Exact solutions for analog Hawking effect in dielectric media

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In the framework of the analog Hawking radiation for dielectric media, we analyze a toy model and also the 2D reduction of the Hopfield model for a specific monotone and realistic profile for the refractive index. We are able to provide exact solutions, which do not require any weak dispersion approximation. The theory of Fuchsian ordinary differential equations is the basic tool for recovering exact solutions, which are rigorously identified and involve the so-called generalized hypergeometric functions ${}_4F_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4; \beta_1, \beta_2, \beta_3; z)$. A complete set of connection formulas are available, both for the subcritical case and for the transcritical one, and also the Stokes phenomenon occurring in the problem is fully discussed. From the physical point of view, we focus on the problem of thermality. Under suitable conditions, the Hawking temperature is deduced, and we show that it is in full agreement with the expression deduced in other frameworks under various approximations.

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I. INTRODUCTION

We are interested to focus our attention on analytical calculations of the analog Hawking effect in dielectric media and in the presence of dispersion. Analytical calculations for the analog Hawking effect, introduced in the seminal paper [1] for nondispersive media, and also in the dispersive case, have been largely discussed in literature [see, e.g., the following (nonexhaustive) list of papers [2–23]]; for weak dispersion and the transcritical case, a rather general mathematical framework, able to encompass in an unified picture very relevant models even for the experiments [24–35], has been discussed in [36–38]. In [39], the authors introduced a new mathematical perspective in the analog Hawking effect by relating the problem to the solution of a fourth-order Fuchsian equation for the subcritical case. As a remarkable example of the possibilities offered by Fuchsian equations, we provide here an *exact* solution of a particular scattering problem inside a dielectric. We stress that this represents a very relevant achievement, both on the physical side and on the mathematical one. Indeed, as far as the Hawking effect in dielectric media is concerned, no exact solution has been provided in the physical literature before. Exact analytical solutions represent an actually very hard task in this field and allow one to explore physical situations that are forbidden in approximate solutions, and as happens in any physical field at hand, they constitute a strong conceptual reference for further studies. We shall provide this exact solution under a specific but physically very relevant choice of the background pulse giving rise to the Hawking radiation (see below), and we are also able to corroborate existing results in approximate models by means of suitable limits of our solutions. Furthermore, a deepening of our general comprehension of the Hawking effect in the presence of dispersion is made possible by our analysis. In particular, we show that nontrivial connection formulas between in- and outgoing states are associated with the presence of pair creation, and, moreover, this nontriviality is in turn related to a Stokes phenomenon. This relevance is mostly evident in the subcritical case, where, e.g., a naive WKB approach would fail to provide any pair-creation process. From a mathematical point of view, our achievement is also important, as we provide exact solutions in a well-grounded physical model to a fourth-order Fuchsian equation, which is also of great interest on the mathematical side.

We choose a monotonic refractive index profile traveling inside the dielectric at a fixed velocity. This kind of background represents a fundamental setting for a good understanding of the pair-creation process, and monotonic backgrounds are often considered in analytical computations in the transcritical regime and in numerical simulations of analog systems (see, e.g., [40]). We stress that exact solutions for the dispersive case of the analog Hawking effect are very hard to be obtained, as even solutions of ordinary differential equations of the fourth order. The only other example we can find is contained in the paper by Philbin [18], which provides exact solutions for the Corley model [3,9]. Therein solutions are obtained at the price of introducing an interesting but somehow unrealistic linear velocity profile $v(x) = -\alpha x$.

The plan of the paper is the following. In Sec. II, we present the model and the monotonic background we are going to consider. In Sec. III, starting from the equation of motion for our model, we show that, by means of a suitable change of the independent spatial variable in the comoving frame of the pulse, we are able to obtain a fourth-order Fuchsian equation. We provide a detailed characterization of its local monodromy and spectral type. In Sec. IV, we provide the exact solution and recover rigorously that the generalized hypergeometric functions ${}_4F_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4; \beta_1, \beta_2, \beta_3; z)$ are involved. Furthermore, we provide a study of the Stokes phenomenon, and we study some physical consequences for the scattering problem at hand. In Sec. V, we consider a generalization of the previous analysis to the case of the original model (the so-called $\phi\psi$ model) to which the Hopfield model reduces in the 2D case, and again we consider the scattering problem and the thermality of the spectrum, which is recovered to coincide with the one deduced in the weak dispersion limit discussed in [36] under suitable conditions. In Sec. VI, we summarize our achievements and display future perspectives for our analysis.

II. THE CAUCHY MODEL AND THE CHOICE OF BACKGROUND

The model we consider is the modified $\phi - \psi$ model (or "Cauchy model") introduced by the authors in [39]. In the laboratory frame, it is expressed by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_{t_l}\phi)^2 + \frac{1}{2}((\partial_{t_l}\psi)^2 + \mu^2\psi^2) + g\phi\partial_{x_l}\psi - \frac{\lambda}{4!}\psi^4. \tag{1}$$

As we shall discuss further on in the following, this model has its ratio in the fact that it provides us the simplest model for analyzing the Hawking effect in dielectrics, due to its simple dispersion relation and for the simplifications it provides in analytical calculations. The Cauchy model at hand simulates an effective description of the interaction between the electromagnetic field and a dielectric medium and it is inspired by the physical model discussed in Sec. V, which is the two-dimensional reduction of the Hopfield model: in that model, the field ϕ represents the electromagnetic field, and the field ψ takes the role of the polarization field. The Lagrangian (1) involves a coupling term between ϕ and ψ and a nonlinear term in the polarization field, and its structure is aimed to reproduce the Cauchy-like dispersion relation [Eq. (4)], i.e., the simplest dispersion relation for the electromagnetic field in dielectrics. Even though its structure is oversimplified, we can show that this model provides a very interesting benchmark for analytical studies and also preserves all the basic features of the more tricky physical model discussed in Sec. V.

The linearized equations of motion (EOMs) around a background solution ψ_B, in the lab frame, are

$$\partial_{t_l}^2\phi - g\partial_{x_l}\psi = 0, \tag{2}$$

$$\partial_{t_l}^2\psi + g\partial_{x_l}\psi - \mu^2\phi + \frac{\lambda}{2}\psi_B^2\psi = 0. \tag{3}$$

The free-field solutions (for $\lambda = 0$) are plane waves $e^{i\omega_{\rm lab}t_l - ik_{\rm lab}x_l}$ which satisfy

$$n_0^2(\omega_{\rm lab}) \coloneqq \frac{k_{\rm lab}^2}{\omega_{\rm lab}^2} = \frac{\mu^2}{g^2} + \frac{\omega_{\rm lab}^2}{g^2} \eqqcolon A + B\omega_{\rm lab}^2. \tag{4}$$

The dispersion relation, in a reference frame moving with velocity V with respect to the lab frame, has four solutions (see Fig. 1): expanding $k(\omega)$ for $\omega \to 0$, the four modes have the following expressions:

$$k_H = \frac{\mu - gV}{g - \mu V}\omega + O(\omega^3), \tag{5}$$

$$k_B = -\frac{\mu + gV}{g + \mu V}\omega + O(\omega^3), \tag{6}$$

$$k_P = \frac{\sqrt{g^2 - \mu^2V^2}}{\gamma V^2} - \left(\frac{1}{V} + \frac{g^2}{\gamma^2 V(g^2 - \mu^2V^2)}\right)\omega - \frac{g^2(2g^2 + \mu^2V^2)}{2\gamma(g^2 - \mu^2V^2)^{5/2}}\omega^2 + O(\omega^3), \tag{7}$$

$$k_N = -\frac{\sqrt{g^2 - \mu^2V^2}}{\gamma V^2} - \left(\frac{1}{V} + \frac{g^2}{\gamma^2 V(g^2 - \mu^2V^2)}\right)\omega + \frac{g^2(2g^2 + \mu^2V^2)}{2\gamma(g^2 - \mu^2V^2)^{5/2}}\omega^2 + O(\omega^3). \tag{8}$$

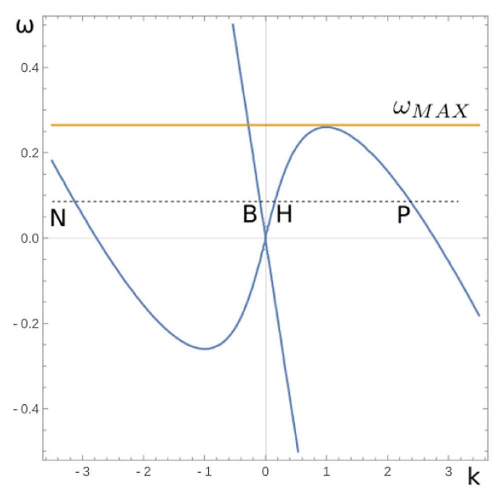


FIG. 1. The dispersion relation (4) represented in the comoving frame with the background, with $g = 1$, $\mu = 1.2$; for $0 < \omega < \omega_{MAX}$ there are four real solutions.

The reason for the choice of the model lies in the simple expression of the dispersion relation: however, in many cases we will refer to $\mathrm{DR}(k)$ as a generic fourth-order polynomial, so the choice of a particular dispersion relation is not really crucial. We are going to solve the linearized equations (2) and (3) with a particular choice of background. In the experiments the background field is represented by a laser pulse, which is naturally *localized* and travels rigidly at a certain velocity V. We will instead consider a monotonic background

$$\psi_b(x - Vt)^2 = 1 - \tanh(\beta(x - Vt)). \tag{9}$$

We claim that this is a good model for the right side of a laser pulse; moreover, a monotonic background represents a better model for an event horizon and allows one to better understand the nature of Hawking radiation. Examples of applications for such kinds of monotonic profiles to refractive index perturbations in dielectric media can be found, e.g., in [14] (see also the associated Supplemental Material of this reference for more details), and cf. also Chap. 10 in [41], as far as the original $\phi\psi$ model, to be discussed in Sec. V, is concerned. We remark that the refractive index perturbation δn one is able to associate with the aforementioned model in the Cauchy dispersion relation regime is substantially proportional to ψ_b^2. Monotonic backgrounds of this type were also used in some previous studies of analog black holes and white holes (see, for example, [40]). Even if the most realistic description of a laser pulse perturbation in a dielectric is associated with a profile involving both a rising part (white hole) of the dielectric perturbation and also a decreasing side (black hole) of it [42], suitable settings can properly simulate an interaction with only, e.g., the rising part, described by the monotone perturbation (see also the discussion in [14]).

The linearized equations (2) and (3) can be put together to a fourth-order equation of generalized Orr-Sommerfeld type. It is convenient to write these equations in the comoving coordinates $t = \gamma(t_l - Vx_l)$, $x = \gamma(x_l - Vt_l)$. Since the potential term is independent of the comoving time, we seek a solution in the form $\psi = e^{-i\omega t}f(x)$, $\phi = e^{-i\omega t}g(x)$. By applying $(v^\mu\partial_\mu)^2$ to the second equation, we obtain a single fourth-order equation for $f(x)$ only,

$$\begin{aligned}0 &= V^4\gamma^4 f^{(4)}(x) + 4iV^3\omega\gamma^4 f^{(3)}(x)\\ &+ \frac{1}{2}\gamma^2 f''(x)\left[-\lambda V^2\tanh(\tilde\beta x) + 2g^2 + \lambda V^2 - 2\mu^2V^2 - 12V^2\omega^2\gamma^2\right]\\ &+ i\gamma^2 V f'(x)\left[i\tilde\beta\lambda V\operatorname{sech}^2(\tilde\beta x) + \omega\left(-\lambda\tanh(\tilde\beta x) + 2g^2 + \lambda - 2\mu^2 - 4\omega^2\gamma^2\right)\right]\\ &+ \frac{1}{2}\gamma^2 f(x)\left[\omega^2\left(\lambda\tanh(\tilde\beta x) - 2g^2V^2 - \lambda + 2\mu^2 + 2\omega^2\gamma^2\right) + 2\tilde\beta\lambda V\operatorname{sech}^2(\tilde\beta x)(\tilde\beta V\tanh(\tilde\beta x) - i\omega)\right],\end{aligned} \tag{10}$$

where $\tilde\beta = \frac{\beta}{\gamma}$.

III. REDUCTION TO A FUCHSIAN EQUATION: MONODROMY AND RIEMANN SCHEME

As in [39], we perform the following change of variables on Eq. (10):

$$z = -e^{2\tilde\beta x}, \tag{11}$$

which implies

$$\partial_x = 2\tilde\beta\theta_z \coloneqq 2\tilde\beta z\frac{d}{dz}. \tag{12}$$

By defining the rescaled parameters $G = \frac{g}{2\tilde\beta}$, $\Omega = \frac{\omega}{2\tilde\beta}$, $M = \frac{\mu}{2\tilde\beta}$, and $\Lambda = \frac{\lambda}{4\tilde\beta^2}$, we end up with the following equation:

$$\begin{aligned}0 &= V^4\gamma^4z^4f^{(4)}(z) + f^{(3)}(z)(6V^4\gamma^4z^3 - 4iV^3\Omega\gamma^4z^3) + f''(z)\left(G^2\gamma^2z^2 - V^2\gamma^2z^2\left(\frac{\Lambda}{z-1} + M^2\right)\right.\\ &\left.- 6V^2\Omega^2\gamma^4z^2 - 12iV^3\Omega\gamma^4z^2 + 7V^4\gamma^4z^2\right) + f'(z)\left(-2iG^2V\Omega\gamma^2z + G^2\gamma^2z + V^2\gamma^2z\left(\frac{\Lambda}{(1-z)^2} - M^2\right)\right.\\ &\left.- 2iV\Omega\gamma^2z\left(\frac{\Lambda}{(1-z)} - M^2\right) + \frac{2\Lambda V^2\gamma^2z^2}{(1-z)^2} - 6V^2\Omega^2\gamma^4z - 4iV^3\Omega\gamma^4z + V^4\gamma^4z + 4iV\Omega^3\gamma^4z\right)\\ &+ f(z)\left(-G^2V^2\Omega^2\gamma^2 + \Omega^2\gamma^2\left(\frac{\Lambda}{1-z} + M^2\right) + \frac{2\Lambda V^2\gamma^2z^2}{(1-z)^3} + \frac{\Lambda V^2\gamma^2z}{(1-z)^3} - \frac{2i\Lambda V\Omega\gamma^2z}{(1-z)^2} + \Omega^4\gamma^4\right).\end{aligned} \tag{13}$$

This equation is of Fuchsian type, with three singular points, $z = 0, 1, \infty$.

As an alternative form, we can write the EOMs (2) and (3) as a system of first order and perform the same change of variables as before. With this procedure we obtain the system

$$\frac{dU}{dz} = A(z)U, \tag{14}$$

where

$$U = \begin{pmatrix} g(z) \\ g'(z) \\ f(z) \\ f'(z) \end{pmatrix}, \tag{15}$$

$$A(z) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\Omega^2}{V^2 z^2} & -\frac{V^2\gamma^2 z - 2iV\Omega\gamma^2 z}{V^2\gamma^2 z^2} & \frac{iG\Omega}{V\gamma z^2} & -\frac{G}{V^2\gamma z} \\ 0 & 0 & 0 & 1 \\ -\frac{iG\Omega}{V\gamma z^2} & \frac{G}{V^2\gamma z} & -\frac{-\frac{\Lambda}{1-z} - M^2 - \Omega^2\gamma^2}{V^2\gamma^2 z^2} & -\frac{V^2\gamma^2 z - 2iV\Omega\gamma^2 z}{V^2\gamma^2 z^2} \end{pmatrix}. \tag{16}$$

We can reduce (14) to a "Fuchsian system of normal form" [43] by changing variables to

$$Y(z) = P(z)U(z), \tag{17}$$

$$P(z) = \mathrm{diag}\left(\frac{1}{z}, 1, \frac{1}{z}, 1\right). \tag{18}$$

The system has now the form

$$\frac{dY}{dz} = \left(\frac{A_1}{z} + \frac{A_2}{z-1}\right)Y, \tag{19}$$

$$A_1 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ \frac{\Omega^2}{V^2} & \frac{-V-2i\Omega}{V} & -\frac{iG\Omega}{V\gamma} & -\frac{G}{V^2\gamma} \\ 0 & 0 & -1 & 1 \\ \frac{iG\Omega}{V\gamma} & \frac{G}{V^2\gamma} & \frac{\Lambda+M^2+\Omega^2\gamma^2}{V^2\gamma^2} & \frac{-V-2i\Omega}{V} \end{pmatrix}, \tag{20}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\Lambda}{V^2\gamma^2} & 0 \end{pmatrix}. \tag{21}$$

The matrices A_1 and A_2 are constant and they are, respectively, the residue at the simple poles $z = 0$ and $z = 1$. We may also define

$$A_0 \coloneqq -A_1 - A_2, \tag{22}$$

which corresponds to the residue at the simple pole $z = \infty$.

A. The local solutions and monodromy

We start looking for local solutions of (13) around $z = \infty$. After changing variables to $t = 1/z$, we can look for a solution in the form

$$f(t) = t^{-i\alpha} \sum_{n=0}^{\infty} c_n t^n. \tag{23}$$

The characteristic equation for the exponent $k \coloneqq 2\tilde{\beta}\alpha$ is

$$\mathrm{DR}(k) \coloneqq \gamma^2(\mu^2(kV+\omega)^2 - g^2(k+V\omega)^2 + (kV+\omega)^4\gamma^2) = 0, \tag{24}$$

which is nothing but the dispersion relation (4) as written in the comoving reference frame with the background. Equation (24) has, in general, four distinct complex solutions, so we find four independent solutions in the form (23): the spectral type of the equation at $z = \infty$ is thus (1111). There is the possibility of emergence of a resonant case, where the difference between eigenvalues is an integer, still in a zero measure set in the space of available parameters appearing in our model.

A similar behavior is found at $z = 0$: we find four independent local solutions of the form

$$f(z) = z^{i\tilde{\alpha}} \sum_{n=0}^{\infty} c_n z^n, \tag{25}$$

where $\tilde{k} \coloneqq 2\tilde{\beta}\,\tilde{\alpha}$ satisfies

$$\mathrm{DR}_0(k) := \gamma^2\Big(-g^2(\tilde{k}+V\omega)^2 + (\tilde{k}V+\omega)^2(\lambda+\mu^2+(\tilde{k}V+\omega)^2\gamma^2)\Big) = 0. \quad (26)$$

Equation (26) is equivalent to (24) if one maps $\mu^2 \mapsto \mu^2 + \lambda$. The spectral type at $z = 0$ is again (1111), again almost everywhere in the space of available parameters appearing in our model. Some interesting formulas concerning (24) and (26) are discussed in Appendix A.

The situation at $z = 1$ is different. After defining $y = z - 1$, the characteristic equation for solutions of the form

$$f(y) = y^a \sum_{n=0}^{\infty} c_n y^n \quad (27)$$

has four integer solutions $a = 0, 1, 2, 3$. This situation is known in literature as the resonant case (cf., e.g., [43]) and requires a particular study. We refer mostly to [44], where still the discussion is left incomplete, and, particularly, to the thorough analysis appearing in [45], and also to [46]. As suggested in the aforementioned literature, we apply the so-called Frobenius method for the analysis of solutions at a Fuchsian singularity, also in the resonant case, and we can also verify if there are logarithmic contributions (even in the resonant case, they might also not appear).

By means of the Frobenius method, we obtain three independent integer solutions

$$u_1(y) = y^3 + y^4\left[-\frac{3}{2} + \frac{\Lambda}{12\gamma^2V^2} + i\frac{\Omega}{V}\right] + o(y^4), \quad (28)$$

$$u_2(y) = y^2 + y^3\left[-2 + \frac{\Lambda}{6\gamma^2V^2} + i\frac{4\Omega}{3V}\right] + o(y^3), \quad (29)$$

$$u_3(y) = y + y^2\left[-3 + \frac{\Lambda}{2\gamma^2V^2} + i\frac{2\Omega}{V}\right] + o(y^2), \quad (30)$$

and one logarithmic solution

$$u_0(y) = 1 + y\left[-6 - \frac{\Lambda}{V^2\gamma^2} + \frac{4i\Omega}{V}\right] + o(y) + \log(y)(R_1u_1(y) + R_2u_2(y) + R_3u_3(y)), \quad (31)$$

where

$$R_3 = \frac{\Lambda}{V^2\gamma^2}, \quad (32)$$

$$R_2 = \frac{\Lambda(5V - 2i\Omega)}{4V^3\gamma^2}, \quad (33)$$

$$R_1 = \frac{\Lambda(9V^2 - 6i\Omega V - \Omega^2)}{18V^4\gamma^2}. \quad (34)$$

The study of the monodromy of the solution is important for the characterization of the equation [43]. Starting from a basis of solutions $(u_1(\bar{z}), \ldots, u_4(\bar{z}))$ evaluated at some $\bar{z} \in \mathbb{C}$, we can prolong these solutions along a path that goes around a singular point $a \in \mathbb{C}$ and closes back to $\bar{z}$ (without enclosing other singular points): the new vector $(u_1'(\bar{z}), \ldots, u_4'(\bar{z}))$ that results from this transformation is related to the initial one by a matrix M_a. Such matrix is independent on the point $\bar{z}$ and is called the "monodromy" matrix of the solutions $(u_1, \ldots, u_4)$ at the point a. The monodromy matrix of the solutions $(u_0(y), u_1(y), u_2(y), u_3(y))$ of (28)–(31) at $z = 1$ is easily computed as

$$M_1 = \begin{pmatrix} 1 & 2\pi i R_1 & 2\pi i R_2 & 2\pi i R_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (35)$$

whose Jordan form is

$$J_{M_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (36)$$

The monodromy at $z = 0$ and $z = \infty$ are even more easy to determine and they are represented, respectively, by the diagonal matrices

$$M_0 = \mathrm{diag}(e^{i2\pi\tilde{\alpha}_1}, e^{i2\pi\tilde{\alpha}_2}, e^{i2\pi\tilde{\alpha}_3}, e^{i2\pi\tilde{\alpha}_4}), \quad (37)$$

$$M_\infty = \mathrm{diag}(e^{-i2\pi\alpha_1}, e^{-i2\pi\alpha_2}, e^{-i2\pi\alpha_3}, e^{-i2\pi\alpha_4}), \quad (38)$$

whose Jordan form is $J_{M_0} = J_{M_\infty} = \mathbb{I}_4$. From the Jordan form we can infer that the spectral type of the equation at $z = 1$ is (3,1) (see [43]). The spectral type of the equation is thus [(1111),(31),(1111)]: this spectral type is classified as "rigid." Without entering into mathematical details, an equation is called rigid if the local monodromy class of its solutions uniquely determines also the "global" monodromy class. Another way of expressing the same concept is that the equation only depends on its "local data" (i.e., the characteristic exponents) and there are no "accessory parameters." We can also calculate the so-called index of rigidity [43]

$$\imath = \sum_{j=0}^{p} \dim Z(M_j) - (p-1)n^2, \quad (39)$$

where $Z(M_j)$ is the centralizer of the matrix M_j (i.e., the dimension of the vector space of matrices that commute with M_j) and $p + 1$ is the number of distinct singular points of the equation. A known result says that a Fuchsian system is rigid if and only if $\imath = 2$. In our case, a simple computation leads to

$$\begin{aligned} Z(M_0) + Z(M_1) + Z(M_\infty) - (4)^2 &= 4 + 10 + 4 - 16 \\ &= 2. \end{aligned} \tag{40}$$

Rigid equations have thus a simple structure and there are many results available for their characterization and solution. The rigidity of the equation allows one, in principle, to find integral representations of the solutions and write exact expressions of connection coefficients for the local solutions at the different singular points: for the physical problem of the scattering. Another consequence of rigidity is that the local monodromy classes uniquely determine the global monodromy: this is interesting for physics, since the action of the global monodromy can be interpreted as the result of the scattering of a wave, so the scattering coefficients may be derived from the monodromy matrices [47,48].

B. Gauge transformation and Riemann scheme

We start again from Eq. (13) and we perform a so-called gauge transformation in order to put to zero as one of the characteristic exponents. We look for a solution of the form

$$f(z) := z^{i\frac{\tilde{k}_1}{2\tilde{\beta}}}(z-1)u(z), \tag{41}$$

where $\tilde{k}_1$ satisfies

$$\mathrm{DR}_0(\tilde{k}_1) = 0.$$

The exponent of $(z-1)$ was chosen to lower the order of the singularities at $z = 1$. The function $u(z)$ now satisfies

$$\begin{aligned} & u(z)\Big[\gamma^2\Big(G^2(\tilde{k}_1+V\Omega)^2-(\tilde{k}_1V+\Omega)^2\Big(\gamma^2(\tilde{k}_1V+\Omega)^2+\Lambda+M^2\Big)\Big) \\ &\quad +\gamma^2 z\Big(-G^2(\tilde{k}_1+V\Omega-i)^2+M^2(\Omega+(\tilde{k}_1-i)V)^2+\gamma^2(\Omega+(\tilde{k}_1-i)V)^4\Big)\Big] \\ &\quad +u'(z)\Big[\gamma^2 z\Big(V(2ik_1V+V+2i\Omega)(-i\gamma^2((1+i)k_1V-iV+(1+i)\Omega)((1+i)k_1V+V+(1+i)\Omega)+L+M^2) \\ &\quad -iG^2(2k_1+2V\Omega-i)\Big) \\ &\quad +\gamma^2z^2\Big(-iV(2\Omega+(2k_1-3i)V)(M^2+\gamma^2((-5+2k_1(k_1-3i))V^2+2(2k_1-3i)V\Omega+2\Omega^2))2iG^2k_1+2iG^2V\Omega+3G^2\Big)\Big] \\ &\quad +u''(z)\Big[\gamma^2z^2\Big(-G^2+6\tilde{k}_1^2V^4\gamma^2+12\tilde{k}_1V^3\Omega\gamma^2-12i\tilde{k}_1V^4\gamma^2+\Lambda V^2+M^2V^2+6V^2\Omega^2\gamma^2-12iV^3\Omega\gamma^2-7V^4\gamma^2\Big) \\ &\quad +\gamma^2z^3\Big(G^2-6\tilde{k}_1^2V^4\gamma^2-12(\tilde{k}_1-2i)V^3\Omega\gamma^2+24i\tilde{k}_1V^4\gamma^2-M^2V^2-6V^2\Omega^2\gamma^2+25V^4\gamma^2\Big)\Big] \\ &\quad +u^{(3)}(z)\Big[\gamma^2z^4\Big(4i\tilde{k}_1V^4\gamma^2+4iV^3\Omega\gamma^2+10V^4\gamma^2\Big)+\gamma^2z^3\Big(-4i\tilde{k}_1V^4\gamma^2-4iV^3\Omega\gamma^2-6V^4\gamma^2\Big)\Big] \\ &\quad +u^{(4)}(z)z^4(z-1)V^4\gamma^4=0. \end{aligned} \tag{42}$$

The last equation can be written in a more convenient form using (24) and (26),

$$\begin{aligned} & u(z)[-\mathrm{DR}_0(\tilde{k}_1)+z\mathrm{DR}(\tilde{k}_1-i)]+u'(z)\Big[z(\mathrm{DR}_0(\tilde{k}_1)-\mathrm{DR}_0(\tilde{k}_1-i))-z^2(\mathrm{DR}(\tilde{k}_1-i)-\mathrm{DR}(\tilde{k}_1-2i))\Big] \\ &\quad +u''(z)\Big[\frac{1}{2}z^3(\mathrm{DR}(\tilde{k}_1-i)-2\mathrm{DR}(\tilde{k}_1-2i)+\mathrm{DR}(\tilde{k}_1-3i))-\frac{1}{2}z^2(\mathrm{DR}_0(\tilde{k}_1)-2\mathrm{DR}_0(\tilde{k}_1-i)+\mathrm{DR}_0(\tilde{k}_1-2i))\Big] \\ &\quad +u^{(3)}(z)\Big[\frac{1}{6}z^3(\mathrm{DR}_0(\tilde{k}_1)-3\mathrm{DR}_0(\tilde{k}_1-i)+3\mathrm{DR}_0(\tilde{k}_1-2i)-\mathrm{DR}_0(\tilde{k}_1-3i)) \\ &\quad -\frac{1}{6}z^4(\mathrm{DR}(\tilde{k}_1-i)-3\mathrm{DR}(\tilde{k}_1-2i)+3\mathrm{DR}(\tilde{k}_1-3i)-\mathrm{DR}(\tilde{k}_1-4i))\Big] \\ &\quad +u^{(4)}(z)\Big[\frac{1}{24}z^5(\mathrm{DR}(\tilde{k}_1-i)-4\mathrm{DR}(\tilde{k}_1-2i)+6\mathrm{DR}(\tilde{k}_1-3i)-4\mathrm{DR}(\tilde{k}_1-4i)+\mathrm{DR}(\tilde{k}_1-5i)) \\ &\quad -\frac{1}{24}z^4(\mathrm{DR}_0(\tilde{k}_1)-4\mathrm{DR}_0(\tilde{k}_1-i)+6\mathrm{DR}_0(\tilde{k}_1-2i)-4\mathrm{DR}_0(\tilde{k}_1-3i)+\mathrm{DR}_0(\tilde{k}_1-4i))\Big]=0, \end{aligned} \tag{43}$$

where $\mathrm{DR}_0(\tilde{k}_j) = 0$ and $\mathrm{DR}(k_j) = 0$.

It is easy to verify, by studying the local solutions as in Sec. III A, that the characteristic exponents of Eq. (43) are

$$\begin{bmatrix} z=0 & z=1 & z=\infty \\ 0 & 0 & 1-\frac{i}{2\tilde{\beta}}(k_1-\tilde{k}_1) \\ \frac{i}{2\tilde{\beta}}(\tilde{k}_2-\tilde{k}_1) & 1 & 1-\frac{i}{2\tilde{\beta}}(k_2-\tilde{k}_1) \\ \frac{i}{2\tilde{\beta}}(\tilde{k}_3-\tilde{k}_1) & 2 & 1-\frac{i}{2\tilde{\beta}}(k_3-\tilde{k}_1) \\ \frac{i}{2\tilde{\beta}}(\tilde{k}_4-\tilde{k}_1) & -1 & 1-\frac{i}{2\tilde{\beta}}(k_4-\tilde{k}_1) \end{bmatrix}. \tag{44}$$

Equation (44) is the so-called Riemann scheme of the equation: the Riemann P scheme is usually written as

$$\mathcal{P}\begin{Bmatrix} w' & w'' & w''' & \\ a_1 & b_1 & c_1 & \\ a_2 & b_2 & c_2 & ;z \\ a_3 & b_3 & c_3 & \\ a_4 & b_4 & c_4 & \end{Bmatrix} \tag{45}$$

and indicates independently the equations and the solutions. By defining

$$\alpha_i \coloneqq 1-\frac{i}{2\tilde{\beta}}(k_i-\tilde{k}_1), \quad i=1,2,3,4, \tag{46}$$

$$\beta_j \coloneqq 1-\frac{i}{2\tilde{\beta}}(\tilde{k}_{j+1}-\tilde{k}_1), \quad j=1,2,3, \tag{47}$$

we can write Eq. (44) as

$$\begin{bmatrix} z=0 & z=1 & z=\infty \\ 0 & 0 & \alpha_1 \\ 1-\beta_1 & 1 & \alpha_2 \\ 1-\beta_2 & 2 & \alpha_3 \\ 1-\beta_3 & -\beta_4 & \alpha_4 \end{bmatrix}, \tag{48}$$

which corresponds to the Riemann scheme of the hypergeometric function

$${}_4F_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4;\beta_1,\beta_2,\beta_3;z)$$

in the standard form [49]. The exponent β_4 in the hypergeometric function is defined by $\sum_{i=0}^4 \alpha_i = \sum_{i=0}^4 \beta_i$ and is indeed equal to 1. Therefore, the spectral type and the Riemann scheme of our fourth-order equation coincide with those of the hypergeometric function ${}_4F_3$. Since the system is rigid, Eq. (42) has to be equivalent to the hypergeometric equation ${}_4E_3$ [50], and ${}_4F_3$ has to be a solution, as we are now going to show.

IV. THE EXACT SOLUTION: HYPERGEOMETRIC ${}_4F_3$, STOKES PHENOMENON, AND CONNECTION FORMULAS

We look for a locally holomorphic solution of Eq. (43) around $z=0$,

$$u(z) = 1 + c_1 z + c_2 z^2 + \cdots. \tag{49}$$

We are going to prove the following proposition that gives the explicit expression of the coefficients c_n:

Proposition 1. Given any two fourth-order polynomials $\mathrm{DR}(k)$ and $\mathrm{DR}_0(k)$, let $\tilde{k}_1$ be one of the roots of DR_0. Let $u(z)$ be a meromorphic function which solves Eq. (43) and suppose that $u(z)$ is locally holomorphic around $z=0$. Then, the general term of the series expansion (49) satisfies

$$c_n = \frac{\prod_{r=1}^n \mathrm{DR}(\tilde{k}_1 - ri)}{\prod_{s=1}^n \mathrm{DR}_0(\tilde{k}_1 - si)}. \tag{50}$$

Proof. See Appendix B. ■

Using the definitions (46) and (47) and writing the dispersion relations in terms of their roots as in (A1) and (A2), we easily find

$$c_n = \frac{\prod_{i=1}^4 \alpha_i(\alpha_i+1)\ldots(\alpha_i+n)}{n!\prod_{j=1}^3 \beta_j(\beta_j+1)\ldots(\beta_j+n)}. \tag{51}$$

This is precisely the general term of the hypergeometric function ${}_4F_3$. So we can say that

$$u(z) = {}_4F_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4;\beta_1,\beta_2,\beta_3;z) \tag{52}$$

is an exact solution of (42), and

$$f(z) = z^{i\frac{\tilde{k}_1}{2\tilde{\beta}}}(z-1)\,{}_4F_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4;\beta_1,\beta_2,\beta_3;z) \tag{53}$$

is a solution of (13). Solving the scattering problems now just amounts to writing the connection coefficients of the hypergeometric function between $z=0$ and $z=\infty$: for example, the connection coefficient $\tilde{k}_1 \to k_1$ is

$$C_{\tilde{k}_1\to k_1} = \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_3-\alpha_1)\Gamma(\alpha_4-\alpha_1)}{\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_1)\Gamma(\beta_3-\alpha_1)}. \tag{54}$$

Notice that in the generic case (excluding resonances) a basis of solution is automatically obtained replacing $\tilde{k}_1$ (and k_1) with any of the $\tilde{k}_j$ (and k_j). Indeed, the equation is invariant under permutation of the j's. To be more explicit, we have the following basis of solutions:

$$(f_1(z), f_2(z), f_3(z), f_4(z)), \tag{55}$$

where $f_j(z)$, with $j > 1$, are just obtained from (53) by replacing $\tilde{k}_1$ (and k_1) with any of the $\tilde{k}_j$ (and k_j). As a consequence, we also obtain the general solution of our equation of motion as follows:

$$F(z) = \sum_{i=1}^{4} D_i f_i(z), \tag{56}$$

where the constants D_i have to be fixed according to the scattering process one is considering. It is remarkable that the basis is already diagonal in the $\tilde{k}_i$, in the sense that the physical modes on the left side (corresponding to $x \to -\infty$, see also the following subsection) are asymptotically represented by just the element of the basis with index j: $f_j(x) \propto e^{i\tilde{k}_j x}$ as $x \to -\infty$.

Some physical considerations are mandatory. The aforementioned connection coefficients are responsible for the phenomenon of mode conversion in the scattering process, i.e., they show that, from passing from the left, i.e., at $x = -\infty$, with input mode $\tilde{k}_1$, to the right, i.e., $x = \infty$, with potential output modes k_j, $j = 1, 2, 3, 4$, the S matrix is not, in general, diagonal, as output modes with $j \neq 1$ are allowed. In making this possible, a fundamental role is played by the Stokes phenomenon, which is discussed in the following subsection. The following point is to be stressed: the Stokes phenomenon is present when at least an irregular singularity appears (see, e.g., [43]). In the present case, the equation with z as independent variable displays three Fuchsian singularities, as seen, i.e., three regular singular points $z = 0, z = 1, z = \infty$. Still, by coming back to the original variable x, which is the relevant one for the physical problem, one finds that, actually, $x = \pm\infty$ on the real axis corresponds to irregular singularities, as essential singularities in $\tanh(\beta x)$ and in $\cosh^{-2}(\beta x)$ appear in the coefficients of the equation itself. This fact is at the root of the Stokes phenomenon in the physical problem at hand.

A. Integral representation and Stokes phenomenon

By using the integral representation of the hypergeometric function and changing variable back to x, we can write the selected solution of the EOM as

$$\begin{aligned} f(x) = & \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}{2\pi i \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)} e^{i\tilde{k}_1 x}(1 + e^{2\tilde{\beta}x}) \\ & \times \int_{\gamma - i\infty}^{\gamma + i\infty} ds \frac{\Gamma(s)\Gamma(\alpha_1 - s)\Gamma(\alpha_2 - s)\Gamma(\alpha_3 - s)\Gamma(\alpha_4 - s)}{\Gamma(\beta_1 - s)\Gamma(\beta_2 - s)\Gamma(\beta_3 - s)} \\ & \times (-1)^{-s} e^{-2\tilde{\beta}xs}, \end{aligned} \tag{57}$$

with $0 < \gamma < 1$. The integrand function has simple poles in the s plane that are disposed on five lines parallel to the real axis. The poles are found at

$$\begin{aligned} s &= \tilde{s}_n \coloneqq -n, \\ s &= s_{1,n} \coloneqq \alpha_1 + n, \\ s &= s_{2,n} \coloneqq \alpha_2 + n, \\ s &= s_{3,n} \coloneqq \alpha_3 + n, \\ s &= s_{4,n} \coloneqq \alpha_4 + n, \end{aligned}$$

with $n = 0, 1, 2, 3\ldots$. The poles are represented in Fig. 2, where the relative position of the poles is fixed by the following identification of the modes (see Fig. 1):

$$\text{“1”} = H, \qquad \text{“2”} = B, \qquad \text{“3”} = P, \qquad \text{“4”} = N. \tag{58}$$

We can analytically continue the function f in the complex x plane in order to study the behavior for $x \to \infty e^{i\theta}$ for different angles θ. By writing

$$x = |x|e^{i\theta}, \qquad s = |s|e^{i\phi},$$

we see that the integral is convergent only in the half-plane defined by

$$\cos\theta\cos\phi - \sin\theta\sin\phi \geq 0,$$

which defines a half-plane delimited by the line $\tan\phi = \cot\theta$. Namely, as the angle θ varies, the corresponding half-plane in the variable s is defined by $\phi \in [\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}]$.

Starting from $\theta = \pi$, which represents the solution at $x < 0$ (inside the horizon), the integral is defined for $\phi \in [\frac{\pi}{2}, \frac{3}{2}\pi]$: in this sector, the integral reduces to the sum of the residues at $s = \tilde{s}_n$. The sum of the residues gives the series

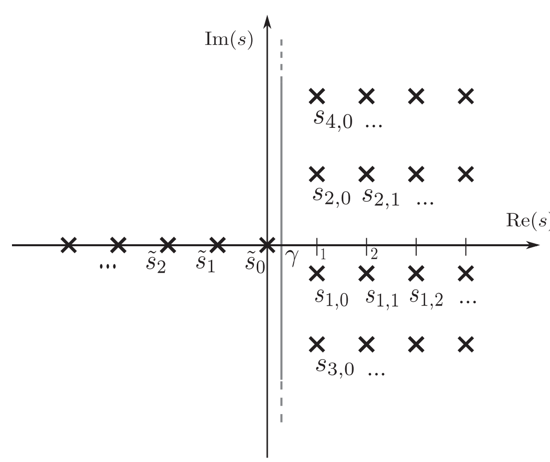


FIG. 2. The poles of the integrand function of Eq. (57). The gray line represents the path of integration. This figure holds true just when k_j and $\tilde{k}_j$ are real for all $j = 1, 2, 3, 4$, i.e., in the subcritical case.

$$f(x<0) = e^{i\tilde{k}_1 x}(1+e^{2\tilde{\beta}x}) \times \left(1+\frac{\alpha_1\alpha_2\alpha_3\alpha_4}{\beta_1\beta_2\beta_3}e^{2\tilde{\beta}x}+\sum_{n\geq 2}O((e^{2\tilde{\beta}x})^n)\right) \overset{x\to-\infty}{\sim} e^{i\tilde{k}_1 x}. \tag{59}$$

As we move θ, the asymptotic expansion (59) remains valid until we encounter new poles in the corresponding half-plane in s: this happens, as one can see from Fig. 3, when $\phi = \arg\alpha_3$ or $\phi = \arg\alpha_4$. As we pass those lines, a new term appears in the asymptotic expansion, corresponding to the residue at the pole $s_{3,0} = \alpha_3$ or $s_{4,0} = \alpha_4$. The appearance of new terms in the asymptotic expansion is known as the Stokes phenomenon: by the previous analysis, we thus identified a first Stokes sector, given by

$$\theta \in \left[\frac{\pi}{2}+\arg\alpha_4, \frac{3}{2}\pi+\arg\alpha_3\right],$$

and the boundaries of this sector are two Stokes lines. As we move θ past the Stokes line $\theta = \frac{3}{2}\pi + \arg\alpha_3$, we include a new pole, $s_{3,0}$, in the contour: the asymptotic expansion becomes

$$f(x) = e^{i\tilde{k}_1 x}(1+O(e^{-2\tilde{\beta}|x|})) + C_{\tilde{k}_1\to k_3}e^{ik_3 x},$$

so the new term introduces mode mixing. This expansion is true until we reach the next pole.

Now, we note that the residues at the poles $s_{j,n}$ are

$$\mathrm{Res}_{s=s_{j,n}} \sim e^{(i\tilde{k}_1+2\tilde{\beta}-2\tilde{\beta}s_{j,n})x} = e^{ik_j x}e^{-nx}.$$

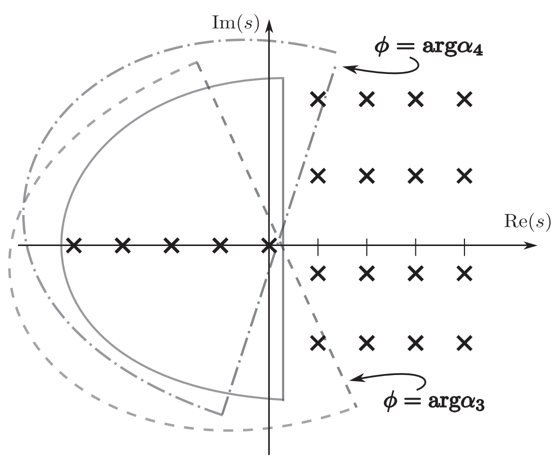


FIG. 3. Three contours of integration representing three values of θ. Full line, $\theta = \pi$; dashed line, $\theta = \frac{3}{2}\pi + \arg\alpha_3$; dash-dotted line, $\theta = \frac{\pi}{2} + \arg\alpha_4$. The latest two are Stokes lines, because the corresponding contour in the s plane (defined by the ϕ angle) includes a new pole, giving rise to an additional term in the asymptotic expansion. Also this figure holds true just for the subcritical case.

Therefore, the contributions of the poles with $n \geq 1$ are negligible as long as we are interested in the asymptotic expansion ($|x| \to \infty$). From this consideration we understand that the only poles that are related to the Stokes phenomenon are $s = s_{j,0}$. The next Stokes lines are thus met at $\theta = \frac{3}{2}\pi + \arg\alpha_1$ or $\theta = \frac{\pi}{2} + \arg\alpha_2$.

It is now easy to figure out, by continuing the argument exposed above, all the Stokes lines of the function $f(x)$, which, ordered by increasing θ, correspond to

$$\theta_1 = \frac{\pi}{2}+\arg\alpha_3, \quad \theta_2 = \frac{\pi}{2}+\arg\alpha_1, \quad \theta_3 = \frac{\pi}{2},$$
$$\theta_4 = \frac{\pi}{2}+\arg\alpha_2, \quad \theta_5 = \frac{\pi}{2}+\arg\alpha_4, \quad \theta_6 = \frac{3}{2}\pi+\arg\alpha_3,$$
$$\theta_7 = \frac{3}{2}\pi+\arg\alpha_1, \quad \theta_8 = \frac{3}{2}\pi,$$
$$\theta_9 = \frac{3}{2}\pi+\arg\alpha_2, \quad \theta_{10} = \frac{3}{2}\pi+\arg\alpha_4.$$

The value of $\arg\alpha_j$ depends on the values of the momenta k_j and $\tilde{k}_1$. The momenta k_j, being unperturbed by the background, are always real. On the other hand, as we will discuss also in Sec. IV C, $\tilde{k}_1$ is real in the subcritical regime and complex in the transcritical regime: in that case we have $\mathrm{Im}\tilde{k}_1 < 0$. We can thus evaluate $\arg\alpha_j$ as

$$\arg\alpha_j = \begin{cases} \arctan\left(-\frac{k_j-\tilde{k}_1}{2\tilde{\beta}}\right), & \text{subcritical regime,} \\ \arctan\left(-\frac{k_j-\mathrm{Re}\tilde{k}_1}{2\tilde{\beta}-\mathrm{Im}\tilde{k}_1}\right), & \text{transcritical regime.} \end{cases}$$

B. Subcritical scattering

The solution (53), for $z \to 0$ (which corresponds to the left asymptotic region $x \to -\infty$) is

$$f \sim z^{i\frac{\tilde{k}_1}{2\tilde{\beta}}} = e^{i\tilde{k}_1 x}. \tag{60}$$

At right infinity $x \to +\infty$ ($z \to \infty$) it splits into a sum of plane waves

$$f \sim \sum_{j=1}^{4} C_j z^{i\frac{ik_j}{2\tilde{\beta}}} = \sum_{j=1}^{4} C_j e^{ik_j x}, \tag{61}$$

where the connection coefficient

$$C_j \coloneqq C_{\tilde{k}_1\to k_j}$$

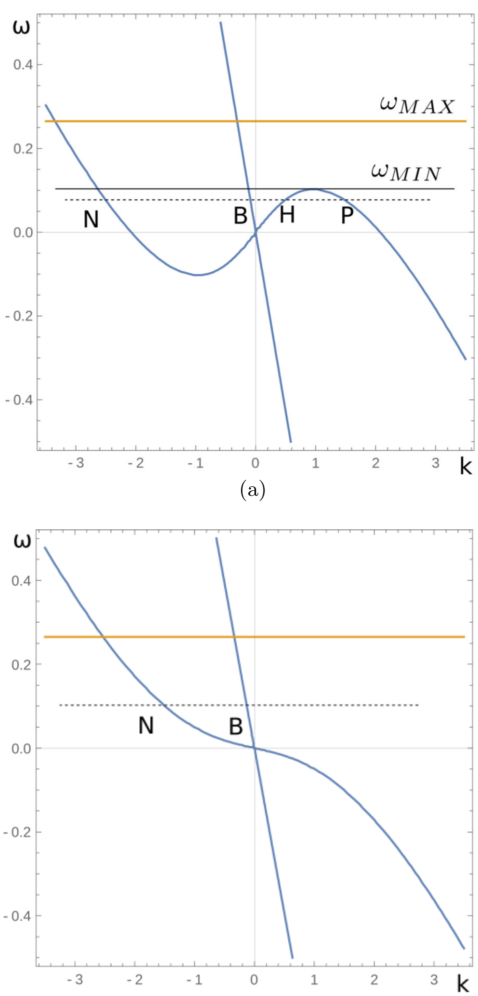
can be obtained from (54) by switching $\alpha_1 \leftrightarrow \alpha_j$. If we put

$$\tilde{k}_1 = \tilde{k}_H,$$

the function $f(x)$ represents the scattering of the ingoing modes H from left infinity and P, N, and B from right infinity, which produces an outgoing H mode at right infinity. This is the process that, since Hawking's seminal work, is usually considered in black hole physics to deduce the spontaneous particle creation. Following backward the outgoing H mode, we find that it originates from a mixture of modes: in particular, the coefficient C_N represents the mixing with the negative-norm N mode. As it is shown in [39], the expected number of spontaneously created Hawking particles is

$$|N| = \left| \frac{|C_N|^2 v_N \partial_\omega \mathrm{DR}(\omega, k)|_{k_N}}{|C_H|^2 v_H \partial_\omega \mathrm{DR}(\omega, k)|_{k_H}} \right|. \tag{62}$$

Notice that the function $\mathrm{DR}(k)$ defined in (24) is slightly different from the function $\mathrm{DR}(\omega, k)$ that appears in (62),



(b)

FIG. 4. (a) The dispersion relation (26) in the perturbed region ($x \to -\infty$) for $\lambda < \lambda_{\mathrm{crit}}$. The modes $0 < \omega < \omega_{MIN}$ do not experience an event horizon (*subcritical regime*). (b) The dispersion relation (26) for $\lambda > \lambda_{\mathrm{crit}}$. In this case, for any ω, the modes H and P become imaginary as they experience an event horizon: this is referred to as *transcritical regime*.

which derives from the normalization of the quantum theory: they differ by a global factor $(\omega + Vk)^2$.

The momenta $\tilde{k}_j(\omega)$ (which correspond to the normal modes at $x \to -\infty$) depend on the background amplitude λ. For large enough values of λ, the momenta $\tilde{k}_H(\omega)$ and $\tilde{k}_P(\omega)$ can become imaginary, as can be seen from Figs. 4(a) and 4(b). In the current literature, the distinction between the subcritical and transcritical regimes is governed by the presence or absence of a horizon, corresponding to a turning point in the differential equation of motion: the subcritical case is when no such turning point appears. On the other hand, the two regimes can be characterized also in a different way: one identifies the transcritical case by the fact that, in the asymptotic dispersion relation, two roots which are real in the unperturbed asymptotic region become complex conjugates in the perturbed region. This is exactly the criterion we adopt in our approach to the problem; see also the discussion in Sec. V E.

We start considering the case where all $\tilde{k}_j$ are real (i.e., subcritical regime). In this case, the square module of (54) can be written explicitly,

$$\begin{aligned} |C_1|^2 = {} & \frac{(\tilde{k}_1 - \tilde{k}_2)(\tilde{k}_1 - \tilde{k}_3)(\tilde{k}_1 - \tilde{k}_4)(\tilde{k}_2 - k_1)(k_1 - \tilde{k}_3)(k_1 - \tilde{k}_4)}{(\tilde{k}_1 - k_2)(\tilde{k}_1 - k_3)(\tilde{k}_1 - k_4)(k_1 - k_2)(k_1 - k_3)(k_1 - k_4)} \\ & \times \frac{\sinh(\frac{\pi\gamma(\tilde{k}_1 - k_2)}{2\beta}) \sinh(\frac{\pi\gamma(\tilde{k}_1 - k_3)}{2\beta}) \sinh(\frac{\pi\gamma(\tilde{k}_1 - k_4)}{2\beta})}{\sinh(\frac{\pi\gamma(\tilde{k}_1 - \tilde{k}_2)}{2\beta}) \sinh(\frac{\pi\gamma(\tilde{k}_1 - \tilde{k}_3)}{2\beta}) \sinh(\frac{\pi\gamma(\tilde{k}_1 - \tilde{k}_4)}{2\beta})} \\ & \times \frac{\sinh(\frac{\pi\gamma(\tilde{k}_2 - k_1)}{2\beta}) \sinh(\frac{\pi\gamma(\tilde{k}_3 - k_1)}{2\beta}) \sinh(\frac{\pi\gamma(\tilde{k}_4 - k_1)}{2\beta})}{\sinh(\frac{\pi\gamma(k_1 - k_2)}{2\beta}) \sinh(\frac{\pi\gamma(k_1 - k_3)}{2\beta}) \sinh(\pi(k_1 - k_4))}, \end{aligned} \tag{63}$$

and similarly $|C_j|^2$ are obtained by rotations of the indices. We want to compare these results with the perturbative expansion we made in [39]. We start by writing explicitly

$$\begin{aligned} |N| = {} & \frac{(\tilde{k}_H - k_N)(k_N - \tilde{k}_P)(k_N - \tilde{k}_N)(k_N - \tilde{k}_B)(\omega + k_H V)^2}{(\tilde{k}_H - k_H)(k_H - \tilde{k}_P)(k_H - \tilde{k}_N)(k_H - \tilde{k}_B)(\omega + k_N V)^2} \\ & \times \frac{\sinh(\frac{\pi\gamma(\tilde{k}_H - k_H)}{2\beta}) \sinh(\frac{\pi\gamma(\tilde{k}_P - k_N)}{2\beta}) \sinh(\frac{\pi\gamma(\tilde{k}_N - k_N)}{2\beta})}{\sinh(\frac{\pi\gamma(\tilde{k}_H - k_N)}{2\beta}) \sinh(\frac{\pi\gamma(\tilde{k}_P - k_H)}{2\beta}) \sinh(\frac{\pi\gamma(\tilde{k}_N - k_H)}{2\beta})} \\ & \times \frac{\sinh(\frac{\pi\gamma(\tilde{k}_B - k_N)}{2\beta}) \sinh(\frac{\pi\gamma(k_H - k_P)}{2\beta}) \sinh(\frac{\pi\gamma(k_H - k_B)}{2\beta})}{\sinh(\frac{\pi\gamma(\tilde{k}_B - k_H)}{2\beta}) \sinh(\frac{\pi\gamma(k_P - k_N)}{2\beta}) \sinh(\frac{\pi\gamma(k_N - k_B)}{2\beta})}, \end{aligned} \tag{64}$$

where we have used the expression for the flux factors

$$v_i(\omega)\partial_\omega \mathrm{DR}|_{k_i} = -\frac{\gamma^2 V^4}{\omega_{\mathrm{lab}}^2|_{k_i}} \prod_{j \neq i} (k_i(\omega) - k_j(\omega)), \tag{65}$$

which was deduced in the Appendix of [39]. We use the low-frequency expressions of the momenta $k_j(\omega)$, written in (5)–(8). Notice that the expressions of $\tilde{k}_j(\omega)$ are simply obtained by the shift $\mu^2 \mapsto \mu^2 + \lambda$.

We stress that, up to this point, our results are exact, in the sense that no approximation has been made. Still, in order to provide analytical expressions to the moments $k_j, \tilde{k}_j$, we are forced to introduce some approximations, indeed $k_j, \tilde{k}_j$ are roots of a fourth-degree equation: one might provide explicit expressions for the corresponding roots, but at the price of writing very long and by no means perspicuous expressions. As a consequence, for these roots we use approximate expressions for low ω, as discussed in the previous sections. With the help of Wolfram's *Mathematica*, we compute the leading order of $|N|$ both in λ and in ω: we checked that the two limits commute, so the order of the two expansions makes no difference. We obtain

$$|N| = \left(\frac{\pi^2\lambda^2\omega g(g+\mu V)\sinh^2\left(\frac{\pi\sqrt{g^2-\mu^2V^2}}{2\beta V^2}\right)}{16\beta^2\gamma\mu(g-\mu V)(g^2-\mu^2V^2)^{3/2}} + O(\omega^2)\right) + O(\lambda^3). \tag{66}$$

We notice that the qualitative behavior is the same as in [39], and in particular we have $N \sim \omega$: this behavior confirms what was found for the subcritical case also in [20]. This is a strong confirmation that such a behavior should be expected in subcritical systems, and it seems not to depend on particular approximations nor on the characteristics of the background function.

An even more interesting comparison is the estimation of the "effective temperature" that the authors found in [39] for the subcritical case. The ratio $\frac{|P|}{|N|}$, in the case $\tilde{k}_j \in \mathbb{R}$ as before, becomes

$$\begin{aligned}\frac{|P|}{|N|} = &\frac{(\tilde{k}_H-k_P)(k_P-\tilde{k}_P)(k_P-\tilde{k}_N)(k_P-\tilde{k}_B)(k_NV+\Omega)^2}{(\tilde{k}_H-k_N)(\tilde{k}_P-k_N)(\tilde{k}_N-k_N)(\tilde{k}_B-k_N)(k_PV+\Omega)^2}\\ &\times\frac{\sinh(\frac{\pi\gamma(\tilde{k}_H-k_N)}{2\beta})\sinh(\frac{\pi\gamma(\tilde{k}_N-k_P)}{2\beta})\sinh(\frac{\pi\gamma(\tilde{k}_P-k_P)}{2\beta})}{\sinh(\frac{\pi\gamma(\tilde{k}_H-k_P)}{2\beta})\sinh(\frac{\pi\gamma(\tilde{k}_P-k_N)}{2\beta})\sinh(\frac{\pi\gamma(\tilde{k}_N-k_N)}{2\beta})}\\ &\times\frac{\sinh(\frac{\pi\gamma(\tilde{k}_B-k_P)}{2\beta})\sinh(\frac{\pi\gamma(k_H-k_N)}{2\beta})\sinh(\frac{\pi\gamma(k_N-k_B)}{2\beta})}{\sinh(\frac{\pi\gamma(\tilde{k}_B-k_N)}{2\beta})\sinh(\frac{\pi\gamma(k_H-k_P)}{2\beta})\sinh(\frac{\pi\gamma(k_P-k_B)}{2\beta})}.\end{aligned} \tag{67}$$

To estimate the temperature we compute the leading order in ω of $\log(\frac{|P|}{|N|})$. We use the low-ω expressions of the modes k_j that we have written in Eqs. (5)–(8). The momenta $\tilde{k}_j$ can be obtained by the switch $\mu^2 \to \mu^2+\lambda$: they can be also written as follows:

$$\tilde{k}_N = \frac{\sqrt{\mu^2+\lambda}-gV}{g-\sqrt{\mu^2+\lambda}V}\omega + O(\omega^3), \tag{68}$$

$$\tilde{k}_B = -\frac{\sqrt{\mu^2+\lambda}+gV}{g+\sqrt{\mu^2+\lambda}V}\omega + O(\omega^3), \tag{69}$$

$$\begin{aligned}\tilde{k}_P = &\frac{\sqrt{\lambda_{\text{crit}}-\lambda}}{\gamma V} - \left(\frac{1}{V}+\frac{g^2}{\gamma^2V^3(\lambda_{\text{crit}}-\lambda)}\right)\omega\\ &-\frac{g^2(2g^2+(\mu^2+\lambda)V^2)}{2\gamma V^5(\lambda_{\text{crit}}-\lambda)^{5/2}}\omega^2 + O(\omega^3),\end{aligned} \tag{70}$$

$$\begin{aligned}\tilde{k}_H = &\frac{\sqrt{\lambda_{\text{crit}}-\lambda}}{\gamma V} - \left(\frac{1}{V}+\frac{g^2}{\gamma^2V^3(\lambda_{\text{crit}}-\lambda)}\right)\omega\\ &+\frac{g^2(2g^2+(\mu^2+\lambda)V^2)}{2\gamma V^5(\lambda_{\text{crit}}-\lambda)^{5/2}}\omega^2 + O(\omega^3),\end{aligned} \tag{71}$$

where

$$\lambda_{\text{crit}} = \frac{g^2-\mu^2V^2}{V^2}. \tag{72}$$

These expressions make clear that the subcritical regime (i.e., $\tilde{k}_j \in \mathbb{R}$) corresponds to $\lambda < \lambda_{\text{crit}}$.

The leading order of $\log(\frac{|P|}{|N|})$ is

$$\begin{aligned}\log\left(\frac{|P|}{|N|}\right) = &\frac{\pi g\omega}{\beta\gamma V(g^2-\mu^2V^2)^{3/2}(g^2-V^2(\lambda+\mu^2))}\Bigg[g\lambda V^2\sqrt{g^2-\mu^2V^2}\\ &\times\left(\coth\left(\frac{\pi(\sqrt{g^2-\mu^2V^2}-\sqrt{g^2-V^2(\lambda+\mu^2)})}{2\beta V^2}\right)+\coth\left(\frac{\pi(\sqrt{g^2-V^2(\lambda+\mu^2)}+\sqrt{g^2-\mu^2V^2})}{2\beta V^2}\right)\right)\\ &+2\left(g^2V\sqrt{(\lambda+\mu^2)(g^2-\mu^2V^2)}-2gV^2(\lambda+\mu^2)\sqrt{g^2-\mu^2V^2}\right)\\ &-2\left(\mu^2V^3\sqrt{(\lambda+\mu^2)(g^2-\mu^2V^2)}+2g^3\sqrt{g^2-\mu^2V^2}\right)\coth\left(\frac{\pi\sqrt{g^2-\mu^2V^2}}{2\beta V^2}\right)\Bigg] + O(\omega^2).\end{aligned} \tag{73}$$

In order to compare it to the perturbative result, we take the limit $\lambda \to 0$,

$$\frac{2g\omega\left(\pi(2g+\mu V)\sqrt{g^2-\mu^2V^2}\coth\left(\frac{\pi\sqrt{g^2-\mu^2V^2}}{2\beta V^2}\right)+2\beta g V^2\right)}{\beta\gamma V(g^2-\mu^2V^2)^{3/2}}. \tag{74}$$

For $\beta \sim 0$, which amounts physically to considering small values for the derivative of the dielectric pulse, we find

$$\log\left(\frac{|P|}{|N|}\right) \approx \beta_{pert}\omega, \tag{75}$$

$$\beta_{pert} = \frac{2\pi g(2g+\mu V)}{\beta\gamma V(g^2-\mu^2V^2)} = \frac{\pi\gamma}{\beta}\lim_{\omega\to 0}\frac{2k_H - k_P - k_N}{\omega}. \tag{76}$$

These results confirm the validity of the prediction made in [39] and that the value of T_{pert} is not strongly dependent on the peculiarities of the background. We notice, however, that the expression (73) allows one to study how the temperature depends on λ: in particular, for $\lambda = \lambda_{\rm crit}$, we have

$$T(\lambda = \lambda_{\rm crit}) = \frac{\omega}{\log(P/N)}\bigg|_{\lambda=\lambda_{\rm crit}} = 0. \tag{77}$$

The vanishing of the temperature at $\lambda = \lambda_{\rm crit}$ is very puzzling, in the sense that thermal particle creation may be found both in the subcritical and in the transcritical case, whereas a discontinuous behavior between the two regimes is just suggested by such a result when $\lambda = \lambda_{\rm crit}$. It has also been stressed that, in the above scheme of approximation for low ω, for $\lambda = \lambda_{\rm crit}$ one finds a degeneracy, at least at the leading order, of $\tilde{k}_P$ with $\tilde{k}_N$, and a singular behavior of the subleading ones. This kind of phenomenon will be investigated in future analysis.

C. Transcritical scattering

We now consider the solution in the transcritical case, that is $\lambda > \lambda_{\rm crit}$. In this case, the $\tilde{k}_H$ and $\tilde{k}_P$ become complex, as it is shown in Fig. 4(b). This fact is the direct consequence of the presence of an event horizon: these modes cannot propagate to the left infinity. The low-ω expressions are found from Eqs. (68)–(71) for $\lambda > \lambda_{\rm crit}$: notice, however, that the modes have now the wrong label, since the mode that is labeled N becomes complex, while the H mode is real, in contradiction with the visual interpretation of Fig. 4(b). Thus, for the transcritical regime, we need to rename the modes in the following way:

$$\tilde{k}_N = \frac{\sqrt{\mu^2+\lambda}-gV}{g-\sqrt{\mu^2+\lambda}V}\omega + O(\omega^3), \tag{78}$$

$$\tilde{k}_B = -\frac{\sqrt{\mu^2+\lambda}+gV}{g+\sqrt{\mu^2+\lambda}V}\omega + O(\omega^3), \tag{79}$$

$$\tilde{k}_P = i\frac{\sqrt{\lambda-\lambda_{\rm crit}}}{\gamma V} - \left(\frac{1}{V} - \frac{g^2}{\gamma^2V^3(\lambda-\lambda_{\rm crit})}\right)\omega + i\frac{g^2(2g^2+(\mu^2+\lambda)V^2)}{2\gamma V^5(\lambda-\lambda_{\rm crit})^{5/2}}\omega^2 + O(\omega^3), \tag{80}$$

$$\tilde{k}_H = -i\frac{\sqrt{\lambda-\lambda_{\rm crit}}}{\gamma V} - \left(\frac{1}{V} - \frac{g^2}{\gamma^2V^3(\lambda-\lambda_{\rm crit})}\right)\omega - i\frac{g^2(2g^2+(\mu^2+\lambda)V^2)}{2\gamma V^5(\lambda-\lambda_{\rm crit})^{5/2}}\omega^2 + O(\omega^3). \tag{81}$$

Notice that it holds

$$(\tilde{k}_H)^* = \tilde{k}_P. \tag{82}$$

This is not true just in the low-ω limit, but for all ω. Indeed, the modes $\tilde{k}$ are the roots of a fourth-order polynomial with real coefficients: since the roots $\tilde{k}_N$ and $\tilde{k}_B$ are always real, the other two roots must be either real or complex conjugates. Another very relevant observation is that, being the basis (55) asymptotically diagonal in the $\tilde{k}_i$, as discussed in the previous sections, we have also the possibility to get rid of the unwanted complex and exponentially growing mode, say $\tilde{k}_4$ (about the growing mode cf., e.g., the discussion in [9]), simply by imposing that the corresponding coefficient D_4 is zero. Actually, in our following discussion, we put only $D_1 \neq 0$. Note also that the connection coefficients, being connecting $\tilde{k}_1$ to k_i, $i = 1, 2, 3, 4$, cannot resume the aforementioned growing mode.

Thanks to (82), we can simplify the expression of $\frac{|P|}{|N|}$: indeed, when computing $\frac{C_P C_P^*}{C_N C_N^*}$, one finds a factor

$$\frac{\Gamma\left(1+\frac{i(\tilde{k}_H-k_P)\gamma}{2\beta}\right)\Gamma\left(-\frac{i(\tilde{k}_P-k_N)\gamma}{2\beta}\right)}{\Gamma\left(1+\frac{i(\tilde{k}_H-k_N)\gamma}{2\beta}\right)\Gamma\left(-\frac{i(\tilde{k}_P-k_P)\gamma}{2\beta}\right)}\frac{\Gamma\left(1+\frac{i(k_P-\tilde{k}_H^*)\gamma}{2\beta}\right)\Gamma\left(\frac{i(\tilde{k}_P^*-k_N)\gamma}{2\beta}\right)}{\Gamma\left(1+\frac{i(k_N-i\tilde{k}_H^*)\gamma}{2\beta}\right)\Gamma\left(\frac{i\gamma(\tilde{k}_P^*-k_P)}{2\beta}\right)},$$

which, using (82) and recalling $\Gamma(1+z)/\Gamma(z) = z$, reduces to

$$\frac{(k_P-\tilde{k}_P)(\tilde{k}_P^*-k_P)}{(\tilde{k}_P-k_N)(\tilde{k}_P^*-k_N)} = \frac{|k_P-\tilde{k}_P|^2}{|k_N-\tilde{k}_P|^2}.$$

The final exact expression is

$$\frac{|P|}{|N|} = \frac{(k_P - \tilde{k}_N)(k_P - \tilde{k}_B)|k_P - \tilde{k}_P|^2(k_N V + \omega)^2}{(k_N - \tilde{k}_N)(k_N - \tilde{k}_B)|k_N - \tilde{k}_P|^2(k_P V + \omega)^2} \frac{\sinh\left(\frac{\pi\gamma(\tilde{k}_N - k_P)}{2\beta}\right)\sinh\left(\frac{\pi\gamma(\tilde{k}_B - k_P)}{2\beta}\right)\sinh\left(\frac{\pi\gamma(k_H - k_N)}{2\beta}\right)\sinh\left(\frac{\pi\gamma(k_N - k_B)}{2\beta}\right)}{\sinh\left(\frac{\pi\gamma(\tilde{k}_N - k_N)}{2\beta}\right)\sinh\left(\frac{\pi\gamma(\tilde{k}_B - k_N)}{2\beta}\right)\sinh\left(\frac{\pi\gamma(k_H - k_P)}{2\beta}\right)\sinh\left(\frac{\pi\gamma(k_P - k_B)}{2\beta}\right)}. \tag{83}$$

As we did in the previous section, we compute, to the leading order in ω,

$$\log\left(\frac{|P|}{|N|}\right) = \frac{2\pi\omega g^2 \coth\left(\frac{\pi\sqrt{g^2-\mu^2V^2}}{2\beta V^2}\right)}{\beta\gamma V(g^2-\mu^2V^2)(1-\lambda_{\rm crit}/\lambda)} + O(\omega^2). \tag{84}$$

We can now write the Hawking temperature

$$T(\lambda) = \frac{\omega}{\log\left(\frac{|P|}{|N|}\right)} = \frac{\beta\gamma V(g^2-\mu^2V^2)}{2\pi g^2 \coth\left(\frac{\pi\sqrt{g^2-\mu^2V^2}}{2\beta\gamma V^2}\right)}\left(1-\frac{\lambda_{\rm crit}}{\lambda}\right). \tag{85}$$

This result confirms what was found for the subcritical case, that is

$$T(\lambda_{\rm crit}) = 0.$$

More interestingly, in the far critical case $\lambda \gg \lambda_{\rm crit}$, if we consider $\beta \sim 0$ as previously done, we find

$$T_H = \frac{\beta\gamma V(g^2-\mu^2V^2)}{2\pi g^2}, \tag{86}$$

which coincides with the far critical limit that was obtained in [39] using the Orr-Sommerfeld approach.

In Fig. 5(b) we plot the temperature in units of T_H, namely,

$$\frac{T(\lambda)}{T_H} = \frac{\omega}{T_H \log\left(\frac{|P|}{|N|}\right)}, \tag{87}$$

for various values of λ, both in the subcritical and transcritical cases. As in the perturbative approach of the previous section, for the plot we choose $g = 1$, $\mu = 1.2$; we then choose a near critical pulse velocity $V = 0.8$ and a low value $\beta = 0.02$. We clearly observe what was predicted in [39] using a perturbative approach: the effective temperature computed for $\lambda \ll \lambda_{\rm crit}$ (subcritical regime) is

$$T(\lambda \sim 0) \approx \frac{T_H}{3}. \tag{88}$$

Starting from this value, the temperature decreases for increasing λ until it reaches zero at $\lambda = \lambda_{\rm crit}$. For $\lambda > \lambda_{\rm crit}$ the temperature starts growing again, and for $\lambda \gg \lambda_{\rm crit}$ it stabilizes at the value T_H.

So far, the exact solution we provided has confirmed the predictions that were made using different approximations in different regimes. In the future, an even deeper study of this solution may allow one to describe precisely the transition between the subcritical and the transcritical regime, the onset of thermality and formation of the event horizon.

V. THE ORIGINAL $\phi\psi$ MODEL, EXACT SOLUTIONS, AND THERMALITY

In this section, we provide results concerning a reduction of the so-called Hopfield model, which represents an effective description of the interaction between the electromagnetic field and a dielectric medium. In particular, atoms and molecules of the dielectric are replaced by a mesoscopic polarization field, still providing an efficient physical description. The electromagnetic Lagrangian for the full Hopfield model is quite involved and has been discussed, by means of different theoretical tools, in [19,51]. A simplified model, introduced in [16], can be related to the two-dimensional reduction of the Hopfield model

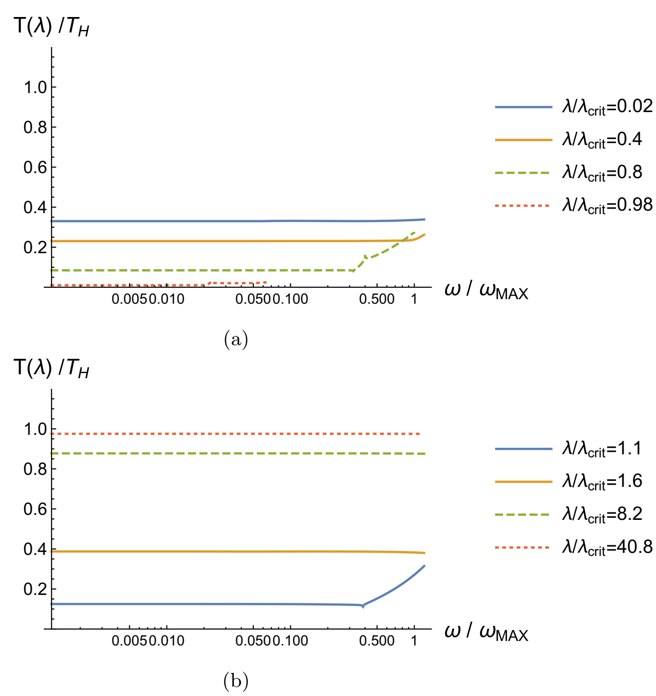


FIG. 5. The temperature $T(\lambda)$ in units of T_H, as defined in (87), for different values of λ, in (a) subcritical regime and (b) transcritical regime. The plots are made for a near critical pulse velocity and a low value of the parameter β. Starting from $T(\lambda \sim 0) \approx T_H/3$, the temperature decreases until reaching zero for $\lambda = \lambda_{\rm crit}$; for $\lambda > 0$ (transcritical) the temperature increases again, reaching T_H for $\lambda \gg \lambda_{\rm crit}$.

adopted in [52] and is such that the electromagnetic field and the polarization field are simulated by a pair of scalar fields, φ and ψ, respectively, in the so-called $\varphi\psi$ model. Despite its simplification, it is still set up in such a way that we get exactly the same dispersion relation and, moreover, we can simulate the same coupling as in the full case. Its Lagrangian is

$$\mathcal{L}_{\varphi\psi} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) + \frac{1}{2\chi\omega_0^2}[(v^\alpha\partial_\alpha\psi)^2 - \omega_0^2\psi^2] - \frac{g}{c}(v^\alpha\partial_\alpha\psi)\varphi + \frac{\lambda}{4!}\psi^4, \tag{89}$$

where χ plays the role of the dielectric susceptibility, v^μ is the usual four-velocity vector of the dielectric, ω_0 is the proper frequency of the medium, and g is the coupling constant between the fields. The latter constant is henceforth put equal to 1, as its original motivation (see [16]) can be relaxed without problems in a more advanced discussion (cf. also [36]). As shown in [38], we may introduce the above fourth-order nonlinear term in the polarization field ψ in the Lagrangian. Herein, we assume $\lambda > 0$.

By extending our analysis, and on the grounds of the previous sections, we adopt a phenomenological model where we can leave room for a spacetime dependence of the microscopic parameters χ, ω_0, in such a way that $\chi\omega_0^2$ is a constant. The equations of motion are

$$\Box\varphi + \frac{1}{c}(v^\alpha\partial_\alpha\psi) = 0, \tag{90}$$

$$\frac{1}{\chi\omega_0^2}(v^\alpha\partial_\alpha)^2\psi + \frac{1}{\chi}\psi - \frac{1}{c}v^\alpha\partial_\alpha\varphi = 0. \tag{91}$$

In particular, we define

$$\epsilon^2 \coloneqq \frac{1}{\chi\omega_0^2}, \tag{92}$$

which corresponds to the parameter appearing in the Orr-Sommerfeld-like equation (master equation, cf. [36]). We can separate the above system, obtaining equations involving only one of the fields φ, ψ. We can also separate the equations for φ, ψ, and, in order to maintain the same line of reasoning as in the previous sections, we focus on ψ, obtaining

$$\epsilon^2\Box(v^\alpha\partial_\alpha)^2\psi + \Box\frac{1}{\chi}\psi + \frac{1}{c^2}(v^\alpha\partial_\alpha)^2\psi = 0. \tag{93}$$

Let us also choose, as usual, the comoving frame; if we put $\psi(t,x) = e^{-i\omega t}h(x)$, we obtain

$$\epsilon^2 h^{(4)}(x) + 2i\epsilon^2\frac{\omega}{v}h^{(3)}(x) + \left(\frac{1-\gamma^2\frac{v^2}{c^2}\chi(x)}{\gamma^2v^2\chi(x)} - \epsilon^2\frac{w^2}{\gamma^2v^2}\right)h^{(2)}(x) + \left(-\frac{2i\omega}{c^2v} - \frac{2\chi'(x)}{\gamma^2v^2\chi^2(x)} + \frac{2\epsilon^2 i\omega^3}{c^2v}\right)h^{(1)}(x) + \left(\frac{\omega^2}{c^2\gamma^2v^2\chi(x)}(1+\gamma^2\chi(x)) + \frac{2\chi'^2(x) - \chi(x)\chi''(x)}{\gamma^2v^2\chi^3(x)} - \epsilon^2\frac{\omega^4}{v^2c^2}\right)h(x) = 0, \tag{94}$$

where $\chi'(x), \chi''(x)$ indicate the first and the second derivative with respect to x. We stress that, with respect to [36], we do not eliminate the third-order term, as we do not need to grant an Orr-Sommerfeld form for our equation of motion, as we are going to compute exact solutions, i.e., solutions that do not depend on the smallness of the parameter ϵ. In the following, we assume the monotone profile in the comoving frame,

$$\frac{1}{\chi(x)} = \frac{1}{\chi_0} - \frac{1}{2}\lambda(1 - \tanh(\tilde{\beta}x)), \tag{95}$$

where χ_0 is a constant value of the dielectric susceptibility and we define the parameter $\tilde{\beta} = \frac{\beta}{\gamma}$ as in the previous model. We discuss some physical consequences of our choice in Appendix C.

A. A Fuchsian framework for the equation of motion

We consider the following change of variable: $z = -e^{2\tilde{\beta}x}$. As a consequence, we obtain

$$\frac{1}{\chi(z)} = \frac{1}{\chi_0} - \frac{\lambda}{1-z}, \tag{96}$$

and

$$16\epsilon^2\tilde{\beta}^4 z^4 h^{(4)}(z) + 16\epsilon^2\frac{(6\tilde{\beta}v + i\omega)}{v}\tilde{\beta}^3 z^3 h^{(3)}(z) + 4\left(\frac{-1+\epsilon^2\omega^2}{c^2} + \frac{1 - z + \chi_0(-\lambda + \epsilon^2\gamma^2(28\tilde{\beta}^2v^2 + 12i\tilde{\beta}v\omega - \omega^2))(1+z)}{\chi_0\gamma^2v^2(1-z)}\right)\tilde{\beta}^2 z^2 h^{(2)}(z) + 4\left(4\tilde{\beta}^3\epsilon^2 + \frac{4i\tilde{\beta}^2\epsilon^2\omega}{v} + \frac{\tilde{\beta}(-1+\epsilon^2\omega^2)}{c^2} + \frac{i\omega(-1+\epsilon^2\omega^2)}{c^2v} + \frac{\tilde{\beta}}{\gamma^2v^2}\left(\frac{1}{\chi_0} - \frac{\lambda(1+z)}{(1-z)^2} - \epsilon^2\gamma^2\omega^2\right)\right)\tilde{\beta}z h^{(1)}(z) + \left(-\frac{4\tilde{\beta}^2\lambda z(z+1)}{\gamma^2v^2(1-z)^3} - \frac{\epsilon^2\omega^4}{v^2c^2} + \frac{\omega^2(1 - z - \chi_0\lambda)}{c^2\chi_0\gamma^2v^2(1-z)} + \frac{\omega^2}{c^2v^2}\right)h(z) = 0. \tag{97}$$

We start looking for local solutions, along the path sketched for (13), around $z = \infty$. By introducing $t = 1/z$, a series expansion for the solutions can be provided in the following form:

$$f(t) = t^{-i\alpha} \sum_{n=0}^{\infty} c_n t^n. \tag{98}$$

The characteristic equation for the exponent $k \coloneqq 2\tilde{\beta}\alpha$ is

$$\mathrm{DR}(k) \coloneqq (\omega^2 - \chi_0 \gamma^2 (kv + \omega)^2 (-1 + \epsilon^2 \omega^2) + c^2 k^2 (-1 + \chi_0 \epsilon^2 \gamma^2 (kv + \omega)^2)) = 0. \tag{99}$$

As in the previous sections, except possibly for a zero measure set in the space of available parameters in our model, we have four distinct roots that, moreover, not differ from each other by an integer value. As a consequence, the spectral type is (1111).

At $z = 0$, we get four independent local solutions of the form

$$f(z) = z^{i\tilde{\alpha}} \sum_{n=0}^{\infty} c_n z^n, \tag{100}$$

where $\tilde{k} \coloneqq 2\tilde{\beta}\,\tilde{\alpha}$ satisfies

$$\mathrm{DR}_0(\tilde{k}) \coloneqq (\omega^2 - \chi_0 \lambda \omega^2 - \chi_0 \gamma^2 (\tilde{k} v + \omega)^2 (-1 + \epsilon^2 \omega^2) + c^2 \tilde{k}^2 (-1 + \chi_0 (\lambda + \epsilon^2 \gamma^2 (\tilde{k} v + \omega)^2))) = 0. \tag{101}$$

The spectral type at $z = 0$ is again (1111), again almost everywhere in the space of available parameters appearing in our model.

Also in this case, at $z = 1$ the so-called resonant case [43] is verified. Let us define $y \coloneqq z - 1$. Then, the characteristic equation for solutions of the form

$$f(y) = y^a \sum_{n=0}^{\infty} c_n y^n \tag{102}$$

has four integer solutions $a = 0, 1, 2, 3$. This situation, again, requires a particular study, which we perform by means of the Frobenius method.

We can show that there exist three independent integer solutions

$$u_1(y) = y^3 + y^4 \left[-\frac{3}{2} - \frac{\lambda}{48\tilde{\beta}^2 \epsilon^2 \gamma^2 v^2} - i\frac{\omega}{4\tilde{\beta} v} \right] + o(y^4), \tag{103}$$

$$u_2(y) = y^2 + y^3 \left[-2 - \frac{\lambda}{24\tilde{\beta}^2 \epsilon^2 \gamma^2 v^2} - i\frac{\omega}{3\tilde{\beta} v} \right] + o(y^3), \tag{104}$$

$$u_3(y) = y + y^2 \left[-3 - \frac{\lambda}{8\tilde{\beta}^2 \epsilon^2 \gamma^2 v^2} - i\frac{\omega}{2\tilde{\beta} v} \right] + o(y^2), \tag{105}$$

and one logarithmic solution

$$u_0(y) = 1 + y \left[-6 + \frac{\lambda}{4\tilde{\beta}^2 \epsilon^2 \gamma^2 v^2} - i\frac{\omega}{\tilde{\beta} v} \right] + o(y) + \log(y) \Big(R_1 u_1(y) + R_2 u_2(y) + R_3 u_3(y) \Big), \tag{106}$$

where

$$R_3 = -\frac{\lambda}{4\tilde{\beta}^2 \epsilon^2 \gamma^2 v^2}, \tag{107}$$

$$R_2 = -\frac{5\lambda}{16\tilde{\beta}^2 \epsilon^2 \gamma^2 v^2}, \tag{108}$$

$$R_1 = -\frac{\lambda (36\tilde{\beta}^2 c^2 + \omega^2)}{288\tilde{\beta}^4 c^2 \epsilon^2 \gamma^2 v^2}. \tag{109}$$

B. Monodromy and rigidity

The monodromy matrix of the solutions $(u_0(y), u_1(y), u_2(y), u_3(y))$ at $z = 1$ is easily computed as

$$M_1 = \begin{pmatrix} 1 & 2\pi i R_1 & 2\pi i R_2 & 2\pi i R_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{110}$$

whose Jordan form is

$$J_{M_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{111}$$

The monodromies at $z = 0$ and $z = \infty$ are, also in this case, represented, respectively, by the diagonal matrices

$$M_0 = \mathrm{diag}(e^{i2\pi\tilde{\alpha}_1}, e^{i2\pi\tilde{\alpha}_2}, e^{i2\pi\tilde{\alpha}_3}, e^{i2\pi\tilde{\alpha}_4}), \tag{112}$$

$$M_\infty = \mathrm{diag}(e^{-i2\pi\alpha_1}, e^{-i2\pi\alpha_2}, e^{-i2\pi\alpha_3}, e^{-i2\pi\alpha_4}), \tag{113}$$

whose Jordan form is $J_{M_0} = J_{M_\infty} = \mathbb{I}_4$. As a consequence, the spectral type of the equation at $z = 1$ is (3,1), and the spectral type of the equation is [(1111),(31),(1111)], i.e., it is rigid as in the case discussed in the previous section.

C. Gauge transformation and Riemann scheme

As in the previous sections for the simplified model, we can obtain a solution of the form

$$f(z) \coloneqq z^{i\frac{\tilde{k}_1}{2\tilde{\beta}}} (z - 1) u(z), \tag{114}$$

where $\tilde{k}_1$ satisfies

$$\mathrm{DR}_0(\tilde{k}_1) = 0.$$

The function $u(z)$ now satisfies an equation that is analog to (42), which we avoid writing explicitly, as it is quite long. What happens is that one may verify that for the solution $u(z)$, also in the case of the standard $\phi\psi$ model, the same Eq. (43) holds true, where now the dispersion relations are the ones in (99) and in (101), respectively. Also the so-called Riemann scheme of the equation is the same as in (44). Also in this case, by letting

$$\alpha_i \coloneqq 1 - \frac{i}{2\tilde{\beta}}(k_i - \tilde{k}_1), \quad i = 1,2,3,4, \tag{115}$$

$$\beta_j \coloneqq 1 - \frac{i}{2\tilde{\beta}}(\tilde{k}_{j+1} - \tilde{k}_1), \quad j = 1,2,3, \tag{116}$$

we can write Eq. (44) as

$$\begin{bmatrix} z=0 & z=1 & z=\infty \\ 0 & 0 & \alpha_1 \\ 1-\beta_1 & 1 & \alpha_2 \\ 1-\beta_2 & 2 & \alpha_3 \\ 1-\beta_3 & -\beta_4 & \alpha_4 \end{bmatrix}, \tag{117}$$

which, as discussed in the previous sections, corresponds to the Riemann scheme of the hypergeometric function

$${}_4F_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4;\beta_1,\beta_2,\beta_3;z)$$

in the standard form.

D. The Hawking temperature

The scattering coefficients C_j for the Hopfield model have the same form as the ones of the previous sections, apart for the different expressions of the momenta $k_j(\omega)$ and $\tilde{k}_j(\omega)$. As in the previous section, one is able to deal with analytical expressions for the momenta only in suitable limits, and this is the strategy that is adopted also in the present case.

In the low-ω limit, the right-infinity modes are

$$k_H = \frac{\chi_0\gamma^2 v + \sqrt{c^2(1+\chi_0)}}{c^2 - \chi_0\gamma^2 v^2}\omega + o(\omega), \tag{118}$$

$$k_B = \frac{\chi_0\gamma^2 v - \sqrt{c^2(1+\chi_0)}}{c^2 - \chi_0\gamma^2 v^2}\omega + o(\omega), \tag{119}$$

$$k_P = \frac{\sqrt{c^2 - \chi_0\gamma^2 v^2}}{c\sqrt{\chi_0}\epsilon\gamma v} + \frac{c^2\omega}{v(\chi_0\gamma^2 v^2 - c^2)} + o(\omega), \tag{120}$$

$$k_N = -\frac{\sqrt{c^2 - \chi_0\gamma^2 v^2}}{c\sqrt{\chi_0}\epsilon\gamma v} + \frac{c^2\omega}{v(\chi_0\gamma^2 v^2 - c^2)} + o(\omega), \tag{121}$$

while the left-infinity modes are given by

$$\tilde{k}_H = \frac{\chi_0\gamma^2 v + \sqrt{c^2(1+\chi_0+\chi_0^2(\lambda-1)\lambda - 2\chi_0\lambda)}}{c^2 - \chi_0\gamma^2 v^2 - c^2\chi_0\lambda}\omega + o(\omega), \tag{122}$$

$$\tilde{k}_B = \frac{\chi_0\gamma^2 v - \sqrt{c^2(1+\chi_0+\chi_0^2(\lambda-1)\lambda - 2\chi_0\lambda)}}{c^2 - \chi_0\gamma^2 v^2 - c^2\chi_0\lambda}\omega + o(\omega), \tag{123}$$

$$\tilde{k}_P = +\frac{\sqrt{\chi_0(\lambda_{\mathrm{crit}} - \lambda)}}{\epsilon\gamma v} - \frac{(1-\chi_0\lambda)\omega}{v\chi_0(\lambda_{\mathrm{crit}} - \lambda)}, \tag{124}$$

$$\tilde{k}_N = -\frac{\sqrt{\chi_0(\lambda_{\mathrm{crit}} - \lambda)}}{\epsilon\gamma v} - \frac{(1-\chi_0\lambda)\omega}{v\chi_0(\lambda_{\mathrm{crit}} - \lambda)}. \tag{125}$$

The transition between subcritical and transcritical regime happens at $\lambda = \lambda_{\mathrm{crit}}$, which in this case is

$$\lambda_{\mathrm{crit}} = \frac{1 - \chi_0\gamma^2\frac{v^2}{c^2}}{\chi_0}. \tag{126}$$

We assume the condition $\lambda_{\mathrm{crit}} > 0$.

As we did in Secs. IV B and IV C, we expand the factor $\frac{|P|}{|N|}$ in order to find an expression of the Hawking temperature. The flux factors that appear in the expressions of $|P|$ and $|N|$ [see Eq. (62)] for the Hopfield model are given by

$$v_i(\omega)\partial_\omega \mathrm{DR}|_{k_i} = -\frac{\gamma^2 V^4}{\omega^2 - k_i(\omega)^2}\prod_{j\neq i}(k_i(\omega) - k_j(\omega)). \tag{127}$$

1. Subcritical case

In the subcritical case, expanding for low-ω and low-λ we find

$$\log\left(\frac{|P|}{|N|}\right) = (\beta_{\mathrm{sub}} + o(\lambda))\omega + o(\omega), \tag{128}$$

where

$$\beta_{\mathrm{sub}} = \frac{2\left(\pi(1-v^2(\chi_0+1))(\gamma^2 v^2\chi_0 + v\sqrt{\chi_0+1}+1)\coth\left(\frac{\pi\gamma\sqrt{1-v^2(\chi_0+1)}}{2\beta v\sqrt{\chi_0}\epsilon}\right) + 2\beta\gamma v^3\chi_0{}^{3/2}\epsilon\sqrt{1-v^2(\chi_0+1)}\right)}{\beta\gamma v((\chi_0+1)v^2-1)^2}. \tag{129}$$

Once again, for $\beta \sim 0$ we find

$$\beta_{\text{sub}} \approx \frac{2\pi(\gamma^2 v^2 \chi_0 + v\sqrt{\chi_0 + 1} + 1)}{\beta\gamma v(1 - v^2(\chi_0 + 1))}, \tag{130}$$

which is the same result that was found in [39] with a perturbative approach. Notice that, for this model, taking the limit $\beta \sim 0$ in (129) is the same as taking the limit $\epsilon \sim 0$: this corresponds to the weak dispersion limit, a situation that is often studied in literature.

We can expand $\log\frac{|P|}{|N|}$ for $\lambda \lesssim \lambda_{\text{crit}}$, and we find

$$\log\frac{|P|}{|N|} = \left(\frac{A}{\frac{\lambda_{\text{crit}}}{\lambda} - 1} + B + O\left(\frac{\lambda_{\text{crit}}}{\lambda} - 1\right)\right)\omega + o(\omega), \tag{131}$$

with determined factors A and B. From this expansion we deduce the temperature in the near critical regime is

$$T_{nc}(\lambda) \approx \frac{\beta(1 - v^2(\chi_0 + 1))\tanh\left(\frac{\pi\gamma\sqrt{1 - v^2(\chi_0 + 1)}}{2\beta v\sqrt{\chi_0}\epsilon}\right)}{4\pi\gamma v\chi_0}\left(\frac{\lambda_{\text{crit}}}{\lambda} - 1\right). \tag{132}$$

In particular, notice as before that

$$T(\lambda = \lambda_{\text{crit}}) = 0.$$

The same considerations as in the previous section hold true about this point.

2. Transcritical case

The transcritical regime is reached for $\lambda > \lambda_{\text{crit}}$. In this case, the low-$\omega$ expansion gives a much simpler result,

$$\log\frac{|P|}{|N|} = \beta(\lambda)\omega + o(\omega), \tag{133}$$

where

$$\beta(\lambda) = \frac{2\pi\gamma v\chi_0 \coth\left(\frac{\pi\gamma\sqrt{1 - v^2(\chi_0 + 1)}}{2\beta v\sqrt{\chi_0}\epsilon}\right)}{\beta(1 - v^2(\chi_0 + 1))(1 - \frac{\lambda_{\text{crit}}}{\lambda})}. \tag{134}$$

Thus, the Hawking temperature in transcritical regime is

$$T_c(\lambda) = \frac{\beta(1 - v^2(\chi_0 + 1))\tanh\left(\frac{\pi\gamma\sqrt{1 - v^2(\chi_0 + 1)}}{2\beta v\sqrt{\chi_0}\epsilon}\right)}{2\pi\gamma v\chi_0}\left(1 - \frac{\lambda_{\text{crit}}}{\lambda}\right). \tag{135}$$

In the limit $\lambda \gg \lambda_{\text{crit}}$ (see also Appendix C), we reach the limit temperature

$$T_H = \frac{\beta(1 - v^2(\chi_0 + 1))\tanh\left(\frac{\pi\gamma\sqrt{1 - v^2(\chi_0 + 1)}}{2\beta v\sqrt{\chi_0}\epsilon}\right)}{2\pi\gamma v\chi_0}. \tag{136}$$

This result, for $\beta \sim 0$, again coincides with what was found in [39] using the Orr-Sommerfeld approach. It is also to be stressed, as for the subcritical case, that if, in place of $\beta \sim 0$, one considers in the last equation $\epsilon \sim 0$, i.e., the usual weak dispersion limit which is commonly adopted in the literature on the dispersive analog Hawking effect, we get the same result. In order to provide a more extensive comparison with the Orr-Sommerfeld approach and some more insights, in the following subsection we sketch the basic calculations involved.

E. The transcritical case in the Orr-Sommerfeld picture

The separated equation of motion for the spatial part of the polarization field $h(x)$, has been displayed in (94), where, in the present case, the specific profile (95) is understood. We eliminate, as usual [36], the third-order term by putting $h(x) = \exp(-2i\frac{\omega}{v}x)f(x)$. Then we obtain the following equation:

$$\epsilon^2 f^{(4)}(x) + \sum_{i=0}^{2} p_{3-i}(x,\epsilon)f^{(2-i)}(x) = 0, \tag{137}$$

where the coefficients $p_i(x,\epsilon)$ are, in the Orr-Sommerfeld approach, analytic functions in ϵ,

$$p_i(x,\epsilon) = \sum_{n=0}^{\infty}\epsilon^n p_{\text{in}}(x).$$

A real turning point $x = x_{tp}$, i.e., a horizon, is found when $p_{30}(x_{tp}) = 0$. See [36] and references therein, with particular focus on the papers by Nishimoto. We get

$$p_3(x,\epsilon) = \left[\left(1 - \chi_0\gamma^2\frac{v^2}{c^2}\right) - \frac{\lambda}{2\gamma^2 v^2}\left(1 - \tanh(\tilde{\beta}x)\right)\right] + \epsilon^2\omega^2\frac{1}{2v^2}\left(1 + 2\frac{v^2}{c^2}\right). \tag{138}$$

We can easily identify the turning point by solving $p_{30}(x = x_{tp}) = 0$. We find

$$\tilde{\beta}x_{tp} = \text{arctanh}\left(1 - 2\frac{\lambda_{\text{crit}}}{\lambda}\right). \tag{139}$$

It interesting to notice that, by assuming λ and λ_{crit} both positive, as in our previous analysis, the condition $\lambda > \lambda_{\text{crit}}$ amounts to the reality of the critical point at hand, i.e., to $(1 - 2\frac{\lambda_{\text{crit}}}{\lambda}) \in (-1, 1)$. This is in agreement with the assumption of the transcritical case and with the interpretation of λ_{crit}. We also have [36]

$$T_H = \frac{\gamma^2 v^2 |n'(x_{tp})|}{2\pi}, \tag{140}$$

where the refractive index is given by $n(x) = \sqrt{1+\chi(x)}$. Then, after restoring the parameter $\beta = \gamma\tilde{\beta}$, we obtain

$$T_H = \frac{\beta c^2 \left(1 - (1+\chi_0)\frac{v^2}{c^2}\right)(1 - \frac{\lambda_{\rm crit}}{\lambda})}{2\pi v \gamma \chi_0}, \tag{141}$$

which is in perfect agreement with (135) if one considers the limit as $\epsilon \to 0$ in (135), because, trivially, the contribution of the factor involving the hyperbolic tangent goes to 1 in that limit.

VI. CONCLUSIONS

In the framework of the analog Hawking effect in dielectric media, we have taken into account both the Cauchy model, which has the characteristic to be as simple as possible, and the original $\phi\psi$ model, with the explicit aim to find out exact solutions for the scattering problem for a suitable but physically meaningful monotone profile for the dielectric refractive index perturbation. On the one hand, this has required us to embed the physical problem, from a mathematical point of view, in the framework of Fuchsian equations on the Riemann sphere. We have first introduced the complex variable z and obtained a fourth-order equation displaying three regular singular points $z = 0, 1, \infty$. We have determined the monodromy properties of the solutions near the aforementioned singular points and also found that our equations satisfy the so-called rigidity properties, which have eventually allowed us to conclude that exact global solutions are available and involve the generalized hypergeometric function ${}_4F_3$. For this hypergeometric function, a study of the Mellin-Barnes integral representation has allowed us to reach two fundamental goals: a complete analysis of the Stokes phenomenon and also a complete set of connection formulas, which are at the root of the description of the S-matrix for the scattering process associated with the analog Hawking effect.

On the other hand, we have taken into account some fundamental physical problems, which, of course, involve, as a focal point, the determination of the analog Hawking temperature. This part of the analysis has required some approximations, as fully analytical calculations are hard to be managed successfully. In particular, for the asymptotic expressions of the momenta of the modes involved in the scattering, we have adopted an expansion for low frequencies ω, which is still standard in analytical calculations in literature. We have also considered both the subcritical regime and the transcritical one, and found explicit expressions for the Hawking temperature that are compatible both with the ones obtained in a perturbative framework in [39] and, in the limit of weak dispersive effects, in [36]. The aforementioned analysis, from a physical point of view, is just a very interesting but still incomplete one, as other regimes (beyond the low-frequency one) can be investigated, and further amplitudes can be calculated, for a complete description of the full scattering matrix involved in the problem. We deserve a deepening and an extension of our study to future investigations.

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APPENDIX A: USEFUL RELATIONS

We write (24) and (26) as

$$\mathrm{DR}(k) = \gamma^4 V^4 (k-k_1)(k-k_2)(k-k_3)(k-k_4), \tag{A1}$$

$$\mathrm{DR}_0(k) = \gamma^4 V^4 (k-\tilde{k}_1)(k-\tilde{k}_2)(k-\tilde{k}_3)(k-\tilde{k}_4), \tag{A2}$$

where k_j is a solution of $\mathrm{DR}(k) = 0$ and $\tilde{k}_j$ is a solution of $\mathrm{DR}_0(\tilde{k}) = 0$. By confronting (24) with (A1) we deduce the following useful relations:

$$\frac{1}{(2\beta)^4} k_1 k_2 k_3 k_4 = \frac{\Omega^2(-G^2V^2 + M^2 + \Omega^2\gamma^2)}{V^4\gamma^2}, \tag{A3}$$

$$\frac{1}{(2\beta)^3} \sum_{i\neq j\neq l} k_i k_j k_l = -\frac{2\Omega(-G^2 + M^2 + 2\Omega^2\gamma^2)}{V^3\gamma^2}, \tag{A4}$$

$$\frac{1}{(2\beta)^2} \sum_{i\neq j} k_i k_j = \frac{V^2(M^2 + 6\Omega^2\gamma^2) - G^2}{V^4\gamma^2}, \tag{A5}$$

$$\frac{1}{(2\beta)} \sum_i k_i = -\frac{4\Omega}{V}, \tag{A6}$$

and similarly from (26) and (A2),

$$\frac{1}{(2\beta)^4} \tilde{k}_1 \tilde{k}_2 \tilde{k}_3 \tilde{k}_4 = \frac{\Omega^2(-G^2V^2 + \Lambda + M^2 + \Omega^2\gamma^2)}{V^4\gamma^2}, \tag{A7}$$

$$\frac{1}{(2\beta)^3} \sum_{i\neq j\neq l} \tilde{k}_i \tilde{k}_j \tilde{k}_l = -\frac{2\Omega(-G^2 + \Lambda + M^2 + 2\Omega^2\gamma^2)}{V^3\gamma^2}, \tag{A8}$$

$$\frac{1}{(2\beta)^2} \sum_{i\neq j} \tilde{k}_i \tilde{k}_j = \frac{V^2(\Lambda + M^2 + 6\Omega^2\gamma^2) - G^2}{V^4\gamma^2}, \tag{A9}$$

$$\frac{1}{(2\beta)} \sum_i \tilde{k}_i = -\frac{4\Omega}{V}. \tag{A10}$$

APPENDIX B: PROOF OF PROPOSITION 1

We prove the theorem by induction. It is easy to verify that Eq. (50) holds for $n = 1, 2, 3, 4$: indeed, substituting (49) into (43) and truncating at order 4 [using $\mathrm{DR}_0(\tilde{k}_1) = 0$] gives

$$\begin{aligned}0 &= z\Big(-c_1\mathrm{DR}_0(\tilde{k}_1 - i) + \mathrm{DR}(\tilde{k}_1 - i)\Big) + z^2\Big(c_1\mathrm{DR}(\tilde{k}_1 - 2i) + 3c_1\mathrm{DR}_0(\tilde{k}_1 - i) - c_2\mathrm{DR}_0(\tilde{k}_1 - 2i) - 3\mathrm{DR}(\tilde{k}_1 - i)\Big)\\ &+ z^3\Big(-3c_1\Big(\mathrm{DR}(\tilde{k}_1 - 2i) + \mathrm{DR}_0(\tilde{k}_1 - i)\Big) + c_2\mathrm{DR}(\tilde{k}_1 - 3i) + 3c_2\mathrm{DR}_0(\tilde{k}_1 - 2i) - c_3\mathrm{DR}_0(\tilde{k}_1 - 3i) + 3\mathrm{DR}(\tilde{k}_1 - i)\Big)\\ &+ z^4\Big(3c_1\mathrm{DR}(\tilde{k}_1 - 2i) + c_1\mathrm{DR}_0(\tilde{k}_1 - i) - 3c_2\Big(\mathrm{DR}(\tilde{k}_1 - 3i) + \mathrm{DR}_0(\tilde{k}_1 - 2i)\Big) + c_3\mathrm{DR}(\tilde{k}_1 - 4i) + 3c_3\mathrm{DR}_0(\tilde{k}_1 - 3i)\\ &- c_4\mathrm{DR}_0(\tilde{k}_1 - 4i) - \mathrm{DR}(\tilde{k}_1 - i)\Big) + O(z^5),\end{aligned} \tag{B1}$$

from which one can compute $c_1, ..., c_4$ by annihilating the coefficient of each order. Even though it is not necessary for the sake of the proof, we can verify (50) also for some further n, by using the following identity, that is true for any fourth-order polynomial $P_4(k)$,

$$0 = \sum_{m=0}^{n} \binom{n}{m} (-1)^m P_4(-im), \qquad n > 4. \tag{B2}$$

The identity holds more generally for any polynomial of order k,

$$\sum_{m=0}^{n} \binom{n}{m} (-1)^m P_k(m) = 0, \quad n > k,$$

and it derives immediately from the following property of binomial coefficients:

$$\sum_{m=0}^{n} \binom{n}{m} (-1)^m m^k = 0, \qquad n > k.$$

The identity (B2) allows one to write $\mathrm{DR}(\tilde{k}_1 - \mathrm{in})$ $(n \geq 5)$ as a linear combination of $\mathrm{DR}(\tilde{k}_1 - i)$, $\mathrm{DR}(\tilde{k}_1 - 2i)$, $\mathrm{DR}(\tilde{k}_1 - 3i)$, and $\mathrm{DR}(\tilde{k}_1 - 4i)$, and similarly for DR_0.

Now, for n generic, assume that c_n, c_{n+1}, c_{n+2}, c_{n+3} satisfy (50). Take any two fourth-degree polynomials

$$\mathrm{DR}(k) = a_0 + a_1k + a_2k^2 + a_3k^3 + a_4k^4,$$
$$\mathrm{DR}_0(k) = b_0 + b_1k + b_2k^2 + b_3k^3 + b_4k^4.$$

Substituting (49) into (43), we find that the coefficient c_{n+4} satisfies the recurrence relation

$$A_0c_n + A_1c_{n+1} + A_2c_{n+2} + A_3c_{n+3} + A_4c_{n+4} = 0, \tag{B3}$$

where

$$\begin{aligned}A_0 &= (b_0 - ib_1(n+1) - b_2(n+1)^2 + ib_3(n+1)^3 + b_4(n+1)^4)\\ &+ k_1(b_1 - 2ib_2(n+1) - 3b_3(n+1)^2 + 4ib_4(n+1)^3)\\ &+ k_1^2(b_2 - 3ib_3(n+1) - 6b_4(n+1)^2) + k_1^3(b_3 - 4ib_4(n+1)) + b_4k_1^4,\end{aligned} \tag{B4}$$

$$\begin{aligned}A_1 &= -(a_0 - ia_1(n+1) - a_2n^2 - 2a_2n - a_2 + ia_3n^3 + 3ia_3n^2 + 3ia_3n + ia_3 + a_4n^4\\ &+ 4a_4n^3 + 6a_4n^2 + 4a_4n + a_4 + 3b_0 - 3ib_1n - 6ib_1 - 3b_2n^2 - 12b_2n - 12b_2 + 3ib_3n^3\\ &+ 18ib_3n^2 + 36ib_3n + 24ib_3 + 3b_4n^4 + 24b_4n^3 + 72b_4n^2 + 96b_4n + 48b_4)\\ &- k_1(a_1 - 2ia_2n - 2ia_2 - 3a_3n^2 - 6a_3n - 3a_3 + 4ia_4n^3 + 12ia_4n^2 + 12ia_4n + 4ia_4\\ &+ 3b_1 - 6ib_2n - 12ib_2 - 9b_3n^2 - 36b_3n - 36b_3 + 12ib_4n^3 + 72ib_4n^2 + 144ib_4n + 96ib_4)\\ &- k_1^2(a_2 - 3ia_3n - 3ia_3 - 6a_4n^2 - 12a_4n - 6a_4 + 3b_2 - 9ib_3n - 18ib_3 - 18b_4n^2 - 72b_4n - 72b_4)\\ &- k_1^3(a_3 - 4ia_4n - 4ia_4 + 3b_3 - 12ib_4n - 24ib_4) - k_1^4(a_4 + 3b_4),\end{aligned} \tag{B5}$$

$$
\begin{aligned}
A_2 = {} & 3(a_0 - ia_1(n+2) - a_2n^2 - 4a_2n - 4a_2 + ia_3n^3 + 6ia_3n^2 + 12ia_3n + 8ia_3 + a_4n^4 \\
& + 8a_4n^3 + 24a_4n^2 + 32a_4n + 16a_4 + b_0 - ib_1n - 3ib_1 - b_2n^2 - 6b_2n - 9b_2 + ib_3n^3 + 9ib_3n^2 \\
& + 27ib_3n + 27ib_3 + b_4n^4 + 12b_4n^3 + 54b_4n^2 + 108b_4n + 81b_4) \\
& + 3k_1(a_1 - 2ia_2n - 4ia_2 - 3a_3n^2 - 12a_3n - 12a_3 + 4ia_4n^3 + 24ia_4n^2 + 48ia_4n \\
& + 32ia_4 + b_1 - 2ib_2n - 6ib_2 - 3b_3n^2 - 18b_3n - 27b_3 + 4ib_4n^3 + 36ib_4n^2 + 108ib_4n + 108ib_4) \\
& + 3k_1^2(a_2 - 3ia_3n - 6ia_3 - 6a_4n^2 - 24a_4n - 24a_4 + b_2 - 3ib_3n - 9ib_3 - 6b_4n^2 - 36b_4n - 54b_4) \\
& + 3k_1^3(a_3 - 4ia_4n - 8ia_4 + b_3 - 4ib_4n - 12ib_4) + 3k_1^4(a_4 + b_4),
\end{aligned} \tag{B6}
$$

$$
\begin{aligned}
A_3 = {} & -3(3a_0 - 3ia_1(n+3) - 3a_2n^2 - 18a_2n - 27a_2 + 3ia_3n^3 + 27ia_3n^2 + 81ia_3n + 81ia_3 \\
& + 3a_4n^4 + 36a_4n^3 + 162a_4n^2 + 324a_4n + 243a_4 + b_0 - ib_1n - 4ib_1 - b_2n^2 - 8b_2n - 16b_2 \\
& + ib_3n^3 + 12ib_3n^2 + 48ib_3n + 64ib_3 + b_4n^4 + 16b_4n^3 + 96b_4n^2 + 256b_4n + 256b_4) \\
& - k_1(3a_1 - 6ia_2n - 18ia_2 - 9a_3n^2 - 54a_3n - 81a_3 + 12ia_4n^3 + 108ia_4n^2 + 324ia_4n \\
& + 324ia_4 + b_1 - 2ib_2n - 8ib_2 - 3b_3n^2 - 24b_3n - 48b_3 + 4ib_4n^3 + 48ib_4n^2 + 192ib_4n + 256ib_4) \\
& - k_1^2(3a_2 - 9ia_3n - 27ia_3 - 18a_4n^2 - 108a_4n - 162a_4 + b_2 - 3ib_3n - 12ib_3 \\
& - 6b_4n^2 - 48b_4n - 96b_4) \\
& - k_1^3(3a_3 - 12ia_4n - 36ia_4 + b_3 - 4ib_4n - 16ib_4) - k_1^4(3a_4 + b_4),
\end{aligned} \tag{B7}
$$

$$
\begin{aligned}
A_4 = {} & (a_0 - ia_1(n+4) - a_2(n+4)^2 + ia_3(n+4)^3 + a_4(n+4)^4) \\
& + k_1(a_1 - 2ia_2(n+4) - 3a_3(n+4)^2 + 4ia_4(n+4)^3) \\
& + k_1^2(a_2 - 3ia_3(n+4) - 6a_4(n+4)^2) + k_1^3(a_3 - 4ia_4(n+4)) + a_4k_1^4.
\end{aligned} \tag{B8}
$$

Substituting the expressions for $c_n, ..., c_{n+3}$ into Eq. (B3), we find that c_{n+4} satisfies (50) if and only if

$$
\begin{aligned}
& A_0\mathrm{DR}_0(\tilde{k}_1 - i(n+1))\mathrm{DR}_0(\tilde{k}_1 - i(n+2))\mathrm{DR}_0(\tilde{k}_1 - i(n+3))\mathrm{DR}_0(\tilde{k}_1 - i(n+4)) \\
& + A_1\mathrm{DR}(\tilde{k}_1 - i(n+1))\mathrm{DR}_0(\tilde{k}_1 - i(n+2))\mathrm{DR}_0(\tilde{k}_1 - i(n+3))\mathrm{DR}_0(\tilde{k}_1 - i(n+4)) \\
& + A_2\mathrm{DR}(\tilde{k}_1 - i(n+1))\mathrm{DR}(\tilde{k}_1 - i(n+2))\mathrm{DR}_0(\tilde{k}_1 - i(n+3))\mathrm{DR}_0(\tilde{k}_1 - i(n+4)) \\
& + A_3\mathrm{DR}(\tilde{k}_1 - i(n+1))\mathrm{DR}(\tilde{k}_1 - i(n+2))\mathrm{DR}(\tilde{k}_1 - i(n+3))\mathrm{DR}_0(\tilde{k}_1 - i(n+4)) \\
& + A_4\mathrm{DR}(\tilde{k}_1 - i(n+1))\mathrm{DR}(\tilde{k}_1 - i(n+2))\mathrm{DR}(\tilde{k}_1 - i(n+3))\mathrm{DR}(k_1 - i(n+4)) = 0,
\end{aligned} \tag{B9}
$$

which indeed is true for any n, as can be checked by direct algebra or using Wolfram's *Mathematica*. ■

APPENDIX C: PHYSICAL CONSEQUENCES OF ASSUMPTION (95)

We define, as in the nondispersive case and in the weakly dispersive one, the refractive index to be

$$n(x) = \sqrt{\chi(x) + 1}. \tag{C1}$$

Also, it is useful to define $n_0^2 \coloneqq \chi_0 + 1$. We can rewrite (95) as follows:

$$n^2(x) - 1 = \frac{n_0^2 - 1}{1 - \frac{\lambda}{2}(n_0^2 - 1)(1 - \tanh(\beta x))}. \tag{C2}$$

We obtain

$$\lim_{x\to+\infty}(n^2(x) - 1) = n_0^2 - 1, \tag{C3}$$

$$\lim_{x\to-\infty}(n^2(x) - 1) = \frac{n_0^2 - 1}{1 - \lambda(n_0^2 - 1)}. \tag{C4}$$

In standard materials we expect $n^2 > 1$. As a consequence, (C3) implies $n^2 > 1$ as $x \to +\infty$ for $n_0^2 > 1$. The same request leads to $1 - \lambda(n_0^2 - 1) > 0$, which means $\lambda < \lambda_{\mathrm{sup}}$, where

$$\lambda_{\rm sup} := \frac{1}{n_0^2 - 1} = \frac{1}{\chi_0}. \tag{C5}$$

We can also wonder if we are assuming a black hole condition [decreasing $n(x)$] or a white hole one [increasing $n(x)$] (cf., e.g., [36]). Given our monotone profile, we find that, by assuming, as we did, $\lambda > 0$ we obtain a black hole geometry. A white hole one would be allowed by a negative λ. It is also to be noted that it is possible to satisfy both $\lambda < \lambda_{\rm sup}$ and $\lambda \gg \lambda_{\rm crit}$, as in the discussion following (135), indeed we have

$$\lambda_{\rm crit} = \lambda_{\rm sup}\gamma^2 \left(1 - n_0^2 \frac{v^2}{c^2}\right), \tag{C6}$$

so that, for n_0^2 very near $\frac{c^2}{v^2} - \delta$, for $0 < \delta \ll 1$, we get $\lambda_{\rm sup} \gg \lambda_{\rm crit}$, and then $\lambda_{\rm sup} > \lambda \gg \lambda_{\rm crit}$ is allowed.

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