

A CHARACTERIZATION OF SOPHIE GERMAIN PRIMES

PAOLO LEONETTI

ABSTRACT. Let $n \geq 5$ be an odd integer. It is shown that $\{1^{\sigma(1)}, \dots, n^{\sigma(n)}\}$ is a complete residue system modulo n for some permutation σ of $\{1, \dots, n\}$ if and only if $\frac{1}{2}(n-1)$ is a Sophie Germain prime. Partial results are obtained also for the case n even.

1. INTRODUCTION

The aim of this article is to study an invariance property of complete residue systems modulo n , which turns out to be related to Sophie Germain primes. We recall that a prime p is a *Sophie Germain prime* if $2p+1$ is prime too, with the associated prime $2p+1$ which is then called a *safe prime*. These special primes have applications in public key cryptography, pseudorandom number generation, and primality testing; see, for example, [1, 4, 6]. Originally, they have been used also in the investigation of cases of Fermat's last theorem [3, § 3.2]. It has been conjectured that there exist infinitely many Sophie Germain primes, but this remains unproven; cf., for instance, [5, § 5.5.5].

Hereafter, we say that an integer $n \geq 2$ is *nice* if $\{1^{\sigma(1)}, \dots, n^{\sigma(n)}\}$ is a complete residue system modulo n for some permutation σ of $\{1, \dots, n\}$. Then, our main result follows:

Theorem 1. *Let $n \geq 5$ be an odd integer. Then n is nice if and only if n is a safe prime.*

Partial results have been obtained also for the case n even:

Theorem 2. *Let $n \geq 4$ be a nice even integer. Then $n = 2p$ for some prime p such that $p-1$ is squarefree. Conversely, if $n = 2p$, for some safe prime $p \geq 7$, then n is nice.*

Note that, according to Theorem 2, 10 is not a nice integer and, on the other hand, it is the double of a safe prime. However, the above results suggest the following:

Conjecture 1. *An integer $n \geq 11$ is nice if and only if n or $\frac{1}{2}n$ is a safe prime.*

Proofs of Theorem 1 and 2 follow in §§ 3 and 4, respectively.

1.1. Notations and conventions. We let \mathbf{Z} be the set of integers (endowed with its usual structure of ordered ring), \mathbf{N} the non-negative integers, and $\mathbf{N}^+ = \mathbf{N} \setminus \{0\}$ the positive integers. Also, the set of (positive rational) primes $\{2, 3, 5, \dots\}$ is denoted by \mathbf{P} .

2010 *Mathematics Subject Classification.* Primary 11A07; Secondary 11A15, 11A41.

Key words and phrases. Complete residue system, permutations, safe primes, Sophie Germain primes.

Unless noted otherwise, the letters n, m, i, j, k, t and z , with or without subscripts, will stand for positive integers, the letters p and q for primes, and the Greek letters σ and η for permutations.

Given an integer $n \geq 2$, we denote by \mathbf{Z}_n the quotient ring between \mathbf{Z} and its ideal $n\mathbf{Z}$; by an abuse of notation, sometimes we identify integers with its residue classes in \mathbf{Z}_n . The radical of n , that is, the product of the pairwise distinct primes which divide n , will be denoted by $\text{rad}(n)$. Moreover, given $p \in \mathbf{P}$, the p -adic valuation of n is $v_p(n)$, i.e., the greatest exponent $e \in \mathbf{N}$ for which p^e divides n .

Given integers $n, k \geq 2$, we denote by $\mathcal{A}_{n,k}$ the set of integers $m \in \{1, \dots, n-1\}$ which are divisible by k , and by $\mathcal{Q}_{n,k}$ the set of (possibly zero) k -th power residues in \mathbf{Z}_n . The set of quadratic residues $\mathcal{Q}_{n,2}$ will be shortened with \mathcal{Q}_n .

Lastly, we write $\#S$ for the cardinality of a set S . We refer to [2] for basic aspects of number theory (including notation not defined here).

2. PRELIMINARIES

Let us start settling down the cases of small values of n .

Lemma 1. *Every integer $n \in \{2, \dots, 7\}$ is nice.*

Proof. It is enough to choose the permutation σ according to the following table:

	$\sigma(1)$	$\sigma(2)$	$\sigma(3)$	$\sigma(4)$	$\sigma(5)$	$\sigma(6)$	$\sigma(7)$
$n = 2$	1	2	*	*	*	*	*
$n = 3$	2	1	3	*	*	*	*
$n = 4$	2	1	3	4	*	*	*
$n = 5$	2	5	1	3	4	*	*
$n = 6$	2	1	4	5	3	6	*
$n = 7$	6	2	1	5	7	3	4

■

Accordingly, let us assume hereafter that $n \geq 8$.

Lemma 2. *Let $n \geq 8$ be a nice integer. Then $n = r$ or $n = 2r$ or $n = 4r$ for some odd squarefree integer $r \geq 3$.*

Proof. Let σ be the associated permutation of $\{1, \dots, n\}$. Note that n divides $n^{\sigma(n)}$ and that $m^{\sigma(m)}$ is divisible by $\text{rad}(n)$ for all $m \in \mathcal{A}_{n,\text{rad}(n)}$.

Since n is nice by hypothesis, i.e., $\{1^{\sigma(1)}, \dots, n^{\sigma(n)}\}$ is a complete residue system in \mathbf{Z}_n , then n does not divide $m^{\sigma(m)}$ for each $m \in \mathcal{A}_{n,\text{rad}(n)}$. Moreover, since $\text{rad}(n)$ divides m by construction, then n does not divide $\text{rad}(n)^{\sigma(m)}$ for these integers m . In particular, n does not divide $\text{rad}(n)^{\#\mathcal{A}_{n,\text{rad}(n)}}$.

This implies that there exists $p \in \mathbf{P}$ which divides n and

$$v_p(n) \geq 1 + \#\mathcal{A}_{n,\text{rad}(n)} = \frac{n}{\text{rad}(n)} = \prod_{q \in \mathbf{P}, q|n} q^{v_q(n)-1} \geq p^{v_p(n)-1}.$$

If $p = 2$, it follows that $v_2(n) = 1$ or $v_2(n) = 2$, and $v_q(n) = 1$ for all other primes q which divide n . Lastly, if $p \geq 3$, then $v_q(n) = 1$ for all primes q which divide n , i.e., n is squarefree. ■

To conclude the section, we obtain a lower bound for the number of quadratic residues of a nice integer.

Lemma 3. *Let $n \geq 2$ be a nice integer. Then $\#\mathcal{Q}_n \geq \lfloor \frac{1}{2}n \rfloor$.*

Proof. Since n is nice, the number of quadratic residues in $\{1^{\sigma(1)}, \dots, n^{\sigma(n)}\}$ has to be $\#\mathcal{Q}_n$. In particular, $\#\mathcal{Q}_n$ is greater than or equal to the number of even integers in $\{\sigma(1), \dots, \sigma(n)\}$, that is, $\#\{1, \dots, n\} \cap 2\mathbf{N} = \lfloor \frac{1}{2}n \rfloor$. ■

3. PROOF OF THEOREM 1

The proof will be splitted into two main parts.

3.1. Only if part. Note that 5 and 7 are safe primes and, at the same time, are nice integers by Lemma 1. Hence, we can assume hereafter that n is a nice odd integer ≥ 9 .

CLAIM 1. Let $n \geq 9$ be a nice odd integer. Then n is prime.

Proof. According to Lemma 2, there are pairwise distinct odd primes q_1, \dots, q_k such that $n = q_1 \cdots q_k$. Note that, by the Chinese remainder theorem, the function $\mathbf{N}^+ \rightarrow \mathbf{N}^+$ defined by $n \mapsto \#\mathcal{Q}_n$ is multiplicative. Therefore, by Lemma 3, we obtain

$$\#\mathcal{Q}_1 \cdots \#\mathcal{Q}_k \geq \frac{1}{2}(q_1 \cdots q_k - 1),$$

which simplifies to

$$\prod_{i=1}^k \left(1 + \frac{1}{q_i}\right) \geq 2^{k-1} \left(1 - \frac{1}{q_1 \cdots q_k}\right).$$

Considering that $n \geq 8$, it follows that

$$\left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{5}\right)^{k-1} \geq \prod_{i=1}^k \left(1 + \frac{1}{q_i}\right) \geq 2^{k-1} \left(1 - \frac{1}{8}\right),$$

which is satisfied only for $k = 1$. ■

Claim 1 will be refined further, by obtaining additional properties of nice primes.

CLAIM 2. Let $p \geq 11$ be a nice prime. Then $p - 1$ is squarefree.

Proof. Let σ be the associated permutation of $\{1, \dots, p\}$ and suppose, for the sake of contradiction, that there exists a prime q such that q^2 divides $p - 1$. Then, note that if m is a q -th power residue or if $\sigma(m)$ is divisible by q , then $m^{\sigma(m)}$ is a q -th power in \mathbf{Z}_p .

Since the number of q -th powers in $\{1^{\sigma(1)}, \dots, p^{\sigma(p)}\}$ has to be $\#\mathcal{Q}_{p,q}$ and $\#\mathcal{A}_{p,q}$ is smaller than $\#\mathcal{Q}_{p,q}$, it follows that $m \in \mathcal{Q}_{p,q}$ whenever $\sigma(m) \in \mathcal{A}_{p,q}$. In particular, $m^{\sigma(m)}$ is a q^2 -th power in \mathbf{Z}_p . In turn, this implies that

$$\frac{p-1}{q} = \#\mathcal{A}_{p,q} \leq \#\mathcal{Q}_{p,q^2} = 1 + \frac{p-1}{q^2} \leq 1 + \frac{p-1}{2q}$$

This is a contradiction because, on one hand, $q \geq \frac{1}{2}(p-1)$ by the above inequality, and, the other hand, $q \leq \sqrt{p-1}$ by the fact that q^2 divides $p-1$. \blacksquare

Without loss of generality, it can be assumed that, if p is a nice (odd) prime with associated permutation σ , then

$$\sigma(1) = p-1. \quad (1)$$

Indeed, by Fermat's little theorem, $m^{p-1} = 1$ in \mathbf{Z}_p for each $m \in \{1, \dots, p-1\}$, implying that necessarily $\sigma(1) = p-1$ or $\sigma(p) = p-1$. On the other hand, if p is a nice prime with associated permutation σ , then p is a nice prime with another associated permutation $\tilde{\sigma}$ defined by $\tilde{\sigma}(p) = \sigma(1)$, $\tilde{\sigma}(1) = \sigma(p)$, and $\tilde{\sigma}(m) = \sigma(m)$ for each $m \in \{2, \dots, p-1\}$.

To conclude the first part of the proof, it is enough to show the following:

CLAIM 3. Let $p \geq 11$ be a nice prime. Then p is a safe prime.

Proof. Let $p \geq 11$ be a nice prime with associated permutation σ . According to Claim 2, $p-1$ is squarefree, i.e., there exist pairwise distinct odd $q_1, \dots, q_k \in \mathbf{P}$ such that $p-1 = 2q_1 \cdots q_k$ (note that $k \geq 1$ by the fact that $\frac{1}{2}(p-1) \geq 5$). Then, we claim that $k = 1$.

Let us suppose, for the sake of contradiction, that $k \geq 2$ and define the (even) integers

$$z_1 = \frac{p-1}{q_1} \quad \text{and} \quad z_2 = \frac{p-1}{q_2}.$$

Then, at least one between z_1 and z_2 does not divide $\sigma(p)$. Indeed, in the opposite case, $p-1 = \text{lcm}(z_1, z_2)$ would divide $\sigma(p)$. On the other hand, since $\sigma(p)$ belongs to $\{1, \dots, p\}$, then we have necessarily $\sigma(p) = p-1$, which contradicts (1). Hence, there exists an integer in $\{z_1, z_2\}$, let us say z , which does not divide $\sigma(p)$, that is, $\sigma(p)$ does not belong to $\mathcal{A}_{p,z}$.

At this point, since $m^{\sigma(m)}$ is a z -th power in \mathbf{Z}_p whenever $\sigma(m)$ belongs to $\mathcal{A}_{p,z}$ and $\#\mathcal{Q}_{p,z} = 1 + \#\mathcal{A}_{p,z}$, then $\mathcal{Q}_{p,z} \setminus \{0\} = \mathcal{A}_{p,z}$ in \mathbf{Z}_p . Denoting by ξ a primitive root of \mathbf{Z}_p , it follows that there exists a permutation η of $\{z, 2z, \dots, p-1\}$ such that

$$\{\xi^{z\eta(z)}, \xi^{(2z)\eta(2z)}, \dots, \xi^{(p-1)\eta(p-1)}\} = \{\xi^z, \xi^{2z}, \dots, \xi^{p-1}\}$$

in \mathbf{Z}_p . By Fermat's little theorem and the fact that p is nice, we have by force $\eta(p-1) = p-1$. Therefore

$$\{\xi^{z\eta(z)}, \xi^{(2z)\eta(2z)}, \dots, \xi^{(p-1-z)\eta(p-1-z)}\} = \{\xi^z, \xi^{2z}, \dots, \xi^{p-1-z}\}$$

in \mathbf{Z}_p , with the consequence that

$$\{z\eta(z), (2z)\eta(2z), \dots, (p-1-z)\eta(p-1-z)\} = \{z, 2z, \dots, p-1-z\}$$

in \mathbf{Z}_{p-1} . Moreover, dividing all elements by z and denoting by q the prime $\frac{1}{z}(p-1)$, it follows that

$$\{\eta(z), 2\eta(2z), \dots, (q-1)\eta(p-1-z)\} = \{1, 2, \dots, q-1\}$$

in \mathbf{Z}_q . In particular, the products of the elements of each set must be the same in \mathbf{Z}_q . This is a contradiction, indeed the product of the set on the right is $(q-1)! \equiv -1 \pmod{q}$ by Wilson's

theorem, while on the left side

$$(q-1)! \prod_{j=1}^{q-1} \eta(zj) = (q-1)! \prod_{j=1}^{q-1} zj = (q-1)!^2 z^{q-1} \equiv 1 \pmod{q},$$

by Fermat's little theorem and the fact that $\gcd(q, z) = 1$. ■

3.2. If part. Let $p \geq 5$ be a Sophie Germain prime. We claim that the prime $n = 2p + 1$ is nice.

Let ξ and τ be generators of (the group of units of) \mathbf{Z}_n and \mathbf{Z}_{2p} , respectively. Note that τ is odd. To conclude the proof of Theorem 1, we have to construct an explicit permutation σ of $\{1, \dots, n\}$ such that $\{1^{\sigma(1)}, \dots, n^{\sigma(n)}\}$ is equal to $\{1, \dots, n\}$ in \mathbf{Z}_n .

To this aim, it is enough to set

$$\sigma(1) = 2p, \quad \sigma(2p) = p, \quad \sigma(2p+1) = 2p+1,$$

together with

$$\sigma\left(\xi^{(j\tau)^i}\right) = \begin{cases} ((j\tau)^i \pmod{2p}) & \text{if } i = 0, \dots, \frac{1}{2}(p-3) \\ ((j\tau)^{i+1} \pmod{2p}) & \text{if } i = \frac{1}{2}(p-1), \dots, p-2 \end{cases},$$

for each $j \in \{1, 2\}$, where $(x \pmod{2p})$ denotes the integer $y \in \{1, \dots, 2p\}$ such that $2p$ divides $x - y$.

Finally, let us check that this permutation really works. Define the sets

$$\mathcal{A}_j = \{\xi^{(j\tau)^0}, \xi^{(j\tau)^1}, \dots, \xi^{(j\tau)^{p-2}}\}$$

in \mathbf{Z}_n , for each $j \in \{1, 2\}$, and note that $\{1, 2p, 2p+1\} \cup \mathcal{A}_1 \cup \mathcal{A}_2$ is equal to $\{1, \dots, n\}$ in \mathbf{Z}_n . Then, it is easy to see that, for each $j \in \{1, 2\}$, the map

$$\mathcal{A}_j \rightarrow \mathcal{A}_j : m \mapsto m^{\sigma(m)}$$

is actually a bijection. Indeed, for each $j \in \{1, 2\}$, it holds

$$\left(\xi^{(j\tau)^i}\right)^{\sigma\left(\xi^{(j\tau)^i}\right)} = \begin{cases} \xi^{(j\tau)^{2i}} & \text{if } i = 0, \dots, \frac{1}{2}(p-3) \\ \xi^{(j\tau)^{2i+1}} & \text{if } i = \frac{1}{2}(p-1), \dots, p-2 \end{cases}.$$

This completes the proof. (Straightforward details are left to the reader.)

4. PROOF OF THEOREM 2

4.1. First part. Note that 4 and 6 are nice integers by Lemma 1 and both of them are in the form $2p$ for some prime p such that $p-1$ is squarefree. Hence, let us hereafter that n is a nice even integer ≥ 8 . In the same spirit of Claim 1, we will prove that $\frac{1}{2}n \in \mathbf{P}$.

CLAIM 4. Let $n \geq 8$ be a nice even integer. Then $n = 2p$ for some prime p .

Proof. According to Lemma 2, there exist $\alpha \in \{1, 2\}$ and pairwise distinct odd primes q_1, \dots, q_k , with $k \geq 1$, such that $n = 2^\alpha q_1 \cdots q_k$. Moreover, by Lemma 3 and the multiplicativity of $n \mapsto \#\mathcal{Q}_n$, we obtain

$$\#\mathcal{Q}_{2^\alpha} \prod_{i=1}^k \#\mathcal{Q}_{q_i} \geq 2^{\alpha-1} q_1 \cdots q_k.$$

Considering that $\#\mathcal{Q}_{2^\alpha} = 2$ for $\alpha \in \{1, 2\}$ and $\#\mathcal{Q}_q = \frac{1}{2}(q+1)$ for each odd $q \in \mathbf{P}$, the above inequality simplifies to

$$\prod_{i=1}^k \left(\frac{1}{2} + \frac{1}{2q_i} \right) \geq \frac{1}{2^{2-\alpha}}.$$

On the other hand, note that, for all integers $k \geq 2$, it holds

$$\prod_{i=1}^k \left(\frac{1}{2} + \frac{1}{2q_i} \right) \leq \left(\frac{1}{2} + \frac{1}{2 \cdot 3} \right)^k \leq \left(\frac{2}{3} \right)^2 < \frac{1}{2}.$$

It follows that $\alpha = 1$ and $k = 1$, i.e., $n = 2p$ for some prime p . ■

To complete the first part of the proof of Theorem 2, it will be enough to show that $p-1$ is squarefree. Accordingly, we will first show that 4 does not divide $p-1$ and, then, that q^2 does not divide $p-1$ for each odd prime q .

Let σ be a permutation associated to $2p$. Note that the number of quadratic residues in \mathbf{Z}_{2p} is $p+1$, and, on the other hand, the number of even positive integers $\leq 2p$ is p . It follows that m has to be a quadratic residue whenever $\sigma(m)$ is even. Moreover, the residue modulo $2p$ of $m^{\sigma(m)}$ will be uniquely determined by the Chinese remainder theorem, given its residues modulo p and modulo 2 (in this respect, note that $m^k \equiv m \pmod{2}$ for all $m, k \in \mathbf{N}^+$).

CLAIM 5. Let $p \geq 5$ be a prime such that $2p$ is nice. Then 4 does not divide $p-1$.

Proof. Let us suppose, for the sake of contradiction, that 4 divides $p-1$. Then $p-1$ and $2p-1$ are quadratic residues in \mathbf{Z}_{2p} . By the above observations, at least one between $p-1$ and $2p-1$ has an even image under σ . This would contradict the fact that $1^{\sigma(1)} \equiv 1 \pmod{2p}$ and $(p+1)^{\sigma(p+1)} \equiv p+1 \pmod{2p}$ since, for all $k \in \mathbf{N}^+$, we have $(2p-1)^{2k} \equiv 1 \pmod{2p}$ and $(p-1)^{2k} \equiv p+1 \pmod{2p}$. ■

We conclude with the following:

CLAIM 6. Let $p \geq 7$ be a prime such that $2p$ is nice. Then $p-1$ is squarefree.

Proof. Note $\frac{1}{2}(p-1)$ is odd by Claim 5 and, by hypothesis, ≥ 3 . Hence, let us suppose, for the sake of contradiction, that there exists an odd prime q such that q^2 divides $p-1$.

In addition, we have $\#\mathcal{Q}_{2p,q} = 2 + \frac{2}{q}(p-1)$ which is greater, on the other hand, than $\#\mathcal{A}_{2p,q} = \frac{2}{q}(p-1)$. With a reasoning similar to Claim 2, the number of q^2 -th power residues in \mathbf{Z}_{2p} has to be greater than or equal to the number of multiples of q in $\{1, \dots, 2p\}$, implying that

$$\frac{2(p-1)}{q} = \#\mathcal{A}_{2p,q} \leq \#\mathcal{Q}_{2p,q^2} = 2 + \frac{p-1}{q^2} \leq 2 \left(1 + \frac{p-1}{6q} \right).$$

It follows that $q \geq \frac{5}{6}(p-1)$. This is a contradiction because $q \leq \sqrt{p-1}$ by the fact that q^2 divides $p-1$ while, on the other hand, $\frac{5}{6}(p-1) > \sqrt{p-1}$ for all primes $p \geq 7$. ■

4.2. Second part. Let $p \geq 7$ be a Sophie Germain prime. We claim that the integer $n = 2(2p+1)$ is nice. The proof follows the same lines of reasoning in § 3.2, therefore we provide here only a sketch.

Let ξ and τ be generators of \mathbf{Z}_n and \mathbf{Z}_{2p} , respectively. Then, define the permutation σ of $\{1, \dots, n\}$ by

$$\sigma(1) = 2p, \sigma(2p) = p, \sigma(2p+1) = 4p+1, \sigma(2p+2) = 4p, \sigma(4p+1) = 3p, \sigma(4p+2) = 4p+2,$$

together with

$$\sigma\left((t\xi)^{(j\tau)^i}\right) = \begin{cases} 2p(t-1) + ((j\tau)^i \bmod 2p) & \text{if } i = 0, \dots, \frac{1}{2}(p-3) \\ 2p(t-1) + ((j\tau)^{i+1} \bmod 2p) & \text{if } i = \frac{1}{2}(p-1), \dots, p-2 \end{cases},$$

for each $t, j \in \{1, 2\}$, where $(x \bmod 2p)$ represents the integer $y \in \{1, \dots, 2p\}$ such that $2p$ divides $x-y$.

Finally, we have to check that this permutation really works. For each $t, j \in \{1, 2\}$ define the sets

$$\mathcal{A}_{t,j} = \{(t\xi)^{(j\tau)^0}, (t\xi)^{(j\tau)^1}, \dots, (t\xi)^{(j\tau)^{p-2}}\}$$

in \mathbf{Z}_n . Again, it is not difficult to check that, for each $t, j \in \{1, 2\}$, the map

$$\mathcal{A}_{t,j} \rightarrow \mathcal{A}_{t,j} : m \mapsto m^{\sigma(m)}$$

is actually a bijection. Indeed, for each $t, j \in \{1, 2\}$, it holds

$$\left((t\xi)^{(j\tau)^i}\right)^{\sigma\left(\xi^{(j\tau)^i}\right)} = \begin{cases} (t\xi)^{(j\tau)^{2i}} & \text{if } i = 0, \dots, \frac{1}{2}(p-3) \\ (t\xi)^{(j\tau)^{2i+1}} & \text{if } i = \frac{1}{2}(p-1), \dots, p-2 \end{cases},$$

which completes the proof.

5. ACKNOWLEDGEMENTS

The author is grateful to Salvatore TRINGALI (University of Graz) for suggestions which improved the readability of the article.

REFERENCES

- [1] M. Agrawal, N. Kayal, and N. Saxena, *PRIMES is in P*, Ann. of Math. **160** (2004), No. 2, 781–793.
- [2] T.M. Apostol, *Introduction to Analytic Number Theory*, Undergrad. Texts Math., Springer-Verlag: New York, 1976.
- [3] H.M. Edwards, *Fermat's Last Theorem: A Genetic Introduction to Algebraic Number Theory*, Springer, 2000.
- [4] R.A.J. Matthews, *Maximally periodic reciprocals*, Bull. Inst. Math. Appl. **28** (1992), 147–148.
- [5] V. Shoup, *A Computational Introduction to Number Theory and Algebra*, Cambridge University Press, 2009.
- [6] W.-S. Yap, S.L. Yeo, S.-H. Heng, and M. Henricksen, *Security analysis of GCM for communication*, Security Comm. Networks **7** (2014), No. 5, 854–864.

UNIVERSITÀ L. BOCCONI, VIA ROENTGEN 1, 20136 MILANO, ITALY.

E-mail address: `leonetti.paolo@gmail.com`