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New Monotonic Properties for Solutions of Odd-Order Advanced Nonlinear Differential Equations

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Abstract: The present paper studies the asymptotic and oscillatory properties of solutions of odd-order differential equations with advanced arguments and in a noncanonical case. By providing new and effective relationships between the corresponding function and the solution, we present strict and new criteria for testing whether the studied equation exhibits oscillatory behavior or converges to zero. Our results contribute uniquely to oscillation theory by presenting some theorems that improve and expand upon the results found in the existing literature. We also provide an example to corroborate the validity of our proposed criteria.

Keywords: odd order; oscillation; advanced differential equations

MSC: 34C10; 34K1

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1. Introduction

In this paper, we are concerned with the asymptotic and oscillatory behavior of solutions of higher-order differential equations with advanced arguments:

$$\left(\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha}\right)' + \Phi(\top)x^{\beta}(\omega(\top)) = 0, \tag{1}$$

where α and β are the quotients of odd positive integers, $\beta \geq \alpha$ and

$$v(\top) = x(\top) + \varrho(\top)x(\zeta(\top)).$$

We assume the following.

Hypothesis 1. *$\varsigma \in C^1[\top_0, \infty), \Phi, \omega, \zeta, \varrho \in C[\top_0, \infty)$ and*

$$\int_{\top_0}^{\infty} \frac{ds}{\varsigma^{1/\alpha}(s)} < \infty; \tag{2}$$

Hypothesis 2. $\varsigma(\top) > 0, \varsigma'(\top) \geq 0, \omega(\top) > \top, \omega'(\top) \geq \omega_0 > 0, \zeta(\top) < \top, \zeta'(\top) \geq \zeta_0 > 0, \Phi(\top) > 0, 0 \leq \varrho(\top) \leq \varrho_0 < \infty.$

Definition 1. *By a solution* $x(\top)$ *of (1), we mean a function* $x \in C^{\iota-1}([\Upsilon_x, \infty), \mathbb{R})$ *for some* $\Upsilon_x \geq \top_0$ *such that* $\varsigma(v^{\iota-1})^{\alpha} \in C^1[\Upsilon_x, \infty)$*, that satisfies (1) on* $[\Upsilon_x, \infty)$*. We focus only on those solutions of (1) that satisfy*

$$\sup\{|x(\top)| : \top \geq \Upsilon\} > 0$$

for all $\Upsilon \geq \Upsilon_x$.

Definition 2. *A solution* $x(\top)$ *of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. The equation itself is said to be oscillatory if all its solutions oscillate.*

Oscillation theory, a key concept in physics and engineering, examines periodic oscillation in systems like pendulums and electrical circuits. Characterized by amplitude, frequency, and phase, these oscillations offer insights into stability, resonance, and energy transfer. By studying these patterns, scientists and engineers can predict behaviors, design stable structures, and innovate technologies. Understanding oscillation theory is crucial for progress in mechanical engineering, electronics, and biological systems [1–6].

Odd-order differential equations are a type of differential equation that contains only odd derivatives. These equations play a crucial role in many scientific and engineering fields, as they can be used to describe a variety of natural phenomena and technological applications. This type of equation is a powerful tool for analyzing dynamic systems that change over time or with other variables, such as mechanical vibrations, fluid flow, and heat transfer. The study of oscillatory solutions of odd-order differential equations is an important area of research, as it helps in understanding how systems stabilize and respond to disturbances. Many mathematical and theoretical techniques are relied upon to analyze these solutions and ensure their oscillation, such as spectral analysis and numerical methods. The relationship between symmetry and odd-order differential equations is pivotal for simplifying and solving these equations. By understanding the symmetries, we can transform, reduce, and sometimes even directly solve these equations, making the concept of symmetry a powerful tool in mathematical analysis and physics.

Delayed differential equations are considered an important and interesting branch in the field of applied mathematics and systems analysis. These equations are characterized by considering not only the current state of the system but also its states at previous moments in time. This makes them a powerful tool for describing systems affected by their past, such as biological, economic, engineering, and other systems. A delayed differential equation includes variables that depend on their values at previous times. This type of delay can be constant or variable, and the delay may be finite or distributed over a period of time. These equations show clear importance in many fields; in biology, for example, they can be used to model biological systems where time delay is a crucial element, such as in cellular processes or epidemics. In engineering, they are used to model systems that involve elements like the time required for signal transmission or the time interval for control. One of the principal challenges confronting these equations is their complexity; exact solutions are rare, and reliance is often placed on approximate or numerical solutions. This requires the use of advanced mathematical methods and, sometimes, specialized software. As for the more complex nonlinear delayed differential equations, finding a closed-form solution becomes an extremely difficult task (see [7–14]).

Advanced arguments in differential equations typically refer to the presence of terms in the equation where the independent variable (often t) is incremented by some positive constant. These types of equations are a subset of functional differential equations and are called "differential equations with advanced arguments" or "forward delay differential equations". Advanced differential equations are critical in the modern era due to their widespread applications across various fields. Models like the SIR (Susceptible, Infected, Recovered) model use differential equations to predict the spread of diseases and the impact of interventions. They help in studying population growth, predator–prey interactions, and ecological systems. Furthermore, techniques like MRI and CT scans rely on solving differential equations to reconstruct images from raw data. Moreover, they allow scientists and engineers to predict the behavior of complex systems under various conditions and provide a deeper understanding of natural and man-made systems, facilitating innovation and discovery [15–19].

The authors in [20] discussed the criteria ensuring that all solutions oscillate in functional differential equations:

$$v^{(\iota)}(\top) + \Phi(\top)v^{\alpha}(\omega(\top)) = 0. \tag{3}$$

Baculíková and Džurina [21] explored the asymptotic properties and oscillation of nth-order advanced differential equations:

$$\left(\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha}\right)' + \Phi(\top)x^{\alpha}(\omega(\top)) = 0, \tag{4}$$

They obtained oscillation results based on the Riccati transformation under conditions

$$\int_{\top_0}^{\infty} \frac{\mathrm{d}s}{\varsigma^{1/\alpha}(s)} = \infty \tag{5}$$

and

$$\omega(\top) \geq \top.$$

The authors in [7,22,23] established some oscillation criteria and solutions of the following higher-order differential equations:

$$\left(\varsigma\left(x^{(\iota-1)}\right)^{\alpha}\right)'(\top) + \Phi(\top)x^{\beta}(\omega(\top)) = 0. \tag{6}$$

Zhang et al. [24] studied the oscillatory behavior of the solutions of Equation (6) and obtained sufficient conditions to ensure the oscillation of the solutions of Equation (6) under the conditions

$$\omega(\top) < \top,\ \beta \geq \alpha$$

and

$$\int_{\top_0}^{\infty} \frac{\mathrm{d}s}{\varsigma^{1/\alpha}(u)} < \infty.$$

Special cases of (1) have been discussed as less general equations, of which we mention, for example, Yao et al. [25], who studied some results of the oscillation of the equation of third-order differential equations with advanced arguments

$$\left(\varsigma_2(\top)\left(\left((\varsigma_1(\top)v'(\top))^{\alpha}\right)'\right)^{\beta}\right)' + \Phi(\top)x^{\alpha}(\omega(\top)) = 0, \tag{7}$$

They provide criteria to ensure the asymptotic or oscillatory behavior of solutions of Equation (7), where

$$\int_{\top_0}^{\infty} \frac{\mathrm{d}s}{\varsigma_1^{1/\alpha}(u)} < \infty \text{ and } \int_{\top_0}^{\infty} \frac{\mathrm{d}s}{\varsigma_2^{1/\alpha}(u)} < \infty.$$

On the other hand, Dzurina and Baculikova [26] studied a less general case of (7) of the following form:

$$\left(\varsigma(\top)(v'(\top))^{\alpha}\right)'' + \Phi(\top)x(\omega(\top)) = 0,$$

when (5) holds.

In [27], some new criteria for the oscillation of third-order functional differential equations of the form

$$\left(\varsigma(\top)(v'(\top))^{\alpha}\right)'' = \Phi_1(\top)f(x(\omega_1(\top))) + \Phi_2(\top)f(x(\omega_2(\top))),$$

where

$$\omega_1(\top) < \top, \omega_2(\top) > \top,$$

and $-f(-xy) \geq f(xy) \geq f(x)f(y)$, were obtained.

The possibility of obtaining criteria that guarantee the oscillation of solutions to Equation (1) has been very limited compared to differential equations when $\varrho(\top) = 0$ because of the difficulty of establishing applicable relationships between the corresponding function and the solution when $0 \leq \varrho(\top) \leq \varrho_0 < \infty$. The purpose of this paper is to provide new capabilities for identifying certain conditions to ensure the emergence of oscillatory behavior for solutions of Equation (1). We introduce multiple new relationships that link the solution to the corresponding function, thereby crossing them to reach new criteria that guarantee the oscillation of solutions of Equation (1).

This paper is organized as follows: Initially, we present the equation targeted by this study along with the necessary conditions for the study, in addition to some relevant information about the field of study and related previous studies that led to Equation (1). In Section 2, we present various lemmas drawn from different references that will be used to prove our main results. In Section 3, we present some results that include the key relationships we later used. This is followed by a presentation of some of the results we obtained, through which we were able to ensure the oscillation of the solutions to Equation (1). In Section 4, we provide some examples that support and confirm the validity of our results. Finally, in Section 4, we provide a brief explanation of the study covered in this paper and the methods used, and then propose an idea for future work that may benefit researchers and those interested in the field.

2. Preliminaries and Existing Results

We offer some auxiliary results that we need to obtain the next important results of the Equation (1).

Lemma 1 ([28])**.** *Assume that* $\theta_1, \theta_2 \in [0, \infty)$. *Then*

$$\frac{(\theta_1 + \theta_2)^{\alpha}}{\theta_1^{\alpha} + \theta_2^{\alpha}} \leq \mu := \begin{cases} 2^{\alpha-1} & \text{if } \alpha \in [1, \infty) \\ 1 & \text{if } \alpha \in (0, 1] \end{cases}.$$

Lemma 2 ([22], Lemma 2.2.3)**.** *Suppose that* $\Psi \in C^{\iota}([\top_0, \infty), \mathbb{R}^+)$. *If there exists* $\top_1 \geq \top_0$ *such that*

$$\Psi^{(\iota-1)}(\top)\Psi^{(\iota)}(\top) \leq 0,$$

where $\Psi^{(\iota)}$ *is of fixed sign on* $[\top_0, \infty)$ *for all* $\top \geq \top_1$ *and*

$$\lim_{\top \to \infty} \Psi(\top) \neq 0,$$

then there $\exists\, \top_\lambda \in [\top_1, \infty)$ *such that*

$$(\iota - 1)!\Psi(\top) \geq \lambda \left| \Psi^{(\iota-1)}(\top) \right| \top^{\iota-1}$$

satisfies, for every $\lambda \in (0, 1)$ *and for all* $\top \in [\top_\lambda, \infty)$.

Lemma 3 ([29])**.** *Assume that* γ *is defined as the ratio between two odd positive integers,*

$$\rho \in C[\top_0, \infty), h \in C^1[\top_0, \infty), \rho(\top) > 0, h(\top) < \top, \lim_{\top \to \infty} h(\top) = \infty, \text{ and } h'(\top) \geq 0. \quad (8)$$

If the first-order delay differential inequality

$$y'(\top) + \rho(\top) y^{\gamma}(h(\top)) \leq 0$$

has a solution $y(\top) > 0$, then the delay equation

$$y'(\top) + \rho(\top)y^{\gamma}(h(\top)) = 0 \tag{9}$$

also has a positive solution.

Lemma 4 ([30,31])**.** *Let $y(\top)$ be a solution of Equations (9) and (8) satisfies. If one of the hypotheses satisfies*
(I) *$\gamma = 1$ and*

$$\liminf_{\top\to\infty} \int_{h(\top)}^{\top} \rho(s)\mathrm{d}s > \frac{1}{e};$$

(II) *Suppose $\gamma > 1$ and $\exists\, \Omega \in C^1[\top_0, \infty)$ such that*

$$\Omega'(\top) > 0 \text{ and } \lim_{\top\to\infty} \Omega(\top) = \infty,$$

$$\limsup_{\top\to\infty} \frac{\gamma\Omega'(h(\top))h'(\top)}{\Omega'(\top)} < 1$$

and

$$\liminf_{\top\to\infty} \frac{\rho(\top)e^{-\Omega(\top)}}{\Omega'(\top)} > 0,$$

then every solution of delay differential Equation (9) is oscillatory.

Lemma 5 ([32,33])**.** *Assume that γ is a ratio of odd positive integers,*

$$\sigma, \rho \in C[\top_0, \infty), \rho(\top) > 0, \sigma(\top) > \top \text{ and } \sigma'(\top) \geq 0.$$

(I) *If the advanced inequality*

$$y'(\top) - \rho(\top)y^{\gamma}(\sigma(\top)) \geq 0 \tag{10}$$

has a solution $y(\top) > 0$, then the advanced equation

$$y'(\top) - \rho(\top)y^{\gamma}(\sigma(\top)) = 0$$

also has an eventually positive solution.
(II) *Suppose that $\gamma = 1$. If*

$$\liminf_{\top\to\infty} \int_{\top}^{\sigma(\top)} \rho(s)\mathrm{d}s > \frac{1}{e},$$

then every solution of advanced differential inequality (10) has no positive solution.

Lemma 6. *Let $x(\top) > 0$ be a solution of (1) belong to $[\top_0,\infty)$. Then there is $\top_1 \geq \top_0$,*

$$v(\top) > 0, \left(\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha}\right)' < 0,$$

and v satisfies either

$$\textbf{(i)}\;\; v^{(\iota-1)}(\top) > 0,\ v^{(\iota)}(\top) < 0$$

or

$$\textbf{(ii)}\;\; v^{(\iota-2)}(\top) > 0,\ v^{(\iota-1)}(\top) < 0,$$

for $\top \geq \top_1$, $\top_1$ is large enough.

Proof. Assume that (1) has a nonoscillatory solution $x(\top) > 0$ such that $v > 0$, and

$$\lim_{\top\to\infty} v(\top) \neq 0.$$

From (1), we get

$$\left(\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha}\right)' = -\Phi(\top)x^{\beta}(\omega(\top)),$$

that is

$$\left(\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha}\right)' < 0.$$

Thus, $\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha}$ is decreasing and has fixed sign, so $v^{(\iota-1)}(\top)$ has fixed sign. Therefore, $v^{(\iota-1)}(\top) > 0$ or $v^{(\iota-1)}(\top) < 0$. Let $v^{(\iota-1)}(\top) > 0$, we see that

$$0 > \left(\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha}\right)'$$

that is,

$$\begin{aligned}\left(\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha}\right)' &= \left(v^{(\iota-1)}(\top)\right)^{\alpha}\varsigma'(\top) + \alpha v^{(\iota)}(\top)\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha-1} \\ &\geq \alpha v^{(\iota)}(\top)\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha-1},\end{aligned}$$

which implies $v^{(\iota)}(\top) < 0$ eventually. Now, let $v^{(\iota-1)}(\top) < 0$, we see that $v^{(\iota-2)}(\top)$ is decreasing, so $v^{(\iota-2)}(\top)$ has eventually fixed sign; however, $v^{(\iota-2)}(\top) < 0$ and $v^{(\iota-1)}(\top) < 0$ imply the contradiction $\lim_{\top\to\infty} v(\top) = -\infty$. So, there exist only two possible cases (**i**) and (**ii**). This completes the proof. □

Notations and Definitions

Throughout this paper, we use the following notations:

$$\begin{aligned}L_1 v(\top) &= \left(\frac{1}{\omega_0}\varsigma\left(\omega^{-1}(\top)\right)\left(v^{(\iota-1)}\left(\omega^{-1}(\top)\right)\right)^{\alpha} + \frac{\varrho_0^{\alpha}}{\omega_0\zeta_0}\varsigma\left(\omega^{-1}(\zeta(\top))\right)\left(v^{(\iota-1)}\left(\omega^{-1}(\zeta(\top))\right)\right)^{\alpha}\right)', \\ L_2 v(\top) &= \left(\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha} + \frac{\varrho_0^{\alpha}}{\zeta_0}\varsigma(\zeta(\top))\left(v^{(\iota-1)}((\zeta(\top)))\right)^{\alpha}\right)'.\end{aligned}$$

Moreover assume that there exist a function $\varphi \in C[\top_0, \infty)$, $\varphi > \top$, $\varphi(\top) \geq \varphi_0 > 0$ and $\lim_{\top\to\infty}\varphi(\top) = \infty$ such that

$$L_3 v(\top) = \left(\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha} + \frac{\varrho_0^{\beta}}{\varphi_0}\varsigma(\varphi(\top))\left(v^{(\iota-1)}((\varphi(\top)))\right)^{\alpha}\right)',$$

also

$$\begin{aligned}\widetilde{\Phi}_1(\top) &= \min\left\{\Phi\left(\omega^{-1}(\top)\right), \Phi\left(\omega^{-1}(\zeta(\top))\right)\right\}, \\ \widetilde{\Phi}_2(\top) &= \min\{\Phi(\top), \Phi(\zeta(\top))\}\end{aligned}$$

and

$$\widetilde{\Phi}_3(\top) = \min\{\Phi(\top), \Phi(\varphi(\top))\}.$$

Remark 1. *All the functional inequalities presented in this manuscript are supposed to hold eventually, that is, they are held for all* $\top$ *large enough.*

Remark 2. *It should be observed that if* y *represents a solution to Equation (1), then* $-y$ *also represents a solution to Equation (1). Consequently, regarding nonoscillatory solutions of Equation (1), it suffices to restrict our attention to positive ones.*

Definition 3. *We say that (1) is almost oscillatory if every solution v of (1) is oscillatory or satisfies* $\lim_{\top\to\infty} v(\top) = 0$.

3. Oscillation Results

Lemma 7. *Let $x > 0$ be a solution of (1). Then*

$$L_1 v(\top) \leq -\frac{1}{\mu}\widetilde{\Phi}_1(\top) v^{\beta}(\top). \tag{11}$$

Moreover,
(**i**) *If $\zeta \circ \omega = \omega \circ \zeta$, $\omega'(\top) \geq \omega_0 > 0$ and $\zeta'(\top) \geq \zeta_0 > 0$, then*

$$L_2 v(\top) \leq -\frac{1}{\mu}\widetilde{\Phi}_2(\top) v^{\beta}(\omega(\top)); \tag{12}$$

(**ii**) *If $\zeta(\top) \leq \varphi(\top)$, then*

$$L_3 v(\top) \leq -\frac{1}{\mu}\widetilde{\Phi}_3(\top) v^{\beta}(\omega(\top)). \tag{13}$$

Proof. Suppose t $x(\top) > 0$ is a solution of (1). From Lemma 1, it is easy to note that

$$(x(\top) + \varrho_0 x(\zeta(\top)))^{\beta} \leq \mu\left(x(\top)^{\beta} + \varrho_0^{\beta} x(\zeta(\top))^{\beta}\right),$$

which implies that

$$\frac{1}{\mu} v^{\beta}(\top) \leq x(\top)^{\beta} + \varrho_0^{\beta} x^{\beta}(\zeta(\top)). \tag{14}$$

So,

$$\frac{1}{\mu} v^{\beta}(\omega(\top)) \leq x(\omega(\top))^{\beta} + \varrho_0^{\beta} x(\zeta(\omega(\top)))^{\beta}. \tag{15}$$

From (1), we have

$$\frac{1}{\omega_0}\left(\varsigma\left(\omega^{-1}(\top)\right)\left(v^{(\iota-1)}\left(\omega^{-1}(\top)\right)\right)^{\alpha}\right)' + \Phi\left(\omega^{-1}(\top)\right) x^{\beta}(\top) \leq 0. \tag{16}$$

Since $\omega'(\top) \geq \omega_0 > 0$ and $\zeta'(\top) \geq \zeta_0 > 0$, we get

$$\frac{\varrho_0^{\beta}}{\omega_0 \zeta_0}\left(\varsigma\left(\omega^{-1}(\zeta(\top))\right)\left(v^{(\iota-1)}\left(\omega^{-1}(\zeta(\top))\right)\right)^{\alpha}\right)' \leq -\varrho_0^{\beta}\Phi\left(\omega^{-1}(\zeta(\top))\right) x^{\beta}(\zeta(\top)). \tag{17}$$

Combining (16) and (17), and using (14), we obtain

$$\begin{aligned} &\left(\frac{\varsigma(\omega^{-1}(\top))}{\omega_0}\left(v^{(\iota-1)}\left(\omega^{-1}(\top)\right)\right)^{\alpha} + \frac{\varrho_0^{\beta}\varsigma(\omega^{-1}(\zeta(\top)))}{\omega_0\,\zeta_0}\left(v^{(\iota-1)}\left(\omega^{-1}(\zeta(\top))\right)\right)^{\alpha}\right)' \\ \leq\ & -\frac{1}{\mu}\widetilde{\Phi}_1(\top) v^{\beta}(\top). \end{aligned}$$

Since $\zeta \circ \omega = \omega \circ \zeta$ and $\zeta'(\top) \geq \zeta_0 > 0$, we obtain

$$\frac{\varrho_0^{\beta}}{\zeta_0}\left(\varsigma(\zeta(\top))\left(v^{(\iota-1)}(\zeta(\top))\right)^{\alpha}\right)' \leq -\varrho_0^{\beta}\Phi(\zeta(\top)) x^{\beta}(\zeta(\omega(\top))). \tag{18}$$

Now, combining (1) and (18), and using (15), we have

$$\left(\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha} + \frac{\varrho_0^{\beta}}{\zeta_0}\varsigma(\zeta(\top))\left(v^{(\iota-1)}((\zeta(\top)))\right)^{\alpha}\right)' \leq -\frac{1}{\mu}\widetilde{\Phi}_2(\top) v^{\beta}(\omega(\top)).$$

Finally, from (1), we see that

$$\frac{\varrho_0^\beta}{\varphi_0}\left(\varsigma(\varphi(\top))\left(v^{(\iota-1)}(\varphi(\top))\right)^\alpha\right)' \leq -\varrho_0^\beta\Phi(\varphi(\top))x^\beta(\varphi(\omega(\top))). \tag{19}$$

Combining (1) and (19), we find

$$\begin{aligned}\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^\alpha + \frac{\varrho_0^\beta}{\varphi_0}\left(\varsigma(\varphi(\top))\left(v^{(\iota-1)}(\varphi(\top))\right)^\alpha\right)' \quad &\leq \quad -\Phi(\top)x^\beta(\omega(\top)) \\ &\quad -\varrho_0^\beta\Phi(\varphi(\top))x^\beta(\varphi(\omega(\top))).\end{aligned}$$

Thus,

$$\left(\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^\alpha + \frac{\varrho_0^\beta}{\varphi_0}\varsigma(\varphi(\top))\left(v^{(\iota-1)}(\varphi(\top))\right)^\alpha\right)' \leq -\frac{1}{\mu}\widetilde{\Phi}_3(\top)v^\beta(\omega(\top)).$$

The proof is complete. □

Remark 3. *The previous lemma includes the relationships that connect solution of Equation (1) with the corresponding function $v(\top)$. These relationships enable us to overcome the condition*

$$0 \leq \varrho(\top) \leq \varrho_0 < \infty$$

and thus obtain highly effective oscillatory conditions.

Theorem 1. *Assume that $\iota \geq 3$ is odd, $\zeta(\top) < \omega^{-1}(\top)$, $\left(\omega^{-1}(\top)\right)' > 0$ and*

$$\int_{\top_0}^{\infty}\eta^{\iota-2}\left[\frac{1}{\varsigma(\eta)}\frac{\varphi_0}{\mu\left(\varrho_0^\beta + \varphi_0\right)}\int_\eta^\infty\widetilde{\Phi}_3(s)\mathrm{d}s\right]^{1l\alpha}\mathrm{d}\eta = \infty. \tag{20}$$

Moreover, the delay differential equation

$$W'(\top) + \frac{1}{\mu}\widetilde{\Phi}_1(\top)\left[\left(\frac{\omega_0\,\zeta_0}{\zeta_0 + \varrho_0^\alpha}\right)^{\frac{1}{\alpha}}\frac{\lambda_0\top^{\iota-1}}{(\iota-1)!\varsigma^{1l\alpha}(\top)}\right]^\beta W^{\beta l\alpha}(\omega(\zeta(\top))) = 0 \tag{21}$$

is oscillatory for some constant $0 < \lambda_0 < 1$. If

$$y'(\top) - \frac{1}{\mu}\widetilde{\Phi}_1(\top)\left(\frac{\omega_0\,\zeta_0}{\zeta_0 + \varrho_0^\alpha}\right)^{\beta l\alpha}\left(\frac{\lambda_1}{(\iota-2)!}\top^{\iota-2}\psi(\top)\right)^\beta y^{\beta l\alpha}(\omega(\top)) \geq 0 \tag{22}$$

is oscillatory for some constant $0 < \lambda_1 < 1$, where $\psi(\top) := \int_\top^\infty\left(1/\varsigma^{1l\alpha}(s)\right)\mathrm{d}s$, then (1) is almost oscillatory.

Proof. Suppose that case (**i**) holds. $v'(\top) > 0$ for all large $\top$. From (1), we obtain

$$\begin{aligned}\lim_{\top\to\infty}\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^\alpha \quad &= \quad c,\ c \text{ is a nonnegative constant,} \\ \lim_{\top\to\infty}v(\top) \quad &= \quad a,\ a \text{ is a positive constant,}\end{aligned}$$

and

$$\lim_{\top\to\infty}v^{(\iota-1)}(\top) = b,\ b \text{ is a nonnegative constant.}$$

Thus,

$$\lim_{\top\to\infty} v^{(n)}(\top) = 0 \text{ for } n = 1, 2, ..., \iota - 1.$$

Integrating (12) and since $\lim_{\top\to\infty} \varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha} \geq 0$ is finite, we have

$$-\left[\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha} + \frac{\varrho_0^{\beta}}{\varphi_0}\varsigma(\varphi(\top))\left(v^{(\iota-1)}((\varphi(\top)))\right)^{\alpha}\right] \leq -\frac{1}{\mu}\widetilde{\Phi}_3(\top)v^{\beta}(\omega(\top)),$$

$$-\varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha}\left(1 + \frac{\varrho_0^{\beta}}{\varphi_0}\right) \leq -\frac{1}{\mu}\int_{\top}^{\infty}\widetilde{\Phi}_3(s)v^{\beta}(\omega(s))\mathrm{d}s,$$

that is

$$-\left(v^{(\iota-1)}(\top)\right)^{\alpha} \leq -\frac{1}{\varsigma(\top)}\frac{\varphi_0}{\mu\left(\varrho_0^{\beta} + \varphi_0\right)}\int_{\top}^{\infty}\widetilde{\Phi}_3(s)v^{\beta}(\omega(s))\mathrm{d}s,$$

$$-\left(v^{(\iota-1)}(\top)\right)^{\alpha} \leq -\frac{1}{\varsigma(\top)}\frac{\varphi_0}{\mu\left(\varrho_0^{\beta} + \varphi_0\right)}\int_{\top}^{\infty}\widetilde{\Phi}_3(s)v^{\beta}(\omega(s))\mathrm{d}s,$$

or

$$-v^{(\iota-1)}(\top) \leq -\left[\frac{1}{\varsigma(\top)}\frac{\varphi_0}{\mu\left(\varrho_0^{\beta} + \varphi_0\right)}\int_{\top}^{\infty}\widetilde{\Phi}_3(s)v^{\beta}(\omega(s))\mathrm{d}s\right]^{1l\alpha}. \tag{23}$$

Integrating (23) from $\top$ to ∞ for $(\iota - 2)$ times, then integrating the resulting inequality from $\top_1$ to ∞ and using

$$\lim_{\top\to\infty} v^{(n)}(\top) = 0 \text{ for } 1 \leq n \leq \iota - 1,$$

we have

$$\int_{\top_1}^{\infty}\frac{1}{(\iota-2)!}(\eta - \top_1)^{\iota-2}\left[\frac{1}{\varsigma(\eta)}\frac{\varphi_0}{\mu\left(\varrho_0^{\beta} + \varphi_0\right)}\int_{\eta}^{\infty}\widetilde{\Phi}_3(s)v^{\beta}(\omega(s))\mathrm{d}s\right]^{1l\alpha}\mathrm{d}\eta < \infty,$$

it is

$$\int_{\top_1}^{\infty}\eta^{\iota-2}\left[\frac{1}{\varsigma(\eta)}\frac{\varphi_0}{\mu\left(\varrho_0^{\beta} + \varphi_0\right)}\int_{\eta}^{\infty}\widetilde{\Phi}_3(s)\mathrm{d}s\right]^{1l\alpha}\mathrm{d}\eta < \infty.$$

This contradicts (20). Set

$$W(\top) = \frac{1}{\omega_0}\varsigma\left(\omega^{-1}(\top)\right)\left(v^{\iota-1}\left(\omega^{-1}(\top)\right)\right)^{\alpha} + \frac{\varrho_0^{\alpha}}{\omega_0\zeta_0}\varsigma\left(\omega^{-1}(\zeta(\top))\right)\left(v^{(\iota-1)}\left(\omega^{-1}(\zeta(\top))\right)\right)^{\alpha}, \tag{24}$$

and

$$g(\top) = \varsigma(\top)\left(v^{(\iota-1)}(\top)\right)^{\alpha}.$$

That is,

$$W(\top) = \frac{1}{\omega_0}g\left(\omega^{-1}(\top)\right) + \frac{\varrho_0^{\alpha}}{\omega_0\zeta_0}g\left(\omega^{-1}(\zeta(\top))\right). \tag{25}$$

Since $g'(\top) < 0, \zeta(\top) \leq \top$ and $\left(\omega^{-1}(\top)\right)' < 0$, it is easy to see that

$$g\left(\omega^{-1}(\zeta(\top))\right) < g\left(\omega^{-1}(\top)\right).$$

So,

$$W(\top) \leq \left(\frac{1}{\omega_0} + \frac{\varrho_0^{\alpha}}{\omega_0\,\zeta_0}\right)g\left(\omega^{-1}(\top)\right),$$

and

$$W^{\beta l\alpha}(\omega(\zeta(\top))) \leq \left(\frac{\zeta_0 + \varrho_0^{\alpha}}{\omega_0\,\zeta_0}\right)^{\beta l\alpha} g^{\beta l\alpha}(\zeta(\top)). \tag{26}$$

In view of Lemma 2, we obtain

$$v^{\beta}(\top) \geq \left(\frac{\lambda}{(\iota-1)!\varsigma^{1l\alpha}(\top)}\top^{\iota-1}\right)^{\beta}\left(\varsigma^{1l\alpha}(\top)v^{(\iota-1)}(\top)\right)^{\beta}, \text{ for every } \lambda \in (0,1),$$

it is

$$v^{\beta}(\top) \geq \left(\frac{\lambda\top^{\iota-1}}{(\iota-1)!\varsigma^{1l\alpha}(\top)}\right)^{\beta} g^{\beta l\alpha}(\top). \tag{27}$$

According to the fact that $v'(\top) > 0$, (11) becomes

$$\begin{aligned}
&\left(\frac{1}{\omega_0}\varsigma\left(\omega^{-1}(\top)\right)\left(v^{(\iota-1)}\left(\omega^{-1}(\top)\right)\right)^{\alpha} + \frac{\varrho_0^{\alpha}}{\omega_0\zeta_0}\varsigma\left(\omega^{-1}(\zeta(\top))\right)\left(v^{(\iota-1)}\left(\omega^{-1}(\zeta(\top))\right)\right)^{\alpha}\right)' \\
&+\frac{1}{\mu}\widetilde{\Phi}_1(\top)^{\beta}v(\zeta(\top)) \\
\leq\ & 0.
\end{aligned}$$

From (24), (26) and (27), we have

$$\begin{aligned}
W'(\top) &\leq -\frac{1}{\mu}\widetilde{\Phi}_1(\top)\left(\frac{\lambda\top^{\iota-1}}{(\iota-1)!\varsigma^{1l\alpha}(\top)}\right)^{\beta} g^{\beta l\alpha}(\top) \\
&\leq -\frac{1}{\mu}\widetilde{\Phi}_1(\top)\left(\frac{\lambda\top^{\iota-1}}{(\iota-1)!\varsigma^{1l\alpha}(\top)}\right)^{\beta}\left(\frac{\omega_0\,\zeta_0}{\zeta_0+\varrho_0^{\alpha}}\right)^{\beta l\alpha} W^{\beta l\alpha}(\omega(\zeta(\top))).
\end{aligned} \tag{28}$$

Therefore, we see that $W > 0$ is a solution of (28). By Lemma 3, (21) also has a positive solution.

Now, assume that v satisfies case $(\mathbf{ii})$. Using the monotonicity of $\varsigma\left(v^{(\iota-1)}\right)^{\alpha}$, we note that

$$\frac{v^{(\iota-1)}(s)}{v^{(\iota-1)}(\top)} \leq \frac{\varsigma^{1l\alpha}(\top)}{\varsigma^{1l\alpha}(s)},\ s \geq \top \geq \top_1. \tag{29}$$

Integrating (29) from $\top$ to ℓ, we obtain

$$v^{(\iota-2)}(\ell) \leq v^{(\iota-2)}(\top) + v^{(\iota-1)}(\top)\varsigma^{1l\alpha}(\top)\int_{\top}^{\ell}\frac{\mathrm{d}s}{\varsigma^{1l\alpha}(s)}.$$

Letting $\ell \to \infty$, we get

$$v^{(\iota-2)}(\top) \geq -\varsigma^{1l\alpha}(\top)v^{(\iota-1)}(\top)\psi(\top). \tag{30}$$

Moreover, by Lemma 2, we have

$$v(\top) \geq \frac{\lambda}{(\iota-2)!}\top^{\iota-2}v^{(\iota-2)}(\top) \text{ fore very } \lambda \in (0,1). \tag{31}$$

Combining (30) and (31), we see that

$$v(\top) \geq -\frac{\lambda}{(\iota-2)!}\top^{\iota-2}\psi(\top)\varsigma^{1l\alpha}(\top)v^{(\iota-1)}(\top).$$

Put $u := \varsigma\left(v^{(\iota-1)}\right)^{\alpha}$, that is

$$v^{\beta}(\top) \geq -\left(\frac{\lambda}{(\iota-2)!}\top^{\iota-2}\psi(\top)\right)^{\beta} u^{\beta/\alpha}(\top). \tag{32}$$

Define the function

$$\widetilde{W}(\top) = \frac{1}{\omega_0} u\left(\omega^{-1}(\top)\right) + \frac{\varrho_0^{\alpha}}{\omega_0 \zeta_0} u\left(\omega^{-1}(\zeta(\top))\right).$$

Since $u'(\top) < 0, \zeta(\top) \leq \top$ and $\left(\omega^{-1}(\top)\right)' < 0$, it follows that

$$\left(\frac{\omega_0 \zeta_0}{\zeta_0 + \varrho_0^{\alpha}}\right)^{\beta l \alpha} \widetilde{W}^{\beta l \alpha}(\omega(\top)) \leq u^{\beta l \alpha}(\top). \tag{33}$$

Combining (32) and (11), we have

$$\left(\frac{1}{\omega_0} u\left(\omega^{-1}(\top)\right) + \frac{\varrho_0^{\alpha}}{\omega_0 \zeta_0} u\left(\omega^{-1}(\zeta(\top))\right)\right)' - \frac{1}{\mu}\widetilde{\Phi}_1(\top)\left(\frac{\lambda}{(\iota-2)!}\top^{\iota-2}\psi(\top)\right)^{\beta} u^{\beta/\alpha}(\top) \leq 0.$$

Using (33), we obtain

$$\widetilde{W}'(\top) - \frac{1}{\mu}\widetilde{\Phi}_1(\top)\left(\frac{\lambda}{(\iota-2)!}\top^{\iota-2}\psi(\top)\right)^{\beta}\left(\frac{\omega_0 \zeta_0}{\zeta_0 + \varrho_0^{\alpha}}\right)^{\beta l \alpha} \widetilde{W}^{\beta l \alpha}(\omega(\top)) \leq 0$$

or

$$-\widetilde{W}'(\top) - \frac{1}{\mu}\widetilde{\Phi}_1(\top)\left(\frac{\lambda}{(\iota-2)!}\top^{\iota-2}\psi(\top)\right)^{\beta}\left(\frac{\omega_0 \zeta_0}{\zeta_0 + \varrho_0^{\alpha}}\right)^{\beta l \alpha}\left(-\widetilde{W}^{\beta l \alpha}(\omega(\top))\right) \geq 0.$$

That is $y := -\widetilde{W} > 0$ is a solution of

$$y'(\top) - \frac{1}{\mu}\widetilde{\Phi}_1(\top)\left(\frac{\lambda}{(\iota-2)!}\top^{\iota-2}\psi(\top)\right)^{\beta}\left(\frac{\omega_0 \zeta_0}{\zeta_0 + \varrho_0^{\alpha}}\right)^{\beta l \alpha}\left(y^{\beta l \alpha}(\omega(\top))\right) \geq 0.$$

By Lemma 5, we see that (22) also has a positive solution. This ends the proof. □

Corollary 1. *Let $\iota \geq 3$ be odd, $\alpha = \beta$, $\zeta(\top) \leq \omega^{-1}(\top)$ and $\left(\omega^{-1}(\top)\right)' > 0$. Suppose that (20) holds and*

$$\liminf_{\top \to \infty} \int_{\omega(\zeta(\top))}^{\top} \frac{\left(s^{\iota-1}\right)^{\alpha}}{\varsigma(s)} \widetilde{\Phi}_1(s) \mathrm{d}s > \frac{\mu\left(\zeta_0 + \varrho_0^{\alpha}\right)((\iota-1)!)^{\alpha}}{\lambda^{\alpha}\omega_0 \zeta_0 e}. \tag{34}$$

If

$$\liminf_{\top \to \infty} \int_{\top}^{\omega(\top)} \widetilde{\Phi}_1(s)\left(s^{\iota-2}\right)^{\alpha} \psi^{\alpha}(s) \mathrm{d}s > \frac{\mu\left(\zeta_0 + \varrho_0^{\alpha}\right)((\iota-2)!)^{\alpha}}{\lambda^{\alpha}\omega_0 \zeta_0 e}, \tag{35}$$

then (1) is almost oscillatory.

Proof. The proof is direct based on Lemmas 4 and 5, and Theorem 1. □

Corollary 2. *Let $\iota \geq 3$ be odd and $\alpha < \beta$. Suppose that (20) holds and there exists a continuously differentiable function Ω such that*

$$\Omega'(\top) > 0 \text{ and } \lim_{\top \to \infty} \Omega(\top) = \infty, \tag{36}$$

$$\limsup_{\top\to\infty}\frac{\Omega'(\zeta(\top))\zeta'(\top)}{\Omega'(\top)}<\frac{\alpha}{\beta},\tag{37}$$

and

$$\liminf_{\top\to\infty}\widetilde{\Phi}_1(\top)\frac{\left(\top^{\iota-1}\right)^{\beta}}{\Omega'(\top)\varsigma^{\beta l\alpha}(\top)}e^{-\Omega(\top)}>0.\tag{38}$$

If

$$\liminf_{\top\to\infty}\int_{\top}^{\omega(\top)}\widetilde{\Phi}_1(s)\left(s^{\iota-2}\psi(s)\right)^{\beta}\mathrm{d}s>\frac{\mu\left(\zeta_0+\varrho_0^{\alpha}\right)^{\beta l\alpha}((\iota-2)!)^{\beta}}{(\omega_0\,\zeta_0)^{\beta l\alpha}e}\tag{39}$$

or

$$\int_{\top_0}^{\infty}\widetilde{\Phi}_1(s)\left(s^{\iota-2}\psi(s)\right)^{\beta}\mathrm{d}s=\infty,\tag{40}$$

then (1) is almost oscillatory.

Proof. In view of Lemma 4, we see that (21) is oscillatory. Suppose that (22) has nonoscillatory solution, say y on $[\top_0,\infty)$. Since $y'(\top)>0$, there exists a constant $M>0$ such that

$$y(\omega(\top))\geq M>0.$$

Integrating (22), we obtain

$$\begin{aligned}y(\top)&\geq\frac{1}{\mu}\left(\frac{\omega_0\zeta_0}{\zeta_0+\varrho_0^{\alpha}}\right)^{\beta l\alpha}\int_{\top_1}^{\top}\widetilde{\Phi}_1(s)\left(\frac{\lambda_1}{(\iota-2)!}s^{\iota-2}\psi(s)\right)^{\beta}y^{\beta/\alpha}(\omega(s))\mathrm{d}s\\&\geq\frac{M^{\beta/\alpha}}{\mu}\left(\frac{\omega_0\zeta_0}{\zeta_0+\varrho_0^{\alpha}}\right)^{\beta l\alpha}\int_{\top_1}^{\top}\widetilde{\Phi}_1(s)\left(\frac{\lambda_1}{(\iota-2)!}s^{\iota-2}\psi(s)\right)^{\beta}\mathrm{d}s.\end{aligned}$$

In view of condition (39), we see that $\lim_{\top\to\infty}y(\top)=\infty$. Hence we have

$$-y^{\beta/\alpha}(\omega(\top))=-y(\omega(\top))y^{(\beta-\alpha)/\alpha}(\omega(\top))\leq-y(\omega(\top)),$$

from which follows by (22) that y is a positive solution of the differential inequality

$$y'(\top)-\frac{1}{\mu}\widetilde{\Phi}_1(\top)\left(\frac{\lambda_1}{(\iota-2)!}\top^{\iota-2}\psi(\top)\right)^{\beta}\left(\frac{\omega_0\,\zeta_0}{\zeta_0+\varrho_0^{\alpha}}\right)^{\beta l\alpha}y(\omega(\top))\geq0.\tag{41}$$

From Lemma 5, condition (39) ensures that the inequality (41) has no positive solution for a suitable constant $\lambda_1\in(0,1)$. This is a contradiction, and hence (22) is oscillatory. It is well known that condition (40) guarantees that (22) is oscillatory. Therefore, by Theorem 1, every solution v of (1) is almost oscillatory. The proof is complete. □

Example 1. *Consider the third-order differential equation*

$$\left(e^{\top}(x(\top)+\varrho_0x(\top-2))''\right)'+4e^{\top+2}x(\top+1)=0,\ \textit{where}\ \top\geq1.\tag{42}$$

From (42), we note that

$$\begin{aligned}\zeta(\top)&=(\top-2),\omega(\top)=\top+1,\varrho(\top)=\varrho_0,\varsigma(\top)=e^{\top},\Phi(\top)=4e^{\top+2},\alpha=\beta=1\\,\mu&=1\ \text{and}\ \iota=3.\end{aligned}$$

According to Corollary 1, (20) implies

$$\int_{\top_0}^{\infty}\eta\left[\frac{1}{\varsigma(\eta)}\frac{\varphi_0}{(\varrho_0+\varphi_0)}\int_{\eta}^{\infty}\widetilde{\Phi}_3(s)\mathrm{d}s\right]\mathrm{d}\eta=\frac{4\varphi_0e^2}{(\varrho_0+\varphi_0)}\int_{\top_0}^{\infty}\eta\left(\frac{1}{e^{\eta}}\int_{\eta}^{\infty}e^{s}\mathrm{d}s\right)\mathrm{d}\eta,$$

that is

$$\frac{4e^2\varphi_0}{\left(\varrho_0^\beta+\varphi_0\right)}\int_1^\infty \eta\left(\frac{1}{e^\eta}[e^\infty-e^\eta]\right)\mathrm{d}\eta=\infty.$$

Also, (34) implies

$$\begin{aligned}\liminf_{\top\to\infty}\int_{\omega(\zeta(\top))}^{\top}\frac{s^2}{\varsigma(s)}\widetilde{\Phi}_1(s)\mathrm{d}s &= \liminf_{\top\to\infty}\int_{\top-1}^{\top}\frac{(s^2)}{e^s}4e^{s+2}\mathrm{d}s\\ &= \liminf_{\top\to\infty}4e^2\int_{\top-1}^{\top}s^2\mathrm{d}s=4e^2[\frac{s^3}{3}]_{\top-1}^{\top}\\ &= \liminf_{\top\to\infty}\frac{4}{3}e^2\left(\top^2-\top+\frac{1}{3}\right),\end{aligned}$$

that is

$$\liminf_{\top\to\infty}\frac{4}{3}e^2\left(\top^2-\top+\frac{1}{3}\right)>\frac{2(\zeta_0+\varrho_0)}{\omega_0\zeta_0 e}.$$

So, by (35), we have

$$\begin{aligned}\liminf_{\top\to\infty}\int_{\top}^{\omega(\top)}\widetilde{\Phi}_1(s)(s)\psi(s)\mathrm{d}s &= \liminf_{\top\to\infty}\int_{\top}^{\top+1}4e^{s+2}(s)\left(\int_s^\infty(1/e^\eta)\mathrm{d}\eta\right)\mathrm{d}s\\ &= \liminf_{\top\to\infty}\int_{\top}^{\top+1}4e^{s+2}(s)e^{-s}\mathrm{d}s\\ &= \liminf_{\top\to\infty}4e^2\int_{\top}^{\top+1}s\mathrm{d}s\\ &= \liminf_{\top\to\infty}4e^2\left((\top+1)^2-\top^2\right)\\ &= \liminf_{\top\to\infty}4e^2\left(\top+\frac{1}{2}\right)=\infty,\end{aligned}$$

that is

$$\liminf_{\top\to\infty}\int_{\top}^{\top+1}4e^{s+2}(s)\left(\int_s^\infty(1/e^\eta)\mathrm{d}\eta\right)\mathrm{d}s>\frac{\zeta_0+\varrho_0}{\omega_0\zeta_0 e}.$$

Now, all conditions of Corollary 1 are satisfied. Therefore, (42) is almost oscillatory.

Remark 4. *We observe that the behavior of the solutions of (42) either oscillates or approaches to zero, and that one of its solutions is* $x(\top)=e^{-2\top}$.

Example 2.

$$\left(e^{\top}(x(\top)+\varrho_0 x(\top-3))''\right)'+2e^{\top+3}x(\top+2)=0,\ \textit{where}\ \top\geq 1. \tag{43}$$

From (42), we note that

$$\zeta(\top)=(\top-3),\omega(\top)=\top+2,\varrho(\top)=\varrho_0,\varsigma(\top)=e^{\top},\Phi(\top)=2e^{\top+3},\omega(\zeta(\top))=\omega(\top)=\top-1,\iota=3.$$

Applying condition (20), we obtain

$$\begin{aligned}\int_{\top_0}^\infty\eta\left[\frac{1}{e^\eta}\frac{\varphi_0}{(\varrho_0+\varphi_0)}\int_\eta^\infty 2e^{s+3}\mathrm{d}s\right]\mathrm{d}\eta &= \frac{2\varphi_0 e^3}{(\varrho_0+\varphi_0)}\int_{\top_0}^\infty\eta\left[\frac{1}{e^\eta}\int_\eta^\infty e^s\mathrm{d}s\right]\mathrm{d}\eta\\ &= \frac{2\varphi_0 e^3}{(\varrho_0+\varphi_0)}\int_{\top_0}^\infty\eta\left[\frac{1}{e^\eta}e^\eta\right]\mathrm{d}\eta=\infty.\end{aligned}$$

Also, by condition (34), we have

$$\begin{aligned}\liminf_{\top\to\infty}\int_{\omega(\zeta(\top))}^{\top}\frac{s^2}{\varsigma(s)}\widetilde{\Phi}_1(s)\mathrm{d}s &= \liminf_{\top\to\infty}\int_{\top-1}^{\top}\frac{(s^2)}{e^s}2e^{\top+3}\mathrm{d}s\\ &= \liminf_{\top\to\infty}2e^3\int_{\top-1}^{\top}s^2\mathrm{d}s = 2e^3[\frac{s^3}{3}]_{\top-1}^{\top}\\ &= \liminf_{\top\to\infty}2e^3\left(\frac{\top^3}{3}-\frac{(\top-1)^3}{3}\right),\end{aligned}$$

that is

$$\liminf_{\top\to\infty}2e^3\left(\frac{\top^3}{3}-\frac{(\top-1)^3}{3}\right) > \frac{(\zeta_0+\varrho_0)}{\omega_0\zeta_0 e}.$$

So, by (35), we have

$$\begin{aligned}\liminf_{\top\to\infty}\int_{\top}^{\omega(\top)}\widetilde{\Phi}_1(s)(s)\psi(s)\mathrm{d}s &= \liminf_{\top\to\infty}\int_{\top}^{\top+2}2e^{s+3}(s)\left(\int_s^{\infty}(1/e^{\eta})\mathrm{d}\eta\right)\mathrm{d}s\\ &= \liminf_{\top\to\infty}\int_{\top}^{\top+2}2e^{s+3}(s)e^{-s}\mathrm{d}s\\ &= \liminf_{\top\to\infty}2e^3\int_{\top}^{\top+2}s\mathrm{d}s\\ &= \liminf_{\top\to\infty}\left(4e^3\top+4e^3\right) = \infty,\end{aligned}$$

that is

$$\liminf_{\top\to\infty}\int_{\top}^{\top+2}2e^{s+3}(s)\left(\int_s^{\infty}(1/e^{\eta})\mathrm{d}\eta\right)\mathrm{d}s > \frac{\zeta_0+\varrho_0}{\omega_0\zeta_0 e}.$$

Therefore, (43) is almost oscillatory.

4. Conclusions

This note presents a new study on the asymptotic properties and oscillatory of a specific class of odd-order advanced differential equations in a noncanonical case. we have obtained a new comparison theorem for deducing the oscillation property of (1) from the oscillation of double first-order differential equations. By providing new relationships to link the solutions of the studied equation to the corresponding function, we have established new and effective criteria for examining whether the solutions of the equation exhibit oscillatory behavior or tend to zero. Our results clearly contribute to enhancing the understanding of the behavior of the solutions of the studied equation and expanding and completing the study found in the previous literature. On the other hand, the possibility of expanding this study remains an inspiring research point for researchers as a direction to benefit from the existing techniques to establish criteria that define the oscillatory behavior of solutions for wider classes of advanced higher-order differential equations of the form

$$\left(\varsigma(\top)\left(\left(x(\top)+\varrho(\top)x(\zeta(\top))\right)^{(\iota-1)}\right)^{\alpha}\right)' + \sum_{i=1}^{m}\Phi(\top)x^{\beta}(\omega_i(\top)) = 0,$$

thereby continuing to enhance progress in this direction.

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