MOST NUMBERS ARE NOT NORMAL

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Abstract. We show, from a topological viewpoint, that most numbers are not normal in a strong sense. More precisely, the set of numbers $x \in (0,1]$ with the following property is comeager: for all integers $b \ge 2$ and $k \ge 1$, the sequence of vectors made by the frequencies of all possibile strings of length k in the b-adic representation of x has a maximal subset of accumulation points, and each of them is the limit of a subsequence with an index set of nonzero asymptotic density. This extends and provides a streamlined proof of the main result given by Olsen in [Math. Proc. Cambridge Philos. Soc. **137** (2004), 43–53]. We provide analogues in the context of analytic P-ideals and regular matrices.

1. Introduction

A real number $x \in (0,1]$ is normal if, informally, for each base $b \ge 2$, its b-adic expansion contains every finite string with the expected uniform limit frequency (the precise definition is given in the next few lines). It is well known that most numbers x are normal from a measure theoretic viewpoint, see e.g. [5] for history and generalizations. However, it has been recently shown that certain subsets of nonnormal numbers may have full Hausdorff dimension, see e.g. [1, 4]. The aim of this work is to show that, from a topological viewpoint, most numbers are not normal in a strong sense. This provides another nonanalogue between measure and category, cf. [25].

For each $x \in (0,1]$, denote its unique nonterminating b-adic expansion by

$$x = \sum\nolimits_{n \ge 1} \frac{d_{b,n}(x)}{b^n}, \tag{1}$$

with each digit $d_{b,n}(x) \in \{0,1,\ldots,b-1\}$, where $b \ge 2$ is a given integer. Then, for each string $\boldsymbol{s} = s_1 \cdots s_k$ with digits $s_j \in \{0,1,\ldots,b-1\}$ and each $n \ge 1$, write $\pi_{b,\boldsymbol{s},n}(x)$ for the proportion of strings $\boldsymbol{s}$ in the b-adic expansion of x which start at some position $\le n$, i.e.,

$$\pi_{b,\boldsymbol{s},n}(x) := \frac{\#\{i \in \{1,\ldots,n\} : d_{b,i+j-1}(x) = s_j \text{ for all } j = 1,\ldots,k\}}{n}.$$

In addition, let S_b^k be the set of all possible strings $\boldsymbol{s} = s_1 \cdots s_k$ in base b of length k, hence $\#S_b^k = b^k$, and denote by $\boldsymbol{\pi}_{b,n}^k(x)$ the vector $(\pi_{b,\boldsymbol{s},n}(x) : \boldsymbol{s} \in S_b^k)$. Of course, $\boldsymbol{\pi}_{b,n}^k(x)$ belongs to the (b^k-1)-dimensional simplex for each n. However, the components of $\boldsymbol{\pi}_{b,n}^k(x)$ satisfy an additional requirement: if $k \ge 2$ and $\boldsymbol{s} = s_1 \cdots s_{k-1}$ is a string in S_b^{k-1}, then

$$\pi_{b,\boldsymbol{s},n}(x) = \sum\nolimits_{s_k} \pi_{b,\boldsymbol{s}s_k,n}(x) = \sum\nolimits_{s_0} \pi_{b,s_0\boldsymbol{s},n}(x) + O(1/n) \qquad \text{as } n \to \infty,$$

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where $s_0\boldsymbol{s}$ and $\boldsymbol{s}s_k$ stand for the concatened strings (indeed, the above identity is obtained by a double counting of the occurrences of the string $\boldsymbol{s}$ as the occurrences of all possible strings $\boldsymbol{s}s_k$; or, equivalently, as the occurrences of all possible strings $s_0\boldsymbol{s}$, with the caveat of counting them correctly at the two extreme positions, hence with an error of at most 1). It follows that the set $\mathrm{L}_b^k(x)$ of accumulation points of the sequence of vectors $(\boldsymbol{\pi}_{b,n}^k(x) : n \ge 1)$ is contained in Δ_b^k, where

$$\Delta_b^k := \Big\{ (p_{\boldsymbol{s}})_{\boldsymbol{s}\in S_b^k} \in \mathbf{R}^{b^k} : \sum\nolimits_{\boldsymbol{s}} p_{\boldsymbol{s}} = 1, \, p_{\boldsymbol{s}} \ge 0 \text{ for all } \boldsymbol{s} \in S_b^k, \\ \text{and } \sum\nolimits_{s_0} p_{s_0\boldsymbol{s}} = \sum\nolimits_{s_k} p_{\boldsymbol{s}s_k} \text{ for all } \boldsymbol{s} \in S_b^{k-1} \Big\}.$$

Then x is said to be *normal* if

$$\forall b \ge 2, \forall k \ge 1, \forall \boldsymbol{s} \in S_b^k, \quad \lim_{n\to\infty} \pi_{b,\boldsymbol{s},n}(x) = 1/b^k.$$

Hence, if x is normal, then $\mathrm{L}_b^k(x) = \{(1/b^k, \dots, 1/b^k)\}$. Olsen proved in [23] that the subset of nonnormal numbers with maximal set of accumulation points is topologically large:

Theorem 1.1. *The set* $\{x \in (0,1] : \mathrm{L}_b^k(x) = \Delta_b^k$ *for all* $b \ge 2, k \ge 1\}$ *is comeager.*

First, we strenghten Theorem 1.1 by showing that the set of accumulation points $\mathrm{L}_b^k(x)$ can be replaced by the much smaller subset of accumulation points $\boldsymbol{\eta}$ such that every neighborhood of $\boldsymbol{\eta}$ contains "sufficiently many" elements of the sequence, where "sufficiently many" is meant with respect to a suitable ideal $\mathcal{I}$ of subsets of the positive integers $\mathbf{N}$; see Theorem 2.1. Hence, Theorem 1.1 corresponds to the case where $\mathcal{I}$ is the family of finite sets.

Then, for certain ideals $\mathcal{I}$ (including the case of the family of asymptotic density zero sets), we even strenghten the latter result by showing that each accumulation point $\boldsymbol{\eta}$ can be chosen to be the limit of a subsequence with "sufficiently many" indexes (as we will see in the next Section, these additional requirements are not equivalent); see Theorem 2.3. The precise definitions, together with the main results, follow in Section 2.

2. Main results

An ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbf{N})$ is a family closed under finite union and subsets. It is also assumed that $\mathcal{I}$ contains the family of finite sets Fin and it is different from $\mathcal{P}(\mathbf{N})$. Every subset of $\mathcal{P}(\mathbf{N})$ is endowed with the relative Cantor-space topology. In particular, we may speak about G_δ-subsets of $\mathcal{P}(\mathbf{N})$, F_σ-ideals, meager ideals, analytic ideals, etc. In addition, we say that $\mathcal{I}$ is a P-ideal if it is σ-directed modulo finite sets, i.e., for each sequence (S_n) of sets in $\mathcal{I}$ there exists $S \in \mathcal{I}$ such that $S_n \setminus S$ is finite for all $n \in \mathbf{N}$. Lastly, we denote by $\mathcal{Z}$ the ideal of asymptotic density zero sets, i.e.,

$$\mathcal{Z} = \{S \subseteq \mathbf{N} : \mathsf{d}^\star(S) = 0\}, \tag{2}$$

where $\mathsf{d}^\star(S) := \limsup_n \frac{1}{n}\#(S \cap [1,n])$ stands for the upper asymptotic density of S, see e.g. [20]. We refer to [14] for a recent survey on ideals and associated filters.

Let $x = (x_n)$ be a sequence taking values in a topological vector space X. Then we say that $\eta \in X$ is an *$\mathcal{I}$-cluster point* of x if $\{n \in \mathbf{N} : x_n \in U\} \notin \mathcal{I}$ for all open neighborhoods U of η. Note that Fin-cluster points are the ordinary accumulation points. Usually $\mathcal{Z}$-cluster points are referred to as *statistical cluster points*, see e.g. [13]. It is worth noting that $\mathcal{I}$-cluster points have been studied much before under a different name. Indeed, as it follows by [19, Theorem 4.2] and [16, Lemma 2.2], they correspond to classical "cluster points" of a filter (depending on x) on the underlying space, cf. [7, Definition 2, p.69].

With these premises, for each $x \in (0,1]$ and for all integers $b \ge 2$ and $k \ge 1$, let $\Gamma_b^k(x,\mathcal{I})$ be the set of $\mathcal{I}$-cluster points of the sequence $(\boldsymbol{\pi}_{b,n}^k(x) : n \ge 1)$.

Theorem 2.1. *The set* $\{x \in (0,1] : \Gamma_b^k(x,\mathcal{I}) = \Delta_b^k \text{ for all } b \ge 2, k \ge 1\}$ *is comeager, provided that* $\mathcal{I}$ *is a meager ideal.*

The class of meager ideals is really broad. Indeed, it contains Fin, $\mathcal{Z}$, the summable ideal $\{S \subseteq \mathbf{N} : \sum_{n \in S} 1/n < \infty\}$, the ideal generated by the upper Banach density, the analytic P-ideals, the Fubini sum Fin $\times$ Fin, the random graph ideal, etc.; cf. e.g. [3, 14]. Note that $\Gamma_b^k(x,\mathcal{I}) = \mathrm{L}_b^k(x)$ if $\mathcal{I} = \mathrm{Fin}$. Therefore Theorem 2.1 significantly strenghtens Theorem 1.1.

Remark 2.2. It is not difficult to see that Theorem 2.1 does not hold without any restriction on $\mathcal{I}$. Indeed, if $\mathcal{I}$ is a maximal ideal (i.e., the complement of a free ultrafilter on $\mathbf{N}$), then for each $x \in (0,1]$ and all integers $b \ge 2$, $k \ge 1$, we have that the sequence $(\boldsymbol{\pi}_{b,n}^k(x) : n \ge 1)$ is bounded, hence it is $\mathcal{I}$-convergent so that $\Gamma_b^k(x,\mathcal{I})$ is a singleton.

On a similar direction, if $x = (x_n)$ is a sequence taking values in a topological vector space X, then $\eta \in X$ is an $\mathcal{I}$-*limit point* of x if there exists a subsequence (x_{n_k}) such that $\lim_k x_{n_k} = \eta$ and $\mathbf{N} \setminus \{n_1, n_2, \ldots\} \in \mathcal{I}$. Usually $\mathcal{Z}$-limit points are referred to as *statistical limit points*, see e.g. [13]. Similarly, for each $x \in (0,1]$ and for all integers $b \ge 2$ and $k \ge 1$, let $\Lambda_b^k(x,\mathcal{I})$ be the set of $\mathcal{I}$-limit points of the sequence $(\boldsymbol{\pi}_{b,n}^k(x) : n \ge 1)$. The analogue of Theorem 2.1 for $\mathcal{I}$-limit points follows.

Theorem 2.3. *The set* $\{x \in (0,1] : \Lambda_b^k(x,\mathcal{I}) = \Delta_b^k \text{ for all } b \ge 2, k \ge 1\}$ *is comeager, provided that* $\mathcal{I}$ *is an analytic P-ideal or an* F_σ*-ideal.*

It is known that every $\mathcal{I}$-limit point is always an $\mathcal{I}$-cluster point, however they can be highly different, as it is shown in [2, Theorem 3.1]. This implies that Theorem 2.3 provides a further improvement on Theorem 2.1 for the subfamily of analytic P-ideals.

It is remarkable that there exist F_σ-ideals which are not P-ideals, see e.g. [11, Section 1.11]. Also, the family of analytic P-ideals is well understood and has been characterized with the aid of lower semicontinuous submeasures, cf. Section 3. The results in [6] suggest that the study of the interplay between the theory of analytic P-ideals and their representability may have some relevant yet unexploited potential for the study of the geometry of Banach spaces.

Finally, recalling that the ideal $\mathcal{Z}$ defined in (2) is an analytic P-ideal, an immediate consequence of Theorem 2.3 (as pointed out in the abstract) follows:

Corollary 2.4. *The set of* $x \in (0,1]$ *such that, for all* $b \ge 2$ *and* $k \ge 1$, *every vector in* Δ_b^k *is a statistical limit point of the sequence* $(\boldsymbol{\pi}_{b,n}^k(x) : n \ge 1)$ *is comeager.*

It would also be interesting to investigate to what extend the same results for nonnormal points belonging to self-similar fractals (as studied, e.g., by Olsen and West in [24] in the context of iterated function systems) are valid.

We leave as open question for the interested reader to check whether Theorem 2.3 can be extended for all $F_{\sigma\delta}$-ideals including, in particular, the ideal $\mathcal{I}$ generated by the upper Banach density (which is known to not be a P-ideal, see e.g. [12, p.299]).

3. Proofs of the main results

Proof of Theorem 2.1. Let $\mathcal{I}$ be a meager ideal on $\mathbf{N}$. It follows by Talagrand's characterization of meager ideals [28, Theorem 21] that it is possible to define a partition $\{I_1, I_2, \ldots\}$ of $\mathbf{N}$ into nonempty finite subsets such that $S \notin \mathcal{I}$ whenever $I_n \subseteq S$ for infinitely many n. Moreover, we can assume without loss of generality that $\max I_n < \min I_{n+1}$ for all $n \in \mathbf{N}$.

The claimed set can be rewritten as $\bigcap_{b\ge 2}\bigcap_{k\ge 1} X_b^k$, where $X_b^k := \{x \in (0,1] : \Gamma_b^k(x,\mathcal{I}) = \Delta_b^k\}$. Since the family of meager subsets of $(0,1]$ is a σ-ideal, it is enough to show that the complement of each X_b^k is meager. To this aim, fix $b \ge 2$ and $k \ge 1$ and denote by $\|\cdot\|$ the Euclidean norm on $\mathbf{R}^{b^k}$. Considering that $\{\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots\} := \Delta_b^k \cap \mathbf{Q}^{b^k}$ is a countable dense subset of Δ_b^k and that $\Gamma_b^k(x,\mathcal{I})$ is a closed subset of Δ_b^k by [19, Lemma 3.1(iv)], it follows that

$$
\begin{aligned}
(0,1] \setminus X_b^k &= \bigcup\nolimits_{t\ge 1} \{x \in (0,1] : \boldsymbol{\eta}_t \notin \Gamma_b^k(x,\mathcal{I})\} \\
&= \bigcup\nolimits_{t\ge 1} \{x \in (0,1] : \exists \varepsilon > 0, \{n \in \mathbf{N} : \|\boldsymbol{\pi}_{b,n}^k(x) - \boldsymbol{\eta}_t\| < \varepsilon\} \in \mathcal{I}\} \\
&\subseteq \bigcup\nolimits_{t,p,m\ge 1} \{x \in (0,1] : \forall q \ge p, \exists n \in I_q, \|\boldsymbol{\pi}_{b,n}^k(x) - \boldsymbol{\eta}_t\| \ge 1/m\}.
\end{aligned}
$$

Denote by $S_{t,m,p}$ the set in the latter union. Thus it is sufficient to show that each $S_{t,p,m}$ is nowhere dense. To this aim, fix $t, p, m \in \mathbf{N}$ and a nonempty relatively open set $G \subseteq (0,1]$. We claim there exists a nonempty open set U contained in G and disjoint from $S_{t,p,m}$. Since G is nonempty and open in $(0,1]$, there exists a string $\tilde{\boldsymbol{s}} = s_1 \cdots s_j \in S_b^j$ such that $x \in G$ whenever $d_{b,i}(x) = s_i$ for all $i = 1, \ldots, j$. Now, pick $x^\star \in (0,1]$ such that $\lim_n \boldsymbol{\pi}_{b,n}^k(x^\star) = \boldsymbol{\eta}_t$, which exists by [22, Theorem 1]. In addition, we can assume without loss of generality that $d_{b,i}(x^\star) = s_i$ for all $i = 1, \ldots, j$. Since $\boldsymbol{\pi}_{b,n}^k(x^\star)$ is convergent to $\boldsymbol{\eta}_t$, there exists $q \ge p + j$ such that $\|\boldsymbol{\pi}_{b,n}^k(x^\star) - \boldsymbol{\eta}_t\| < 1/m$ for all $n \ge \min I_q$. Define $V := \{x \in (0,1] : d_{b,i}(x) = d_{b,i}(x^\star)$ for all $i = 1, \ldots, \max I_q + k\}$ and note that $V \subseteq G$ because $d_{b,i}(x) = s_i$ for all $i \le j$ and $x \in V$, and $V \cap S_{t,m,p} = \emptyset$ because, for each $x \in V$, the required property is not satisfied for this choice of q since $\boldsymbol{\pi}_{b,n}^k(x) = \boldsymbol{\pi}_{b,n}^k(x^\star)$ for all $n \le \max I_q$. Clearly, V has nonempty interior, hence it is possible to choose such $U \subseteq V$.

This proves that each $S_{t,m,p}$ is nowhere dense, concluding the proof. □

Before we proceed to the proof of Theorem 2.3, we need to recall the classical Solecki's characterization of analytic P-ideals. A lower semicontinuous submeasure (in short, lscsm) is a monotone subadditive function $\varphi : \mathcal{P}(\mathbf{N}) \to [0,\infty]$ such that $\varphi(\emptyset) = 0$, $\varphi(\{n\}) < \infty$, and $\varphi(A) = \lim_m \varphi(A \cap [1,m])$ for all $A \subseteq \mathbf{N}$ and $n \in \mathbf{N}$. It follows by [26, Theorem 3.1] that an ideal $\mathcal{I}$ is an analytic P-ideal if and only if there exists a lscsm φ such that

$$
\mathcal{I} = \{A \subseteq \mathbf{N} : \|A\|_\varphi = 0\}, \quad \|\mathbf{N}\|_\varphi = 1, \quad \text{and} \quad \varphi(\mathbf{N}) < \infty. \tag{3}
$$

Here, $\|A\|_\varphi := \lim_n \varphi(A \setminus [1,n])$ for all $A \subseteq \mathbf{N}$. Note that $\|A\|_\varphi = \|B\|_\varphi$ whenever the symmetric difference $A \triangle B$ is finite, cf. [11, Lemma 1.3.3(b)]. Easy examples of lscsms are $\varphi(A) := \#A$ or $\varphi(A) := \sup_n \frac{1}{n}\#(A \cap [1,n])$ for all $A \subseteq \mathbf{N}$ which lead, respectively, to the ideals Fin and $\mathcal{Z}$ through the representation (3).

Proof of Theorem 2.3. First, let us suppose that $\mathcal{I}$ is an F_σ-ideal. We obtain by [2, Theorem 2.3] that $\Lambda_b^k(x,\mathcal{I}) = \Gamma_b^k(x,\mathcal{I})$ for each $b \ge 2$, $k \ge 1$, and $x \in (0,1]$. Therefore the claim follows by Theorem 2.1.

Then, we assume hereafter that $\mathcal{I}$ is an analytic P-ideal generated by a lscsm φ as in (3). Fix integers $b \ge 2$ and $k \ge 1$, and define the function

$$\mathfrak{u} : (0,1] \times \Delta_b^k \to \mathbf{R} : (x, \boldsymbol{\eta}) \mapsto \lim_{t\to\infty} \|\{n \in \mathbf{N} : \|\boldsymbol{\pi}_{b,n}^k(x) - \boldsymbol{\eta}\| \le 1/t\}\|_\varphi$$

where $\|\cdot\|$ stands for the Euclidean norm on $\mathbf{R}^{b^k}$. It follows by [2, Lemma 2.1] that every section $\mathfrak{u}(x,\cdot)$ is upper semicontinuous, so that the set

$$\Lambda_b^k(x,\mathcal{I},q) := \{\boldsymbol{\eta} \in \Delta_b^k : \mathfrak{u}(x,\boldsymbol{\eta}) \ge q\}$$

is closed for each $x \in (0,1]$ and $q \in \mathbf{R}$.

At this point, we prove that, for each $\boldsymbol{\eta} \in \Delta_b^k$, the set $X(\boldsymbol{\eta}) := \{x \in (0,1] : \mathfrak{u}(x,\boldsymbol{\eta}) \ge 1/2\}$ is comeager. To this aim, fix $\boldsymbol{\eta} \in \Delta_b^k$ and notice that

$$\begin{aligned}(0,1] \setminus X(\boldsymbol{\eta}) &= \bigcup\nolimits_{t\ge 1} \{x \in (0,1] : \|\{n \in \mathbf{N} : \|\boldsymbol{\pi}_{b,n}^k(x) - \boldsymbol{\eta}\| \le 1/t\}\|_\varphi < 1/2\} \\ &= \bigcup\nolimits_{t\ge 1} \{x \in (0,1] : \lim_{h\to\infty} \varphi(\{n \ge h : \|\boldsymbol{\pi}_{b,n}^k(x) - \boldsymbol{\eta}\| \le 1/t\}) < 1/2\} \\ &= \bigcup\nolimits_{t,h\ge 1} \{x \in (0,1] : \varphi(\{n \ge h : \|\boldsymbol{\pi}_{b,n}^k(x) - \boldsymbol{\eta}\| \le 1/t\}) < 1/2\}.\end{aligned}$$

Denoting by $Y_{t,h}$ the inner set above, it is sufficient to show that each $Y_{t,h}$ is nowhere dense. Hence, fix $G \subseteq (0,1]$, $\tilde{\boldsymbol{s}} \in S_b^j$, and $x^\star \in (0,1]$ as in the proof of Theorem 2.1. Considering that $\|\cdot\|_\varphi$ is invariant under finite sets, it follows that

$$\varphi(\{n \ge j' : \|\boldsymbol{\pi}_{b,n}^k(x^\star) - \boldsymbol{\eta}\| \le 1/t\}) \ge \|\{n \ge j' : \|\boldsymbol{\pi}_{b,n}^k(x^\star) - \boldsymbol{\eta}\| \le 1/t\}\|_\varphi = \mathfrak{u}(x^\star, \boldsymbol{\eta}) = 1,$$

where $j' := j + h$. Since φ is lower semicontinuous, there exists an integer $j'' > j'$ such that

$$\varphi(\{n \in [j', j''] : \|\boldsymbol{\pi}_{b,n}^k(x^\star) - \boldsymbol{\eta}\| \le 1/t\}) \ge 1/2.$$

Define $V := \{x \in (0,1] : d_{b,i}(x) = d_{b,i}(x^\star) \text{ for all } i = 1, \ldots, j''\}$. Similarly, note that $V \subseteq G$ because $d_{b,i}(x) = s_i$ for all $i \le j$ and $x \in V$, and $V \cap Y_{t,h} = \emptyset$ because $\varphi(\{n \ge h : \|\boldsymbol{\pi}_{b,n}^k(x) - \boldsymbol{\eta}\| \le 1/t\})$ is at least $\varphi(\{n \in [j', j''] : \|\boldsymbol{\pi}_{b,n}^k(x) - \boldsymbol{\eta}\| \le 1/t\}) \ge 1/2$ for all $x \in V$. Since V has nonempty interior, it is possible to choose $U \subseteq V$ with the required property.

Finally, let E be a countable dense subset of Δ_b^k. Considering that $X := \{x \in (0,1] : E \subseteq \Lambda_b^k(x,\mathcal{I},1/2)\}$ is equal to $\bigcap_{\boldsymbol{\eta}\in E} X(\boldsymbol{\eta})$, it follows that the set X is comeager. However, considering that

$$\Lambda_b^k(x,\mathcal{I}) = \bigcup\nolimits_{q>0} \Lambda_b^k(x,\mathcal{I},q)$$

by [2, Theorem 2.2] and that $\Lambda_b^k(x,\mathcal{I},1/2)$ is a closed subset such that $E \subseteq \Lambda_b^k(x,\mathcal{I},1/2) \subseteq \Lambda_b^k(x,\mathcal{I}) \subseteq \Delta_b^k$ for all $x \in X$, we obtain that $\Lambda_b^k(x,\mathcal{I},1/2) = \Lambda_b^k(x,\mathcal{I}) = \Delta_b^k$ for all $x \in X$. In particular, the claimed set contains X, which is comeager. This concludes the proof. □

4. Applications

4.1. **Hausdorff and packing dimensions.** We refer to [10, Chapter 3] for the definitions of the Hausdorff dimension and the packing dimension.

Proposition 4.1. *The sets defined in Theorem 2.1 and Theorem 2.3 have Hausdorff dimension* 0 *and packing dimension* 1*.*

Proof. Reasoning as in [23], the claimed sets are contained in the corresponding ones with ideal Fin, which have Hausdorff dimension 0 by [22, Theorem 2.1]. In addition, since all sets are comeager, we conclude that they have packing dimension 1 by [10, Corollary 3.10(b)]. □

4.2. **Regular matrices.** We extend the main results contained in [15, 27]. To this aim, let $A = (a_{n,i} : n, i \in \mathbf{N})$ be a *regular matrix*, that is, an infinite real-valued matrix such that, if $\boldsymbol{z} = (\boldsymbol{z}_n)$ is a $\mathbf{R}^d$-valued sequence convergent to $\boldsymbol{\eta}$, then $A_n\boldsymbol{z} := \sum_i a_{n,i}\boldsymbol{z}_i$ exists for all $n \in \mathbf{N}$ and $\lim_n A_n\boldsymbol{z} = \boldsymbol{\eta}$, see e.g. [9, Chapter 4]. Then, for each $x \in (0,1]$ and integers $b \ge 2$ and $k \ge 1$, let $\Gamma_b^k(x, \mathcal{I}, A)$ be the set of $\mathcal{I}$-cluster points of the sequence of vectors $\left(A_n\boldsymbol{\pi}_b^k(x) : n \ge 1\right)$, where $\boldsymbol{\pi}_b^k(x)$ is the sequence $(\boldsymbol{\pi}_{b,n}^k(x) : n \ge 1)$.

In particular, $\Gamma_b^k(x, \mathcal{I}, A) = \Gamma_b^k(x, \mathcal{I})$ if A is the infinite identity matrix.

Theorem 4.2. *The set* $\{x \in (0,1] : \Gamma_b^k(x, \mathcal{I}, A) \supseteq \Delta_b^k \text{ for all } b \ge 2, k \ge 1\}$ *is comeager, provided that* $\mathcal{I}$ *is a meager ideal and* A *is a regular matrix.*

Proof. Fix a regular matrix $A = (a_{n,i})$ and a meager ideal $\mathcal{I}$. The proof goes along the same lines as the proof of Theorem 2.1, replacing the definition of $S_{t,m,p}$ with

$$S'_{t,m,p} := \{x \in (0,1] : \forall q \ge p, \exists n \in I_q, \|A_n\boldsymbol{\pi}_b^k(x) - \boldsymbol{\eta}_t\| \ge 1/m\}.$$

Recall that, thanks to the classical Silverman–Toeplitz characterization of regular matrices, see e.g. [9, Theorem 4.1, II] or [8], we have that $\sup_n \sum_i |a_{n,i}| < \infty$. Since $\lim_n \boldsymbol{\pi}_{b,n}^k(x^\star) = \boldsymbol{\eta}_t$, it follows that there exist sufficiently large integers $q \ge p + j$ and $j_A \ge j$ such that, if $d_{b,i}(x) = d_{b,i}(x^\star)$ for all $i = 1, \dots, j_A + k$, then

$$\begin{aligned} \|A_n\boldsymbol{\pi}_b^k(x) - \boldsymbol{\eta}_t\| &\le \|A_n\boldsymbol{\pi}_{b,}^k(x^\star) - \boldsymbol{\eta}_t\| + \left\|\sum\nolimits_i a_{n,i}(\boldsymbol{\pi}_{b,i}^k(x) - \boldsymbol{\pi}_{b,i}^k(x^\star))\right\| \\ &\le \|A_n\boldsymbol{\pi}_b^k(x^\star) - \boldsymbol{\eta}_t\| + \sum\nolimits_i |a_{n,i}|\, \|\boldsymbol{\pi}_{b,i}^k(x) - \boldsymbol{\pi}_{b,i}^k(x^\star)\| \\ &\le \|A_n\boldsymbol{\pi}_b^k(x^\star) - \boldsymbol{\eta}_t\| + \sum\nolimits_{i > j_A} |a_{n,i}| < \frac{1}{m} \end{aligned} \tag{4}$$

for all $n \in I_q$. We conclude analogously that $S'_{t,m,p}$ is nowhere dense. □

The main result in [27] corresponds to the case $\mathcal{I} = \text{Fin}$ and $k = 1$, although with a different proof; cf. also Example 4.10 below.

At this point, we need an intermediate result which is of independent interest. For each bounded sequence $\boldsymbol{x} = (\boldsymbol{x}_n)$ with values in $\mathbf{R}^k$, let K-core$(\boldsymbol{x})$ be the *Knopp core* of $\boldsymbol{x}$, that is, the convex hull of the set of accumulation points of $\boldsymbol{x}$. In other words, K-core$(\boldsymbol{x}) = \operatorname{co} \mathrm{L}_{\boldsymbol{x}}$, where $\operatorname{co} S$ is the convex hull of $S \subseteq \mathbf{R}^k$ and $\mathrm{L}_{\boldsymbol{x}}$ is the set of accumulation points of $\boldsymbol{x}$. The ideal version of the Knopp core has been studied in [18, 16]. The classical Knopp theorem states that, if $k = 2$ and A is a nonnegative regular matrix, then

$$\text{K-core}(A\boldsymbol{x}) \subseteq \text{K-core}(\boldsymbol{x}) \tag{5}$$

for all bounded sequences $\boldsymbol{x}$, where $A\boldsymbol{x} = (A_n\boldsymbol{x} : n \ge 1)$, see [17, p. 115]; cf. [9, Chapter 6] for a textbook exposition. A generalization in the case $k = 1$ can be found in [21]. We show, in particular, that a stronger version of Knopp's theorem holds for every $k \in \mathbf{N}$.

Proposition 4.3. *Let $\boldsymbol{x} = (\boldsymbol{x}_n)$ be a bounded sequence taking values in $\mathbf{R}^k$, and fix a regular matrix A such that $\lim_n \sum_i |a_{n,i}| = 1$. Then inclusion (5) holds.*

Proof. Define $\kappa := \sup_n \|\boldsymbol{x}_n\|$ and let $\boldsymbol{\eta}$ be an accumulation point of $A\boldsymbol{x}$. It is sufficient to show that $\boldsymbol{\eta} \in K := \text{K-core}(\boldsymbol{x})$. Possibly deleting some rows of A, we can assume without loss of generality that $\lim A\boldsymbol{x} = \boldsymbol{\eta}$. For each $m \in \mathbf{N}$, let K_m be the closure of $\text{co}\{x_m, x_{m+1}, \ldots\}$, hence $K \subseteq K_m$. Define $d(\boldsymbol{a}, C) := \min_{\boldsymbol{b} \in C} \|\boldsymbol{a} - \boldsymbol{b}\|$ for all $\boldsymbol{a} \in \mathbf{R}^k$ and nonempty compact sets $C \subseteq \mathbf{R}^k$. In addition, for each $m \in \mathbf{N}$, let $Q_m(\boldsymbol{a}) \in K_m$ be the unique vector such that $d(\boldsymbol{a}, K_m) = \|\boldsymbol{a} - Q_m(\boldsymbol{a})\|$. Similarly, let $Q(\boldsymbol{a})$ be the vector in K which minimizes its distance with $\boldsymbol{a}$. Then, notice that, for all $n, m \in \mathbf{N}$, we have

$$\begin{aligned} d(A_n\boldsymbol{x}, K) &\le \inf\nolimits_{\boldsymbol{b} \in K} \inf\nolimits_{\boldsymbol{c} \in \mathbf{R}^k} (\|A_n\boldsymbol{x} - \boldsymbol{c}\| + \|\boldsymbol{c} - \boldsymbol{b}\|) \\ &\le \inf\nolimits_{\boldsymbol{c} \in K_m} \inf\nolimits_{\boldsymbol{b} \in K} (\|A_n\boldsymbol{x} - \boldsymbol{c}\| + \|\boldsymbol{c} - \boldsymbol{b}\|) \\ &\le \inf\nolimits_{\boldsymbol{c} \in K_m} \|A_n\boldsymbol{x} - \boldsymbol{c}\| + \sup\nolimits_{\boldsymbol{y} \in K_m} \inf\nolimits_{\boldsymbol{b} \in K} \|\boldsymbol{y} - \boldsymbol{b}\| \\ &= d(A_n\boldsymbol{x}, K_m) + \sup\nolimits_{\boldsymbol{y} \in K_m} d(\boldsymbol{y}, K) \end{aligned}$$

Since $d(\boldsymbol{\eta}, K) = \lim_n d(A_n\boldsymbol{x}, K)$ by the continuity of $d(\cdot, K)$, it is sufficient to show that both $d(A_n\boldsymbol{x}, K_m)$ and $\sup_{\boldsymbol{y} \in K_m} d(\boldsymbol{y}, K)$ are sufficiently small if n is sufficiently large and m is chosen properly.

To this aim, fix $\varepsilon > 0$ and choose $m \in \mathbf{N}$ such that $\sup_{\boldsymbol{y} \in K_m} d(\boldsymbol{y}, K) \le \varepsilon/2$. Indeed, it is sufficient to choose $m \in \mathbf{N}$ such that $d(\boldsymbol{x}_n, \mathrm{L}_{\boldsymbol{x}}) < \varepsilon/2$ for all $n \ge m$: indeed, in the opposite, the subsequence $(\boldsymbol{x}_j)_{j \in J}$, where $J := \{n \in \mathbf{N} : d(\boldsymbol{x}_n, \mathrm{L}_{\boldsymbol{x}}) \ge \varepsilon/2\}$, would be bounded and without any accumulation point, which is impossible. Now pick $\boldsymbol{y} \in K_m$ so that $\boldsymbol{y} = \sum_j \lambda_{i_j} \boldsymbol{x}_{i_j}$ for some strictly increasing sequence (i_j) of positive integers such that $i_1 \ge m$ and some real nonnegative sequence (λ_{i_j}) with $\sum_j \lambda_{i_j} = 1$. It follows that

$$d(\boldsymbol{y}, K) \le \left\|\boldsymbol{y} - \sum\nolimits_j \lambda_{i_j} Q(\boldsymbol{x}_{i_j})\right\| \le \sum\nolimits_j \lambda_{i_j} \left\|\boldsymbol{x}_{i_j} - Q(\boldsymbol{x}_{i_j})\right\| \le \sum\nolimits_j \lambda_{i_j} d(\boldsymbol{x}_{i_j}, L_{\boldsymbol{x}}) \le \frac{\varepsilon}{2}.$$

Suppose for the moment that A has nonnegative entries. Since A is regular, we get $\lim_n \sum_i a_{n,i} = 1$ and $\lim_n \sum_{i<m} a_{n,i} = 0$ by the Silverman–Toeplitz characterization, hence $\lim_n \sum_{i \ge m} a_{n,i} = 1$ and there exists $n_0 \in \mathbf{N}$ such that $\sum_{i \ge m} a_{n,i} \ge 1/2$ for all $n \ge n_0$. Thus, for each $n \ge n_0$, we obtain that $d(A_n\boldsymbol{x}, K_m) = \|A_n\boldsymbol{x} - Q_m(A_n\boldsymbol{x})\| \le \alpha_n + \beta_n + \gamma_n$, where

$$\alpha_n := \left\|A_n\boldsymbol{x} - \frac{A_n\boldsymbol{x}}{\sum_i a_{n,i}}\right\|, \quad \beta_n := \left\|\frac{A_n\boldsymbol{x}}{\sum_i a_{n,i}} - Q_m\left(\frac{A_n\boldsymbol{x}}{\sum_i a_{n,i}}\right)\right\|,$$

and

$$\gamma_n := \left\|Q_m\left(\frac{A_n\boldsymbol{x}}{\sum_i a_{n,i}}\right) - Q_m(A_n\boldsymbol{x})\right\|.$$

Recalling that $\kappa = \sup_n \|\boldsymbol{x}_n\|$, it is easy to see that

$$\gamma_n \le \alpha_n \le \kappa \sum\nolimits_i |a_{n,i}| \cdot \left(1 - \frac{1}{\sum_i a_{n,i}}\right).$$

In addition, setting $t_n := \sum_{i \ge m} a_{n,i} / \sum_i a_{n,i} \in [0,1]$ for all $n \ge n_0$, we get

$$\begin{aligned}\beta_n &\le \left\| \frac{\sum_i^\star}{\sum_i a_{n,i}} - \frac{\sum_{i \ge m}^\star}{\sum_{i \ge m} a_{n,i}} \right\| \\ &= \frac{1}{\sum_{i \ge m} a_{n,i} \sum_i a_{n,i}} \left\| \sum_{i \ge m} a_{n,i} \left(\sum_{i<m}^\star + \sum_{i \ge m}^\star \right) - \sum_i a_{n,i} \sum_{i \ge m}^\star \right\| \\ &= \frac{1}{\sum_{i \ge m} a_{n,i}} \left\| t_n \sum_{i<m}^\star + (1-t_n) \sum_{i \ge m}^\star \right\| \\ &\le 2\kappa \left(t_n \sum_{i<m} |a_{n,i}| + (1-t_n) \sum_i |a_{n,i}| \right).\end{aligned} \tag{6}$$

where $\sum_{i \in I}^\star$ stands for $\sum_{i \in I} a_{n,i} \boldsymbol{x}_i$. Note that the hypothesis that the entries of A are nonnegative has been used only in the first line of (6), so that $\sum_{i \ge m}^\star / \sum_{i \ge m} a_{n,i} \in K_m$. Since $\lim_n \sum_{i<m} |a_{n,i}| = 0$, $\lim_n t_n = 1$, and $\sup_n \sum_i |a_{n,i}| < \infty$ by the regularity of A, it follows that all $\alpha_n, \beta_n, \gamma_n$ are smaller than $\varepsilon/6$ if n is sufficiently large. Therefore $d(A_n\boldsymbol{x}, K) \le \varepsilon$ and, since ε is arbitrary, we conclude that $\boldsymbol{\eta} = \lim_n A_n \boldsymbol{x} \in K$.

Lastly, suppose that A is a regular matrix such that $\lim_n \sum_i |a_{n,i}| = 1$ and let $B = (b_{n,i})$ be the nonnegative regular matrix defined by $b_{n,i} = |a_{n,i}|$ for all $n, i \in \mathbf{N}$. Considering that

$$d(A_n\boldsymbol{x}, K_m) \le \|A_n\boldsymbol{x} - B_n\boldsymbol{x}\| + d(B_n\boldsymbol{x}, K_m) \le \kappa \sum_i |a_{n,i} - |a_{n,i}|| + \varepsilon,$$

and that $\lim_n \sum_i |a_{n,i} - |a_{n,i}|| = 0$ because $\lim_n \sum_i a_{n,i} = \lim_n \sum_i |a_{n,i}| = 1$, we conclude that $d(A_n\boldsymbol{x}, K_m) \le 2\varepsilon$ whenever n is sufficiently large. The claim follows as before. □

The following corollary is immediate:

Corollary 4.4. *Let $\boldsymbol{x} = (\boldsymbol{x}_n)$ be a bounded sequence taking values in $\mathbf{R}^k$, and fix a nonnegative regular matrix A. Then inclusion (5) holds.*

Remark 4.5. Inclusion (5) fails for an arbitrary regular matrix: indeed, let $A = (a_{n,i})$ be the matrix defined by $a_{n,2n} = 2$, $a_{n,2n-1} = -1$ for all $n \in \mathbf{N}$, and $a_{n,i} = 0$ otherwise. Set also $k = 1$ and let x be the sequence such that $x_n = (-1)^n$ for all $n \in \mathbf{N}$. Then A is regular and $\lim Ax = 3 \notin \{-1, 1\} = \text{K-core}(x)$.

Remark 4.6. Proposition 4.3 keeps holding on a (possibly infinite dimensional) Hilbert space X with the following provisoes: replace the definition of K-core($\boldsymbol{x}$) with the *closure* of $\operatorname{co} \mathrm{L}_{\boldsymbol{x}}$ (this coincides in the case that $X = \mathbf{R}^k$) and assume that the sequence $\boldsymbol{x}$ is contained in a compact set (so that K-core($\boldsymbol{x}$) is also nonempty).

With these premises, we can strenghten Theorem 4.2 as follows.

Theorem 4.7. *The set $\{x \in (0,1] : \Gamma_b^k(x, \mathcal{I}, A) = \Delta_b^k \text{ for all } b \ge 2, k \ge 1\}$ is comeager, provided that $\mathcal{I}$ is a meager ideal and A is a regular matrix such that $\lim_n \sum_i |a_{n,i}| = 1$.*

Proof. Let us suppose that $A = (a_{n,i})$ is nonnegative regular matrix, i.e., $a_{n,i} \ge 0$ for all $n, i \in \mathbf{N}$, and fix a meager ideal $\mathcal{I}$, a real $x \in (0,1]$, and integers $b \ge 2$, $k \ge 1$. Thanks to Theorem 4.2, it is sufficient to show that every accumulation point of the sequence $(A_n \boldsymbol{\pi}_b^k(x) : n \ge 1)$ is contained in the convex hull of the set of accumulation points of $(\boldsymbol{\pi}_{b,n}^k(x) : n \ge 1)$, which is in turn contained into Δ_b^k. This follows by Proposition 4.3. □

Since the family of meager sets is a σ-ideal, the following is immediate by Theorem 4.7.

Corollary 4.8. *Let $\mathscr{A}$ be a countable family of regular matrices such that $\lim_n \sum_i |a_{n,i}| = 1$. Then the set $\{x \in (0,1] : \Gamma_b^k(x, \mathcal{I}, A) = \Delta_b^k \text{ for all } b \geq 2, k \geq 1, \text{ and all } A \in \mathscr{A}\}$ is comeager, provided that $\mathcal{I}$ is a meager ideal.*

It is worth to remark that the main result [15] is obtained as an instance of Corollary 4.8, letting $\mathscr{A}$ be the set of iterates of the Cesàro matrix (note that they are nonnegative regular matrices), and setting $k = 1$ and $\mathcal{I} = \mathrm{Fin}$. The same holds for the iterates of the Hölder matrix and the logarithmic Riesz matrix as in [24, Sections 3 and 4].

Next, we show that the hypothesis $\lim_n \sum_i |a_{n,i}| = 1$ for the entries of the regular matrix in Theorem 4.7 cannot be removed.

Example 4.9. Let $A = (a_{n,i})$ be the matrix such that $a_{n,(2n-1)!} = -1$ and $a_{n,(2n)!} = 2$ for all $n \in \mathbf{N}$, and $a_{n,i} = 0$ otherwise. It is easily seen that A is regular. Then, set $b = 2$, $k = 1$, and $\mathcal{I} = \mathrm{Fin}$. We claim that the set of all $x \in (0,1]$ such that 2 is an accumulation point of the sequence $\pi_{2,1}(x) = (\pi_{2,1,n}(x) : n \geq 1)$ is comeager. Indeed, its complement can be rewritten as $\bigcup_{m,p} S_{m,p}$, where

$$S_{m,p} := \{x \in (0,1] : |A_n \pi_{2,1}(x) - 2| \geq 1/m \text{ for all } n \geq p\}.$$

Let $x^\star \in (0,1]$ such that $d_{2,n}(x^\star) = 1$ if and only if $(2i-1)! \leq n < (2i)!$ for some $i \in \mathbf{N}$. Then it is easily seen that $\lim_n \pi_{2,1,n}(x^\star) = 2$. Along the same lines of the proof of Theorem 4.2, it follows that each $S_{m,p}$ is meager. We conclude that $\{x \in (0,1] : \Gamma_2^1(x, \mathrm{Fin}, A) = \Delta_2^1\}$ is meager, which proves that the condition $\lim_n \sum_i |a_{n,i}| = 1$ in the statement of Theorem 4.7 cannot be removed.

In addition, the main result in [27] states that Theorem 4.2, specialized to the case $\mathcal{I} = \mathrm{Fin}$ and $k = 1$, can be further strengtened so that the set $\{x \in (0,1] : \Gamma_b^1(x, \mathrm{Fin}, A) \supseteq \Delta_b^1 \text{ for all } b \geq 2 \text{ and all regular } A\}$ is comeager. Taking into account the argument in the proof of Theorem 4.7, this would imply that the set

$$\{x \in (0,1] : \Gamma_b^1(x, \mathrm{Fin}, A) = \Delta_b^1 \text{ for all } b \geq 2 \text{ and all nonnegative regular } A\} \tag{7}$$

should be comeager. However, this is false as it is shown in the next example.

Example 4.10. For each $y \in (0,1]$, let $(e_{y,k} : k \geq 1)$ be the increasing enumeration of the infinite set $\{n \in \mathbf{N} : d_{2,n}(y) = 1\}$. Then, let $\mathscr{A} = \{A_y : y \in (0,1]\}$ be family of matrices $A_y = (a_{n,i}^{(y)})$ with entries in $\{0,1\}$ so that $a_{n,i}^{(y)} = 1$ if and only if $e_{y,n} = i$ for all $y \in (0,1]$ and all $n, i \in \mathbf{N}$. Then each A_y is a nonnegative regular matrix. It follows, for each ideal $\mathcal{I}$,

$$\{x \in (0,1] : \Gamma_2^1(x, \mathcal{I}, A) = \Delta_2^1 \text{ for all } A \in \mathscr{A}\} = \emptyset.$$

Indeed, for each $x \in (0,1]$, the sequence $\boldsymbol{\pi}_2^1(x) = (\boldsymbol{\pi}_{2,n}^1(x) : n \geq 1)$ has an accumulation point $\boldsymbol{\eta} \in \Delta_2^1$. Hence there exists a subsequence $(\boldsymbol{\pi}_{2,n_k}^1(x) : k \geq 1)$ which is convergent to $\boldsymbol{\eta}$. Equivalently, $\lim A_y \boldsymbol{\pi}_2^1(x) = \boldsymbol{\eta}$, where $y \in (0,1]$ is defined such that $e_{y,k} = n_k$ for all $k \in \mathbf{N}$. Therefore $\{\boldsymbol{\eta}\} = \Gamma_2^1(x, \mathcal{I}, A_y) \neq \Delta_2^1$. in particular, the set defined in (7) is empty.

Lastly, the analogues of Theorem 4.2 and Theorem 4.7 hold for $\mathcal{I}$-limit points, if $\mathcal{I}$ is an F_σ-ideal or an analytic P-ideal. Indeed, denoting with $\Lambda_b^k(x, \mathcal{I}, A)$ the set of $\mathcal{I}$-limit points of the sequence $(A_n \boldsymbol{\pi}_b^k(x) : n \geq 1)$, we obtain:

Theorem 4.11. *Let A be a regular matrix and let $\mathcal{I}$ be an F_σ-ideal or an analytic P-ideal. Then the set $\{x \in (0,1] : \Lambda_b^k(x,\mathcal{I},A) \supseteq \Delta_b^k$ for all $b \ge 2, k \ge 1\}$ is comeager.*

Moreover, the set $\{x \in (0,1] : \Lambda_b^k(x,\mathcal{I},A) = \Delta_b^k$ for all $b \ge 2, k \ge 1\}$ is comeager if, in addition, A satisfies $\lim_n \sum_i |a_{n,i}| = 1$.

Proof. The first part goes along the same lines of the proof of Theorem 2.3. Here, we replace $\boldsymbol{\pi}_b^k(x)$ with $(A_n\boldsymbol{\pi}_b^k(x) : n \ge 1)$ and using the chain of inequalities (4): more precisely, we consider $j'' \in \mathbf{N}$ such that $\varphi(\{n \in [j', j''] : \|A_n\boldsymbol{\pi}_b^k(x') - \boldsymbol{\eta}\| \le 1/2t\}) \ge 1/2$, and, taking into considering (4), we define $V := \{x \in (0,1] : d_{b,i}(x) = d_{b,i}(x^\star)$ for all $i = 1, \dots, k + j'''\}$, where j''' is a sufficiently large integer such that $\sum_{i > j'''} |a_{n,i}| \le 1/2t$ for all $n \in [j', j'']$.

The second part follows, as in Theorem 4.7, by the fact that every accumulation point of $(A_n\boldsymbol{\pi}_b^k(x) : n \ge 1)$ belongs to Δ_b^k. □

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