i\beta V \tanh(\beta x/\gamma))^2 + \omega^2 \gamma^2 (-g^2 V^2 + \mu^2 + \omega^2 \gamma^2)]. \tag{21}$$

We can further manipulate the equation by performing the following change of variables, usual for the Pöschl-Teller potential,

$$z = -e^{2\tilde\beta x}, \qquad \tilde\beta := \frac{\beta}{\gamma}, \tag{22}$$

which implies

$$\partial_x = 2\tilde\beta \theta_z := 2\tilde\beta z \frac{d}{dz}. \tag{23}$$

By defining the rescaled parameters $G = \frac{g}{2\tilde\beta}$, $\Omega = \frac{\omega}{2\tilde\beta}$, and $M = \frac{\mu}{2\tilde\beta}$, we end up with the following equation:

$$0 = (V^4 \gamma^4) z^4 f^{(4)} + (6V^4 \gamma^4 - 4iV^3 \Omega \gamma^4) z^3 f^{(3)} + \left(G^2 \gamma^2 - M^2 V^2 \gamma^2 + 7V^4 \gamma^4 - 12iV^3 \Omega \gamma^4 - 6V^2 \Omega^2 \gamma^4 - \frac{\eta}{\tilde\beta^2} \frac{V^2 \gamma^2 z}{(1-z)^2}\right) z^2 f^{(2)}$$

$$+ \left(G^2 \gamma^2 - M^2 V^2 \gamma^2 - 2iG^2 V\Omega \gamma^2 + 2iM^2 V\Omega \gamma^2 + V^4 \gamma^4 - 4iV^3 \Omega \gamma^4 - 6V^2 \Omega^2 \gamma^4 + 4iV\Omega^3 \gamma^4\right.$$

$$+ \frac{\eta}{\tilde\beta^2} \left(\frac{2iV\Omega \gamma^2 z}{(1-z)^2} - \frac{V^2 \gamma^2 z(3+z)}{(1-z)^3}\right)\right) z f^{(1)} + \left(M^2 \Omega^2 \gamma^2 - G^2 V^2 \Omega^2 \gamma^2 + \Omega^4 \gamma^4\right.$$

$$+ \frac{\eta}{\tilde\beta^2} \left(\frac{\Omega^2 \gamma^2 z}{(1-z)^2} + \frac{2iV\Omega \gamma^2 z}{(1-z)^3} - \frac{V^2 \gamma^2 z(1+4z+z^2)}{(1-z)^4}\right)\right) f. \tag{24}$$

Equation (24) is of Fuchsian type. The complete characterization of the equation is not in the interest of this work and will be the subject of some future publication. In the present work, we propose a perturbative method for the solution of the equation, which will be exposed in the next section.

IV. PERTURBATIVE METHOD

The solitonic solution (11) has no free parameter, except for the velocity V: its amplitude, in particular, is fixed by the constants of the Lagrangian. Nevertheless, as in the previous section, we linearize the field equations for our nonlinear theory around *weak* solitonlike backgrounds (whichever produced experimentally), with the aim to consider the perturbation induced by the solitonlike background as "small." We stress that we have the freedom to choose the spatial dependence of the perturbation in the refractive index in such a way that it can be reproduced using linearization around a given background solution. This attitude is corroborated by the experimental fact that strong laser pulses may induce in dielectrics a nonlinear perturbation $\delta n(x)$ (in the comoving frame) of the refractive index which is typically orders of magnitude lower than the leading term (e.g. order of 10^{-3} compared with one of the leading terms).

There is an immediate consequence of this approach: as all the nonhomogeneity is associated with the

refractive index $n(x)$ is due to the correction $\delta n(x)$, and the latter one is treated perturbatively in η which is associated directly with the amplitude of the solitonlike background, we are considering a situation where at the lowest order homogeneity occurs, and then no horizon can appear; i.e. we are automatically in a subcritical regime. This is the main difference with respect to the perturbative approach where the perturbation parameter ϵ is associated with a weak dispersion, and where instead a horizon may appear in the leading order equation (in the form of a real turning point).

The perturbative method for Fuchsian equations that we apply was proposed in [24]. We start by eliminating the third-order derivative by the change of variable

$$f(z) = z^{-\frac{3}{2}+i\frac{\Omega}{V}}u(z)$$

that gives

$$u^{(4)} + [v_1(z) + \eta w_1(z)]u^{(2)} + [v_2(z) + \eta w_2(z)]u^{(1)}$$
$$+ [v_3(z) + \eta w_3(z)]u = 0, \tag{25}$$

where

$$v_1(z) = \frac{2G^2 - 2M^2V^2 + 5V^4\gamma^2}{2V^4\gamma^2 z^2},$$

$$v_2(z) = \frac{-2G^2V + 2M^2V^3 + 2iG^2\Omega - 2iG^2V^2\Omega - 5V^5\gamma^2}{V^5\gamma^2 z^3},$$

$$v_3(z) = \frac{36G^2V^2 - 36M^2V^4 - 48iG^2V\Omega + 48iG^2V^3\Omega}{16V^6\gamma^2 z^4}$$
$$- \frac{16G^2\Omega^2 + 32G^2V^2\Omega^2 - 16G^2V^4\Omega^2 + 81V^6\gamma^2}{16V^6\gamma^2 z^4},$$

$$w_1(z) = -\frac{1}{\tilde{\beta}^2 V^2\gamma^2(z-1)^2 z},$$

$$w_2(z) = \frac{4}{\tilde{\beta}^2 V^2\gamma^2(z-1)^3 z},$$

$$w_3(z) = \frac{-1 + 2z - 25z^2}{4\tilde{\beta}^2 V^2\gamma^2(z-1)^4 z^3}.$$

Now we consider η as a small parameter and formally expand the solution as

$$u(z) = u_0(z) + \eta u_1(z) + \eta^2 u_2(z) + \cdots, \tag{26}$$

which allows us to obtain a regular perturbative expansion (to be compared with the singular perturbation expansion one obtains by expanding with respect to a low dispersion parameter ϵ; cf. Sec. VII). Herein, we compute the first-order solution u_1, and we will discuss the possibility of taking $\eta \to 1$.

Solving the unperturbed equation ($\eta = 0$) is very easy, and it gives

$$u_0(z) = z^{i\alpha}, \tag{27}$$

where α satisfies a fourth-degree algebraic equation. By putting $i\alpha = i\frac{k}{2\beta} + \frac{3}{2} + i\frac{\Omega}{V}$, we find that k is one of the four solutions of the dispersion relation (5) as written in the comoving frame:

$$\gamma(-g^2(k+V\omega)^2 + (kV+\omega)^2(\mu^2 + (kV+\omega)^2\gamma^2)) = 0. \tag{28}$$

By substituting (26) into (25) we find the set of equations

$$u_1^{(4)} + v_1(z)u_1^{(2)} + v_2(z)u_1^{(1)} + v_3(z)u_1 = w_1(z)u_0^{(2)} + w_2(z)u_0^{(1)} + w_3(z)u_0,$$
$$\ldots, \tag{29}$$

$$u_n^{(4)} + v_1(z)u_n^{(2)} + v_2(z)u_n^{(1)} + v_3(z)u_n = w_1(z)u_{(n-1)}^{(2)} + w_2(z)u_{(n-1)}^{(1)} + w_3(z)u_{(n-1)}. \tag{30}$$

Thus every u_n satisfies a linear differential equation with a source term that depends on $u_{(n-1)}$; the associated homogeneous equation is the unperturbed equation satisfied by u_0.

Let us consider the equation for u_1. We can explicitly solve it by applying the method of variation of constants. By defining

$$r_1(z) := w_1(z)u_0^{(2)} + w_2(z)u_0^{(1)} + w_3(z)u_0, \tag{31}$$

we compute the quantities

$$W(z) = \det \begin{pmatrix} z^{i\alpha_1} & z^{i\alpha_2} & z^{i\alpha_3} & z^{i\alpha_4} \\ (z^{i\alpha_1})' & (z^{i\alpha_2})' & (z^{i\alpha_3})' & (z^{i\alpha_4})' \\ (z^{i\alpha_1})'' & (z^{i\alpha_2})'' & (z^{i\alpha_3})'' & (z^{i\alpha_4})'' \\ (z^{i\alpha_1})''' & (z^{i\alpha_2})''' & (z^{i\alpha_3})''' & (z^{i\alpha_4})''' \end{pmatrix},$$

$$W_1(z) = r_1(z) \det \begin{pmatrix} z^{i\alpha_2} & z^{i\alpha_3} & z^{i\alpha_4} \\ (z^{i\alpha_2})' & (z^{i\alpha_3})' & (z^{i\alpha_4})' \\ (z^{i\alpha_2})'' & (z^{i\alpha_3})'' & (z^{i\alpha_4})'' \end{pmatrix},$$

$$W_2(z) = -r_1(z) \det \begin{pmatrix} z^{i\alpha_1} & z^{i\alpha_3} & z^{i\alpha_4} \\ (z^{i\alpha_1})' & (z^{i\alpha_3})' & (z^{i\alpha_4})' \\ (z^{i\alpha_1})'' & (z^{i\alpha_3})'' & (z^{i\alpha_4})'' \end{pmatrix},$$

$$W_3(z) = r_1(z) \det \begin{pmatrix} z^{i\alpha_1} & z^{i\alpha_2} & z^{i\alpha_4} \\ (z^{i\alpha_1})' & (z^{i\alpha_2})' & (z^{i\alpha_4})' \\ (z^{i\alpha_1})'' & (z^{i\alpha_2})'' & (z^{i\alpha_4})'' \end{pmatrix},$$

$$W_4(z) = -r_1(z) \det \begin{pmatrix} z^{i\alpha_1} & z^{i\alpha_2} & z^{i\alpha_3} \\ (z^{i\alpha_1})' & (z^{i\alpha_2})' & (z^{i\alpha_3})' \\ (z^{i\alpha_1})'' & (z^{i\alpha_2})'' & (z^{i\alpha_3})'' \end{pmatrix}.$$

A particular solution to the first equation of (29) is given by

$$u_1(z) = \sum_{j=1}^{4} z^{i\alpha_j} \int dz' \frac{W_j(z')}{W(z')}. \tag{32}$$

In principle, we can recursively iterate the procedure to obtain particular solutions to u_n. Notice that the general solution for each order n is obtained by adding a solution to the homogeneous equation, which is the same for every n, so we can say that the general solution $u(z)$ is obtained by adding to the iterative solution a combination of the solutions to the unperturbed equation: the coefficients of the combination will be set by the boundary condition of the scattering.

V. BOUNDARY CONDITIONS FOR THE SCATTERING

For the subcritical case, it is standard to consider a white hole configuration, in which an initial state, representing a Hawking mode, approaches the white hole–like perturbation at early times, and four modes emerge at late times: three backward modes P, N, B, where N is the only negative norm mode appearing in the scattering, and a transmitted-mode T representing the fraction of the Hawking mode H which is transmitted beyond the perturbation. This is nothing but what happens in the presence of a real white hole horizon in the transcritical case, apart for the transmitted mode. For the black hole case, an analytical study is trickier, as the black hole–like configuration, in principle, is not related to the white hole–like one by time reversal: in the presence of a black hole horizon, one has three entering

initial modes P, N, B which are converted in a scattered emerging mode H. When the scattering is subcritical, in principle, given three initial modes P, N, B, one should consider the possibility to get three transmitted particles, and then it is evident that one does not obtain the time reversal configuration of a white hole subcritical scattering.

We consider a white hole–like configuration, as usual. Thus, in the initial state, we will have only a right-moving H-mode, which is scattered and mode converted in a way such that we obtain the four modes in the final state. The asymptotic form of the solution we seek is thus

$$f(x) \sim \begin{cases} e^{ik_H x} + C_2 e^{ik_P x} + C_3 e^{ik_N x} + C_4 e^{ik_B x}, & x \to -\infty, \\ C_1 e^{ik_H x}, & x \to +\infty, \end{cases} \tag{33}$$

when we recall that the H-mode has a positive group velocity (right-moving), while all other modes have negative group velocity (left-moving). Since the unperturbed solution would be just $f(x) = e^{ik_H x}$ defined everywhere, we expect the coefficient C_1 of the transmitted part of the Hawking mode to be $O(1)$, while all other coefficients must be $O(\varepsilon)$. We can now proceed to the solution of Eq. (29) for $u_1(z)$ considering for the source term the unperturbed solution $u_0(z) = z^{i\alpha_H}$, where $i\alpha_H = i\frac{k_H}{2\beta} + \frac{3}{2} + i\frac{\Omega}{V}$. The detailed computation of the solution is shown in Appendix C: in what follows we will just expose the final expressions.

The asymptotic coefficients of the solution [see Eq. (33)] at first perturbative order are

$$C_1 = \frac{1}{1 + i\eta\tau}, \tag{34}$$

$$C_2 = \frac{i\eta(-1)^{i\frac{k_H-k_P}{2\beta}}}{1 + i\eta\tau} \frac{\pi(k_P V + \omega)^2 \operatorname{csch}\left((k_H - k_P)\frac{\pi\gamma}{2\beta}\right)}{4\beta^2 V^4 (k_P - k_N)(k_P - k_B)}, \tag{35}$$

$$C_3 = \frac{i\eta(-1)^{i\frac{k_H-k_N}{2\beta}}}{1 + i\eta\tau} \frac{\pi(k_N V + \omega)^2 \operatorname{csch}\left((k_H - k_N)\frac{\pi\gamma}{2\beta}\right)}{4\beta^2 V^4 (k_N - k_P)(k_N - k_B)}, \tag{36}$$

$$C_4 = \frac{i\eta(-1)^{i\frac{k_H-k_B}{2\beta}}}{1 + i\eta\tau} \frac{\pi(k_B V + \omega)^2 \operatorname{csch}\left((k_H - k_B)\frac{\pi\gamma}{2\beta}\right)}{4\beta^2 V^4 (k_B - k_P)(k_B - k_N)}, \tag{37}$$

$$\tau = \frac{(k_H V + \omega)^2}{2\beta V^4 \gamma (k_H - k_P)(k_H - k_N)(k_H - k_B)}. \tag{38}$$

VI. RESULTS

Current conservation implies that the following equation holds true:

$$|J_H| = |J_T| + |J_P| - |J_N| + |J_B|, \qquad (39)$$

so that, by defining

$$|T| = \frac{|J_T|}{|J_H|}, \qquad (40)$$

$$|N| = \frac{|J_N|}{|J_H|}, \qquad (41)$$

$$|B| = \frac{|J_B|}{|J_H|}, \qquad (42)$$

we get

$$1 = |T| + |P| - |N| + |B|, \qquad (43)$$

where for the model labeled by K we have $|K| := |C_K|^2 |\frac{v_K \partial_\omega DR|_{k_K}}{v_H \partial_\omega DR|_{k_H}}|$, $v_K(\omega)$ is the group velocity of the K-mode, and $DR(\omega, k)$ is the dispersion relation function defined in Eq. (A3): we provide a derivation of (43) in Appendix B. The ratio $|N|$ corresponds, in the black hole case, to the rate of spontaneous emission of H-mode waves, as it can be argued by using standard Bogoliubov coefficients. In the present case, it represents the rate of pair-production in the subcritical process at hand.

From the scattering coefficients (34) and the expression (A10) for the flux factors, we can write down explicitly the expressions

$$|N| = \frac{\eta^2}{1 + \eta^2 \tau^2} \frac{\pi^2 (k_N V + \omega)^2 (k_H V + \omega)^2 \text{csch}[((k_H - k_N)\gamma\pi)/(2\beta)]^2}{16\beta^4 V^8 (k_N - k_P)(k_N - k_B)(k_H - k_P)(k_H - k_B)}, \qquad (44)$$

$$\frac{|P|}{|N|} = \frac{(k_N - k_B)(k_P - k_H)(k_P V + \omega)^2 (\text{csch}[\pi\gamma(k_H - k_P)/(2\beta)])^2}{(k_P - k_B)(k_N - k_H)(k_N V + \omega)^2 (\text{csch}[\pi\gamma(k_H - k_N)/(2\beta)])^2}, \qquad (45)$$

$$|T| = \frac{1}{1 + \eta^2 \tau^2}, \qquad (46)$$

$$|B| = \frac{\eta^2}{1 + \eta^2 \tau^2} \frac{\pi^2 (k_B V + \omega)^2 (k_H V + \omega)^2 (\text{csch}[\pi\gamma(k_H - k_B)/(2\beta)])^2}{16\beta^4 V^8 (k_B - k_N)(k_B - k_P)(k_H - k_N)(k_H - k_P)}. \qquad (47)$$

As a consequence, we can provide

$$|N| = \frac{1 - |T| - |B|}{\frac{|P|}{|N|} - 1}. \qquad (48)$$

We stress that these expressions are exact except for the perturbative approximation in η: no other approximation has been made throughout the computation, so they are valid for all frequencies, as long as we know $k_j(\omega)$.

For an explicit evaluation we have expanded the modes $k_j(\omega)$ from Eq. (5) near $\omega = 0$, obtaining

$$k_H = \frac{\mu - gV}{g - \mu V} \omega + O(\omega^3), \qquad (49)$$

$$k_B = -\frac{\mu + gV}{g + \mu V} \omega + O(\omega^3), \qquad (50)$$

$$k_P = \frac{\sqrt{g^2 - \mu^2 V^2}}{\gamma V^2} - \left(\frac{1}{V} + \frac{g^2}{\gamma^2 V(g^2 - \mu^2 V^2)}\right)\omega - \frac{g^2(2g^2 + \mu^2 V^2)}{2\gamma(g^2 - \mu^2 V^2)^{5/2}}\omega^2 + O(\omega^3), \qquad (51)$$

$$k_N = -\frac{\sqrt{g^2 - \mu^2 V^2}}{\gamma V^2} - \left(\frac{1}{V} + \frac{g^2}{\gamma^2 V(g^2 - \mu^2 V^2)}\right)\omega + \frac{g^2(2g^2 + \mu^2 V^2)}{2\gamma(g^2 - \mu^2 V^2)^{5/2}}\omega^2 + O(\omega^3), \qquad (52)$$

and we put $g = 1$, $\mu = 1.2$, and $V = 0.5$. As to k_T, due to the fact that the perturbation vanishes very rapidly in the comoving frame both for $x \to \infty$ and for $x \to -\infty$, one easily finds $k_T = k_H$. These values are taken just to obtain qualitatively the same dispersion relation as in an experimental situation of laser pulses in silica, but they are not meant to be quantitatively accurate. In the following analysis of the results we will treat both η and β as independent parameters: although they are uniquely
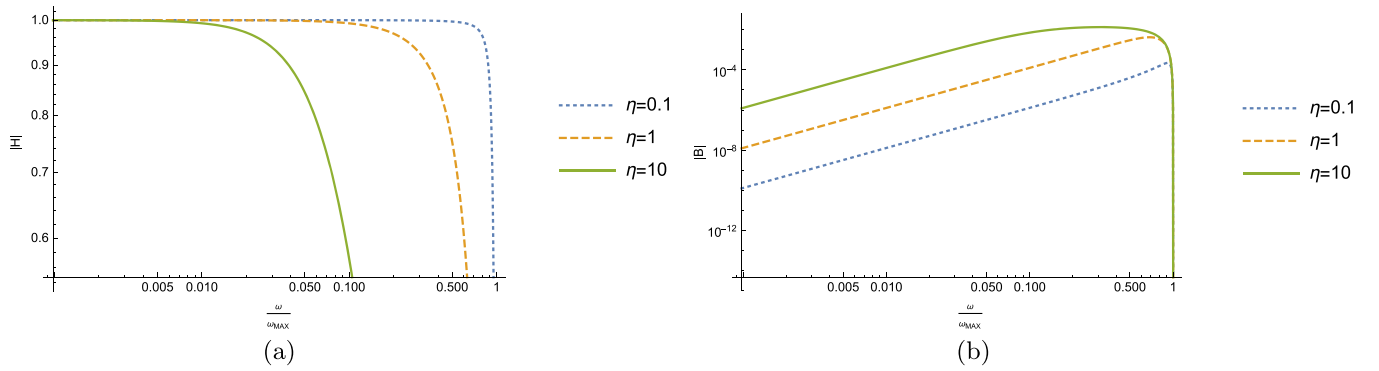
FIG. 2. The coefficients T and B from (46) and (47), for different values of the amplitude of the perturbation η.

determined by the soliton (11), we still think that it is interesting to study the dependence of the radiation on these parameters. We might equivalently say that we are studying a generic perturbation of the form (14) with generic parameters, even though this perturbation might not be a solution of the nonlinear equations of motion (3). The scattering coefficients $|T|$, $|B|$, $|P|$, and $|N|$ are plotted in Figs. 2 and 3, for different values of η. Notice that $|B|$ results

are smaller than the others, as was expected. The transmission coefficient $|T| \sim 1$ for low frequencies, but it decreases more rapidly as η increases: this is a sign that for high perturbation we approximate the formation of an event horizon, and this interpretation is supported by the fact that the frequencies near ω_{MAX} are less transmitted.

The coefficients $|P|$ and $|N|$ are nearly equal at low frequencies and are $\sim\omega$. This behavior is in agreement with
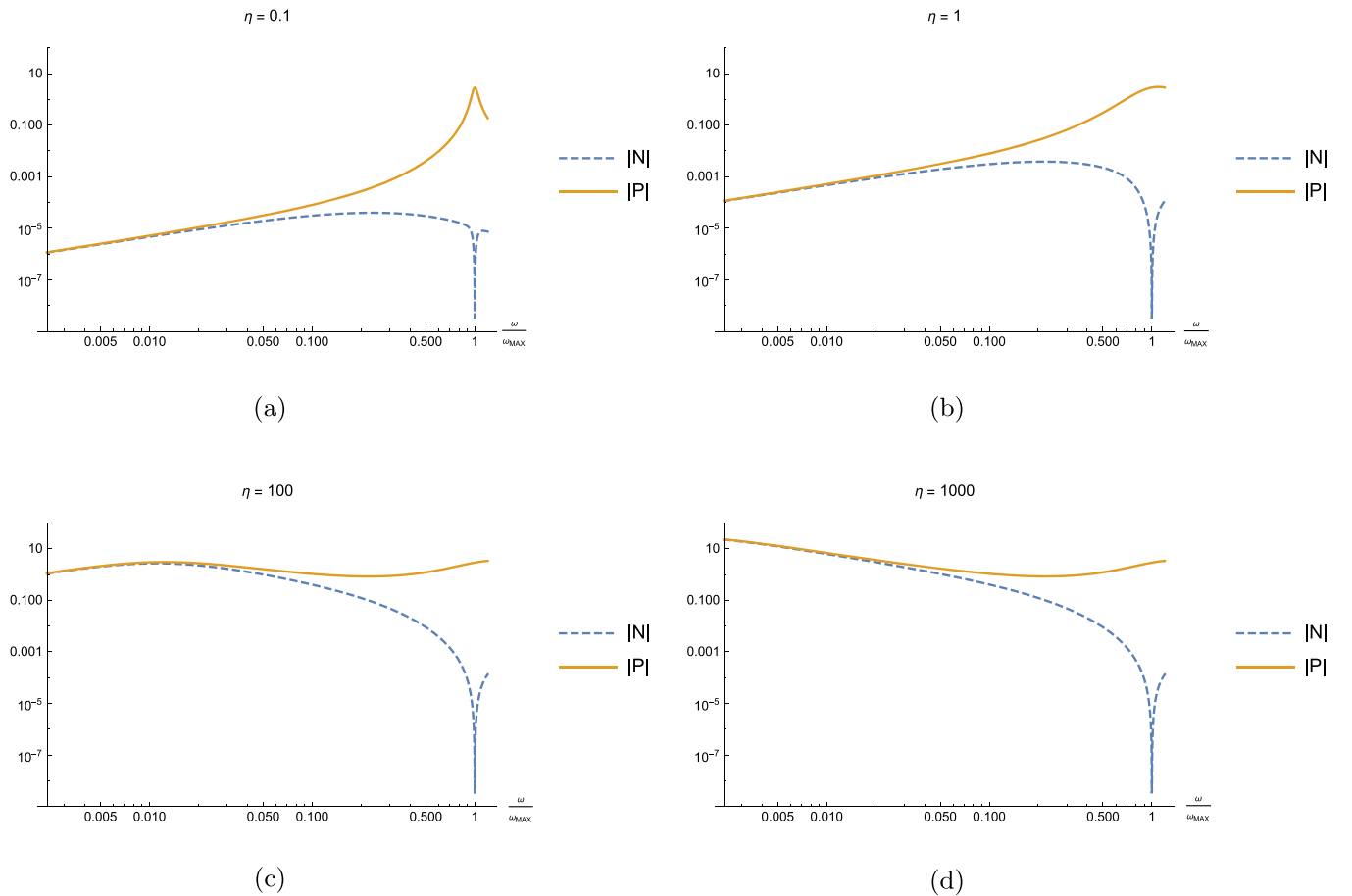

FIG. 3. The coefficients P and N from (44) and (45), for different values of the amplitude of the perturbation η. In (a)–(c) the behavior at small frequencies is $P, N \sim \omega$, which agrees with the findings of [6]; in (d) we show the onset of critical behavior, as $P, N \sim \omega^{-1}$.
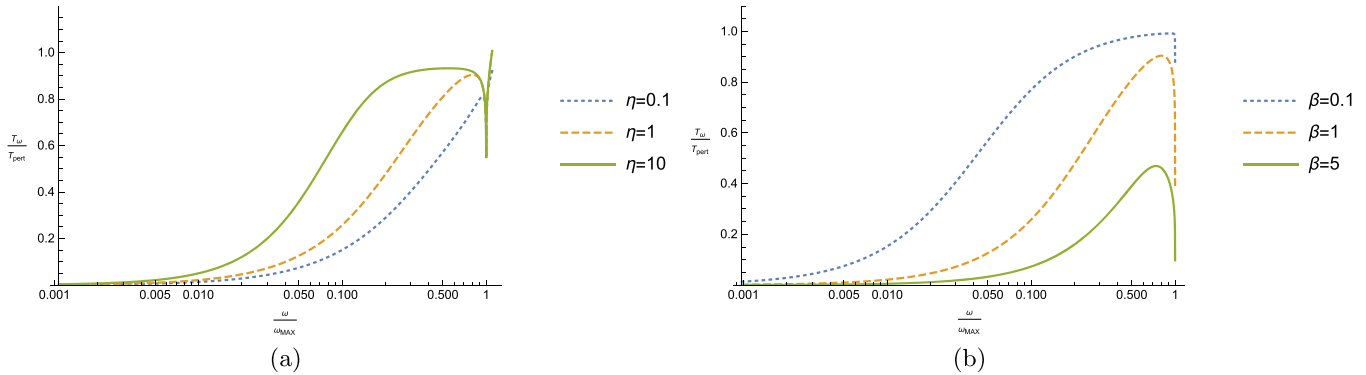
FIG. 4.   The temperature T_ω as defined in (54). (a) Plot at fixed $\beta = 1$, for different η; we see the formation of a plateau at $T_\omega \approx T_{\text{pert}}$ for high η. (b) Plot at fixed $\eta = 1$, for different β; the estimation of the value of the plateau T_{pert} is less accurate for increasing β.

what was found by Coutant and Weinfurtner [6] for a subcritical flow in shallow water.

The ratio $\frac{|P|}{|N|}$ is very close to an exponential function, $\frac{|P|}{|N|} \sim e^{\frac{\omega}{T}}$. This is shown in Fig. 5, where we see that $\omega / \log\left(\frac{|P|}{|N|}\right)$ is almost constant. The temperature T_{pert} (the subscript is for "perturbative") can be estimated analytically from the arguments of the csch in (45): we find

$$\beta_{\text{pert}} = \frac{1}{T_{\text{pert}}} = \frac{2\pi\gamma}{2\beta} \lim_{\omega \to 0} \frac{2k_H - k_P - k_N}{\omega}$$
$$= \frac{2\pi g(2g + \mu V)}{\beta\gamma V(g^2 - \mu^2 V^2)}. \quad (53)$$

We note that $T_{\text{pert}} \propto \beta$, which is proportional to the derivative of the background function (11): this is what was expected based on previous literature about analog Hawking radiation in the critical case. As it is clear from Fig. 5, while the exponential approximation is still valid, our estimation of the temperature is less good as β increases.
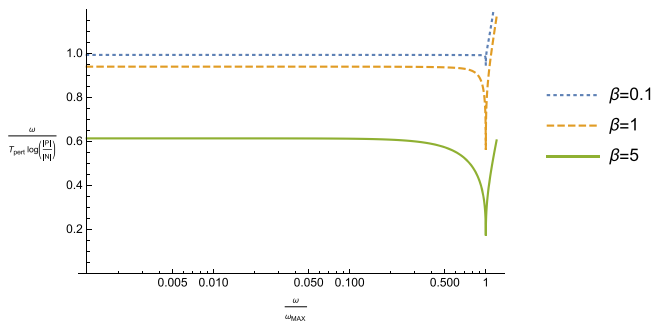

FIG. 5.   The ratio $\omega / \log\left(\frac{|P|}{|N|}\right)$ normalized by the estimated temperature T_{pert} [see Eq. (53)] as a function of β. The ratio is almost constant, which means that $\frac{|P|}{|N|}$ is indeed exponential with the frequency; our estimation of the "temperature" T_{pert} proves to be good at low β, while less accurate for increasing β.

Finally, we define a temperature function T_ω by the relation

$$|N| =: \frac{1}{e^{\frac{\omega}{T_\omega}} - 1}. \quad (54)$$

The function T_ω is plotted in Fig. 4 for different values of η and β. We note T_ω is not constant, showing a lack of thermality in the emission spectrum. However, for higher values of η, a plateau is created for frequencies close to ω_{MAX}, with $T_\omega \approx T_H$. This fact is again interpreted as a sign of the presence of a group horizon for higher frequencies: indeed, a similar behavior was found by a numerical study of the transcritical regime in shallow water [3].

The near criticality of the system for high η is underlined also by the behavior of $|P|$ and $|N|$, which start growing as $\sim \frac{1}{\omega}$ near $\omega = 0$ [see Fig. 3(d)]. This is precisely the behavior that one expects in the critical case. We specify that the results for high values of η should be taken carefully since η is precisely the expansion parameter of our perturbative solution. However, we find it interesting that already at the leading order, the solution shows a near critical behavior for high η; we expect that the same qualitative behavior is present also in the higher order solutions, possibly showing up already at lower values of η.

We stress that the ratio $\omega / \log\left(\frac{|P|}{|N|}\right)$ displayed in Fig. 5 can also be considered as a realistic way for testing thermality in experiments, both in the subcritical case (where thermality is just effective) and in the transcritical one. Compare also [1] for analogous measurements in the case of water. In an optical system such as a dielectric, one can measure the amplitude of the peaks corresponding to the P, N modes in the photon spectrum, as e.g. described in [25] (see in particular the associated supplementary information), and cf. also [26], Chap. 10. Solitonlike laser pulses are e.g. considered in [27], both for diamond and for fused silica, and suitable tuning of the input conditions, such as power, wavelength, and focusing, makes it possible to control, to some extent, the steepening effects and the peak velocity [28]. Steepening of the pulse is, of course,

fundamental in order to enhance the pair-creation process, as the associated temperature is proportional to the spatial derivative of the refractive index in the transcritical case, and this is, to some extent, true also in the subcritical one discussed above, as $T_{\text{pert}} \propto \beta$, where β is still related to the derivative $n'(x)$ in the comoving frame, as remarked above. To provide a realistic estimate of the perturbation in the refractive index one may obtain in experiments, let us also recall that the effective refractive index associated with the Kerr effect is $n \sim n_0 + n_2 I$, where the nonlinear index n_2 in fused silica is $n_2 \sim 3.2 \times 10^{16} \text{ cm}^2/\text{W}$, so that for pulses whose intensity is $I \sim 10^{13} \text{ W/cm}^2$, in fused silica one may obtain $\delta n \coloneqq n_2 I \sim 3 \times 10^{-3}$. Conditions for both the subcritical and the transcritical cases can be matched (cf. also [25]), and then both the situations can be checked experimentally: this can be done more realistically for the model to be discussed in the next section, which is more straightforwardly related to the standard Hopfield model for dielectric media.

VII. THE ORIGINAL $\phi\psi$-MODEL

In the original $\phi\psi$-model, including a ψ^4 term, one has

$$\mathcal{L}_{\varphi\psi} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2\chi\omega_0^2}[(v^\alpha \partial_\alpha \psi)^2 - \omega_0^2 \psi^2]$$
$$+ \frac{1}{c}(v^\alpha \partial_\alpha \psi)\phi - \frac{\lambda}{4!}\psi^4, \tag{55}$$

where ϕ, ψ play the role of the electromagnetic field and polarization field, respectively, χ plays the role of the dielectric susceptibility, v^μ is the usual four-velocity vector of the dielectric, and ω_0 is the proper frequency of the medium [22]. We get the system

$$\Box\phi - \frac{1}{c}(v^\mu \partial_\mu \psi) = 0, \tag{56}$$

$$\left(\frac{1}{\chi\omega_0^2}(v^\mu \partial_\mu)^2 + \frac{1}{\chi}\right)\psi + \frac{1}{c}(v^\mu \partial_\mu \phi) + \frac{\lambda}{3!}\psi^3 = 0. \tag{57}$$

As in [29], we allow the spatial dependence to appear in χ and in ω_0 in such a way that $\chi\omega_0^2 = \text{const}$. Indeed, by linearizing the model around solitonlike solutions, directly written in the comoving frame, one obtains

$$\frac{1}{\chi} \mapsto \frac{1}{\chi} + \frac{\lambda}{2}\psi_0^2(x), \tag{58}$$

$$\omega_0^2 \mapsto \omega_0^2\left(1 + \chi\frac{\lambda}{2}\psi_0^2(x)\right), \tag{59}$$

in such a way that $\chi\omega_0^2$ remains invariant: $\chi\omega_0^2 \mapsto \chi\omega_0^2$.

In this case, we identify the parameter associated with dispersion as follows:

$$\epsilon^2 \coloneqq \frac{1}{\chi\omega_0^2}.$$

This parameter has been considered as the small parameter associated with the model in the limit of low dispersion. In the present case, we adopt a different view where dispersion can also be strong, and the expansion parameter is instead associated with the amplitude of the background solution around which the EOMs are linearized. We limit ourselves to notice that a linearization of the EOMs around a background solitonlike solution $\psi_0(x)$ amounts simply in replacing $\frac{\lambda}{3!}\psi^3 \mapsto \frac{\lambda}{2}\psi_0^2\psi$ in (57).

A. A separated equation for ψ

By applying the operator $\Box$ on the left of equation (57), as shown in [17], we obtain the following fourth-order ordinary differential equation:

$$-\epsilon^2 \partial_x^4 f - 2i\epsilon^2 \frac{\omega}{v}\partial_x^3 f + \frac{1}{\chi\gamma^2 v^2}\left(-\left(1 - \chi\gamma^2 \frac{v^2}{c^2}\right) + \epsilon^2 \chi\omega^2\right)\partial_x^2 f$$
$$+ 2\left(i\frac{\omega}{v}\frac{1}{c^2}(1 - \epsilon^2\omega^2) - \frac{1}{\gamma^2 v^2}\left(\partial_x \frac{1}{\chi}\right)\right)\partial_x f$$
$$+ \left(\epsilon^2 \frac{\omega^4}{v^2 c^2} - \frac{1}{\gamma^2 v^2}\left(\partial_x^2 \frac{1}{\chi}\right) - \frac{\omega^2}{\chi\gamma^2 v^2 c^2} - \frac{\omega^2}{c^2 v^2}\right)f = 0. \tag{60}$$

We also define $f(x) = h(x)\zeta(x)$, with

$$h(x) = A\exp\left(-i\frac{\omega}{2v}x\right), \tag{61}$$

where A is a constant. $h(x)$ is chosen such that the third-order term vanishes, and the procedure is analogous to the Liouville transformation which eliminates the first-order term in a second-order linear ordinary differential equation. This leads to the following quartic equation, which is just of the type "Orr–Sommerfeld":

$$-\epsilon^2 \partial_x^4 \zeta + \left[-\frac{1}{\chi\gamma^2 v^2}\left(1 - \chi\gamma^2 \frac{v^2}{c^2}\right) + \epsilon^2 \frac{1}{\gamma^2 v^2}\left(1 - \frac{3}{2}\gamma^2\right)\omega^2\right]\partial_x^2 \zeta$$
$$+ \left(i\frac{\omega}{v}\frac{1}{\chi\gamma^2 v^2}\left(1 + \chi\gamma^2 \frac{v^2}{c^2}\right) - 2\frac{1}{\gamma^2 v^2}\left(\partial_x \frac{1}{\chi}\right) - i\epsilon^2 \frac{\omega^3}{vc^2}\right)\partial_x \zeta$$
$$+ \left[\frac{1}{\gamma^2 v^2}\left(i\frac{\omega}{v}\left(\partial_x \frac{1}{\chi}\right) - \left(\partial_x^2 \frac{1}{\chi}\right)\right)\right.$$
$$+ \frac{1}{\gamma^2 v^2}\left(\frac{1}{4\chi}\frac{\omega^2}{v^2}\left(1 - \chi\gamma^2 \frac{v^2}{c^2}\right) - \frac{\omega^2}{\chi c^2}\right)$$
$$\left. + \epsilon^2\left(\frac{\omega^4}{v^4}\left(-\frac{1}{16} + \frac{1}{4}\frac{v^2}{c^2}\right)\right)\right]\zeta = 0. \tag{62}$$

The effect of the linearization around a background solitonlike solution $\psi_0(x)$ consists simply in the replacement

$$\frac{1}{\chi} \mapsto \frac{1}{\chi_0} + \frac{\lambda}{2}\psi_0^2(x), \qquad (63)$$

where χ_0 is constant. We choose

$$\psi_0(x) := 2\sqrt{\frac{|\eta|}{|\lambda|}} \frac{1}{\cosh(\tilde{\beta}x)}. \qquad (64)$$

As in the model discussed in the previous sections, we turn to the change of variable

$$z := -\exp(2\tilde{\beta}x). \qquad (65)$$

As a consequence, with some abuse of language, we have $\zeta = f(z)$. Then we obtain

$$\frac{1}{\chi} \mapsto \frac{1}{\chi_0} - 2\eta\frac{z}{(1-z)^2}. \qquad (66)$$

Furthermore, to cancel a new third-order term that appears again after the above-mentioned independent variable change, we set $\zeta(z) = z^{-3/2}h(z)$. Then we obtain the following equation:

$$h^{(4)} + (u_2 + \eta w_2)h^{(2)} + (u_1 + \eta w_1)h^{(1)}$$
$$+ (u_0 + \eta w_0)h = 0, \qquad (67)$$

where

$$u_2 = \frac{\frac{1}{\gamma^2 v^2 \chi_0}\left(1 - \chi_0\gamma^2\frac{v^2}{c^2}\right) + \epsilon^2\left(10\tilde{\beta}^2 - \left(1 - \frac{3}{2}\gamma^2\right)\omega^2\right)}{4\tilde{\beta}^2\epsilon^2 z^2},$$

$$u_1 = \frac{-4\tilde{\beta}vc^2\left(1 - \chi_0\gamma^2\frac{v^2}{c^2}\right) - ic^2\omega\left(1 + \chi_0\gamma^2\frac{v^2}{c^2}\right) + \epsilon^2\chi_0 v(-40\tilde{\beta}^3 v^2 c^2\gamma^2 + 2\tilde{\beta}c^2(2 - 3\gamma^2)\omega^2 + i\gamma^2 v\omega^3)}{8\tilde{\beta}^3 c^2 \chi_0 \epsilon^2 \gamma^2 v^3 z^3},$$

$$u_0 = \frac{144\tilde{\beta}^2 v^2 c^2\left(1 - \chi_0\gamma^2\frac{v^2}{c^2}\right) + 48i\tilde{\beta}vc^2\left(1 + \chi_0\gamma^2\frac{v^2}{c^2}\right) + 4\omega^2((4 + \chi_0\gamma^2)v^2 - c^2) + \epsilon^2\delta}{256\tilde{\beta}^4 c^2 \chi_0 \epsilon^2 \gamma^2 v^4 z^4},$$

$$w_2 = -\frac{1}{2\tilde{\beta}^2\epsilon^2\gamma^2 v^2 z(1-z)^2},$$

$$w_1 = \frac{i\omega(z-1) + 8\tilde{\beta}vz}{4\tilde{\beta}^3\epsilon^2\gamma^2 v^3 z^2(-1+z)^3},$$

$$w_0 = \frac{(c^2 - 4v^2)\omega^2(z-1)^2 + 4i\tilde{\beta}c^2 v\omega(1-z)(5z-1) - 4\tilde{\beta}^2 c^2 v^2(1 - 2z + 25z^2)}{32\tilde{\beta}^4 c^2\epsilon^2\gamma^2 v^4(z-1)^4 z^3},$$

with

$$\delta := \chi_0(1296\tilde{\beta}^4 v^4 c^2\gamma^2 + 72\tilde{\beta}^2 c^2 v^2\omega^2(-2 + 3\gamma^2)$$
$$- 48i\tilde{\beta}\gamma^2 v^3\omega^3 + \gamma^2(c^2 - 4v^2)\omega^4).$$

B. Dispersion relation and its roots

As in the previous case, we can guess that solutions of the zeroth order equation are of the form $z^{i\alpha}$ and that α satisfies the dispersion relation associated with the model. In the comoving frame, the eikonal equation for the model provides us with the following equation:

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\frac{1}{\chi\omega_0^2}(\omega_0^2 - \gamma^2(\omega + vk)^2) - \frac{1}{c^2}\gamma^2(\omega + vk)^2 = 0. \qquad (68)$$

It is a quartic equation whose roots $k_i(\omega)$, $i = 1, 2, 3, 4$ cannot be managed in simple formulas unless some kind of

approximation is provided. Our ansatz is the following. We choose $\delta \to 0$ as an expansion parameter, and we put

$$k =: \frac{1}{\delta}u - \frac{\omega}{v}, \qquad (69)$$

$$\omega_0 =: \frac{1}{\delta}\bar{\omega}_0. \qquad (70)$$

We mean to indicate an expansion where k, unless it is zero, dominates over ω/v; i.e. ω is small relative to a nonzero k, and that also ω_0 is big compared to ω (this kind of approach leads also to the Cauchy approximation; cf. e.g. [26], Chap. 9). As a consequence, we find the following rewriting of (68):

$$\left(u^2 - 2\delta\frac{\omega}{v}u + \delta^2\frac{\omega^2}{\gamma^2 v^2}\right)\left(1 - \frac{\gamma^2 v^2}{\bar{\omega}_0^2}u^2\right) - \chi\gamma^2 v^2 u^2 = 0. \qquad (71)$$

We look for a series solution in u:

$$u = u_0 + \delta u_1 + \delta^2 u_2 + \delta^3 u^3 + \cdots.$$

As a consequence, we expect

$$k = \frac{1}{\delta} u_0 + \left(u_1 - \frac{\omega}{v}\right) + \delta u_2 + \delta^2 u_3 + \cdots.$$

We find at the zeroth order two vanishing degenerate solutions $u_0 = 0$, to be associated with the modes H, B, and also

$$u_{0P} := \frac{\bar{\omega}_0}{\gamma v}\sqrt{1 - \chi\gamma^2\frac{v^2}{c^2}}, \tag{72}$$

$$u_{0N} := -u_{0P}. \tag{73}$$

Corrections for the two nondegenerate solutions in the first order are

$$u_{1P} := -\frac{\omega}{v}\chi\gamma^2\frac{v^2}{c^2}\frac{1}{1 - \chi\gamma^2\frac{v^2}{c^2}}, \tag{74}$$

$$u_{1N} := u_{1P}. \tag{75}$$

Corrections at the first order for the two degenerate zero roots $u_0 = 0$ arise from the second-order contribution to the dispersion relation (as the first-order contribution vanishes identically for $u_0 = 0$): one finds

$$u_{1H} := \frac{\omega}{v}\left(1 + \sqrt{1 + \chi}\frac{v}{c}\right)\frac{1}{1 - \chi\gamma^2\frac{v^2}{c^2}}, \tag{76}$$

$$u_{1B} := \frac{\omega}{v}\left(1 - \sqrt{1 + \chi}\frac{v}{c}\right)\frac{1}{1 - \chi\gamma^2\frac{v^2}{c^2}}. \tag{77}$$

We are not interested in further corrections. It is nice to point out that the expressions we have found are compatible with the Wentzel-Kramers-Brillouin (WKB) behavior of the solutions found in [17].

C. Scattering coefficients

We proceed in the same way as exposed in Secs. IV and V, performing a perturbative expansion of Eq. (67) in the parameter η. We just give the results of the scattering coefficients, analogous to (34):

$$C_1 = \frac{1}{1 + i\eta\tau}, \tag{78}$$

$$C_2 = \frac{i\eta(-1)^{i\frac{k_H - k_P}{2\tilde{\beta}}}}{1 + i\eta\tau}\frac{\pi(k_P^2 - \omega^2)\mathrm{csch}\left(\frac{\pi\gamma(k_H - k_P)}{2\tilde{\beta}}\right)}{2\tilde{\beta}^2 V^2\epsilon^2(k_P - k_N)(k_P - k_B)}, \tag{79}$$

$$C_3 = \frac{i\eta(-1)^{i\frac{k_H - k_N}{2\tilde{\beta}}}}{1 + i\eta\tau}\frac{\pi(k_N^2 - \omega^2)\mathrm{csch}\left(\frac{\pi\gamma(k_H - k_N)}{2\tilde{\beta}}\right)}{2\tilde{\beta}^2 V^2\epsilon^2(k_N - k_P)(k_N - k_B)}, \tag{80}$$

$$C_4 = \frac{i\eta(-1)^{i\frac{k_H - k_B}{2\tilde{\beta}}}}{1 + i\eta\tau}\frac{\pi(k_B^2 - \omega^2)\mathrm{csch}\left(\frac{\pi\gamma(k_H - k_B)}{2\tilde{\beta}}\right)}{2\tilde{\beta}^2 V^2\epsilon^2(k_B - k_P)(k_B - k_N)}, \tag{81}$$

$$\tau = \frac{(k_H^2 - \omega^2)}{\tilde{\beta}V^2\gamma\epsilon^2(k_H - k_P)(k_H - k_N)(k_H - k_B)}. \tag{82}$$

Recalling that, from (68),

$$(k_J^2 - \omega^2) = \frac{\chi\omega_0^2\gamma^2(\omega + Vk_J)^2}{\omega_0^2 - \gamma^2(\omega + Vk_J)^2} \overset{\omega_{\mathrm{lab}}^2 \ll \omega_0^2}{\approx} \chi\gamma^2(\omega + Vk_J)^2, \tag{83}$$

we can see that the expressions of the coefficients (78) basically reduce to (34) at low frequencies. This fact was expected since the Cauchy dispersion relation in an approximation of the Sellmeier for $\omega_{\mathrm{lab}}^2 \ll \omega_0^2$, but it can be viewed also as a check of consistency and robustness of our results.

VIII. COMPARISON WITH THE ORR-SOMMERFELD APPROACH

Equations such as (21) are called of generalized Orr-Sommerfeld type. Such equations emerge often in analog gravity and have been studied extensively in [17]. Here the authors developed a general technique for computing the Hawking spectrum in the *transcritical* case, using a perturbative approach in the low-dispersion parameter. In this section we compare our results to those derived with the Orr-Sommerfeld approach: in particular, we establish a relation between the effective temperature we have defined in Eq. (53) for the subcritical regime, with the Hawking temperature the authors find in [17] for the transcritical case. From Eqs. (5) and (17) we can identify the low-dispersion limit as

$$\epsilon^2 := \frac{1}{g^2} \to 0, \qquad \frac{\mu^2}{g^2} =: \theta = \mathrm{const}, \qquad \frac{\eta}{g^2} =: \rho = \mathrm{const}. \tag{84}$$

Notice that this means that $\mu^2 \propto \epsilon^{-2}$ and $\eta \propto \epsilon^{-2}$. By eliminating the third-order term, Eq. (21) can be written as

$$\epsilon^2 f^{(4)}(x) + p_3(x)f''(x) + p_2(x)f'(x) + p_1(x)f(x) = 0, \tag{85}$$

where

$$p_3(x) = \frac{1 - \theta V^2 + \rho V^2\mathrm{sech}^2(\tilde{\beta}x)}{V^4\gamma^2}, \tag{86}$$

$$p_2(x) = \frac{2i((V^2-1)\omega - 4\tilde{\beta}\rho V^3 \tanh(\tilde{\beta}x)\mathrm{sech}^2(\tilde{\beta}x))}{V^5\gamma^2}, \quad (87)$$

$$p_1(x) = \frac{1}{V^6\gamma^2}(-2\tilde{\beta}^2\rho V^4 \mathrm{sech}^4(\tilde{\beta}x)$$

$$+4\tilde{\beta}^2\rho V^4 \tanh^2(\tilde{\beta}x)\mathrm{sech}^2(\tilde{\beta}x) - (V^2-1)^2\omega^2). \quad (88)$$

Notice that the functions $p_j(x)$ do not depend on ϵ, as in the Corley model. This form is directly comparable with the form in [17]. The position of the horizon is defined by

$$p_3(x) = 0. \quad (89)$$

By changing the variable to $z = -e^{2\tilde{\beta}x}$ as before, we find

$$z_{H\pm} = \frac{g^2 + 2\eta V^2 - \mu^2 V^2 \pm 2\sqrt{g^2\eta V^2 + \eta^2 V^4 - \eta\mu^2 V^4}}{g^2 - \mu^2 V^2}$$

$$= \frac{1 + 2\rho V^2 - \theta V^2 \pm 2\sqrt{\rho V^2 + \rho^2 V^4 - \rho\theta V^4}}{1 - \theta V^2}. \quad (90)$$

The corresponding points $x_{H\pm}$ are real if

$$\left\{\eta < -\frac{(g^2 - \mu^2 V^2)}{V^2} =: -\eta_{\min}\right\} \cup \{\eta > 0\}. \quad (91)$$

In this comparison with the critical regime, we just consider the case of positive Kerr nonlinearity: $\delta n > 0 \Rightarrow \eta < 0$. Thus we neglect positive values of η, and we identify $\eta < -\eta_{\min}$ as the condition of transcriticality of the perturbation. Notice that $\eta_{\min} > 0$. In Fig. 6 we plot the dispersion relation in the critical case.

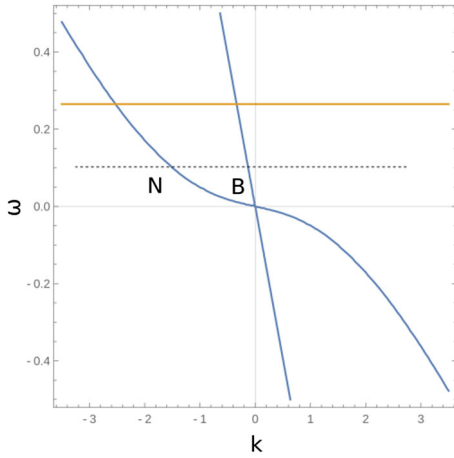In [17] thermal Hawking radiation was predicted with a temperature

FIG. 6. The dispersion relation (18) represented at the peak of the background field for $\eta < -\eta_{\min}$. In this configuration, for any ω, the modes H and P become imaginary as they experience an event horizon: this is referred to as the *transcritical regime*.

$$T_H = \frac{\kappa}{2\pi} = \frac{\gamma^2 V^2 n'(x_H)}{2\pi}, \quad (92)$$

where $n'(x_H)$ is the derivative of the refraction index at the horizon. We compute n' from the Cauchy dispersion formula including the background correction in the comoving frame:

$$n^2(x) = n_0^2 - \frac{\lambda}{2g^2}\psi_B^2(x), \quad (93)$$

where

$$n_0^2 := \frac{\mu^2}{g^2}. \quad (94)$$

Here we do not consider the dependence on ω_{lab} of the refractive index [Eq. (18)], because the Orr-Sommerfeld approach is true in the low-dispersion limit ($\omega_{\mathrm{lab}} \sim 0$): indeed, the event horizon is defined by

$$p_3(x_H) = 0 \quad \Leftrightarrow \quad n^2(x_H) = \frac{1}{V^2}. \quad (95)$$

With these considerations, we find

$$n'(x_H) = \frac{V}{2}(n^2(x_H))' = V\left(-\frac{\lambda}{2g^2}\psi_B^2(x_H)\right)\frac{\psi_B'(x_H)}{\psi_B(x_H)}, \quad (96)$$

and using (93), (95), and (14), we end up with

$$T_H = \frac{\beta\gamma V}{2\pi}\frac{g^2 - \mu^2 V^2}{g^2}\frac{1 + z_H}{1 - z_H}$$

$$= \frac{\beta\gamma V}{2\pi}\frac{g^2 - \mu^2 V^2}{g^2}\sqrt{\frac{|\eta| - \eta_{\min}}{|\eta|}}. \quad (97)$$

This result is surprisingly similar to (53), although they were computed in very different ways, one within the critical regime and the other in the subcritical. We notice that if the expression (53) for T_{pert} is computed for near critical velocity

$$n_0^2 = \frac{\mu^2}{g^2} \lesssim \frac{1}{V^2}, $$

and compare it to (97) in the very critical case ($|\eta| \gg \eta_{\min}$), we find

$$T_{\mathrm{pert}} \approx \frac{T_H}{3}. \quad (98)$$

We are not able to give a quantitative interpretation of the missing factor of 3 with respect to the case of real turning points. Note that we obtain $T_{\mathrm{pert}} \propto T_H$, and the

proportionality coefficient becomes 1/3 just for the suitable limits of near criticality for T_{pert} and very large η for T_H. It is in any case remarkable that our result, qualitatively, is just consistent with the previous literature, as, in the subcritical case, a reduced pair-production, due to the less efficient mode-conversion occurring in the absence of a true horizon (real turning point), must be expected. Moreover, we may suggest that it could be an effect of considering a non-monotonic background such as (11), whereas the results of the Orr-Sommerfeld approach in [17] were computed assuming a monotonic background. In any case, the similarity of the critical and subcritical temperatures is interesting, and it suggests that they are related by some physical mechanism: this will be a matter for future studies.

The same estimations can be done using the $\phi\psi$ model, introduced in Sec. VII. We set $c = 1$. The temperature estimated from the coefficients (78) is

$$\beta_{\text{pert}} = \frac{1}{T_{\text{pert}}} = \frac{2\pi\gamma}{2\beta} \lim_{\omega \to 0} \frac{2k_H - k_P - k_N}{\omega}$$

$$= \frac{2\pi(1 + \gamma^2 V^2 \chi_0 + V\sqrt{\chi_0 + 1})}{\beta\gamma V(1 - V^2(\chi_0 + 1))}. \qquad (99)$$

For near-critical velocity

$$n_0^2 = 1 + \chi_0 \lesssim \frac{1}{V^2}, \qquad (100)$$

we see that the term between brackets in the numerator of (99) is

$$\left(1 + \gamma^2 V^2 \chi_0 + V\sqrt{\chi_0 + 1}\right) \approx 3, \qquad (101)$$

which gives

$$T_{\text{pert}} \approx \frac{\beta\gamma V(1 - V^2(\chi_0 + 1))}{6\pi}. \qquad (102)$$

The analysis of the critical case with the Orr-Sommerfeld approach gives that criticality is found for

$$\eta < -\eta_{\text{min}}, \qquad \eta_{\text{min}} := \frac{1}{\chi_0}(1 - \chi_0 \gamma^2 V^2) > 0,$$

and the Hawking temperature is

$$T_H = \frac{\beta}{2\pi\chi_0 V\gamma}(1 - V^2(1 + \chi_0))\sqrt{\frac{|\eta| - \eta_{\text{min}}}{|\eta|}}$$

$$\approx \frac{\beta\gamma V}{2\pi}(1 - V^2(1 + \chi_0))\sqrt{\frac{|\eta| - \eta_{\text{min}}}{|\eta|}}, \qquad (103)$$

where in the last line we used (100) as before. We see that, once again, the relation between the two temperatures (for $|\eta| \gg \eta_{\text{min}}$) is

$$T_{\text{pert}} \approx \frac{T_H}{3}, \qquad (104)$$

confirming what we found using the Cauchy model.

IX. CONCLUSIONS

We proposed a new approach to Hawking-like radiation in the subcritical case for a particular, but arguably realistic, class of solitonlike backgrounds in a nonlinear dielectric. The method allows a straightforward analytical solution of the scattering problem at the leading perturbation order of the amplitude of the background field, represented by the parameter η. With respect to other existing techniques, our approach does not rely on the approximation of weak dispersion, and indeed our predictions are not restricted to the $\omega \sim 0$ region. We tested our approach on a simplified model of scalar electrodynamics and checked its robustness on the scalar field reduction of the Hopfield model, which strongly corroborates the results obtained in the simplified model. In both cases, we can define an effective temperature associated with the spectrum of the emitted radiation, which, in the limit where the subcritical case approaches the transcritical one, is one-third of the Hawking temperature estimated by other established methods for the transcritical regime. Even though a quantitative interpretation of this fact is not yet available to us, we can stress that a reduced effective temperature is still to be expected, consistently with what is known in literature for the subcritical case, where a reduced pair-production occurs, due to the less efficient mode-conversion occurring in the absence of a true horizon. Future work will be focused on the transition to the critical regime, and it will possibly give us a clearer view.

The perturbative expansion we propose, although being naturally suited for the study of the subcritical regime, is not theoretically limited to this case: the problem of the transition to the transcritical regime is configured, in this context, as possibly a matter of being able to compute enough perturbative orders. This can be corroborated by the fact that a new mathematical perspective on the phenomenon of particle creation was provided, as we related particle creation to the solution of a fourth-order Fuchsian equation. Fuchsian equations have many well-studied properties, and much of that theoretical machinery may be applied to the problem of analog Hawking radiation, potentially extending our analytical comprehension of the phenomenon. This will be a matter for future studies.

APPENDIX A: THE CONSERVED CURRENT

We compute the current vector, evaluated on normal modes $e^{ik_\mu x^\mu}$ in a generic frame moving with velocity V with respect to the laboratory. From Eq. (15) we find that if $\psi(x) = e^{ik_\mu x^\mu}$, then $\phi(x) = -i\frac{gn^\mu k_\mu}{(v^\mu k_\mu)^2}e^{ik_\mu x^\mu}$: substituting these fields into (7) we find

$$J^\mu \propto v^\mu \omega_{\text{lab}}\left(1 + \frac{\mu^2 + \omega_{\text{lab}}^2}{\omega_{\text{lab}}^2}\right) + n^\mu \frac{g^2 k_{\text{lab}}}{\omega_{\text{lab}}^2}, \quad \text{(A1)}$$

where $v^\mu = (\gamma, -V\gamma)$, $n^\mu = (-\gamma V, \gamma)$, $\omega_{\text{lab}} = v^\mu k_\mu = \gamma(\omega + Vk)$, and $k_{\text{lab}} = -n^\mu k_\mu = \gamma(k + V\omega)$. We can verify that J^μ is timelike. Indeed,

$$J^\mu J_\mu = \omega_{\text{lab}}^2\left(1 + \frac{2(\mu^2 + \omega_{\text{lab}}^2)}{\omega_{\text{lab}}^2} + \frac{(\mu^2 + \omega_{\text{lab}}^2)^2}{\omega_{\text{lab}}^4}\right.$$
$$\left. - \frac{g^2(\mu^2 + \omega_{\text{lab}}^2)}{\omega_{\text{lab}}^4}\right), \quad \text{(A2)}$$

which is positive provided that $\mu^2 > g^2$, which is the condition we required at the beginning [see below Eq. (5)]. Thus J^μ is timelike, so the sign of J^0 is constant in every inertial reference frame: since in the lab frame $\text{sign}(J^0) = \text{sign}(\omega_{\text{lab}})$, this must be true in every frame.

We now define the dispersion relation function

$$\text{DR}(\omega, k) := \mu^2 + \omega_{\text{lab}}^2 - \frac{g^2 k_{\text{lab}}^2}{\omega_{\text{lab}}^2}. \quad \text{(A3)}$$

The free normal modes of the theory satisfy $\text{DR}(\omega, k) = 0$. It is easy to show that the following relations hold:

$$J^0 \propto \partial_\omega \text{DR}|_{\text{DR}=0}, \quad \text{(A4)}$$

$$J^x \propto -\partial_k \text{DR}|_{\text{DR}=0}. \quad \text{(A5)}$$

Thus, we can identify the measure of the Hilbert space of free normal modes as

$$d\mu_j := \frac{d\omega}{2\pi \partial_k \text{DR}|_{k=k_j(\omega)}}, \quad \text{(A6)}$$

where we are writing the field theory in the frequency representation and k_j are the different *real* solutions of $\text{DR} = 0$. It is also straightforward to show

$$J^x = v_g(\omega)J^0, \quad \text{(A7)}$$

where $v_g(\omega) = \frac{d\omega}{dk}|_{\text{DR}=0}$ is the group velocity of the normal modes. This relation could be derived also from the implicit function theorem, which states $\frac{\partial_k \text{DR}}{\partial_\omega \text{DR}} = -\frac{d\omega}{dk}|_{\text{DR}=0}$.

We can write the flux J^x also in a more convenient way for future computations. Since we called $k_H(\omega)$, $k_P(\omega)$,

$k_N(\omega)$, and $k_B(\omega)$ the four solutions of the dispersion relation, we can write Eq. (A3) as

$$\text{DR}(\omega, k) = \frac{\gamma^2 V^4}{\omega_{\text{lab}}^2} \prod_{j=H,P,N,B} (k - k_j(\omega)). \quad \text{(A8)}$$

It is now easy to verify that

$$\partial_\omega \text{DR}|_{k=k_i(\omega)} = -\frac{\gamma^2 V^4}{\omega_{\text{lab}}^2|_{k_i}} \frac{\partial k_i}{\partial \omega} \prod_{j \neq i}(k_i(\omega) - k_j(\omega)), \quad \text{(A9)}$$

and then

$$J^x(e^{-i\omega t + ik_i(\omega)x}) = v_i(\omega)\partial_\omega \text{DR}|_{k_i}$$
$$= -\frac{\gamma^2 V^4}{\omega_{\text{lab}}^2|_{k_i}} \prod_{j \neq i}(k_i(\omega) - k_j(\omega)). \quad \text{(A10)}$$

APPENDIX B: THE CONSERVATION LAW

To understand the conservation law (43), we must consider that the initial and final states of the scattering should be *wave packets* rather than plane waves. A normalized wave packet centered around the frequency ω and momentum $k_j(\omega)$ [here $k_j(\omega)$ is any of the solutions of the dispersion relation] has the form

$$\int \frac{d\omega}{2\pi \partial_k \text{DR}|_{k_j(\omega)}} f_\epsilon(\omega) e^{i\omega t - ik_j(\omega)x}, \quad \text{(B1)}$$

where $\frac{d\omega}{2\pi \partial_k \text{DR}|_{k_j(\omega)}}$ is the measure of the Hilbert space defined by plane waves (in the frequency representation) and $f_\epsilon(\omega)$ is a distribution centered around ω with a small bandwidth ϵ, which satisfies

$$\int \frac{d\omega}{2\pi \partial_k \text{DR}|_{k_j(\omega)}} |f_\epsilon(\omega)|^2 = 1.$$

We can see Eq. (B1) represents a wave packet by evaluating it at $x = vt$, for some velocity v: for $t = \pm\infty$ we can say that the dominant contribution comes from the stationary point, which is

$$1 - V\frac{\partial k_j}{\partial \omega} = 0,$$

which means that the function (B1) is traveling at a constant velocity $v = v_j(\omega)$, where $v_j(\omega)$ is the group velocity of the mode k_j.

The asymptotic plane-wave solution (33) corresponds to the wave packets

$$\psi(x,t) \sim \int \frac{d\omega}{2\pi\partial_k \mathrm{DR}|_{k_H(\omega)}} f_\epsilon(\omega)e^{i\omega t}\begin{cases} e^{-ik_H x} + C_2 e^{-ik_P x} + C_3 e^{-ik_N x} + C_4 e^{-ik_B x}, & x \to -\infty, \\ C_1 e^{-ik_H x}, & x \to +\infty. \end{cases} \tag{B2}$$

At $t = -\infty$ the only contribution comes from the ingoing H-mode, so the measure of integration is chosen such that

$$(\psi|\psi)_{t=-\infty} = 1 \equiv \int dx J^0(\psi(x, t = -\infty)).$$

The current conservation implies

$$\frac{d}{dt}\int dx J^0 = \int dx \partial_0 J^0 = \int dx \partial_x J^x = 0. \tag{B3}$$

At $t = +\infty$ the solution splits into four outgoing localized wave packets with central momenta: each mode propagates with a different group velocity, so we can assume that the wave packets are not overlapping at $t = +\infty$. With these considerations, the computation of the norm gives

$$\int \frac{d\omega}{2\pi\partial_k \mathrm{DR}|_{k_H(\omega)}} \frac{d\omega'}{2\pi\partial_k \mathrm{DR}|_{k_H(\omega')}} f_\epsilon(\omega)f_\epsilon^*(\omega') \sum_{i=H,P,N,B} |C_i|^2 2\pi\partial_\omega \mathrm{DR}|_{k_i} \delta(k_H(\omega) - k_H(\omega'))$$

$$= \int \frac{d\omega}{2\pi\partial_k \mathrm{DR}_{k_H}} |f_\epsilon(\omega)|^2 \frac{1}{2\pi\partial_k \mathrm{DR}_{k_H}} \sum_{i=H,P,N,B} |C_i|^2 2\pi v_i \partial_\omega \mathrm{DR}|_{k_i} \equiv 1. \tag{B4}$$

The last equality comes from current conservation. By recalling $\partial_k \mathrm{DR}|_{k_j(\omega)} = -v_j(\omega)\partial_\omega \mathrm{DR}|_{k_j(\omega)}$ (from the implicit function theorem) we see that the sign of each term is determined by the product $v_j(\omega)\omega_{\mathrm{lab}}$. Since the equality must hold for every distribution f_ϵ, we finally obtain the following relation between the absolute values:

$$1 = |C_1|^2 + |C_2|^2\left|\frac{v_P \partial_\omega \mathrm{DR}|_{k_P}}{v_H \partial_\omega \mathrm{DR}|_{k_H}}\right| - |C_3|^2\left|\frac{v_N \partial_\omega \mathrm{DR}|_{k_N}}{v_H \partial_\omega \mathrm{DR}|_{k_H}}\right|$$

$$+ |C_4|^2\left|\frac{v_B \partial_\omega \mathrm{DR}|_{k_B}}{v_H \partial_\omega \mathrm{DR}|_{k_H}}\right|, \tag{B5}$$

which is precisely (43).

APPENDIX C: SOLUTION OF FIRST-ORDER EQUATION

We start computing the first-order solution $u_1(z)$ [Eq. (32)] around $z = 0$ ($x = -\infty$). We proceed as exposed in Sec. IV to derive the expression of the first-order solution. In the scattering of the H-mode, we expect the asymptotic coefficient of the H-mode to be $O(1)$, while all the others should be $O(\eta)$ (see Sec. V): for this reason, the zero-order solution we put into the source term (31) corresponds to a single H-mode, which means

$$u_0(z) = z^{i\alpha_H}, \qquad i\alpha_H = i\frac{k_H}{2\beta} + \frac{3}{2} + i\frac{\Omega}{V}. \tag{C1}$$

From this choice, we get

$$r_1(z) = \frac{z^{-3+i\alpha_H}(4\alpha_H^2(z-1)^2 + 4i\alpha_H(5z^2 - 6z + 1) - 25z^2 + 2z - 1)}{4(z-1)^4 V^2 y^2}, \tag{C2}$$

$$\frac{W_1(z)}{W(z)} = \frac{i(4\alpha_H^2(z-1)^2 + 4i\alpha_H(5z^2 - 6z + 1) - 25z^2 + 2z - 1)}{4(z-1)^4 V^2 \gamma^2(\alpha_H - \alpha_P)(\alpha_H - \alpha_N)(\alpha_H - \alpha_B)}, \tag{C3}$$

$$\frac{W_2(z)}{W(z)} = -\frac{i(4\alpha_H^2(z-1)^2 + 4i\alpha_H(5z^2 - 6z + 1) - 25z^2 + 2z - 1)z^{i(\alpha_H - \alpha_P)}}{4(z-1)^4 V^2 \gamma^2(\alpha_P - \alpha_H)(\alpha_P - \alpha_N)(\alpha_P - \alpha_B)}, \tag{C4}$$

$$\frac{W_3(z)}{W(z)} = -\frac{i(4\alpha_H^2(z-1)^2 + 4i\alpha_H(5z^2 - 6z + 1) - 25z^2 + 2z - 1)z^{i(\alpha_H - \alpha_N)}}{4(z-1)^4 V^2 \gamma^2(\alpha_N - \alpha_H)(\alpha_N - \alpha_P)(\alpha_N - \alpha_B)}, \tag{C5}$$

$$\frac{W_4(z)}{W(z)} = -\frac{i(4\alpha_H^2(z-1)^2 + 4i\alpha_H(5z^2 - 6z + 1) - 25z^2 + 2z - 1)z^{i(\alpha_H - \alpha_B)}}{4(z-1)^4 V^2 \gamma^2(\alpha_B - \alpha_H)(\alpha_B - \alpha_P)(\alpha_B - \alpha_N)}. \tag{C6}$$

Computing the integrals in (32), we get

$$
\begin{aligned}
\tilde{u}_1(z) = \frac{iz^{i\alpha_H}}{4V^2\gamma^2} \Bigg(& \frac{-4\alpha_H^2(z-1)^2 - 4i\alpha_H(5z^2 - 8z + 3) + 25z^2 - 26z + 9}{(z-1)^3(\alpha_H - \alpha_P)(\alpha_H - \alpha_N)(\alpha_H - \alpha_B)} \\
& + \frac{z(i(2\alpha_H + 5i)^2 {}_2F_1(2, i(\alpha_H - \alpha_P) + 1, i(\alpha_H - \alpha_P) + 2; z) + 16(\alpha_H + 3i)_2F_1(3, i(\alpha_H - \alpha_P) + 1, i(\alpha_H - \alpha_P) + 2; z))}{(\alpha_H - \alpha_P)(\alpha_H - \alpha_P - i)(\alpha_P - \alpha_N)(\alpha_P - \alpha_B)} \\
& - \frac{z(24i_2F_1(4, i(\alpha_H - \alpha_P) + 1, i(\alpha_H - \alpha_P) + 2; z))}{(\alpha_H - \alpha_P)(\alpha_H - \alpha_P - i)(\alpha_P - \alpha_N)(\alpha_P - \alpha_B)} \\
& + \frac{z(i(2\alpha_H + 5i)^2 {}_2F_1(2, i(\alpha_H - \alpha_N) + 1, i(\alpha_H - \alpha_N) + 2; z) + 16(\alpha_H + 3i)_2F_1(3, i(\alpha_H - \alpha_N) + 1, i(\alpha_H - \alpha_N) + 2; z))}{(\alpha_H - \alpha_N)(\alpha_H - \alpha_N - i)(\alpha_N - \alpha_P)(\alpha_N - \alpha_B)} \\
& - \frac{z(24i_2F_1(4, i(\alpha_H - \alpha_N) + 1, i(\alpha_H - \alpha_N) + 2; z))}{(\alpha_H - \alpha_N)(\alpha_H - \alpha_N - i)(\alpha_N - \alpha_P)(\alpha_N - \alpha_B)} \\
& + \frac{z(i(2\alpha_H + 5i)^2 {}_2F_1(2, i(\alpha_H - \alpha_B) + 1, i(\alpha_H - \alpha_B) + 2; z) + 16(\alpha_H + 3i)_2F_1(3, i(\alpha_H - \alpha_B) + 1, i(\alpha_H - \alpha_B) + 2; z))}{(\alpha_H - \alpha_B)(\alpha_H - \alpha_B - i)(\alpha_B - \alpha_P)(\alpha_B - \alpha_N)} \\
& - \frac{z(24i_2F_1(4, i(\alpha_H - \alpha_B) + 1, i(\alpha_H - \alpha_B) + 2; z))}{(\alpha_H - \alpha_B)(\alpha_H - \alpha_B - i)(\alpha_B - \alpha_P)(\alpha_B - \alpha_N)} \Bigg).
\end{aligned}
\tag{C7}
$$

This is a *particular* solution of (29): the general expression is obtained by adding a linear combination of free-field solutions (with coefficients of order $O(\eta)$). The coefficients of such combinations will be determined later based on the boundary conditions. From (C7) we can compute the asymptotic expression of $\tilde{u}_1$ around $z = 0$ ($x = -\infty$):

$$
\tilde{u}_1(z \approx 0) = c_H z^{i\alpha_H}(1 + O(z)),
$$

$$
c_H = -\frac{i(-4\alpha_H^2 - 12i\alpha_H + 9)}{4V^2\gamma^2(\alpha_H - \alpha_P)(\alpha_H - \alpha_N)(\alpha_H - \alpha_B)}. \tag{C8}
$$

The function (C7) is defined on the whole complex plane: to write the asymptotic expression at $z = \infty$ ($x = +\infty$) we use the connection formulas of the hypergeometric function

$$
\begin{aligned}
{}_2F_1&(a, b, c; z) \\
&= \frac{(-z)^{-a}\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)} {}_2F_1\left(a, a - c + 1; a - b + 1; \frac{1}{z}\right) \\
&+ \frac{(-z)^{-b}\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)} {}_2F_1\left(b, b - c + 1; -a + b + 1; \frac{1}{z}\right).
\end{aligned}
\tag{C9}
$$

Expanding around $z = \infty$ we find

$$
\tilde{u}_1(z \approx \infty) = -\sum_{J = P, N, B} c_J z^{i\alpha_J}(1 + O(z)),
$$

$$
c_P = -\frac{i\pi(2\alpha_P + 3i)^2(-1)^{i(\alpha_H - \alpha_P)}\operatorname{csch}(\pi(\alpha_H - \alpha_P))}{4V^2\gamma^2(\alpha_P - \alpha_N)(\alpha_P - \alpha_B)},
$$

$$
c_N = -\frac{i\pi(2\alpha_N + 3i)^2(-1)^{i(\alpha_H - \alpha_N)}\operatorname{csch}(\pi(\alpha_H - \alpha_N))}{4V^2\gamma^2(\alpha_N - \alpha_P)(\alpha_N - \alpha_B)},
$$

$$
c_B = -\frac{i\pi(2\alpha_B + 3i)^2(-1)^{i(\alpha_H - \alpha_B)}\operatorname{csch}(\pi(\alpha_H - \alpha_B))}{4V^2\gamma^2(\alpha_B - \alpha_N)(\alpha_B - \alpha_P)}.
$$

$$\tag{C10}$$

From the boundary conditions (33) we see that we do not have the modes P, N, and B at right infinity; so we add to the particular solution the linear combination

$$
c_P z^{i\alpha_P} + c_N z^{i\alpha_N} + c_B z^{i\alpha_B}.
$$

The first-order correction to the solution is thus

$$
u_1(z) = \tilde{u}_1(z) + c_P z^{i\alpha_P} + c_N z^{i\alpha_N} + c_B z^{i\alpha_B},
$$

such that $u_1(z \approx \infty) = 0$. Finally, we can write down the solution $u(z)$ at the first perturbative order and its asymptotic behavior:

$$
u(z \to 0) \sim (1 + \eta c_H)z^{i\alpha_H} + \eta c_P z^{i\alpha_P} + \eta c_N z^{i\alpha_N} + \eta c_B z^{i\alpha_B},
$$

$$\tag{C11}$$

$$
u(z \to \infty) \sim z^{i\alpha_H}. \tag{C12}
$$

The coefficients (34) are obtained dividing (C11) by $(1 + \eta c_H)$ and recalling $i\alpha_j = i\frac{k_j}{2\beta} + \frac{3}{2} + i\frac{\Omega}{V}$.

[1] S. Weinfurtner, E. W. Tedford, M. C. J. Penrice, W. G. Unruh, and G. A. Lawrence, Measurement of Stimulated Hawking Emission in an Analogue System, Phys. Rev. Lett. **106,** 021302 (2011).

[2] S. Weinfurtner, E. W. Tedford, M. C. J. Penrice, W. G. Unruh, and G. A. Lawrence, Classical aspects of Hawking radiation verified in analogue gravity experiment, Lect. Notes Phys. **870,** 167 (2013).

[3] F. Michel and R. Parentani, Probing the thermal character of analogue Hawking radiation for shallow water waves?, Phys. Rev. D **90,** 044033 (2014).

[4] L. P. Euvé, F. Michel, R. Parentani, and G. Rousseaux, Wave blocking and partial transmission in subcritical flows over an obstacle, Phys. Rev. D **91,** 024020 (2015).

[5] S. Robertson, F. Michel, and R. Parentani, Scattering of gravity waves in subcritical flows over an obstacle, Phys. Rev. D **93,** 124060 (2016).

[6] A. Coutant and S. Weinfurtner, The imprint of the analogue Hawking effect in subcritical flows, Phys. Rev. D **94,** 064026 (2016).

[7] H. Bremmer, The W.K.B. approximation as the first term of a geometric-optical series, Commun. Pure Appl. Math. **4,** 105 (1951).

[8] H. Bremmer, The propagation of electromagnetic waves through a stratified medium and its W.K.B. approximation for oblique incidence, Physica (Amsterdam) **15,** 593 (1949).

[9] A. Coutant and S. Weinfurtner, Low-frequency analogue Hawking radiation: The Korteweg–de Vries model, Phys. Rev. D **97,** 025005 (2018).

[10] A. Coutant and S. Weinfurtner, Low frequency analogue Hawking radiation: The Bogoliubov–de Gennes model, Phys. Rev. D **97,** 025006 (2018).

[11] G. Rousseaux, C. Mathis, P. Maïssa, T. G. Philbin, and U. Leonhardt, Observation of negative phase velocity waves in a water tank: A classical analogue to the Hawking effect?, New J. Phys. **10,** 053015 (2008).

[12] L. P. Euvé, F. Michel, R. Parentani, T. G. Philbin, and G. Rousseaux, Observation of Noise Correlated by the Hawking Effect in a Water Tank, Phys. Rev. Lett. **117,** 121301 (2016).

[13] L. P. Euvé, S. Robertson, N. James, A. Fabbri, and G. Rousseaux, Scattering of Co-Current Surface Waves on an Analogue Black Hole, Phys. Rev. Lett. **124,** 141101 (2020).

[14] J. Fourdrinoy, S. Robertson, N. James, A. Fabbri, and G. Rousseaux, Correlations on weakly time-dependent transcritical white-hole flows, Phys. Rev. D **105,** 085022 (2022).

[15] S. Finazzi and I. Carusotto, Quantum vacuum emission in a nonlinear optical medium illuminated by a strong laser pulse, Phys. Rev. A **87,** 023803 (2013).

[16] S. Finazzi and I. Carusotto, Spontaneous quantum emission from analog white holes in a nonlinear optical medium, Phys. Rev. A **89,** 053807 (2014).

[17] F. Belgiorno, S. L. Cacciatori, and A. Viganò, Analog Hawking effect: A master equation, Phys. Rev. D **102,** 105003 (2020).

[18] F. Belgiorno, S. L. Cacciatori, A. Farahat, and A. Viganò, Analog Hawking effect: BEC and surface waves, Phys. Rev. D **102,** 105004 (2020).

[19] T. Nishimoto, On the Orr-Sommerfeld type equations, II Connection formulas, Kōdai Math. Sem. Rep. **29,** 233 (1978).

[20] T. Nishimoto, On the Orr-Sommerfeld type equations, I W.K.B. approximation, Kōdai Math. Sem. Rep. **24,** 281 (1972).

[21] F. Belgiorno, S. L. Cacciatori, and F. Dalla Piazza, Hawking effect in dielectric media and the Hopfield model, Phys. Rev. D **91,** 124063 (2015).

[22] F. Belgiorno and S. L. Cacciatori, Analogous Hawking effect in dielectric media and solitonic solutions, Universe **6,** 127 (2020).

[23] J. J. Hopfield, Theory of the contribution of excitons to the complex dielectric constant of crystals, Phys. Rev. **112,** 1555 (1958).

[24] A. V. Chueshev and V. V. Chueshev, Variational formulas of the monodromy group for a third-order equation on a compact Riemann surface, J. Sib. Fed. Univ. Math. Phys. **15,** 308 (2022).

[25] M. Petev, N. Westerberg, D. Moss, E. Rubino, C. Rimoldi, S. L. Cacciatori, F. Belgiorno, and D. Faccio, Blackbody Emission from Light Interacting with an Effective Moving Dispersive Medium, Phys. Rev. Lett. **111,** 043902 (2013).

[26] F. Belgiorno, S. L. Cacciatori, and D. Faccio, *Hawking Radiation* (World Scientific Publishing Company, Singapore, 2018).

[27] E. Rubino, A. Lotti, F. Belgiorno, S. L. Cacciatori, A. Couairon, U. Leonhardt, and D. Faccio, Soliton-induced relativistic-scattering and amplification, Sci. Rep. **2,** 932 (2012).

[28] E. Rubino, F. Belgiorno, S. L. Cacciatori, and D. Faccio, Laser pulse analogues for gravity, in *Analogue Gravity Phenomenology*, Lecture Notes in Physics Vol. 870, edited by D. Faccio, F. Belgiorno, S. L. Cacciatori, V. Gorini, S. Liberati, and U. Moschella (Springer, New York, 2013), pp. 247—273, ISBN 978-331900265-1.

[29] F. Belgiorno, S. L. Cacciatori, F. Dalla Piazza, and M. Doronzo, Hopfield-Kerr model and analogue black hole radiation in dielectrics, Phys. Rev. D **96,** 096024 (2017).