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Letter

Anti-de Sitter, plane waves and quantum field theory

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ABSTRACT

We present a new plane-wave expansion of the Wightman functions of anti de Sitter scalar fields and showcase its conceptual and technical importance in AdS quantum field theory. We deduce from it a new integral representation of the Feynman propagator which helps in clarifying the relation between AdS Euclidean and Lorentzian Feynman diagrams in concrete examples. The plane-wave expansion makes it possible also to demonstrate numerous new nontrivial formulas for Bessel and Legendre functions, and we provide two examples.

It is hard to overstate the importance of having manifestly covariant momentum space representations of relativistic quantum fields based on plane waves. Such representations provide the physically intuitive ingredients of Feynman diagrams and a great deal of the predictive power of Quantum Field Theory (QFT) depends on them. Plane wave expansions do exist also for de Sitter (dS) quantum fields [1,2] while they still do not in the anti-de Sitter (AdS) case, which may come as a surprise after 50 years of research in AdS. The challenge cannot be tackled within the usual approach to AdS QFT that only focuses on the boundary conditions at spacelike infinity [3,4] but largely ignores the global analyticity properties of the AdS manifold and of the correlation functions of the fields [5]. Disregarding those properties, the topological difficulties associated with the existence of closed timelike curves and the necessity to move to the covering of the manifold cannot be overcome. Here we present a solution for AdS scalar fields in spacetime dimension d and open a new way to deal with AdS QFT. Tensorial and spinorial correlation functions may be derived by applying suitable differential operators to our formulae.

The dS and the AdS spacetimes are real Lorentzian submanifolds of the complex sphere $S_d^{(c)}$ having quite different properties: dS_d is globally hyperbolic but does not admit global timelike Killing vector fields while the contrary happens for AdS_d. These features make the corresponding QFT's very different too, but, as we will show in this letter, the manifestly covariant plane-wave expansions of the Wightman functions are almost identical, the main difference being of a topological nature.

Let us start by recalling the structure of the two-point function of a scalar field in a d-dimensional Minkowski spacetime M_d with metric $\eta_{\mu\nu} = (+,-,\dots,-)$:

$$\mathcal{W}_m^{M_d}(\mathrm{x}_1,\mathrm{x}_2) = \int \Psi_{\vec{\mathrm{p}}}^{(-)}(\mathrm{x}_1)\Psi_{\vec{\mathrm{p}}}^{(+)}(\mathrm{x}_2)d\vec{\mathrm{p}}, \tag{1}$$

$$\Psi_{\vec{\mathrm{p}}}^{(\pm)}(x) = \frac{\exp(\pm ip\cdot \mathrm{x})}{\sqrt{2(2\pi)^{d-1}\omega}},\ \ \mathrm{p}^0 = \sqrt{|\vec{\mathrm{p}}|^2+m^2}. \tag{2}$$

The integral (1) is the superposition of a complete set of *plane waves* (labeled by all possible $(d-1)$-momenta $\vec{p}$) evaluated at a first point x_1 multiplied by their complex conjugates evaluated at a second point x_2: it has a distributional meaning but it invites us to move the first point into the backward tube T_- and the second point into the forward tube T_+ of the complex Minkowski space $M_d^{(c)}$:

$$T_\pm = \{\mathrm{z} = \mathrm{x}+i\mathrm{y} \in M_d^{(c)}\ :\ \pm\mathrm{y} \in V_+\}, \tag{3}$$

$$V_+ = \{\mathrm{y} \in M_d\ :\ \mathrm{y}\cdot\mathrm{y} > 0,\ \mathrm{y}^0 > 0\}. \tag{4}$$

Since the waves are exponentially decreasing in the respective tubes, this move vastly improves the convergence of the integral and shows that the *distribution* (1) is one of a very special kind: it is the boundary value on the reals of a *function* $W_m^{M_d}(\mathrm{z}_1,\mathrm{z}_2)$ holomorphic in the *domain* $T_-\times T_+$, a mathematical property equivalent to the physical requirement that the states of the theory have positive energy in every inertial frame; such fundamental equivalence holds also for the n-point functions of interacting fields [6] and is so significant that the Euclidean formulation of QFT would not exist without it [7].

The (complex) d-dimensional de Sitter space-time

$$dS_d^{(c)} = \{z \in M_{d+1}^{(c)}\ :\ z\cdot z = \eta_{\mu\nu}z^\mu z^\nu = -R^2\} \tag{5}$$

also contains dS invariant *domains of analyticity* [2] describable simply as the intersections of $dS_d^{(c)}$ with the tubes $T_\pm$ of the ambient complex Minkowski space $M_{d+1}^{(c)}$:

$$\mathcal{T}_\pm = dS_d^{(c)} \cap T_\pm = \{x+iy \in dS_d^{(c)}\ :\ y \in \pm V_+\}. \tag{6}$$

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The second observation is that the asymptotic lightcone

$$C_+ = \partial V_+ = \{\xi \in M_{d+1},\ \ \xi^2 = 0,\ \ \xi^0 > 0\}, \tag{7}$$

which describes the boundary at timelike infinity of the real de Sitter manifold, may be interpreted as the space of momentum directions; the fact that timelike geodesics and conserved quantities can be parametrized by the choice of two such vectors confirms this interpretation:

$$x^\mu(\tau) = \frac{R\left(\xi^\mu e^{\frac{\tau}{R}} - \eta^\mu e^{-\frac{\tau}{R}}\right)}{\sqrt{2\xi\cdot\eta}},\ K^{\mu\nu} = \frac{m(\xi\wedge\eta)^{\mu\nu}}{(\xi\cdot\eta)} \tag{8}$$

where τ is the proper time and m is the mass of the classical particle in geodesic motion [8].

What replaces the waves (2) in the de Sitter universe? Given any complex number λ (the mass parameter), dS plane waves [1,2] are constructed as follows

$$\mathcal{T}_\pm \times C_+ \ni z, \xi \to \psi^\pm_\lambda(z,\xi) = (\xi\cdot z)^\lambda = e^{\lambda \log(\xi\cdot z)}. \tag{9}$$

These functions are univalued and holomorphic in the tubular domains $\mathcal{T}_-$ and $\mathcal{T}_+$ and solve there the Klein-Gordon equation $(\Box_{dS} + \mu^2_\lambda)\psi^\pm_\lambda(z) = 0$ with complex squared mass $\mu^2_\lambda = \lambda(1-d-\lambda)$. The "phase" of the above waves is constant on the planes $z\cdot\xi = const$.

For any pair z_1, z_2 of points in the domain $\mathcal{T}_-\times\mathcal{T}_+$ the canonically normalized (so-called) Bunch-Davis dS two-point function (here generalized to complex squared masses and to spacetime dimension d) is a holomorphic superposition of plane waves similar to (1):

$$W^{dS_d}_\lambda(z_1,z_2) = \frac{e^{i\pi\left(\lambda+\frac{d-1}{2}\right)}\Gamma(-\lambda)\Gamma(\lambda+d-1)}{2^{d+1}\pi^d} \times \int_\gamma \psi^-_\lambda(z_1,\xi)\,\psi^+_{1-d-\lambda}(z_2,\xi)\,d\mu_\gamma(\xi) \tag{10}$$

where $d\mu_\gamma(\xi)$ is the restriction of the volume form of the cone C_+ to a complete $(d-1)$-dimensional cycle γ.

Crucial property : the homogeneity degree $(1-d)$ of the integrand implies that the rhs is the integral of a closed differential form and therefore it is manifestly de Sitter invariant; it thus depends only on the invariant scalar product $z_1\cdot z_2$; a simple calculation [1,2] shows that it is proportional to a Legendre function of the first kind:

$$W^{dS_d}_\lambda(z_1,z_2) = \frac{1}{2(2\pi)^{d/2}}\Gamma(-\lambda)\Gamma(\lambda+d-1)\times \times\left((z_1\cdot z_2)^2-1\right)^{\frac{2-d}{4}} P^{-\frac{d-2}{2}}_{\lambda+\frac{d-2}{2}}(z_1\cdot z_2). \tag{11}$$

Formula (11) explicitly provides the analytic continuation of the two-point function to the cut-plane

$$\Delta_{dS} = \{z_1, z_2 \in dS^{(c)}_d\ :\ z_1\cdot z_2 \neq \rho,\ \ \rho \le -1\} \tag{12}$$

which contains all pairs of complex events of $dS^{(c)}_d$ minus the causal cut; it contains in particular all the non-coincident pairs of dS Euclidean points. Eq. (11) also shows that the involution $\lambda\to(1-d-\lambda)$ which leaves μ^2_λ invariant is a symmetry of the two-point function.

The complex d–dimensional AdS manifold can also be defined by its embedding, now in the complex space $\mathbf{C}^{d+1}_2$ which is a copy of $\mathbf{C}^{d+1}$ endowed with the scalar product

$$z_1\cdot z_2 = z^0_1 z^0_2 - z^1_1 z^1_2 - \ldots - z^{d-1}_1 z^{d-1}_2 + z^d_1 z^d_2 \tag{13}$$

(in the following the dot refers to this product):

$$AdS^{(c)}_d = \{z\in\mathbf{C}^{d+1}_2,\ z^2 = z\cdot z = R^2\}. \tag{14}$$

The real null cone of the ambient space

$$C_d = \{\xi\in\mathbf{R}^{d+1}_2,\ \xi^2 = \xi\cdot\xi = 0,\ \xi\neq 0\} \tag{15}$$

describes the boundary at spacelike infinity of the real manifold AdS_d; real null vectors may here be interpreted as *spacelike* momentum directions and used to parametrize spacelike AdS geodesics as in Eq. (8). Timelike geodesics and the associated conserved quantities are instead parametrized by null vectors of the complex cone

$$C^{(c)}_d = \{\zeta\in\mathbf{C}^{d+1}_2,\ \zeta\cdot\zeta = 0,\ \zeta\neq 0\} \tag{16}$$

$$x^\mu(\tau) = \frac{R\,\zeta^\mu e^{\frac{i\tau}{R}} + R\,\zeta^{\mu *} e^{-\frac{i\tau}{R}}}{\sqrt{2(\zeta\cdot\zeta^*)}},\ K^{\mu\nu} = \frac{i\,m(\zeta\wedge\zeta^*)^{\mu\nu}}{(\zeta\cdot\zeta^*)}. \tag{17}$$

This parametrization shows that timelike geodesics which contain an event x also contain the antipodal event $-x$ and that the proper time elapsed between them is πR on each geodesic. To remove the periodicity (but not the periodical focusing - see the discussion given in [9]) of the geodesics and to have well-defined plane waves it is necessary to move to the covering manifold.

In spacetime dimension $d\ge 2$ the covering of the complex manifold $AdS^{(c)}_d$ is $AdS^{(c)}_d$ itself. On the contrary, the real manifold AdS_d admits a nontrivial covering space $\widetilde{AdS_d}$; also the backward and forward *chiral* tuboids [5]

$$\mathcal{Z}_\pm = \{z = x+iy\in AdS^{(c)}_d\ :\ y\cdot y > 0,\ \epsilon(z) = y^0x^d - x^0y^d \gtrless 0\} \tag{18}$$

have nontrivial coverings $\widetilde{\mathcal{Z}}_\pm$. Here again, as in Minkowskian QFT, any local and covariant two-point distribution with positive energy spectrum is the boundary value on $\widetilde{A}dS_d\times\widetilde{A}dS_d$ of a function holomorphic in the domain $\widetilde{\mathcal{Z}}_-\times\widetilde{\mathcal{Z}}_+$. These properties completely determine the two-point functions [5] and, as a consequence, select the possible boundary behaviour of the modes. Analyticity properties of the same kind characterize also the n-point functions of interacting fields and allow for the Euclidean formulation of AdS QFT [5].

The last ingredients that we need are the two AdS-invariant *chiral cones*:

$$C_\pm = \{\zeta = \xi + i\chi\in C^{(c)}_d\ :\ \chi\cdot\chi > 0,\ \epsilon(\zeta)\gtrless 0\} \tag{19}$$

and their punctured closures

$$\overline{C}_\pm = \{\zeta\in C^{(c)}_d\ :\ \chi\cdot\chi\ge 0,\ \epsilon(\zeta)\gtreqqless 0,\ \zeta\neq 0\}. \tag{20}$$

The scalar product $z,\zeta\mapsto z\cdot\zeta$ maps $\mathcal{Z}_-\times\overline{C}_+$ and $\mathcal{Z}_+\times\overline{C}_-$ onto the punctured complex-plane $\mathbf{C}^* = \mathbf{C}\setminus\{0\}$: in the above domains the scalar product $z\cdot\zeta$ cannot vanish. Furthermore, the scalar product can be lifted to a map of $\widetilde{\mathcal{Z}}_-\times\overline{C}_+$ and $\widetilde{\mathcal{Z}}_+\times\overline{C}_-$ onto $\widetilde{\mathbf{C}}^*$ which also never vanish. By abuse of language we denote also the lifted scalar product by $z\cdot\zeta$.

We are now in position to answer the question: what replaces the plane waves (2) in the AdS case? Given any complex number λ, we introduce (unnormalized) AdS plane waves as follows:

$$\widetilde{\mathcal{Z}}_\pm\times\overline{C}_\mp \ni \zeta, z\to\phi^\pm_\lambda(z,\zeta) = (z\cdot\zeta)^\lambda = e^{\lambda\log(z\cdot\zeta)}. \tag{21}$$

A previous attempt [10] was based on a local $i\epsilon$-prescription. Here the waves are, in their global domains of definition, univalued, holomorphic solutions of the KG equation $(\Box_{AdS} + m^2_\lambda)\phi^\pm_\lambda(z,\xi) = 0$ with complex squared mass $m^2_\lambda = -\mu^2_\lambda = \lambda(\lambda+d-1)$.

Let us now consider a family of two-point functions formally identical to (10):

$$W^{AdS_d}_\lambda(z_1,z_2) = c_d(\lambda)\int_\gamma \phi^-_\lambda(z_1,\xi)\,\phi^+_{1-d-\lambda}(z_2,\xi)\,d\mu_\gamma(\xi). \tag{22}$$

When the integral is extended to a complete cycle γ of the real cone C_d the outcome is not AdS invariant because the integrand gets multiplied by $\exp(-4\pi i\lambda)$ after one revolution in the (ξ^0,ξ^d)-plane. Exceptions are $\lambda = l$ integer, but the integral vanishes identically, and $\lambda = l+\frac{1}{2}$ half-integer, which gives the correct AdS-invariant result. For generic λ, integrating on the covering of the cone does not help and is physically unsound as it would over-count the possible momentum directions. Indeed, a complete cycle γ is already too much, and this partly explains the vanishing of the integral for integer $\lambda = l$.

The solution for general λ is given by Eq. (22) with the following specifications: for $\mathrm{Re}\,\lambda > -1$ the cycle $\gamma(z_1)$ should belong to a relative homology class of $H_{d-1}(C_-,\{\zeta\ :\ \zeta\cdot z_1 = 0\})$ (a similar construction based on the second point z_2 gives identical results); the canonically normalized plane-wave expansion of the two-point Wightman function holomorphic in $\mathcal{Z}_- \times \mathcal{Z}_+$ is given by

$$W_\lambda^{AdS_d}(z_1,z_2) = -\frac{\pi^{1-d}\Gamma(\lambda+d-1)}{2^{d+1}\cos\left(\frac{\pi d}{2}\right)\Gamma(\lambda+1)}\times$$
$$\times\int_{\gamma(z_1)}(z_1\cdot\zeta)^\lambda\,(z_2\cdot\zeta)^{1-d-\lambda}\,d\mu_\gamma(\zeta) = \tag{23}$$
$$= \frac{e^{-i\pi\frac{d-2}{2}}}{(2\pi)^{\frac{d}{2}}}((z_1\cdot z_2)^2-1)^{\frac{2-d}{4}}\,Q^{\frac{d-2}{2}}_{\frac{d-2}{2}+\lambda}(z_1\cdot z_2). \tag{24}$$

A symmetric presentation of (23) is possible by using the shadow transformation of the waves that follows from (23) by letting z_2 tend to the boundary (the cone C_d):

$$W_\lambda^{AdS_d}(z_1,z_2) = c'_d(\lambda)\int_{\gamma(z_1)}\int_{\gamma(z_2)}(z_1\cdot\zeta_1)^\lambda\times$$
$$\times(\zeta_1\cdot\zeta_2)^{1-\lambda-d}(z_2\cdot\zeta_2)^\lambda\,d\mu_{\gamma(z_1)}\,d\mu_{\gamma(z_2)}. \tag{25}$$

In odd spacetime dimension (AdS_3, AdS_5, ...) Eq. (23) has to be understood as follows

$$W_\lambda^{AdS_{2n+1}}(z_1,z_2) = \frac{(-1)^{n+1}\Gamma(\lambda+2n)}{(2\pi)^d\Gamma(\lambda+1)}\times$$
$$\times\int_{\gamma(z_1)}(z_1\cdot\zeta)^\lambda(z_2\cdot\zeta)^{-\lambda-2n}\log(\zeta\cdot z_2)d\mu_\gamma(\zeta). \tag{26}$$

Eq. (24) provides the analytic continuation of $W_\lambda^{AdS_d}(z_1,z_2)$ both to the complex λ-plane and to the covering $\widetilde{\Delta}_{AdS}$ of the cut-plane

$$\Delta_{AdS} = \{z_1,z_2\in AdS_d^{(c)};\ z_1\cdot z_2\neq\rho,\ -1\le\rho\le 1\}. \tag{27}$$

Δ_{AdS} contains all pairs of events of $AdS_d^{(c)}$ minus the causal cut; it contains in particular all pairs of non-coincident Euclidean points. Eq. (24) shows that $\lambda\to(1-d-\lambda)$ is not a symmetry of the two-point function.

Eq. (23) is very significant for the study of Legendre functions of the second kind. To give one example, a nontrivial consequence is the, perhaps unknown, multiplication theorem (calculated at $d=2$; a more complicated formula of the same kind holds for general d):

$$Q_\lambda(x\,\mathrm{ch}\,u) = \sum_{n=0}^\infty\frac{\left(1-x^{-2}\right)^n}{x^{1+\lambda}\Gamma(1+n)}\,\frac{Q^n_{n+\lambda}(\mathrm{ch}\,u)}{(2\,\mathrm{sh}\,u)^n}. \tag{28}$$

Let us consider now the Poincaré foliation of $AdS_d^{(c)}$:

$$z(\mathrm{z},u) = \begin{cases} z^\mu & = & \frac{1}{u}\mathrm{z}^\mu\\ z^{d-1} & = & \frac{1-u^2}{2u}+\frac{1}{2u}\mathrm{z}^2\\ z^d & = & \frac{1+u^2}{2u}-\frac{1}{2u}\mathrm{z}^2\end{cases},\ \mathrm{z}^2 = \eta_{\mu\nu}\mathrm{z}^\mu\mathrm{z}^\nu. \tag{29}$$

For real $u>0$ and real $\mathrm{z}^\mu = \mathrm{x}^\mu$ the above coordinates cover one half of AdS_d which we denote AdS_M. Its partial complexification (u is kept real) contains the Euclidean manifold $EAdS_d$; the leaves at fixed $u>0$ are copies of $M^{(c)}_{d-1}$; the Minkowskian tubes $T_\pm$ of the leaves are contained into $\mathcal{Z}_\pm$ (see [5] for more details). The latter property has also an important technical value and allows to compute the Fourier transforms of the waves w.r.t. the coordinates x^μ and to derive, after some pain, a diagonal representation of the two-point function (see also [5,11]):

$$W_\lambda^{AdS_d}(z_1(\mathrm{z}_1,u),z_2(\mathrm{z}_2,u')) = \frac{(uu')^{\frac{d-1}{2}}}{2(2\pi)^{d-2}}\times$$
$$\times\int\theta(\mathrm{p}^0)\theta(\mathrm{p}^2)e^{-i\mathrm{p}(\mathrm{z}_1-\mathrm{z}_2)}J_\nu\left(u\sqrt{\mathrm{p}^2}\right)J_\nu\left(u'\sqrt{\mathrm{p}^2}\right)d\mathrm{p} =$$
$$= \frac{(uu')^{\frac{d-1}{2}}}{2}\int_0^\infty \mathrm{W}_m^{M_{d-1}}(\mathrm{z}_1,\mathrm{z}_2)J_\nu(mu)J_\nu(mu')dm^2 \tag{30}$$

where $\nu(\lambda) = \frac{d-1}{2}+\lambda$ $\mathrm{p\,z} = \eta_{\mu\nu}\mathrm{p}^\mu\mathrm{z}^\nu$ and $\mathrm{p}^2 = \eta_{\mu\nu}\mathrm{p}^\mu\mathrm{p}^\nu$. The first equality is valid for $\mathrm{z}_1\in T_-$ and $\mathrm{z}_2\in T_+$; the second equality (30) has the form of a Källén-Lehmann expansion and holds in the whole cut-plane

$$\Delta_M = \{\mathrm{z}_1,\mathrm{z}_2\in M^{(c)}_{d-1};\ \ \mathrm{z}_1\cdot\mathrm{z}_2\neq\rho,\ \ \rho\ge 0\}.$$

From Eq. (30) it may be deduced a new integral representation of the Feynman propagator for pair of events x_1,x_2 *in the real Poincaré chart* AdS_M:

$$G_\lambda^{AdS_d}(x_1(\mathrm{x}_1,u),x_2(\mathrm{x}_2,u)) = \frac{(uu')^{\frac{d-1}{2}}}{2}\times$$
$$\times\int_0^\infty G_m^{M_{d-1}}(\mathrm{x}_1,\mathrm{x}_2)J_\nu(mu)J_\nu\left(mu'\right)dm^2; \tag{31}$$

here $G_m^{M_{d-1}}(\mathrm{x}_1,\mathrm{x}_2)$ is the Feynman propagator of massive Minkowskian scalar field in spacetime dimension $d-1$:

$$(\Box_{AdS}+m_\lambda^2)G_\lambda^{AdS_d}(x_1(\mathrm{x}_1,u),x_2(\mathrm{x}_2,u)) =$$
$$= (uu')^{\frac{d-1}{2}}\int_0^\infty u^2\delta(\mathrm{z}_1,\mathrm{z}_2)J_\nu(mu)J_\nu\left(mu'\right)mdm$$
$$= u^d\delta(\mathrm{z}_1,\mathrm{z}_2)\delta(u,u') = \delta_{AdS_M}(x_1,x_2). \tag{32}$$

In the last equality we used the inversion theorem for the Hankel transform of order ν [12] and $\sqrt{g(x)} = u^{-d}$.

AdS Feynman Diagrams are usually computed in the Euclidean manifold, often by using the "split" representation of the Euclidean Feynman propagator [13–15]: the latter may be understood as the generalized inverse Mehler-Fock transform with respect to the mass parameter λ of the de Sitter Wightman function (10) taken at purely imaginary events $z_1 = iy_1\in\mathcal{T}_-$ and $z_2 = iy_2\in\mathcal{T}_+$ [16, Appendix A].

The relation of AdS Euclidean diagrams with real space diagrams [10,14,17] remains however poorly understood, also because calculations based on the split representation cannot be Wick-rotated.

Eq. (31) may shed some new light on this issue. In particular it immediately implies that banana diagrams computed in the Euclidean manifold $EAdS_d$ [16] can be Wick-rotated and coincide with the same diagrams calculated in AdS_M.

What is less obvious is whether a more general diagram computed by integrating only in a Poincaré patch AdS_M produces an AdS invariant result. This happens for the elementary one line diagram:

$$\int_{AdS_M}G_\lambda^{AdS_d}(x_1,x)\sqrt{g(x)}\,dx =$$
$$\int_0^\infty\int_0^\infty\left(\frac{u'}{u}\right)^{\frac{d-1}{2}}\frac{J_\nu(mu)J_\nu\left(mu'\right)}{mu}du\,dm.$$

Carrying the integration in the u variable first we get

$$\int_0^\infty\frac{m^{\frac{d-3}{2}}\Gamma\left(\frac{\lambda}{2}\right)u'^{\frac{d-1}{2}}J_{\frac{d-1}{2}+\lambda}(mu')}{2^{\frac{d+1}{2}}\Gamma\left(\frac{1}{2}(d+\lambda+1)\right)}dm =$$
$$= \frac{1}{\lambda(d+\lambda-1)} = \frac{1}{m_\lambda^2}. \tag{33}$$

The result does not depend on u', is AdS invariant and coincides with the Euclidean calculation. A less trivial example is the two-line diagram:

$$F_{\lambda_1\lambda_2}(x_1,x_2) = \int_{AdS_M}G_{\lambda_1}^{AdS_d}(x_1,x)G_{\lambda_2}^{AdS_d}(x,x_2)\sqrt{g(x)}dx.$$

Would we know the AdS invariance of $F_{\lambda_1\lambda_2}$ we could immediately write [16]

$$F_{\lambda\nu}(x_1,x_2) = -\frac{G_{\lambda_1}^{AdS_d}(x_1,x_2)-G_{\lambda_2}^{AdS_d}(x_1,x_2)}{m_{\lambda_1}^2-m_{\lambda_2}^2}. \tag{34}$$

A direct calculation that confirms the above result may be performed by Wick-rotating the identity

$$\int_0^\infty\frac{J_{\nu_1}(mu)J_{\nu_1}\left(mu'\right)-J_{\nu_2}(mu)J_{\nu_2}\left(mu'\right)}{(\nu_2^2-\nu_1^2)(\mathrm{p}^2+m^2)}mdm =$$
$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{abJ_{\nu_1}(au)J_{\nu_1}(av)J_{\nu_2}(bv)J_{\nu_2}\left(bu'\right)}{v\left(a^2+\mathrm{p}^2\right)\left(b^2+\mathrm{p}^2\right)}dv\,da\,db$$

that can be shown at first for $\mathrm{p}^2>0$ by using Eq. (24) and the Euclidean version of Eq. (31). Taking the limit $x_1\to x_2$ we obtain as a consequence the one-loop banana integral calculated in AdS_M, and this coincides with the Euclidean calculation done in [16].

We may also compute Witten diagrams in real space: the external legs are the boundary values of the plane waves (21) and the propagator is once more the one given in Eq. (31). “Ingoing” waves should be boundary values of the waves holomorphic in $\mathcal{Z}_-$ and “outgoing” waves boundary values of the waves holomorphic in $\mathcal{Z}_+$.

We conjecture that also these diagrams integrated over AdS_M give AdS invariant results (see also [10,14,17]) but this is a question that remains at the moment open.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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This letter is dedicated to the memory of Henri Epstein.

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