

CONVERGENT SUBSERIES OF DIVERGENT SERIES

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ABSTRACT. Let \mathcal{X} be the set of positive real sequences $x = (x_n)$ such that the series $\sum_n x_n$ is divergent. For each $x \in \mathcal{X}$, let \mathcal{I}_x be the collection of all $A \subseteq \mathbf{N}$ such that the subseries $\sum_{n \in A} x_n$ is convergent. Moreover, let \mathcal{A} be the set of sequences $x \in \mathcal{X}$ such that $\lim_n x_n = 0$ and $\mathcal{I}_x \neq \mathcal{I}_y$ for all sequences $y = (y_n) \in \mathcal{X}$ with $\liminf_n y_{n+1}/y_n > 0$. We show that \mathcal{A} is comeager and that contains uncountably many sequences x which generate pairwise nonisomorphic ideals \mathcal{I}_x . This answers, in particular, an open question recently posed by M. Filipczak and G. Horbaczewska.

1. INTRODUCTION

Let \mathcal{X} be the set of positive real sequences $x = (x_n)$ with divergent series $\sum_n x_n$. For each $x \in \mathcal{X}$, let \mathcal{I}_x the collection of sets of positive integers A such that the (possibly finite) subseries indexed by A is convergent, that is,

$$\mathcal{I}_x := \left\{ A \subseteq \mathbf{N} : \sum_{n \in A} x_n < \infty \right\}. \quad (1)$$

Note that each \mathcal{I}_x is closed under finite unions and subsets, i.e., it is an *ideal*. Moreover, it contains the collection Fin of finite sets $A \subseteq \mathbf{N}$, and it is different from the power set $\mathcal{P}(\mathbf{N})$. Following [4], a collection of sets of the type (1) is called *summable ideal*. It is not difficult to see that every infinite set of \mathbf{N} contains an infinite subset in \mathcal{I}_x if and only if $\lim_n x_n = 0$. Accordingly, define

$$\mathcal{Z} := \mathcal{X} \cap c_0 = \{x \in \mathcal{X} : \lim_{n \rightarrow \infty} x_n = 0\}.$$

It is known that the families \mathcal{I}_x defined in (1) are “small”, both in the measure-theoretic sense and the categorical sense, meaning that “almost all” subseries diverge, see [3, 6, 13, 16]. Related results in the context of filter convergence have been given in [1, 2, 10]. The set of limits of convergent subseries of a given series $\sum_n x_n$, which is usually called “achievement set”, has been studied in [7, 9, 12]. Of special interest have been specific subseries of the harmonic series $\sum_n \frac{1}{n}$; see, e.g., [11, 14, 15, 17].

Roughly, the question that we are going to answer is the following: Is it true that for each $x \in \mathcal{Z}$ there exists $y \in \mathcal{X}$ such that $\mathcal{I}_x = \mathcal{I}_y$ and y_n “does not oscillates too much”?

Hoping for a characterization of the class of summable ideals \mathcal{I}_x with $x \in \mathcal{Z}$, M. Filipczak and G. Horbaczewska asked recently in [5] the following:

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Question 1.1. Is it true that for each $x \in \mathcal{L}$ there exists $y \in \mathcal{X}$ such that $\mathcal{I}_x = \mathcal{I}_y$ and

$$\forall n \in \mathbf{N}, \quad \frac{y_{n+1}}{y_n} \geq \frac{n}{n+2} ?$$

We show in Theorem 1.3 below that the answer is negative in a strong sense. To this aim, define

$$\mathcal{Y} := \left\{ y \in \mathcal{X} : \liminf_{n \rightarrow \infty} \frac{y_{n+1}}{y_n} > 0 \right\},$$

and let \sim be the equivalence relation on \mathcal{X} so that two sequences are identified if they generate the same ideal, so that

$$\forall x, y \in \mathcal{X}, \quad x \sim y \iff \left(\forall A \subseteq \mathbf{N}, \sum_{n \in A} x_n < \infty \iff \sum_{n \in A} y_n < \infty \right).$$

First, we show that the set of pairs $(x, y) \in \mathcal{X}^2$ such that x is \sim -equivalent to y is topologically well behaved:

Proposition 1.2. *\sim is a coanalytic relation on \mathcal{X} .*

Then, we answer Question 1.1 by showing that:

Theorem 1.3. *There exists $x \in \mathcal{L}$ such that $x \not\sim y$ for all $y \in \mathcal{Y}$.*

In light of the explicit example which will be given in the proof of Theorem 1.3, one may ask about the topological largeness of the set of such sequences. To be precise, is it true that

$$\mathcal{A} := \{x \in \mathcal{L} : \forall y \in \mathcal{Y}, x \not\sim y\}$$

is a set of second Baire category, i.e., not topologically small? Note that the question is really meaningful since \mathcal{L} is completely metrizable (hence by Baire's category theorem \mathcal{L} is not meager in itself): this follows by Alexandrov's theorem [8, Theorem 3.11] and the fact that

$$\mathcal{L} = \bigcap_{n \geq 1} \{x \in c_0 : x_n > 0\} \cap \bigcap_{m \geq 1} \bigcup_{k \geq 1} \{x \in c_0 : x_1 + \dots + x_k > m\}$$

is a G_δ -subset of the Polish space c_0 . With the premises, we show that \mathcal{A} is comeager, that is, $\mathcal{L} \setminus \mathcal{A}$ is a set of first Baire category:

Theorem 1.4. *\mathcal{A} is comeager in \mathcal{L} . In particular, \mathcal{A} is uncountable.*

We remark that Theorem 1.4 gives an additional information on relation \sim . Since it is a coanalytic equivalence relation by Proposition 1.2, we can appeal to the deep theorem by Silver [8, Theorem 35.20] which states that every coanalytic equivalence relation on a Polish space either has countably many equivalence classes or there is a perfect set consisting of non-equivalent pairs. Thanks to Theorem 1.4, the latter holds for the relation \sim in a strong form. Indeed, every pair in $\mathcal{A} \times \mathcal{Y}$ does not belong to \sim , where \mathcal{A} is comeager (hence it contains a G_δ -comeager subset) and \mathcal{Y} is an uncountable F_σ -set. Therefore $\mathcal{A} \times \mathcal{Y}$ contains a product of two perfect sets by [8, Theorem 13.6].

Lastly, on a similar direction, we strenghten the fact that \mathcal{A} is uncountable by proving that exist uncountably many sequences in \mathcal{A} which generate pairwise nonisomorphic ideals (here, recall that two ideals \mathcal{I}, \mathcal{J} are isomorphic if there exists a bijection $f : \mathbf{N} \rightarrow \mathbf{N}$ such that $f[A] \in \mathcal{I}$ if and only if $A \in \mathcal{J}$ for all $A \subseteq \mathbf{N}$).

Theorem 1.5. *There are \mathfrak{c} sequences in \mathcal{A} which generate pairwise nonisomorphic ideals.*

Hereafter, we use the convention that $\sum_{n \geq 1} a_n \ll \sum_{n \geq 1} b_n$, with each $a_n, b_n > 0$, is a shorthand for the existence of $C > 0$ such that $\sum_{n \leq k} a_n \leq C \sum_{n \leq k} b_n$ for all $k \in \mathbf{N}$.

2. PROOF OF PROPOSITION 1.2

Equivalently, we have to show that the set $E := \{(x, y) \in \mathcal{X}^2 : x \not\sim y\}$ is analytic in \mathcal{X}^2 . For, note that E is the projection on \mathcal{X}^2 of $E_1 \cup E_2$, where

$$E_1 := \{(A, x, y) \in \mathcal{P}(\mathbf{N}) \times \mathcal{X}^2 : A \in \mathcal{I}_x \setminus \mathcal{I}_y\}$$

and, similarly,

$$E_2 := \{(A, x, y) \in \mathcal{P}(\mathbf{N}) \times \mathcal{X}^2 : A \in \mathcal{I}_y \setminus \mathcal{I}_x\}.$$

Now, for each $n \in \mathbf{N}$, define the functions $\alpha_n, \beta_n : \mathcal{P}(\mathbf{N}) \times \mathcal{X}^2 \rightarrow \mathbf{R}$ by $\alpha_n(A, x, y) = \sum x_k$ and $\beta_n(A, x, y) = \sum y_k$, where each sum is extended over all $k \in A$ such that $k \leq n$. Since they are continuous, the set $(\alpha_n \leq k) := \{(A, x, y) \in \mathcal{P}(\mathbf{N}) \times \mathcal{X}^2 : \alpha_n(A, x, y) \leq k\}$ is closed and $(\beta_n > k)$ is open for all $n, k \in \mathbf{N}$. Therefore

$$E_1 = \left(\bigcup_{k \geq 1} \bigcap_{n \geq 1} (\alpha_n \leq k) \right) \cap \left(\bigcap_{k \geq 1} \bigcup_{n \geq 1} (\beta_n > k) \right)$$

is the intersection of an F_σ -set and a G_δ -set, hence it is Borel. Analogously, E_2 is Borel. This proves that E is analytic subset of \mathcal{X}^2 .

3. PROOF OF THEOREM 1.3

Define the sequence $x = (x_n)$ so that $x_n = \frac{1}{n}$ if n is even and $x_n = \frac{1}{n \log(n+1)}$ if n is odd. Note that $\lim_n x_n = 0$ and that $\sum_n x_n = \infty$, hence $x \in \mathcal{L}$. At this point, fix $y \in \mathcal{Y}$ such that $\kappa := \liminf_n y_{n+1}/y_n > 0$ and let us show that $\mathcal{I}_x \neq \mathcal{I}_y$.

Let \mathbf{P} be the set of prime numbers, with increasing enumeration (p_n) . By the prime number theorem we have p_n is asymptotically equal to $n \log(n)$ as $n \rightarrow \infty$, hence

$$\sum_{n \in \mathbf{P}} x_n = \sum_{n \geq 1} x_{p_n} \ll \sum_{n \geq 1} \frac{1}{p_n \log(p_n)} \ll \sum_{n \geq 2} \frac{1}{n \log^2(n)} < \infty,$$

with the consequence that $\mathbf{P} \in \mathcal{I}_x$. In addition, $\mathbf{P} - 1 \notin \mathcal{I}_x$ because

$$\sum_{n \in \mathbf{P}-1} x_n = \sum_{n \geq 1} x_{p_n-1} \gg \sum_{n \geq 1} \frac{1}{p_n} \gg \sum_{n \geq 2} \frac{1}{n \log(n)} = \infty.$$

Lastly, suppose for the sake of contradiction that $\mathcal{I}_x = \mathcal{I}_y$. Then we should have that $\mathbf{P} \in \mathcal{I}_y$ and, at the same time, $\mathbf{P} - 1 \notin \mathcal{I}_y$. The latter means that

$$\sum_{n \geq 1} y_{p_n-1} = \infty.$$

However, this implies that

$$\sum_{n \geq 1} y_{p_n} \gg \sum_{n \geq 1} \kappa y_{p_n-1} = \infty, \tag{2}$$

contradicting that $\mathbf{P} \in \mathcal{I}_y$.

4. PROOF OF THEOREM 1.4

Consider the Banach–Mazur game defined as follows: Players I and II choose alternatively nonempty open subsets of \mathcal{Z} as a nonincreasing chain $U_1 \supseteq V_1 \supseteq U_2 \supseteq V_2 \supseteq \dots$, where Player I chooses the sets U_m . Player II has a winning strategy if $\bigcap_m V_m \subseteq \mathcal{A}$. Thanks to [8, Theorem 8.33], Player II has a winning strategy if and only if \mathcal{A} is comeager. Hence, the rest of the proof consists in showing that Player II has a winning strategy.

Note that the open neighborhood of a sequence $x \in \mathcal{Z}$ with radius $\varepsilon > 0$ satisfies

$$B_\varepsilon(x) := \{y \in \mathcal{Z} : \|x - y\| < \varepsilon\} \supseteq \{y \in \mathcal{Z} : \forall n \in \mathbf{N}, |x_n - y_n| < \varepsilon/2\}.$$

Since $x \in \mathcal{Z} \subseteq c_0$, there exists $k_0 = k_0(x, \varepsilon) \in \mathbf{N}$ such that $x_n < \varepsilon/2$ for all $n \geq k_0$. Hence

$$B_\varepsilon(x) \supseteq W_\varepsilon(x) := \{y \in \mathcal{Z} : \forall n \geq k_0(x, \varepsilon), y_n < \varepsilon/2 \text{ and } \forall n < k_0(x, \varepsilon), |x_n - y_n| < \varepsilon/2\}. \quad (3)$$

For each $m \in \mathbf{N}$, suppose that the nonempty open set U_m has been fixed by Player I. Hence, U_m contains an open ball $B_{\varepsilon_m}(x^{(m)})$, for some $x^{(m)} \in \mathcal{Z}$ and $\varepsilon_m > 0$. In particular, thanks to (3), there exists a sufficiently large integer $k_0 = k_0(x^{(m)}, \varepsilon_m) \in \mathbf{N}$ such that $y_n < \varepsilon/2$ for all $y \in W_{\varepsilon_m}(x^{(m)})$ and $n \geq k_0$. Without loss of generality, let us suppose that k_0 is even.

At this point, let x^* be the sequence in \mathcal{A} defined in the proof of Theorem 1.3. Then, for each $m \in \mathbf{N}$, let t_m be an integer such that $\max\{x_{p_m}^*, x_{p_m-1}^*\} < t_m \cdot \frac{\varepsilon_m}{2}$ (we recall that p_m stands for the m th prime number), and define the positive real

$$\delta_m := \min \left\{ \frac{1}{m^2 t_m}, \frac{\varepsilon_m}{2} - \frac{\max\{x_{p_m}^*, x_{p_m-1}^*\}}{t_m} \right\}.$$

Note that $\lim_m \delta_m = 0$. Now, define the set $I_m = \mathbf{N} \cap [k_0(x^{(m)}, \varepsilon_m), k_0(x^{(m)}, \varepsilon_m) + 2t_m)$ and let $z^{(m)}$ be the sequence such that

$$\forall n \in \mathbf{N} \quad z_n^{(m)} = \begin{cases} x_{p_m-1}^*/t_m & \text{if } n \in I_m \text{ and } n \text{ even,} \\ x_{p_m}^*/t_m & \text{if } n \in I_m \text{ and } n \text{ odd,} \\ x_n^{(m)} & \text{if } n \notin I_m. \end{cases}$$

Lastly, set $V_m := B_{\delta_m}(z^{(m)})$ and note that by construction

$$\forall m \in \mathbf{N}, \quad V_m \subseteq W_{\varepsilon_m}(x^{(m)}) \subseteq B_{\varepsilon_m}(x^{(m)}) \subseteq U_m,$$

hence V_m is a nonempty open set contained in U_m . In addition, the sequence of centers $(z^{(m)})$ is a Cauchy sequence in the complete metric space \mathcal{Z} . Hence it is convergent to some $z \in \mathcal{Z}$ and it is straightforward to see that $\{z\} = \bigcap_m V_m$.

To complete the proof, we need to show that $z \in \mathcal{A}$. Set $A := (\bigcup_m I_m) \setminus 2\mathbf{N}$. Proceeding as in the proof of Theorem 1.3, we see that

$$\sum_{n \in A} z_n = \sum_{m \geq 1} \sum_{n \in I_m \setminus 2\mathbf{N}} z_n \leq \sum_{m \geq 1} \sum_{n \in I_m \setminus 2\mathbf{N}} (z_n^{(m)} + \delta_m) \leq \sum_{m \geq 1} |I_m| \left(\frac{x_{p_m}^*}{t_m} + \frac{1}{m^2 t_m} \right) < \infty,$$

hence $A \in \mathcal{I}_z$. Similarly, $A - 1 \notin \mathcal{I}_z$ since

$$\sum_{n \in A-1} z_n \geq \sum_{m \geq 1} \sum_{n \in I_m \cap 2\mathbf{N}} (z_n^{(m)} - \delta_m) \gg \sum_{m \geq 1} |I_m| \left(\frac{x_{p_m-1}^*}{t_m} - \frac{1}{m^2 t_m} \right) = \infty.$$

Now, fix $y \in \mathcal{Y}$ such that $\kappa := \liminf_n y_{n+1}/y_n > 0$ and suppose that $\mathcal{I}_z = \mathcal{I}_y$. Then we would have that $A \in \mathcal{I}_y$ and $A - 1 \notin \mathcal{I}_y$, which is impossible reasoning as in (2).

5. PROOF OF THEOREM 1.5

Let x^* be the sequence defined in the proof of Theorem 1.3. For each $r \in (0, 1]$, let $x^{(r)}$ be the sequence defined by $x_n^{(r)} = (x_n^*)^r$ for all $n \in \mathbf{N}$. Replacing the set of primes \mathbf{P} with $\{\lfloor p_n^{1/r} \rfloor : n \in \mathbf{N}\}$ and reasoning as in the proof of Theorem 1.3, we obtain that $x^{(r)} \in \mathcal{A}$.

To complete the proof, fix reals r, s such that $0 < r < s \leq 1$. Then, it is sufficient to show that the ideals generated by $x^{(r)}$ and $x^{(s)}$ are not isomorphic. To this aim, let $f : \mathbf{N} \rightarrow \mathbf{N}$ be a bijection and assume for the sake contradiction that

$$\forall A \subseteq \mathbf{N}, \quad \sum_{n \in A} x_{f(n)}^{(r)} < \infty \quad \text{if and only if} \quad \sum_{n \in A} x_n^{(s)} < \infty. \quad (4)$$

Fix $t \in (1, s/r)$ and define $S := \{n \in \mathbf{N} : f(n) > n^t\}$ and $T := \mathbf{N} \setminus S$. We have $\sum_{n \in S} \frac{1}{f(n)} \leq \sum_{n \in S} \frac{1}{n^t} < \infty$. Considering that f is a bijection and the harmonic series is divergent, we obtain that $\sum_{n \in T} \frac{1}{f(n)} = \infty$ (in particular, T is infinite). In addition, since $r < 1$ and

$$\frac{1}{f(n)} \ll \frac{1}{(f(n) \log(f(n) + 1))^r} \leq x_{f(n)}^{(r)} \leq \frac{1}{f^r(n)},$$

we get that $\sum_{n \in T} x_{f(n)}^{(r)} = \infty$ and, thanks to (4), also that $\sum_{n \in T} x_n^{(s)} = \infty$. Note that, if $n \in T$, then $f(n) \leq n^t$, which implies that

$$\forall n \in T, \quad \frac{x_n^{(s)}}{x_{f(n)}^{(r)}} \leq \frac{1/n^s}{1/(f(n) \log(f(n) + 1))^r} \ll \frac{(\log n)^r}{n^{s-tr}},$$

which has limit 0 if $n \rightarrow \infty$ (and belongs to T). In particular, for each $k \in \mathbf{N}$, there exists $n_k \in \mathbf{N}$ such that $x_n^{(s)}/x_{f(n)}^{(r)} \leq 1/k^2$ for all $n \geq n_k$. Let (A_k) be a sequence of finite subsets of T defined recursively as follows: for each $k \in \mathbf{N}$, let A_k be a finite subset of T such that $\min A_k \geq n_k + \max A_{k-1}$ and $\sum_{n \in A_k} x_{f(n)}^{(r)} \in (\frac{1}{2}, 1)$ where, by convention, we assume $\max A_0 := 0$ (note that it is really possible to define such sequence). Finally, define $A := \bigcup_k A_k$ so that we obtain

$$\sum_{n \in A} x_{f(n)}^{(r)} = \sum_{k \geq 1} \sum_{n \in A_k} x_{f(n)}^{(r)} = \infty \quad \text{and} \quad \sum_{n \in A} x_n^{(s)} \leq \sum_{k \geq 1} \sum_{n \in A_k} \frac{x_{f(n)}^{(r)}}{k^2} \leq \sum_{k \geq 1} \frac{1}{k^2} < \infty.$$

This contradicts (4), concluding the proof.

6. CONCLUDING REMARKS

We remark that the ideal \mathcal{I}_x defined in the proof above is just (an isomorphic copy of) the Fubini sum $\mathcal{I}_s \oplus \mathcal{I}_t$, where $s, t \in \mathcal{Z}$ are sequences defined by $s_n = \frac{1}{n}$ and $t_n = \frac{1}{n \log(n+1)}$ for all $n \in \mathbf{N}$. Here, we recall that the Fubini sum of two ideals \mathcal{I} and \mathcal{J} on \mathbf{N} is the ideal $\mathcal{I} \oplus \mathcal{J}$ on $\{0, 1\} \times \mathbf{N}$ of all sets A such that $\{n \in \mathbf{N} : (0, n) \in A\} \in \mathcal{I}$ and $\{n \in \mathbf{N} : (1, n) \in A\} \in \mathcal{J}$, cf. e.g. [4, p. 8]

Also, some comments are in order about the simplifications. The assumption that the sequence x has positive elements (instead of nonnegative elements) is rather innocuous. Indeed, in the opposite, if $x_n = 0$ for infinitely many n , then the summable ideal \mathcal{I}_x would be (isomorphic to) the Fubini sum $\mathcal{P}(\mathbf{N}) \oplus \mathcal{I}_y$, for some $y \in \mathcal{X}$. Lastly, also the hypothesis $x \in \mathcal{L}$ in Question 1.1 (instead of $x \in \mathcal{X}$) has a similar justification. Indeed, if $x \in \mathcal{X} \setminus \mathcal{L}$, then \mathcal{I}_x would be the Fubini sum $\text{Fin} \oplus \mathcal{I}_y$, for some $y \in \mathcal{L}$.

We conclude with two open questions:

Question 6.1. Is it true that for each $x \in \mathcal{L}$ there are (possibly infinite) sequences $y^1, y^2, \dots \in \mathcal{Y}$ such that $\mathcal{I}_x = \oplus_i \mathcal{I}_{y^i}$?

Question 6.2. Is it true that \mathcal{A} is an analytic subset of \mathcal{L} ?

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