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Dense lineability and spaceability in certain subsets of ℓ_∞

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Abstract

We investigate dense lineability and spaceability of subsets of ℓ_∞ with a prescribed number of accumulation points. We prove that the set of all bounded sequences with exactly countably many accumulation points is densely lineable in ℓ_∞, thus complementing a recent result of Papathanasiou who proved the same for the sequences with continuum many accumulation points. We also prove that these sets are spaceable. We then consider the same problems for the set of bounded non-convergent sequences with a finite number of accumulation points. We prove that such a set is densely lineable in ℓ_∞ and that it is nevertheless not spaceable. The said problems are also studied in the setting of ideal convergence and in the space $\mathbb{R}^\omega$.

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1 | INTRODUCTION

A subset M of a vector space X is said to be *lineable* (resp., *κ-lineable*, for a cardinal κ) if $M \cup \{0\}$ contains a vector space of infinite dimension (resp., of dimension κ). Lineability problems have been investigated in several areas of Mathematical Analysis; we refer to, for example, [1, 2, 6, 11–13, 24, 25] for a rather non-exhaustive list of results. Let us just quote here the seminal result of Gurariy [14] that the set of continuous, nowhere differentiable functions is lineable in $C([0, 1])$. There are several variants and strengthenings of the above definition. If X is a Banach space (or, more generally, a topological vector space), a subset M of X is *spaceable* if $M \cup \{0\}$ contains a closed infinite-dimensional subspace; M is *densely lineable* in X if $M \cup \{0\}$ contains a linear subspace that is dense in X.

A particular case where these properties have been considered in the literature is when the subset M has the form $X \setminus Y$, where Y is a closed subspace of X; in which setting there are simple and complete results, see [5, 18, 27]. In particular, $X \setminus Y$ is spaceable if and only if $X \setminus Y$ is lineable, if and only if Y has infinite codimension (i.e., X/Y is infinite-dimensional) [27]. Moreover, for separable X, these conditions are equivalent to $X \setminus Y$ being densely lineable in X [5]. For non-separable spaces, Papathanasiou [21] very recently proved that $\ell_\infty \setminus c_0$ is densely lineable in ℓ_∞. It is however most unfortunate that his result is actually consequence of [5]; indeed, the very same proof of [5, Theorem 2.5] gives the complete characterisation that $X \setminus Y$ is densely lineable in X if and only if $\dim(X/Y) \geqslant \mathrm{dens}(X)$. For the sake of completeness, we record this result in Corollary 3.2.

Yet, inspection of the proof in [21] gives the following more precise result: there is a dense subspace V of ℓ_∞ such that every non-zero vector in V has exactly continuum many accumulation points. This result was the starting point of our research, as we were pondering lineability results for subsets of ℓ_∞ with a prescribed number of accumulation points (see [3] for some results in a similar direction). Before we can explain our results, it will be convenient to introduce a piece of notation that we shall use extensively throughout the paper. For a vector $x \in \ell_\infty$, we indicate by L_x the set of its accumulation points. If κ is a cardinal number, $\mathrm{L}(\kappa)$ stands for the set of all $x \in \ell_\infty$ that have exactly κ accumulation points; in other words,

$$\mathrm{L}(\kappa) = \{x \in \ell_\infty : |\mathrm{L}_x| = \kappa\}.$$

In this notation, the result in [21] asserts that $\mathrm{L}(\mathfrak{c})$ is densely lineable in ℓ_∞. As it turns out, this more precise version can also be easily derived from [5, Theorem 2.5], as we can write $\mathrm{L}(\mathfrak{c}) = \ell_\infty \setminus Y$ where Y is the linear subspace $\bigcup_{\kappa \leqslant \omega} \mathrm{L}(\kappa)$ (see Remark 3.5). Similarly, we also show that $\mathrm{L}(\omega)$ is densely lineable in ℓ_∞ (Theorem 3.4). Notice that $\mathrm{L}(\kappa) = \emptyset$ for uncountable $\kappa < \mathfrak{c}$, as L_x is a closed set; hence, these results settle the situation for sequences with infinitely many accumulation points. Next, in Theorem 3.6 we prove that the set $\bigcup_{2 \leqslant n < \omega} \mathrm{L}(n)$ (that is, the set of non-convergent sequences with finitely many accumulation points) is also densely lineable in ℓ_∞. We have to exclude $n = 1$ in the above union, as $\bigcup_{1 \leqslant n < \omega} \mathrm{L}(n)$ clearly is a dense subspace of ℓ_∞.

Having answered the problem for what concerns dense lineability, in Section 4 we turn our attention to spaceability of the said sets. Here, the results cannot be derived from the characterisation mentioned in the second paragraph, as the result in [27, Section 6] only works when Y is a closed subspace of X. This assumption is not available in our setting because the linear subspaces that we consider are $\bigcup_{\kappa \leqslant \omega} \mathrm{L}(\kappa)$ and $\bigcup_{\kappa < \omega} \mathrm{L}(\kappa)$ that are both dense in ℓ_∞. Yet, we give a simple direct proof that $\mathrm{L}(\mathfrak{c})$ and $\mathrm{L}(\omega)$ are spaceable (Theorem 4.7). On the other hand, the main result of the section is of negative nature as it asserts that $\bigcup_{2 \leqslant n < \omega} \mathrm{L}(n)$ is not spaceable (Theorem 4.6).

The proof of the latter relies on a result which we consider to be of independent interest: if $A \subseteq \{2, 3, \dots\}$ is a non-empty finite interval, then $\bigcup_{n \in A} \mathrm{L}(n)$ (that is, the set of bounded sequences with a number of accumulations points prescribed by A) is $|A|$-lineable and, in addition, the lineability constant $|A|$ is sharp (Theorem 4.4). This opens the way, in Section 5, to the search of several finer lineability results, in which we show that the lineability of $\bigcup_{n \in A} \mathrm{L}(n)$ is a much harder task when A is not an interval. To wit, we prove that if $A \subseteq \{2, 3, \dots\}$ is a sufficiently 'sparse' infinite set, then $\bigcup_{n \in A} \mathrm{L}(n)$ is not even 2-lineable; for example, the sets $\bigcup_{2 \leqslant n < \omega} \mathrm{L}(n!)$ and $\bigcup_{1 \leqslant n < \omega} \mathrm{L}(3^n)$

are not 2-lineable (Corollary 5.8). On the other hand, it is also possible that an infinite set A contains no non-trivial intervals and yet $\bigcup_{n\in A} \mathrm{L}(n)$ is $\mathfrak{c}$-lineable. Indeed, we prove in Theorem 5.9 that the set $\bigcup_{1\leqslant n<\omega} \mathrm{L}(2n+1)$ is $\mathfrak{c}$-lineable. Finally, in Section 6 we discuss extensions of our results when we replace convergent sequences and accumulation points with ideal convergent sequences and $\mathcal{I}$-cluster points, respectively; we also discuss the same problems in the space $\mathbb{R}^\omega$ with the pointwise topology, instead of ℓ_∞. Finally, we collect some open problems that arise from our research.

2 PRELIMINARIES

Our notation regarding Topology, Functional Analysis and Set Theory is quite standard, as in most textbooks; we refer, for example, to [9, 10, 17] for unexplained notation and terminology. The unique caveat is that by *subspace* of a normed space we understand a linear subspace, not necessarily closed. This will cause no confusion, as we will almost only consider subspaces that are either closed, or dense; when closedness is assumed, it will be stressed explicitly. For $x = (x(n))_{n\in\omega} \in \ell_\infty$ we define $\mathrm{suppt}(x) := \{n \in \omega : x(n) \neq 0\}$. Given a set Γ, $|\Gamma|$ denotes the cardinality of Γ and $\mathcal{P}(\Gamma)$ denotes the collection of all its subsets. We regard cardinal numbers as initial ordinal numbers; in particular, we write ω for the smallest infinite cardinal. The cardinality of continuum is denoted by $\mathfrak{c}$. When A and B are subsets of Γ, we write $A \subseteq^* B$ to mean that $A \setminus B$ is finite; similarly, $A =^* B$ means that the symmetric difference between A and B is finite. $x\restriction_A$ denotes the restriction of the function x to the subset A of its domain. For a subset $A \subseteq \Gamma$ we denote by $\mathbf{1}_A$ the characteristic function of A. A family $\mathscr{I} \subseteq \mathcal{P}(\omega)$ is *independent* if for any distinct sets $X_0, \dots, X_n, Y_0, \dots, Y_m \in \mathscr{I}$

$$X_0 \cap \cdots \cap X_n \setminus (Y_0 \cup \cdots \cup Y_m) \text{ is infinite.}$$

It is well-known that ω contains an independent family of cardinality $\mathfrak{c}$ (see [17, Lemma 7.7]).

Recall that for a sequence $x \in \ell_\infty$ and $\eta \in \mathbb{R}$, η is an *accumulation point* of x if $\{n \in \omega : |x(n) - \eta| < \varepsilon\}$ is infinite for all $\varepsilon > 0$. Let us record explicitly the following notation that we mentioned already in Section 1.

Notation 2.1. For a vector $x \in \ell_\infty$ and a cardinal number κ, we write

$$\mathrm{L}_x := \{\eta \in \mathbb{R} : \eta \text{ is an accumulation point of } x\}$$

$$\mathrm{L}(\kappa) := \{x \in \ell_\infty : |\mathrm{L}_x| = \kappa\}.$$

Given $x, y \in \ell_\infty$ and $\alpha, \beta \in \mathbb{R}$ it is clear that $\mathrm{L}_{\alpha x+\beta y} \subseteq \{\alpha\xi + \beta\eta : \xi \in \mathrm{L}_x, \eta \in \mathrm{L}_y\}$. For sequences with finitely many accumulation points we have the following simple consequence that we shall use several times.

Lemma 2.2. *Let $x \in \mathrm{L}(k)$, $y \in \mathrm{L}(n)$ and $z \in \mathrm{span}\{x, y\}$. Then $|\mathrm{L}_z| \leqslant kn$. Moreover, if $z = \alpha x + \beta y$ where both α and β are different from 0, then*

$$\max\left\{\frac{n}{k}, \frac{k}{n}\right\} \leqslant |\mathrm{L}_z| \leqslant kn.$$

Proof. If $z = \alpha x + \beta y$, then $\mathrm{L}_z \subseteq \{\alpha\xi + \beta\eta : \xi \in \mathrm{L}_x, \eta \in \mathrm{L}_y\}$ gives $|\mathrm{L}_z| \leqslant kn$. For the 'Moreover' part, we can assume that $k \leqslant n$. As $\beta \neq 0$, $y \in \mathrm{span}\{x, z\}$; hence the first part gives $n = |\mathrm{L}_y| \leqslant |\mathrm{L}_z| \cdot k$, and we are done. □

We conclude the section by giving a convenient representation for a sequence with finitely many accumulation points, that we shall use several times in what follows. We denote by $\sim_{c_0}$ the equivalence relation on ℓ_∞ defined by

$$x \sim_{c_0} y \quad \text{if and only if} \quad x - y \in c_0.$$

Lemma 2.3. *Fix $n \in \omega$ and a sequence $x \in \mathrm{L}(n)$. Then there are a partition $\{S_1, \dots, S_n\}$ of ω in infinite sets and mutually distinct scalars $\xi_1, \dots, \xi_n$ such that*

$$x \sim_{c_0} \xi_1 \mathbf{1}_{S_1} + \dots + \xi_n \mathbf{1}_{S_n}. \tag{2.1}$$

Moreover, such a representation is unique up to the order and finite sets. More precisely, if $\eta_1 \mathbf{1}_{T_1} + \dots + \eta_m \mathbf{1}_{T_m}$ is another representation, then $n = m$ and there is a bijection σ of $\{1, \dots, n\}$ such that $\eta_j = \xi_{\sigma(j)}$ and $T_j =^* S_{\sigma(j)}$, for every $j \in \{1, \dots, n\}$.

Remark 2.4. Note that if x admits a representation as in (2.1), then $\mathrm{L}_x = \{\xi_1, \dots, \xi_n\}$ and $\|x\| \geqslant \max\{|\xi_i| : i \in \{1, \dots, n\}\}$. The shortest way to prove the second formula is to realise that $\max\{|\xi_i| : i \in \{1, \dots, n\}\} = \|q(x)\|_{\ell_\infty/c_0} \leqslant \|x\|$, where $q : \ell_\infty \to \ell_\infty/c_0$ is the quotient map.

Proof. Let $\{\xi_1, \dots, \xi_n\}$ be the accumulation points of x and $\{S_1, \dots, S_n\}$ be a partition of ω in infinite sets such that $\lim_{k \in S_i} x(k) = \xi_i$ for every $i \in \{1, \dots, n\}$. Hence, we get $x \sim_{c_0} \xi_1 \mathbf{1}_{S_1} + \dots + \xi_n \mathbf{1}_{S_n}$. Conversely, if x has the representation (2.1), $\mathrm{L}_x = \{\xi_1, \dots, \xi_n\}$; therefore the scalars $\xi_1, \dots, \xi_n$ are uniquely determined up to the order. Suppose that there exists a second partition $\{T_1, \dots, T_n\}$ such that $x \sim_{c_0} \xi_1 \mathbf{1}_{T_1} + \dots + \xi_n \mathbf{1}_{T_n}$. Then, $\xi_1(\mathbf{1}_{S_1} - \mathbf{1}_{T_1}) + \dots + \xi_n(\mathbf{1}_{S_n} - \mathbf{1}_{T_n}) \in c_0$ and it attains finitely many values; hence such a sequence is eventually equal to zero, whence $S_i =^* T_i$ for every $i \in \{1, \dots, n\}$. □

3 | DENSE LINEABILITY

In this section, we prove that $\mathrm{L}(\omega)$ and $\bigcup_{2 \leqslant n < \omega} \mathrm{L}(n)$ are densely lineable in ℓ_∞, thus complementing the result from [21] that $\mathrm{L}(\mathfrak{c})$ is densely lineable in ℓ_∞. As it turns out, both results are consequence of the extension of [5, Theorem 2.5] that we mentioned already in the Introduction. Therefore, to begin with, we recall [5, Theorem 2.5] in its general version. Even though the proof is essentially the same as in [5], we provide a full argument for convenience of the reader. For a topological vector space X, $\mathrm{dens}(X)$ denotes the density character of X and $\dim(X)$ its linear dimension (namely, the cardinality of an algebraic basis). If Y is a linear subspace of X, the *codimension* of Y in X is $\dim(X/Y)$. The weight of a topological space X is denoted by $\mathrm{w}(X)$.

Lemma 3.1. *Let X be a topological vector space and Y be a linear subspace such that $\mathrm{w}(X) \leqslant \dim(X/Y)$. Then $X \setminus Y$ is densely lineable in X.*

Proof. Let $\kappa := \mathrm{w}(X)$ and $\{B_\alpha\}_{\alpha\in\kappa}$ be a topological basis for X. Assume that every B_α is non-empty. We build by transfinite induction vectors $\{x_\alpha\}_{\alpha\in\kappa}$ such that

$$x_\alpha \in B_\alpha \setminus \mathrm{span}(Y \cup \{x_\gamma\}_{\gamma\in\alpha}) \text{ for all } \alpha < \kappa.$$

As $\mathrm{int}(Y) = \emptyset$, there is $x_0 \in B_0 \setminus Y$. Let $\alpha < \kappa$ and suppose, by transfinite induction, that $x_\beta \in B_\beta \setminus \mathrm{span}(Y \cup \{x_\gamma\}_{\gamma\in\beta})$ has been defined for every $\beta < \alpha$. Let $Y_\alpha := \mathrm{span}(Y \cup \{x_\beta\}_{\beta\in\alpha})$. The assumption that Y has codimension at least κ in X gives $Y_\alpha \subsetneq X$, so $\mathrm{int}(Y_\alpha) = \emptyset$. Hence, there is $x_\alpha \in B_\alpha \setminus Y_\alpha$. This shows the existence of the vectors $\{x_\alpha\}_{\alpha\in\kappa}$. The subset $\{x_\alpha\}_{\alpha\in\kappa}$ is dense in X, therefore $V := \mathrm{span}\{x_\alpha\}_{\alpha\in\kappa}$ is dense in X and it is readily seen that $V \cap Y = \{0\}$. □

Corollary 3.2. *Let X be a metrisable infinite-dimensional topological vector space with $\kappa = \mathrm{dens}(X)$ and Y be a linear subspace. Then the following are equivalent.*

(i) *$X \setminus Y$ is densely lineable in X.*
(ii) *$X \setminus Y$ is κ-lineable.*
(iii) *$\kappa \leqslant \dim(X/Y)$.*

Proof. Every metric space X satisfies $\mathrm{dens}(X) = \mathrm{w}(X)$; hence, (iii) ⇒ (i) follows from Lemma 3.1. (i) ⇒ (ii) is obvious. For (ii) ⇒ (iii), take a subspace V of X with $\dim(V) = \kappa$ and such that $V \cap Y = \{0\}$; let $q : X \to X/Y$ be the canonical quotient map. As $q\restriction_V$ is injective, we have $\kappa = \dim(V) = \dim(q[V]) \leqslant \dim(X/Y)$. □

To build a vector space of dimension $\mathfrak{c}$ inside $\mathrm{L}(\omega)$ we shall exploit the 'strong' linear independence of geometric sequences in order to prevent non-trivial linear combinations to have only finitely many accumulation points. Similar uses of geometric sequences can be found in several places in the literature, for example, [8, 15, 16, 19]. For this purpose, we will use the following standard lemma, see, for example, [8, Proposition 2.1]; its proof is so simple that we give it here.

Lemma 3.3. *Let $\lambda_0, \dots, \lambda_n \in (0,1)$ be mutually distinct scalars and let $\beta_0 \dots, \beta_n \in \mathbb{R}$ not all equal to 0. Then the sequence*

$$\left(\beta_0\lambda_0^j + \cdots + \beta_n\lambda_n^j\right)_{j\in\omega}$$

attains each of its values finitely many times. In particular, its range is an infinite set.

Proof. We can assume that $0 < \lambda_0 < \cdots < \lambda_n < 1$ and that $\beta_i \neq 0$ for every $i \in \{0, \dots, n\}$. Moreover, the conclusion is clearly true when $n = 0$, so we assume $n \geqslant 1$. Towards a contradiction, assume that there are a subsequence $(j_k)_{k\in\omega}$ of ω and $\gamma \in \mathbb{R}$ such that

$$\beta_0\lambda_0^{j_k} + \cdots + \beta_n\lambda_n^{j_k} = \gamma \qquad \text{for every } k \in \omega.$$

Letting $k \to \infty$ shows that $\gamma = 0$. Hence, we can divide by $\lambda_n^{j_k}$ to get

$$\beta_0\left(\frac{\lambda_0}{\lambda_n}\right)^{j_k} + \cdots + \beta_{n-1}\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{j_k} = -\beta_n.$$

As $\lambda_i < \lambda_n$ for $i \in \{0, \dots, n-1\}$, letting $k \to \infty$ gives $\beta_n = 0$, a contradiction. □

Theorem 3.4. $\mathsf{L}(\omega)$ *is densely lineable in* ℓ_∞.

Proof. Let $X := \bigcup_{\kappa \leqslant \omega} \mathsf{L}(\kappa)$ and $Y := \bigcup_{\kappa < \omega} \mathsf{L}(\kappa)$. Then X and Y are linear subspaces of ℓ_∞, X is dense in ℓ_∞, and $X \setminus Y = \mathsf{L}(\omega)$. Therefore, if we prove that $\mathsf{L}(\omega)$ is $\mathfrak{c}$-lineable, Corollary 3.2 would yield us that $\mathsf{L}(\omega)$ is densely lineable in X, hence also in ℓ_∞, which would conclude the proof.

To this aim, take disjoint subsets $(B_j)_{j\in\omega}$ of ω such that each B_j is an infinite set. We can now define, for every $q \in (0,1)$, the following vector in ℓ_∞

$$f_q := \sum_{j=0}^{\infty} q^j \mathbf{1}_{B_j}; \tag{3.1}$$

it is sufficient to prove, as we now do, that no linear combination of $\{f_q : q \in (0,1)\}$ with non-zero scalars belongs to Y. For this aim, take mutually distinct $q_0, \dots, q_N \in (0,1)$ and non-zero scalars $d_0, \dots, d_N \in \mathbb{R}$. Then we can write

$$x := \sum_{n=0}^{N} d_n f_{q_n} = \sum_{j=0}^{\infty} \left(\sum_{n=0}^{N} d_n (q_n)^j \right) \mathbf{1}_{B_j}. \tag{3.2}$$

Lemma 3.3 yields us that the sequence $(h_j)_{j\in\omega}$, defined by

$$h_j := \sum_{n=0}^{N} d_n (q_n)^j \tag{3.3}$$

attains infinitely many distinct values. As each value is attained on the corresponding infinite set B_j, it follows that the sequence x admits infinitely many accumulation points. On the other hand, $h_j \to 0$; thus L_x is the countable set

$$\mathsf{L}_x = \{0, h_j\}_{j\in\omega}.$$

Hence, $x \in \mathsf{L}(\omega)$ and we are done. □

Remark 3.5. A small variation of the above proof gives an alternative argument that $\mathsf{L}(\mathfrak{c})$ is densely lineable in ℓ_∞. Indeed, we now consider $X := \ell_\infty$ and $Y := \bigcup_{\kappa \leqslant \omega} \mathsf{L}(\kappa)$ and we only have to show that $X \setminus Y = \mathsf{L}(\mathfrak{c})$ is $\mathfrak{c}$-lineable. Next, for every $j \in \omega$ let $r_j : \omega \to (0,1)$ be a sequence such that $\mathrm{suppt}(r_j) = B_j$ and $\mathsf{L}_{r_j} = [0,1]$. Then replace the vectors f_q given in (3.1) with

$$f_q := \sum_{j=0}^{\infty} q^j r_j \mathbf{1}_{B_j} \qquad (q \in (0,1)).$$

At this point, if x is as in (3.2) (with the extra factor r_j) and h_j is as in (3.3), take $j \in \omega$ with $h_j \neq 0$. Then $x \restriction_{B_j} = h_j r_j \restriction_{B_j} \in \mathsf{L}(\mathfrak{c})$ (as $\mathsf{L}_{r_j} = [0,1]$). Thus, $x \in \mathsf{L}(\mathfrak{c})$, and we are done.

Finally, we cover the case of $\bigcup_{2 \leqslant n < \omega} \mathsf{L}(n)$.

Theorem 3.6. $\bigcup_{2 \leqslant n < \omega} \mathsf{L}(n)$ *is densely lineable in* ℓ_∞.

Proof. In this case, we consider the linear subspaces of ℓ_∞ given by $X := \bigcup_{n<\omega} \mathrm{L}(n)$ and $Y = \mathrm{L}(1) = c$ and, as above, we only need to prove that $\dim(X/Y) = \mathfrak{c}$. This is consequence of the fact that X/c is dense in ℓ_∞/c, whose density character is $\mathfrak{c}$. Alternatively, one can take an independent family $\mathscr{I} \subseteq \mathcal{P}(\omega)$ of cardinality $\mathfrak{c}$; then it is easy to see that $\mathrm{span}\{\mathbf{1}_A : A \in \mathscr{I}\}$ has dimension equal to $\mathfrak{c}$ and $\mathrm{span}\{\mathbf{1}_A : A \in \mathscr{I}\} \cap \mathrm{L}(1) = \{0\}$. □

4 | SPACEABILITY

In this section, we focus on spaceability results for the sets $\bigcup_{2\leqslant n<\omega} \mathrm{L}(n)$, $\mathrm{L}(\omega)$ and $\mathrm{L}(\mathfrak{c})$. The main result is Theorem 4.6 asserting that $\bigcup_{2\leqslant n<\omega} \mathrm{L}(n)$ is not spaceable. A key ingredient in its proof is Theorem 4.4, where we show that the subspace $\mathrm{L}(n) \cup \cdots \cup \mathrm{L}(n+d)$ is $(d+1)$-lineable but not $(d+2)$-lineable. As a complement to this, we conclude the section with the easy result that $\mathrm{L}(\omega)$ and $\mathrm{L}(\mathfrak{c})$ are spaceable.

The basic idea for the proof of Theorem 4.4 consists in finding certain linear combinations of vectors in a way to suitably increase or decrease the number of accumulation points. This will be achieved by means of the following lemmas. The first one will allow us to reduce the number of accumulation points as much as possible; the second asserts that small perturbations can't decrease the number of accumulation points; the last one claims that if no linear combination of two vectors increases the number of accumulation points, then the partitions associated to the vectors as in Lemma 2.3 must be one finer than the other (modulo finite sets).

Lemma 4.1. *Let $x_1, \dots, x_n \in \mathbb{R}^n$. Then there are scalars $c_1, \dots, c_n \in \mathbb{R}$, not all equal to zero, and $\gamma \in \mathbb{R}$ such that*

$$c_1 x_1 + \cdots + c_n x_n = \gamma(1, \dots, 1).$$

Proof. If the vectors $x_1, \dots, x_n$ are linearly independent, their linear span is $\mathbb{R}^n$, so there exists a linear combination that equals $(1, \dots, 1)$. In the case they are linearly dependent, then there exists a non-trivial linear combination of them that gives $(0, \dots, 0)$. □

Lemma 4.2. *Let $x \in \ell_\infty$ be a sequence with $|\mathrm{L}_x| < \infty$. There is $\varepsilon > 0$ such that for all vectors $y \in \ell_\infty$ with $|\mathrm{L}_y| < \infty$ and $\|y\| < \varepsilon$,*

$$|\mathrm{L}_{x+y}| \geqslant \max\{|\mathrm{L}_x|, |\mathrm{L}_y|\}.$$

Proof. As L_x is a finite set, we may take $\varepsilon > 0$ such that L_x is a 2ε-separated set (i.e., $|\alpha - \beta| \geqslant 2\varepsilon$ for distinct $\alpha, \beta \in \mathrm{L}_x$). Now take any $y \in \ell_\infty$ with $|\mathrm{L}_y| < \infty$ and $\|y\| < \varepsilon$. According to Lemma 2.3, we can write

$$x \sim_{c_0} \xi_1 \mathbf{1}_{S_1} + \cdots + \xi_n \mathbf{1}_{S_n} \qquad \text{and} \qquad y \sim_{c_0} \eta_1 \mathbf{1}_{T_1} + \cdots + \eta_k \mathbf{1}_{T_k}.$$

To check that $|\mathrm{L}_{x+y}| \geqslant |\mathrm{L}_y| = k$, fix $i \in \{1, \dots, k\}$ and take $j_i \in \{1, \dots, n\}$ such that $T_i \cap S_{j_i}$ is infinite. Therefore, $\eta_i + \xi_{j_i}$ is an accumulation point of $x + y$. Hence, if by contradiction $|\mathrm{L}_{x+y}| < |\mathrm{L}_y| = k$, there must be distinct indices $i, l \in \{1, \dots, k\}$ such that $\eta_i + \xi_{j_i} = \eta_l + \xi_{j_l}$. If $j_i = j_l$, we get the absurd that $\eta_i = \eta_l$. On the other hand, if $j_i \neq j_l$, then $2\varepsilon \leqslant |\xi_{j_i} - \xi_{j_l}| = |\eta_i - \eta_l| \leqslant 2\|y\| < 2\varepsilon$, a contradiction. The proof that $|\mathrm{L}_{x+y}| \geqslant |\mathrm{L}_x|$ is similar (starting with $i \in \{1, \dots, n\}$), therefore we omit it. □

Lemma 4.3. *Assume $x \in \mathrm{L}(n)$ and $y \in \mathrm{L}(k)$ have the representation*

$$x \sim_{c_0} \xi_1 \mathbf{1}_{S_1} + \cdots + \xi_n \mathbf{1}_{S_n} \qquad \text{and} \qquad y \sim_{c_0} \eta_1 \mathbf{1}_{T_1} + \cdots + \eta_k \mathbf{1}_{T_k},$$

as in Lemma 2.3. Suppose also that $n \leqslant k$ and that every $z \in \operatorname{span}\{x, y\}$ satisfies $|\mathrm{L}_z| \leqslant k$. Then for every $i \in \{1, \dots, k\}$, there exists $j \in \{1, \dots, n\}$ such that $T_i \subseteq^* S_j$.

Proof. Suppose by contradiction that there is $i \in \{1, \dots, k\}$ such that $T_i \not\subseteq^* S_j$ for every $j \in \{1, \dots, n\}$. Then there are two distinct indices $j_1, j_2 \in \{1, \dots, n\}$ such that $T_i \cap S_{j_1}$ and $T_i \cap S_{j_2}$ are both infinite. According to Lemma 4.2, for sufficiently small $\varepsilon > 0$, $(x + \varepsilon y)\restriction_{T_i}$ has at least two accumulation points (as ξ_{j_1}, ξ_{j_2} are accumulation points of $x\restriction_{T_i}$) and $(x + \varepsilon y)\restriction_{\omega\setminus T_i}$ has at least $k - 1$ accumulation points ($y\restriction_{\omega\setminus T_i}$ has $k - 1$ accumulation points). Moreover, for small $\varepsilon > 0$, the sets of accumulation points of the elements $(x + \varepsilon y)\restriction_{T_i}$ and $(x + \varepsilon y)\restriction_{\omega\setminus T_i}$ are disjoint. Thus, $x + \varepsilon y$ has at least $k + 1$ accumulation points, and we are done. □

We are now ready for the first main result of the section.

Theorem 4.4. *Let $n, d \in \omega$ be such that $n \geqslant 2$. Then $\mathrm{L}(n) \cup \cdots \cup \mathrm{L}(n + d)$ is $(d + 1)$-lineable, but not $(d + 2)$-lineable.*

Proof. We start by showing that $\mathrm{L}(n) \cup \cdots \cup \mathrm{L}(n + d)$ is $(d + 1)$-lineable. We claim that there are vectors $v_1, \dots, v_{n+d} \in \mathbb{R}^{d+1}$ such that, for all non-zero $\alpha := (\alpha_0, \dots, \alpha_d) \in \mathbb{R}^{d+1}$ the set $\{\alpha \cdot v_j\}_{j=1}^{n+d}$ has cardinality at least n ($\alpha \cdot v_j$ is the inner product of the vectors α and v_j in $\mathbb{R}^{d+1}$). As we didn't find a short proof of this claim, we decided to postpone its proof until Proposition 5.1. So, assuming the validity of the claim for now, take vectors $v_1, \dots, v_{n+d} \in \mathbb{R}^{d+1}$ as above and let $(B_j)_{j=1}^{n+d}$ be a partition of ω into infinite sets. For $k \in \{0, \dots, d\}$, define the vector

$$e_k := \sum_{j=1}^{n+d} \mathbf{1}_{B_j} v_j(k)$$

and let $V := \operatorname{span}\{e_k\}_{k=0}^{d}$. As $V \subseteq \operatorname{span}\{\mathbf{1}_{B_1}, \dots, \mathbf{1}_{B_{n+d}}\}$ and the sets B_j are disjoint and infinite, it follows that every vector in V has at most $n + d$ accumulation points. Thus, we only need to prove that every non-zero vector in V has at least n accumulation points. Take scalars $\alpha_0, \dots, \alpha_d$, not all equal to 0, and note that

$$\sum_{k=0}^{d} \alpha_k e_k = \sum_{j=1}^{n+d} \left(\sum_{k=0}^{d} \alpha_k v_j(k) \right) \mathbf{1}_{B_j} = \sum_{j=1}^{n+d} \alpha \cdot v_j \mathbf{1}_{B_j}.$$

Once more, the fact that the sets B_j are disjoint and infinite yields that the accumulation points of $\sum_{k=0}^{d} \alpha_k e_k$ are exactly

$$\{\alpha \cdot v_j\}_{j=1}^{n+d}.$$

By our assumption, such a set has cardinality at least n, as desired.

Next, we shall show that $\mathrm{L}(n) \cup \cdots \cup \mathrm{L}(n+d)$ is not $(d+2)$-lineable. Therefore, we fix $n \geqslant 2$ and $d \in \omega$ and assume, towards a contradiction, that V is a vector space of dimension $d+2$ and $V \subseteq \mathrm{L}(n) \cup \cdots \cup \mathrm{L}(n+d) \cup \{0\}$. Define $N \in \omega$ to be

$$N := \max\{|\mathrm{L}_x| : x \in V\};$$

our assumption yields that $N \leqslant n+d$. Moreover, we can select $e_1 \in V \cap \mathrm{L}(N)$; hence we can find a basis $\{e_1, \tilde{e}_2, \dots, \tilde{e}_{d+2}\}$ of V that contains e_1. For $\varepsilon > 0$ sufficiently small, the vectors $e_k := \tilde{e}_k + \varepsilon e_1$ $(k = 2, \dots, d+2)$ belong to $\mathrm{L}(N)$: indeed, on the one hand, $|\mathrm{L}_{e_k}| \geqslant |\mathrm{L}_{e_1}| = N$ by Lemma 4.2 and, on the other one, $|\mathrm{L}_{e_k}| \leqslant N$ by definition of N. Consequently, the set $\{e_1, \dots, e_{d+2}\}$ forms a basis of V and each e_k belongs to $\mathrm{L}(N)$.

Lemma 2.3 allows us to write

$$e_1 \sim_{c_0} \xi_1 \mathbf{1}_{S_1} + \cdots + \xi_N \mathbf{1}_{S_N} \qquad \text{and} \qquad e_2 \sim_{c_0} \eta_1 \mathbf{1}_{T_1} + \cdots + \eta_N \mathbf{1}_{T_N}.$$

As every vector in the linear span of $\{e_1, e_2\}$ has at most N accumulation points, an appeal to Lemma 4.3 assures us that for every $i \in \{1, \dots, N\}$ there is $j_i \in \{1, \dots, N\}$ such that $S_i \subseteq^* T_{j_i}$. $\{S_1, \dots, S_N\}$ being a partition, we conclude that indeed $S_i =^* T_{j_i}$. Up to a permutation in the representation of e_2, we can assume that $S_i =^* T_i$ for every $i \in \{1, \dots, N\}$. If we repeat the same argument with e_1 and e_k for every $k \in \{3, \dots, d+2\}$, we obtain, for every $k \in \{1, \dots, d+2\}$ mutually distinct scalars $\xi_1(k), \dots, \xi_N(k)$ such that

$$e_k \sim_{c_0} \xi_1(k) \mathbf{1}_{S_1} + \cdots + \xi_N(k) \mathbf{1}_{S_N}.$$

Before we continue, let us observe that necessarily $N \geqslant d+2$. Indeed, if not, the vectors $\{\xi_1(k)\mathbf{1}_{S_1} + \cdots + \xi_N(k)\mathbf{1}_{S_N}\}_{k=1}^{d+2}$ would be linearly dependent, so there would exist scalars $\alpha_1, \dots, \alpha_{d+2}$, not all equal to zero and such that the corresponding linear combination of the vectors $\{\xi_1(k)\mathbf{1}_{S_1} + \cdots + \xi_N(k)\mathbf{1}_{S_N}\}_{k=1}^{d+2}$ would be equal to 0. Hence, $\alpha_1 e_1 + \cdots + \alpha_{d+2} e_{d+2} \in c_0$, a contradiction (note that the vector $\alpha_1 e_1 + \cdots + \alpha_{d+2} e_{d+2}$ cannot be equal to 0, as the vectors e_k are linearly independent by construction).

As $N \geqslant d+2$, we can consider the vectors

$$\big(\xi_1(k), \dots, \xi_{d+2}(k)\big) \in \mathbb{R}^{d+2} \qquad (k \in \{1, \dots, d+2\})$$

and apply Lemma 4.1 to them. This yields us scalars $\alpha_1, \dots, \alpha_{d+2} \in \mathbb{R}$, not all equal to zero, and $\gamma \in \mathbb{R}$ such that

$$\sum_{k=1}^{d+2} \alpha_k \big(\xi_1(k), \dots, \xi_{d+2}(k)\big) = \gamma(1, \dots, 1).$$

Consequently, we have

$$\sum_{k=1}^{d+2} \alpha_k e_k \sim_{c_0} \gamma \mathbf{1}_{S_1 \cup \cdots \cup S_{d+2}} + \left(\sum_{k=1}^{d+2} \alpha_k \xi_{d+3}(k)\right) \mathbf{1}_{S_{d+3}} + \cdots + \left(\sum_{k=1}^{d+2} \alpha_k \xi_N(k)\right) \mathbf{1}_{S_N}.$$

From this equation, we conclude that the non-zero vector $\sum_{k=1}^{d+2} \alpha_k e_k$ (let us recall that $\{e_1, \dots, e_{d+2}\}$ is a linear basis of V) has at most $(N-d-1)$ accumulation points. Finally, recalling that $N \leqslant n+d$, we reach the contradiction that $\sum_{k=1}^{d+2} \alpha_k e_k$ has at most $(n-1)$ accumulation points. □

As a particular case we have the following result. Note that it directly yields that the set $\bigcup_{n\in A} \mathrm{L}(n)$ is never lineable, when A is finite. Also, in case the set A is not an interval, this corollary might fail to be sharp (see Corollary 5.7).

Corollary 4.5. *Let A be a non-empty finite subset of ω such that $\min A \geqslant 2$. Then $\bigcup_{n\in A} \mathrm{L}(n)$ is not $(\mathrm{diam}(A)+2)$-lineable, where $\mathrm{diam}(A) := \max A - \min A$.*

We are now finally in position to pass to spaceability results. We first give the main result of the section concerning $\bigcup_{2\leqslant n<\omega} \mathrm{L}(n)$ and we then conclude the section with the simpler result for $\mathrm{L}(\omega)$ and $\mathrm{L}(\mathfrak{c})$.

Theorem 4.6. *$\bigcup_{2\leqslant n<\omega} \mathrm{L}(n)$ is not spaceable in ℓ_∞.*

Proof. Towards a contradiction, assume that there is a closed, infinite-dimensional subspace Y of ℓ_∞ such that $Y \subseteq \bigcup_{2\leqslant n<\omega} \mathrm{L}(n) \cup \{0\}$. According to Theorem 4.4, Y is contained in $\bigcup_{2\leqslant n\leqslant N} \mathrm{L}(n) \cup \{0\}$ for no $N \in \omega$, so $Y \cap \mathrm{L}(n)$ is non-empty for infinitely many $n \in \omega$. We shall build by induction a sequence $(\varepsilon_k)_{k\in\omega}$ of positive scalars with $\varepsilon_{k+1} \leqslant \frac{1}{2}\varepsilon_k$ for every $k \in \omega$, a sequence $(y_k)_{k\in\omega}$ of unit vectors in Y, and a strictly increasing sequence $(N_k)_{k\in\omega}$ of natural numbers, with the following properties (for every $k \in \omega$).

(i) $\varepsilon_0 y_0 + \cdots + \varepsilon_k y_k \in \mathrm{L}(N_k)$.
(ii) $\varepsilon_0 y_0 + \cdots + \varepsilon_k y_k + y \in \bigcup_{N_k\leqslant n<\omega} \mathrm{L}(n)$ for every $y \in Y$ with $\|y\| \leqslant 2\varepsilon_{k+1}$.

Indeed, to start the induction, we set $\varepsilon_0 := 1$, we take any unit vector $y_0 \in Y$ and we set $N_0 := |\mathrm{L}_{y_0}|$. Assuming inductively to have already found $(\varepsilon_j)_{j\leqslant k}$, $(y_j)_{j\leqslant k}$ and $(N_j)_{j\leqslant k}$ as above, we apply Lemma 4.2 to the vector $\varepsilon_0 y_0 + \cdots + \varepsilon_k y_k$ and we find ε_{k+1} such that $\varepsilon_0 y_0 + \cdots + \varepsilon_k y_k + y$ has at least $\max\{N_k, |\mathrm{L}_y|\}$ accumulation points for every $y \in Y$ with $\|y\| \leqslant 2\varepsilon_{k+1}$; clearly, we can also assume $2\varepsilon_{k+1} \leqslant \varepsilon_k$. As $Y \cap \mathrm{L}(n) \neq \emptyset$ for infinitely many $n \in \omega$, we are now in position to take a unit vector $y_{k+1} \in Y$ with $|\mathrm{L}_{y_{k+1}}| > N_k$. By Lemma 4.2, the cardinality of the accumulation points of $\varepsilon_0 y_0 + \cdots + \varepsilon_{k+1} y_{k+1}$, which we denote by N_{k+1}, is greater than N_k. This concludes the induction step.

Finally, as Y is closed, $y := \sum_{k=0}^{\infty} \varepsilon_k y_k \in Y$. However, for every $k \in \omega$ we have

$$\left\| \sum_{j=k+1}^{\infty} \varepsilon_j y_j \right\| \leqslant \sum_{j=k+1}^{\infty} \varepsilon_j \leqslant \sum_{j=0}^{\infty} 2^{-j} \varepsilon_{k+1} = 2\varepsilon_{k+1}.$$

Hence, if we write

$$y = \varepsilon_0 y_0 + \cdots + \varepsilon_k y_k + \sum_{j=k+1}^{\infty} \varepsilon_j y_j,$$

we see from (ii) that $|\mathrm{L}_y| \geqslant N_k$. As $k \in \omega$ was arbitrary and $N_k \to \infty$ when $k \to \infty$, we conclude that $y \notin \bigcup_{2\leqslant n<\omega} \mathrm{L}(n)$, a contradiction. □

Theorem 4.7. *$\mathrm{L}(\omega)$ and $\mathrm{L}(\mathfrak{c})$ are spaceable in ℓ_∞. More precisely, $\mathrm{L}(\omega) \cup \{0\}$ contains c_0 isometrically and $\mathrm{L}(\mathfrak{c}) \cup \{0\}$ contains ℓ_∞ isometrically.*

Proof. We first consider the case of $\mathrm{L}(\omega)$. Let $(A_{n,k})_{n,k\in\omega}$ be a partition of ω into infinite sets and define the vectors

$$e_n := \sum_{k=0}^{\infty} a_k \cdot \mathbf{1}_{A_{n,k}}, \tag{4.1}$$

where $a_k = 2^{-k}$ for $k \in \omega$. Each e_n is a unit vector and $e_n \in \mathrm{L}(\omega)$ for each $n \in \omega$. Moreover, $\mathrm{suppt}(e_n) = \bigcup_{k\in\omega} A_{n,k}$, hence the vectors e_n are disjointly supported. Thus the map $(\alpha_n)_{n\in\omega} \mapsto \sum_{n=0}^{\infty} \alpha_n e_n$ is an isometry from c_0 onto $Y := \overline{\mathrm{span}}\{e_n\}_{n\in\omega}$ and each non-zero element of Y belongs to $\mathrm{L}(\omega)$. Indeed, if $x := \sum_{n=0}^{\infty} \alpha_n e_n \in Y$, then $\mathrm{L}_x = \{\alpha_n \cdot a_k\}_{n,k\in\omega} \cup \{0\}$ (as both α_n and a_k tend to 0). If additionally $x \in Y \setminus \{0\}$, then some α_n is non-zero, whence $|\mathrm{L}_x| = \omega$, as desired.

For the case of $\mathrm{L}(\mathfrak{c})$, we replace the sequence $a_k = 2^{-k}$ with an enumeration $(a_k)_{k\in\omega}$ of the rationals in (0,1). The definition of the vectors e_n is the same with the unique difference that the series defining e_n only converges in the pointwise topology. Now the subspace Y is defined as

$$Y := \left\{ \sum_{n=1}^{\infty} \alpha_n e_n : (\alpha_n)_{n\in\omega} \in \ell_\infty \right\}$$

(where, as before, the series converges pointwise). As the vectors e_n are disjointly supported unit vectors, the map $(\alpha_n)_{n\in\omega} \mapsto \sum_{n=1}^{\infty} \alpha_n e_n$ defines an isometry of ℓ_∞ onto Y. Finally, as before, we see that $Y \setminus \{0\} \subseteq \mathrm{L}(\mathfrak{c})$, as here $\mathrm{L}_{e_n} = [0,1]$. □

5 | FINER LINEABILITY RESULTS

In this section, we delve deeper into lineability results for the set $\bigcup_{n\in A} \mathrm{L}(n)$, where A is a (finite) subset of ω such that $\min A \geqslant 2$. In the first result we prove Proposition 5.1, whose validity was claimed during the proof of Theorem 4.4. Next, we give some results that show how more complicated the situation is when A is not an interval. In particular, there are infinite sets A such that $\bigcup_{n\in A} \mathrm{L}(n)$ is not 2-lineable (Corollary 5.8) and, on the other hand, there are sets A that do not contain non-trivial intervals and such that $\bigcup_{n\in A} \mathrm{L}(n)$ is $\mathfrak{c}$-lineable (Theorem 5.9).

Proposition 5.1. *Let $n, d \in \omega$ with $n \geqslant 1$. Then there are vectors $\{v_1, \dots, v_{n+d}\} \in \mathbb{R}^{d+1}$ such that, for all non-zero $\alpha := (\alpha_0, \dots, \alpha_d) \in \mathbb{R}^{d+1}$, the set $\{\alpha \cdot v_k\}_{k=1}^{n+d}$ has cardinality at least n.*

We recall that we indicate by $\alpha \cdot v$ the inner product of the vectors $\alpha, v \in \mathbb{R}^{d+1}$. Note that, by Lemma 4.1, there is a non-zero $\alpha := (\alpha_0, \dots, \alpha_d) \in \mathbb{R}^{d+1}$ such that $\{\alpha \cdot v_k\}_{k=1}^{d+1}$ is a singleton. Hence, for such α the set $\{\alpha \cdot v_k\}_{k=1}^{n+d}$ has cardinality at most n, so the above result is sharp.

Proof. The result is trivial for $n = 1$, thus we assume that $n \geqslant 2$. We begin by introducing a piece of notation. Assume that $\mathcal{V} = \{v_1, \dots, v_k\}$ (where $k \geqslant 1$) are vectors in $\mathbb{R}^{d+1}$ and $\{\mathcal{V}_1, \dots, \mathcal{V}_{n-1}\}$ is a partition of $\mathcal{V}$ in exactly $n-1$, possibly empty, sets. For every $j \in \{1, \dots, n-1\}$ such that $\mathcal{V}_j$ is non-empty we define a vector $w_j \in \mathcal{V}_j$ to be $w_j := v_i$, where i is the least index with $v_i \in \mathcal{V}_j$. Roughly speaking, w_j is the 'first' vector in $\mathcal{V}_j$. Moreover, we define sets

$$\mathcal{W}_j := \{v - w_j : v \in \mathcal{V}_j \setminus \{w_j\}\}, \quad \text{when} \quad \mathcal{V}_j \neq \emptyset$$

and $\mathcal{W}_j = \emptyset$ otherwise. Finally, we say that the set $\mathcal{V} = \{v_1, \dots, v_k\}$ has the *many increments property* (MIP, for short) if for every partition $\mathscr{V} = \{\mathcal{V}_1, \dots, \mathcal{V}_{n-1}\}$ of $\mathcal{V}$:

(MIP1) the sets $\{\mathcal{W}_j\}_{j=1}^{n-1}$ are pairwise disjoint, and

(MIP2) setting $\mathcal{W} := \bigcup_{j=1}^{n-1} \mathcal{W}_j$, span $\mathcal{W}$ has dimension at least $\min\{|\mathcal{W}|, d+1\}$.

Claim 5.2. There exists a family $\mathcal{V} = \{v_1, \dots, v_{n+d}\} \subseteq \mathbb{R}^{d+1}$ consisting of mutually distinct vectors and having property (MIP).

Assuming the claim for now, let us show that a family as in the claim also verifies the conclusion of the proposition. In fact, given such a $\mathcal{V}$, for any partition $\mathscr{V} = \{\mathcal{V}_1, \dots, \mathcal{V}_{n-1}\}$ of $\mathcal{V}$, the above set $\mathcal{W}$ satisfies span$(\mathcal{W}) = \mathbb{R}^{d+1}$. Indeed, by (MIP1)

$$|\mathcal{W}| = \sum_{j=1}^{n-1} \left|\mathcal{W}_j\right| \geqslant \sum_{j=1}^{n-1} (|\mathcal{V}_j| - 1) = |\mathcal{V}| - (n-1) = d+1.$$

So, span$(\mathcal{W})$ has dimension $d+1$ by (MIP2). Suppose now that $\alpha := (\alpha_0, \dots, \alpha_d) \in \mathbb{R}^{d+1}$ is such that $\{\alpha \cdot v_k\}_{k=1}^{n+d}$ has cardinality at most $n-1$. Then there is a partition $\{\mathcal{V}_1, \dots, \mathcal{V}_{n-1}\}$ of $\mathcal{V}$ such that $\{\alpha \cdot v : v \in \mathcal{V}_j\}$ is at most a singleton for every $j \in \{1, \dots, n-1\}$ (in order to have exactly $n-1$ elements in the partition, some $\mathcal{V}_j$ might be empty). But this means that

$$\alpha \cdot (v - w_j) = 0 \text{ for all } j \text{ such that } \mathcal{V}_j \neq \emptyset \text{ and all } v \in \mathcal{V}_j \setminus \{w_j\}.$$

In other words, α is orthogonal to all the vectors in $\mathcal{W}$. Therefore, α is orthogonal to span$(\mathcal{W})$, which by our construction is equal to $\mathbb{R}^{d+1}$; thus $\alpha = 0$, as desired.

Therefore, we only need to prove Claim 5.2 and we build the vectors $\{v_1, \dots, v_{n+d}\}$ recursively (recall that n and d are fixed). Set $v_1 := 0$ and note that, up to relabelling, the unique partition $\{\mathcal{V}_1, \dots, \mathcal{V}_{n-1}\}$ of $\{v_1\}$ is given by $\mathcal{V}_1 = \{v_1\}$ and $\mathcal{V}_2 = \cdots = \mathcal{V}_{n-1} = \emptyset$. Hence, $\mathcal{W}_j = \emptyset$ for every j, so the singleton $\{v_1\}$ satisfies (MIP). Suppose now that, for some $k \leqslant n+d-1$, we have already found vectors $\{v_1, \dots, v_k\}$ satisfying property (MIP). We now look for conditions on v_{k+1} so that the property (MIP) holds also for $\{v_1, \dots, v_{k+1}\}$. First of all, we need $v_{k+1} \notin \{v_1, \dots, v_k\}$. Next, assume that $\mathscr{V} = \{\mathcal{V}_1, \dots, \mathcal{V}_{n-1}\}$ is a partition of $\{v_1, \dots, v_{k+1}\}$ and, up to relabelling the indices of the partition, that $v_{k+1} \in \mathcal{V}_{n-1}$. If $\mathcal{V}_{n-1} = \{v_{k+1}\}$, then $\mathcal{W}_{n-1} = \emptyset$, so (MIP1) and (MIP2) are satisfied because of the inductive assumption applied to the partition $\{\mathcal{V}_1, \dots, \mathcal{V}_{n-2}, \emptyset\}$ of $\{v_1, \dots, v_k\}$.

Therefore, we assume that $\mathcal{V}_{n-1}$ is not a singleton, whence $w_{n-1} \neq v_{k+1}$, by definition of w_{n-1}. To satisfy condition (MIP1) for the partition $\mathscr{V}$, the vector $v_{k+1} - w_{n-1}$ should not belong to $\mathcal{W}_j$ for every $j \in \{1, \dots, n-2\}$. As there are only finitely many partitions $\{\mathcal{V}_1, \dots, \mathcal{V}_{n-1}\}$ of $\{v_1, \dots, v_{k+1}\}$, we conclude that v_{k+1} must be chosen outside a finite subset of $\mathbb{R}^{d+1}$. To verify (MIP2), we distinguish two cases. If the vectors in

$$\mathcal{W}_* := \bigcup_{j=1}^{n-2} \mathcal{W}_j \cup \left\{v - w_{n-1} : v \in \mathcal{V}_{n-1} \setminus \{w_{n-1}, v_{k+1}\}\right\} \tag{5.1}$$

are at least $d+1$ in number, then their linear span has dimension at least $d+1$, by the (MIP2) property of $\{v_1, \dots, v_k\}$. *A fortiori*, span $\mathcal{W}$ has dimension at least $d+1$, so no condition is imposed on v_{k+1}.

Otherwise, suppose that $\mathcal{W}_*$ has cardinality at most d. Therefore, the linear span of $\mathcal{W}_*$ is a proper subspace H of $\mathbb{R}^{d+1}$, of dimension exactly $|\mathcal{W}_*|$ by (MIP2). Moreover, $\mathcal{W} = \mathcal{W}_* \cup \{v_{k+1} - w_{n-1}\}$. Hence, the vectors $\{v_1, \dots, v_{k+1}\}$ satisfy (MIP2) if and only if $v_{k+1} - w_{n-1}$ is linearly independent from H. In other words, if and only if v_{k+1} does not belong to the proper affine subspace $w_{n-1} + H$. Consequently, as there are only finitely many partitions $\{\mathcal{V}_1, \dots, \mathcal{V}_{n-1}\}$ of $\{v_1, \dots, v_{k+1}\}$, then the vector v_{k+1} must be chosen outside finitely many proper affine subspaces of $\mathbb{R}^{d+1}$. This yields that it is possible to select $v_{k+1} \notin \{v_1, \dots, v_k\}$ such that $\{v_1, \dots, v_{k+1}\}$ satisfies property (MIP) and concludes the proof. □

For the second part of the section, it will be convenient to introduce the following notation. For each non-empty set $A \subseteq \omega$ with $\min A \geqslant 2$, define

$$\ell(A) := \sup \left\{ m \in \omega : \bigcup_{n \in A} \mathrm{L}(n) \text{ is } m\text{-lineable} \right\}.$$

Note that Theorem 4.4 can be equivalently rewritten as $\ell(A) = |A|$ whenever $A \subseteq \omega$ is a finite non-empty interval with $\min A \geqslant 2$. The same theorem also implies

$$\ell(A) \geqslant \sup\{|I| : I \subseteq A \text{ is an interval}\} \tag{5.2}$$

whenever $A \subseteq \omega$ is a non-empty set with $\min A \geqslant 2$. We will see in the forthcoming results that, when A is not an interval, the inequality (5.2) can be very far from being sharp. Indeed, we will see in Theorem 5.9 that there exist sets A with $\ell(A) = \infty$ and which contain no non-trivial intervals. Before this, we prove the existence of infinite sets A such that $\ell(A) = 1$ (see Corollary 5.8).

Lemma 5.3. *Fix vectors $x \in \mathrm{L}(n)$ and $y \in \mathrm{L}(k)$, for some $n, k \in \omega$, with representations*

$$x \sim_{c_0} \xi_1 \mathbf{1}_{S_1} + \dots + \xi_n \mathbf{1}_{S_n} \quad \text{and} \quad y \sim_{c_0} \eta_1 \mathbf{1}_{T_1} + \dots + \eta_k \mathbf{1}_{T_k},$$

respectively, as in Lemma 2.3. Define

$$\mathcal{E} := \left\{ (i,j) \in \{1, \dots, n\} \times \{1, \dots, k\} : S_i \cap T_j \text{ is infinite} \right\}$$

and suppose that the points in

$$\mathcal{P} := \left\{ (\xi_i, \eta_j) : (i,j) \in \mathcal{E} \right\}$$

are not collinear. Then there is $z \in \operatorname{span}\{x, y\}$ such that

$$\left\lfloor \frac{|\mathcal{E}| + 1}{2} \right\rfloor \leqslant |\mathrm{L}_z| \leqslant |\mathcal{E}| - 1.$$

Proof. We begin with the following combinatorial observation. Let $\mathcal{P}$ be a set of m non-collinear points in the plane. Then there exists a line ℓ, determined by at least two points in $\mathcal{P}$, such that if $\mathcal{L}$ is a set of parallel lines to ℓ and $\mathcal{P} \subseteq \mathcal{L}$, then $|\mathcal{L}| \geqslant \lfloor \frac{m+1}{2} \rfloor$. Indeed, according to [23], there are a line ℓ, determined by at least two points of $\mathcal{P}$, and $\lfloor \frac{m-1}{2} \rfloor$ points in $\mathcal{P}$ whose distances from ℓ are positive and mutually distinct (see the first sentence in [23, Section 2]). Such points necessarily belong to mutually distinct lines from $\mathscr{L} \setminus \{\ell\}$, so $|\mathscr{L}| \geqslant \lfloor \frac{m-1}{2} \rfloor + 1 = \lfloor \frac{m+1}{2} \rfloor$.

Now let $\mathcal{P} := \{(\xi_i, \eta_j) : (i,j) \in \mathcal{E}\}$ and note that $|\mathcal{P}| = |\mathcal{E}|$, as the ξ_i's and the η_j's are mutually distinct. Let ℓ be a line as in the observation above; then there are scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\ell = \{(\xi, \eta) \in \mathbb{R}^2 : \alpha\xi + \beta\eta = \gamma\}$. Therefore,

$$\left\lfloor \frac{|\mathcal{E}|+1}{2} \right\rfloor \leqslant \left|\left\{\alpha\xi_i + \beta\eta_j : (i,j) \in \mathcal{E}\right\}\right| \leqslant |\mathcal{E}| - 1,$$

the right-hand side inequality being true because two distinct points of $\mathcal{P}$ belong to ℓ. The conclusion follows observing that $\mathrm{L}_{\alpha x + \beta y} = \{\alpha\xi_i + \beta\eta_j : (i,j) \in \mathcal{E}\}$. □

Proposition 5.4. *Fix a non-empty finite set $A \subseteq \omega$ with $\min A \geqslant 2$ and fix $k \in \omega$ such that $k > 2\max A$. Then $\ell(A \cup \{k\}) = \ell(A)$.*

Proof. Assume, towards a contradiction, that $\ell(A) < \ell(A \cup \{k\})$ and take a subspace V of $\bigcup_{i \in A \cup \{k\}} \mathrm{L}(i) \cup \{0\}$ of dimension $\ell(A) + 1$. By definition of $\ell(A)$, there exists a vector $y \in V \cap \mathrm{L}(k)$. Moreover, as $\ell(A) + 1 \geqslant 2$ and $\mathrm{L}(k)$ is not 2-lineable by Theorem 4.4, we can take a vector $x \in V$ such that $|\mathrm{L}_x| \in A$. Hence, letting $M := \max A$, we have $|\mathrm{L}_x| \leqslant M$. Thanks to Lemma 2.3, we have the representations

$$x \sim_{c_0} \xi_1 \mathbf{1}_{S_1} + \cdots + \xi_n \mathbf{1}_{S_n} \quad \text{and} \quad y \sim_{c_0} \eta_1 \mathbf{1}_{T_1} + \cdots + \eta_k \mathbf{1}_{T_k}$$

(here $n := |\mathrm{L}_x| \leqslant M$).

Consider the sets $\mathcal{E}$ and $\mathcal{P}$ corresponding to x and y as in the statement of Lemma 5.3. If all points in $\mathcal{P}$ belong to the same line $\{(\xi, \eta) \in \mathbb{R}^2 : \alpha\xi + \beta\eta = \gamma\}$, then the sequence $\alpha x + \beta y \in V$ would be convergent to γ, a contradiction. Hence, the points of $\mathcal{P}$ are not collinear. Moreover, by our assumption on V, every linear combination of x and y has at most k accumulation points. Thus, by Lemma 4.3, there exists a partition $\{I_1, \dots, I_n\}$ of $\{1, \dots, k\}$ such that $S_j =^* \bigcup_{i \in I_j} T_i$ for $j \in \{1, \dots, n\}$. This assures us that $|\mathcal{E}| = k$. Therefore, we can apply Lemma 5.3 and we obtain the existence of a vector $z \in \operatorname{span}\{x, y\}$ such that

$$\left\lfloor \frac{k+1}{2} \right\rfloor \leqslant |\mathrm{L}_z| \leqslant k - 1.$$

However, the assumption $k > 2\max A$ implies $\lfloor \frac{k+1}{2} \rfloor > \max A$, so $\mathrm{L}_z \notin A \cup \{k\}$, a contradiction with the fact that $\operatorname{span}\{x, y\} \subseteq V \subseteq \bigcup_{i \in A \cup \{k\}} \mathrm{L}(i) \cup \{0\}$. □

Remark 5.5. The above proof shows that, if A and k are as in the statement of Proposition 5.4, then every vector space contained in $\bigcup_{n \in A \cup \{k\}} \mathrm{L}(n) \cup \{0\}$ and of dimension at least 2 does not intersect $\mathrm{L}(k)$. This is not true anymore if A and k don't satisfy the condition of the proposition, as the following example shows.

Example 5.6. For each integer $n \geqslant 2$, set $A_n := \{n, n+1, 2n\}$ and take vectors $x, y \in \mathrm{L}(2n)$ with

$$x \sim_{c_0} \xi_1 \mathbf{1}_{S_1} + \cdots + \xi_{2n} \mathbf{1}_{S_{2n}} \quad \text{and} \quad y \sim_{c_0} \eta_1 \mathbf{1}_{S_1} + \cdots + \eta_{2n} \mathbf{1}_{S_{2n}},$$

where $\{S_1, \dots, S_{2n}\}$ is a partition of ω into infinite sets. Further, the two families of distinct scalars $\{\xi_1, \dots, \xi_{2n}\}$ and $\{\eta_1, \dots, \eta_{2n}\}$ are chosen so that, if $P_j := (\xi_j, \eta_j)$, then $\mathcal{P} = \{P_1, \dots, P_{2n}\}$ are the vertices of a regular polygon with $2n$ edges labelled in the clockwise order. Then $V := \operatorname{span}\{x, y\}$ is a 2-dimensional vector space such that $V \subseteq \bigcup_{k\in A_n} \mathrm{L}(k) \cup \{0\}$ and $V \cap \mathrm{L}(k) \neq \emptyset$ for each $k \in A_n$. Indeed, if $\mathscr{L}$ is a set of parallel lines such that $\mathcal{P} \subseteq \mathscr{L}$ and every line in $\mathscr{L}$ contains a point of $\mathcal{P}$, then there are three cases. If every line in $\mathscr{L}$ only contains one point of $\mathcal{P}$, then $|\mathscr{L}| = 2n$; if one line in $\mathscr{L}$ contains P_1 and P_2, then $|\mathscr{L}| = n$; finally, if one line in $\mathscr{L}$ contains P_1 and P_3, then $|\mathscr{L}| = n + 1$. We omit the elementary geometric considerations required to prove that there only are these three cases (and we advise the reader to draw a picture).

We now give two examples of consequences of the above result. The first one implies in particular that, if A is not an interval, Corollary 4.5 might not be sharp:

Corollary 5.7. *Let $n, k \in \omega$ be such that $n \geqslant 2$ and $k > 2n$. Then $\mathrm{L}(n) \cup \mathrm{L}(k)$ is not 2-lineable.*

Proof. We have $\ell(\{n\}) = 1$ by Theorem 4.4. As $k > 2n$, we conclude by Proposition 5.4 that $\ell(\{n, k\}) = \ell(\{n\}) = 1$. □

By iteration of the above argument, we readily obtain the following result. It implies in particular that $\bigcup_{2\leqslant n<\omega} \mathrm{L}(n!)$ and $\bigcup_{1\leqslant n<\omega} \mathrm{L}(3^n)$ are not 2-lineable.

Corollary 5.8. *Let $(a_n)_{n\in\omega}$ be an increasing sequence in ω such that $a_0 \geqslant 2$ and $a_{n+1} > 2a_n$ for all $n \in \omega$. Then $\bigcup_{n\in\omega} \mathrm{L}(a_n)$ is not 2-lineable.*

Proof. Applying inductively Proposition 5.4 to $\{a_0, \dots, a_n\}$ and a_{n+1} we obtain that $\bigcup_{k\leqslant N} \mathrm{L}(a_k)$ is not 2-lineable for every $N \in \omega$. If there exists a 2-dimensional vector space $V \subseteq \bigcup_{k\in\omega} \mathrm{L}(a_k) \cup \{0\}$, then, by Lemma 2.2, $V \subseteq \bigcup_{k\leqslant N} \mathrm{L}(a_k) \cup \{0\}$ for some N, a contradiction. □

These type of results and (5.2) might lead one to conjecture that $\ell(A)$ could be large only if A contains large intervals. The last result of the section gives a strong negative answer to this conjecture. In particular, it follows that the inequality (5.2) is not sharp, even if A is finite.

Theorem 5.9. $\bigcup_{1\leqslant n<\omega} \mathrm{L}(2n + 1)$ *is $\mathfrak{c}$-lineable.*

Proof. Let $\mathscr{A} := \{A^e_\gamma : e \in \{-1, 0, 1\}, \gamma \in \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ be such that $\{A^{-1}_\gamma, A^0_\gamma, A^1_\gamma\}$ is a partition of ω for each $\gamma \in \mathfrak{c}$ and

$$A^{e_1}_{\gamma_1} \cap \dots \cap A^{e_k}_{\gamma_k} \text{ is an infinite set}$$

for all $k \geqslant 1$, all distinct $\gamma_1, \dots, \gamma_k \in \mathfrak{c}$ and all $e = (e_j)^k_{j=1} \in \{-1, 0, 1\}^k$. Let us observe that the existence of such a family $\mathscr{A}$ follows similarly as the existence of independent families (see, e.g., [22, Example 2, p. 10]). Indeed, the set $\mathbb{Q}[x]$ of polynomials with rational coefficients is countable, hence we can construct $\mathscr{A}$ as a subset of $\mathcal{P}(\mathbb{Q}[x])$. Therefore, it is sufficient to define, for each $\gamma \in \mathbb{R}$, $A^{-1}_\gamma := \{p \in P : p(\gamma) \leqslant -1\}$, $A^0_\gamma := \{p \in P : |p(\gamma)| < 1\}$ and $A^1_\gamma := \{p \in P : p(\gamma) \geqslant 1\}$.

At this point, for each $\gamma \in \mathfrak{c}$, define the vector

$$x_\gamma := \mathbf{1}_{A^1_\gamma} - \mathbf{1}_{A^{-1}_\gamma} \tag{5.3}$$

and set $V := \mathrm{span}\{x_\gamma : \gamma \in \mathfrak{c}\}$. We claim that each non-zero $z \in V$ is a non-convergent sequence with an odd number of accumulation points. To this aim, suppose that $z = \sum_{j=1}^k \alpha_j x_{\gamma_j}$ for some non-zero $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ and some distinct $\gamma_1, \dots, \gamma_k \in \mathfrak{c}$.

Note that

$$\mathrm{L}_z = \left\{ \sum_{j=1}^k \alpha_j e_j : e = (e_j)_{j=1}^k \in \{-1,0,1\}^k \right\}.$$

As $-e \in \{-1,0,1\}^k$ whenever $e \in \{-1,0,1\}^k$, we obtain that $\mathrm{L}_z = -\mathrm{L}_z$. Moreover, setting $e = (e_1, 0, \dots, 0)$ with $e_1 \in \{-1,0,1\}$, we get that $\alpha_1 \cdot \{-1,0,1\} \subseteq \mathrm{L}_z$. Hence, $|\mathrm{L}_z| \geqslant 3$ and $0 \in \mathrm{L}_z$. Combining this with $\mathrm{L}_z = -\mathrm{L}_z$, the result follows. □

Remark 5.10. The same argument as above, paired with Lemma 2.2, proves that $\mathrm{span}\{x_0, x_1\}$ is a 2-dimensional vector space contained in $\bigcup_{n \in A} \mathrm{L}(n) \cup \{0\}$, where $A := \{3,5,7,9\}$ and the vectors x_i are defined as in (5.3). Therefore, $\ell(\{3,5,7,9\}) \geqslant 2$.

6 | FINAL REMARKS

In this last section, we collect some observations concerning possible improvements of the results presented in our paper. Let us start with one comment concerning maximal lineability. A subset M of a vector space X is *maximal lineable* if $M \cup \{0\}$ contains a linear subspace V such that $\dim(V) = \dim(X)$. Clearly, every dense subspace V of ℓ_∞ satisfies $\dim(V) = \mathfrak{c}$, merely because $\dim(\ell_\infty) = \mathrm{dens}(\ell_\infty) = \mathfrak{c}$. Consequently, all our results concerning dense lineability in ℓ_∞ automatically are 'maximal dense lineability' results (note that the situation is different if the Banach space X is separable, as maximal lineability would require finding a subspace of dimension continuum, while a dense subspace might have countable dimension).

6.1 | Ideal convergence

Next, we discuss extensions of our results to the setting of ideal convergence. Recall that an ideal $\mathcal{I}$ on ω is a proper subfamily of $\mathcal{P}(\omega)$ that is closed under subsets and finite unions and that contains all singletons of ω. We denote by Fin the ideal of finite sets; hence $\mathrm{Fin} \subseteq \mathcal{I}$ for every ideal $\mathcal{I}$. For each sequence $x \in \ell_\infty$, let $\Gamma_x(\mathcal{I})$ be the set of its $\mathcal{I}$-cluster points, that is, the set of all $\eta \in \mathbb{R}$ such that $\{n \in \omega : |x(n) - \eta| < \varepsilon\} \notin \mathcal{I}$ for all $\varepsilon > 0$. It is easy to see each $\Gamma_x(\mathcal{I})$ is non-empty, closed and contained in $\mathrm{L}_x = \Gamma_x(\mathrm{Fin})$. Given a cardinal κ, define the set

$$\Gamma(\mathcal{I}, \kappa) := \{x \in \ell_\infty : |\Gamma_x(\mathcal{I})| = \kappa\}.$$

Hence, $\Gamma(\mathrm{Fin}, \kappa) = \mathrm{L}(\kappa)$ for all κ and $\Gamma(\mathcal{I}, \kappa) = \emptyset$ for uncountable $\kappa < \mathfrak{c}$.

Let $\mathcal{I}$ be an ideal on ω such that there are disjoint subsets $(B_j)_{j\in\omega}$ of ω with $B_j \notin \mathcal{I}$ for every $j \in \omega$. We stress here that this condition is satisfied by a large class of ideals. Besides the case $\mathcal{I} = \mathrm{Fin}$, it holds for all meagre ideals, as it readily follows from a classical characterisation of meagre filters due to Talagrand [26, Theorem 21], see also [4, Theorem 4.1.2]. Moreover, this condition is satisfied by all ideals that do not contain any isomorphic copy of a maximal ideal. Then, minimal variations in the proofs of Theorem 3.4, Remark 3.5 and Theorem 4.7 give that both $\Gamma(\mathcal{I},\omega)$ and $\Gamma(\mathcal{I},\mathfrak{c})$ are densely lineable in ℓ_∞ and spaceable. We chose to state our main results only for $\mathcal{I} = \mathrm{Fin}$ for the sake of clarity of the exposition, but we now quickly discuss how to prove the more general case.

The spaceability results are obtained from Theorem 4.7 by using a partition $(A_{j,k})_{j,k\in\omega}$ such that $A_{j,k} \notin \mathcal{I}$. The dense lineability of $\Gamma(\mathcal{I},\omega)$ in ℓ_∞ follows from the very same argument as in Theorem 3.4, using again a partition $(B_j)_{j\in\omega}$ such that $B_j \notin \mathcal{I}$. Note that $X := \bigcup_{\kappa\leqslant\omega}\Gamma(\mathcal{I},\kappa)$ and $Y := \bigcup_{\kappa<\omega}\Gamma(\mathcal{I},\kappa)$ are still vector spaces, due to the standard fact that

$$\Gamma_{x+y}(\mathcal{I}) \subseteq \Gamma_x(\mathcal{I}) + \Gamma_y(\mathcal{I}) \tag{6.1}$$

(which readily follows, for example, from [7, Chapter I, Section 7, no. 3, Proposition 8]). For the dense lineability of $\Gamma(\mathcal{I},\mathfrak{c})$ in ℓ_∞, we need sequences $r_j : \omega \to (0,1)$ with $\mathrm{suppt}(r_j) = B_j$ and $\Gamma_{r_j}(\mathcal{I}) = [0,1]$. For this, take disjoint sets $(A_{j,k})_{j,k\in\omega}$ with $A_{j,k} \notin \mathcal{I}$ and define $B_j := \cup_{k\in\omega} A_{j,k}$. Then, let $(q_k)_{k\in\omega}$ be an enumeration of $\mathbb{Q}\cap(0,1)$ and define r_j to be equal to q_k on A_k $(k\in\omega)$ and 0 elsewhere. The same argument as in Remark 3.5, using again (6.1), gives the result.

Finally, we can also modify the proof of Theorem 3.6 to prove that $\bigcup_{2\leqslant n<\omega}\Gamma(\mathcal{I},n)$ is densely lineable in ℓ_∞, for the same class of ideals. Indeed, let $(B_j)_j \in \omega$ be a partition of ω as above and let $\mathscr{I}$ be an independent family on ω of cardinality $\mathfrak{c}$. For $A \in \mathscr{I}$ define $B_A := \bigcup_{j\in A} B_j$ and let $\mathscr{J} := \{B_A : A \in \mathscr{I}\}$. Then $V := \mathrm{span}\{\mathbf{1}_B : B \in \mathscr{J}\}$ is a vector space of dimension $\mathfrak{c}$, every vector in V has finitely many $\mathcal{I}$-cluster points, and $V \cap \Gamma(\mathcal{I},1) = \{0\}$. To prove the last assertion, let $D_0,\dots,D_N \in \mathscr{J}$ and non-zero scalars $d_0,\dots,d_N \in \mathbb{R}$. By definition, the sets

$$D_0 \setminus (D_1 \cup \cdots \cup D_N) \quad \text{and} \quad \omega \setminus (D_0 \cup \cdots \cup D_N)$$

do not belong to $\mathcal{I}$. Hence, $\sum_{j=0}^N d_j \mathbf{1}_{D_j}$ attains the values d_0 and 0 on sets that do not belong to $\mathcal{I}$, thus it is not $\mathcal{I}$-convergent.

6.2 | $\mathbb{R}^\omega$ and pointwise convergence

Although all the paper remained in the realm of Banach spaces, we only used little Banach space theoretic structure of ℓ_∞. Therefore, it is natural to ask whether similar results can be true if we replace ℓ_∞ with the larger space $\mathbb{R}^\omega$ of all scalar sequences. For a sequence $(x(n))_{n\in\omega} \in \mathbb{R}^\omega$, the set L_x of accumulation points of x is now defined as a subset of $\mathbb{R}\cup\{\pm\infty\}$ (if a subsequence of $(x(n))_{n\in\omega}$ diverges to $\pm\infty$, $\pm\infty$ is considered to be an accumulation point of the sequence). The definition of $\mathrm{L}(\kappa)$ is also modified accordingly; for example, every sequence $(x(n))_{n\in\omega}$ that diverges to ∞ belongs to $\mathrm{L}(1)$.

$\mathbb{R}^\omega$ is a separable, completely metrisable topological vector space when endowed with the pointwise topology. (Throughout all the subsection, we always endow $\mathbb{R}^\omega$ with the pointwise topology.) In addition, c_{00} (and, hence, ℓ_∞) is dense in $\mathbb{R}^\omega$. Therefore, our results immediately imply that $\mathrm{L}(\mathfrak{c})$, $\mathrm{L}(\omega)$ and $\bigcup_{2\leqslant n<\omega}\mathrm{L}(n)$ are densely lineable in $\mathbb{R}^\omega$. Note, on the other hand, that the results for

ℓ_∞ are stronger, as the norm topology is substantially finer than the pointwise one; in particular, there is no obvious way to recover the results for ℓ_∞ from the corresponding one for $\mathbb{R}^\omega$.

As regards spaceability, the same argument as in Theorem 4.7 shows that $\mathrm{L}(\mathfrak{c})$ is spaceable in $\mathbb{R}^\omega$. Indeed, if $Y := \overline{\mathrm{span}}\{e_n\}_{n\in\omega}$, where e_n is as in (4.1) and the closure is in the pointwise topology, then

$$Y = \left\{ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \alpha_n a_k \cdot \mathbf{1}_{A_{n,k}} : (\alpha_n)_{n\in\omega} \in \mathbb{R}^\omega \right\}$$

(here, the above series converge in the pointwise topology). Hence, if $x \in Y \setminus \{0\}$, write $x := \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \alpha_n a_k \cdot \mathbf{1}_{A_{n,k}}$ and take $n \in \omega$ with $\alpha_n \neq 0$; thus $\alpha_n \cdot [0,1] \subseteq \mathrm{L}_x$.

On the other hand, the above argument does not extend to prove that $\mathrm{L}(\omega)$ is spaceable in $\mathbb{R}^\omega$ (because the sequence $(\alpha_n)_{n\in\omega} \in \mathbb{R}^\omega$ can create uncountably many accumulation points). Interestingly, it turns out that, differently from Theorem 4.7, $\mathrm{L}(\omega)$ is not spaceable in $\mathbb{R}^\omega$. In the next theorem, we actually prove a more general result.

Theorem 6.1. *For every closed infinite-dimensional subspace Y of $\mathbb{R}^\omega$ there is $x \in Y$ such that $\mathrm{L}_x = \mathbb{R} \cup \{\pm\infty\}$. In particular, $\bigcup_{\kappa \leqslant \omega} \mathrm{L}(\kappa)$ is not spaceable in $\mathbb{R}^\omega$.*

Let us remark that, aside implying the non spaceability of $\mathrm{L}(\omega)$, the result implies that also $\bigcup_{2\leqslant n<\omega} \mathrm{L}(n)$ is not spaceable in $\mathbb{R}^\omega$.

Proof. Notice that $\{x \in Y : n \leqslant \min(\mathrm{suppt}(x))\}$ has finite codimension in Y for every $n \in \omega$. Hence, by the fact that Y is infinite-dimensional, we can find a sequence $(x_n)_{n\in\omega}$ of non-zero vectors of Y such that the sequence $s_n := \min(\mathrm{suppt}(x_n))$ is strictly increasing. Then, let $(q_n)_{n\in\omega}$ be an enumeration of $\mathbb{Q}$. Take $(\alpha(n))_{n\in\omega} \in \mathbb{R}^\omega$ that solves the following system of equations:

$$\sum_{j=0}^{k} \alpha(j) x_j(s_k) = q_k \qquad (k \in \omega).$$

Such a system can indeed be solved recursively, using the fact that $x_k(s_k) \neq 0$ for every $k \in \omega$. We are now in position to define the vectors

$$u_k := \sum_{j=0}^{k} \alpha(j) x_j \in Y \quad \text{and} \quad u := \sum_{j=0}^{\infty} \alpha(j) x_j.$$

Note that the series defining u converges pointwise, due to the assumption that $(s_n)_{n\in\omega}$ is strictly increasing. By the same reason, we also conclude that $u_k \to u$ pointwise, hence $u \in Y$. However, $u(s_k) = q_k$ for every $k \in \omega$, hence $\mathrm{L}_u = \mathbb{R} \cup \{\pm\infty\}$. □

6.3 | Further research

In conclusion of our presentation, we shall highlight some directions for possible further research that seem natural in light of the results presented. Concerning spaceability, recall that the two closed subspaces that we constructed in Theorem 4.7, contained in $\mathrm{L}(\omega) \cup \{0\}$ and $\mathrm{L}(\mathfrak{c}) \cup \{0\}$, are isometric to c_0 and ℓ_∞, respectively. It would be interesting to know whether it is possible to build a non-separable closed subspace also in the case of $\mathrm{L}(\omega)$.

Problem 6.2. Does $\mathrm{L}(\omega) \cup \{0\}$ contain a closed non-separable subspace? Does it contain an isometric copy of ℓ_∞? The same questions could be asked for $\bigcup_{\kappa \leqslant \omega} \mathrm{L}(\kappa)$.

Another possible direction of investigation could consist in digging deeper in the linear structure of the sets $\bigcup_{n \in A} \mathrm{L}(n)$, where $A \subseteq \omega$ and $\min A \geqslant 2$. In Theorem 4.4, we gave a complete result in the case when A is an interval of the form $\{n, n+1, \dots, n+d\}$. On the other hand, we saw in Section 5 that when A is not an interval the situation is less clear. For example, it is quite conceivable that the assumption $k > 2 \max A$ in Proposition 5.4 could be improved. In the same direction, one might try to characterise those finite sets A for which $\ell(A) = \max\{|I| : I \subseteq A \text{ is an interval}\}$.

A slightly different question, that we find particularly interesting, is the following (which ought to be compared with Theorem 5.9).

Problem 6.3. Is $\bigcup_{1 \leqslant n < \omega} \mathrm{L}(2n)$ lineable?

Similarly, we could ask if $\bigcup_{2 \leqslant n < \omega} \mathrm{L}(n^2)$ is lineable. Note that we do not even know if these sets are 2-lineable. In connection with Theorem 5.9, it is also natural to ask the following.

Problem 6.4. Is $\bigcup_{1 \leqslant n < \omega} \mathrm{L}(2n+1)$ densely lineable in ℓ_∞?

Added in proof

After the completion of our research, we were informed of some new results that were motivated by our paper. Menet and Papathanasiou [20] recently obtained several interesting results that in particular solve Problems 6.2 and 6.4 and imply that $\bigcup_{2 \leqslant n < \omega} \mathrm{L}(n^2)$ is not lineable. Although Problem 6.3 seems to be still open, Davide Ravasini recently showed that $\bigcup_{1 \leqslant n < \omega} \mathrm{L}(2n)$ is 2-lineable. We are most grateful to him for allowing us to explain his argument here.

One uses the same notation and construction as in Example 5.6. Let $\mathcal{P}$ be the vertices of a regular 15-gon and let $\mathcal{T} \subseteq \mathcal{P}$ be the vertices of an equilateral triangle. Then the points $\mathcal{P} \setminus \mathcal{T}$ are as desired. Indeed, if $\mathscr{L}$ is a set of parallel lines with $\mathcal{P} \subseteq \mathscr{L}$ and such that every line of $\mathscr{L}$ contains a point of $\mathcal{P}$, then $|\mathscr{L}|$ equals 15 or 8. In the first case, exactly 12 lines are needed to cover $\mathcal{P} \setminus \mathcal{T}$. So, we can assume that $|\mathscr{L}| = 8$. Now, if one edge of $\mathcal{T}$ is parallel to the lines in $\mathscr{L}$, then exactly 2 of the lines in $\mathscr{L}$ don't contain points of $\mathcal{P} \setminus \mathcal{T}$ (the line containing a unique point of $\mathcal{P}$ actually contains a point of $\mathcal{T}$). In the other case, all 8 lines contain points of $\mathcal{P} \setminus \mathcal{T}$. Hence, in order to cover $\mathcal{P} \setminus \mathcal{T}$ one needs 6, 8, or 12 parallel lines, which means that $\mathrm{L}(6) \cup \mathrm{L}(8) \cup \mathrm{L}(12)$ is 2-lineable.

Incidentally, the same construction works for every $(2n-1)$-gon, provided that n is even and $2n-1$ is a multiple of 3. Hence, for every $k \geqslant 1$, $\mathrm{L}(6k) \cup \mathrm{L}(6k+2) \cup \mathrm{L}(12k)$ is 2-lineable.

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