

PRACTICAL NUMBERS AMONG THE BINOMIAL COEFFICIENTS

PAOLO LEONETTI[†] AND CARLO SANNA[‡]

ABSTRACT. A *practical number* is a positive integer n such that every positive integer less than n can be written as a sum of distinct divisors of n . We prove that most of the binomial coefficients are practical numbers. Precisely, letting $f(n)$ denote the number of binomial coefficients $\binom{n}{k}$, with $0 \leq k \leq n$, that are not practical numbers, we show that

$$f(n) < n^{1-(\log 2-\delta)/\log \log n}$$

for all integers $n \in [3, x]$, but at most $O_\gamma(x^{1-(\delta-\gamma)/\log \log x})$ exceptions, for all $x \geq 3$ and $0 < \gamma < \delta < \log 2$. Furthermore, we prove that the central binomial coefficient $\binom{2n}{n}$ is a practical number for all positive integers $n \leq x$ but at most $O(x^{0.88097})$ exceptions. We also pose some questions on this topic.

1. INTRODUCTION

A *practical number* is a positive integer n such that every positive integer less than n can be written as a sum of distinct divisors of n . This property has been introduced by Srinivasan [19]. Estimates for the counting function of practical numbers have been given by Hausman–Shapiro [5], Tenenbaum [20], Margenstern [9], Saias [15], and finally Weingartner [21], who proved that there are asymptotically $Cx/\log x$ practical numbers less than x , for some constant $C > 0$, as previously conjectured by Margenstern [9]. On another direction, Melfi [11] proved that every positive even integer is the sum of two practical numbers, and that there are infinitely many triples $(n, n+2, n+4)$ of practical numbers. Also, Melfi [10] proved that in every Lucas sequence, satisfying some mild conditions, there are infinitely many practical numbers, and Sanna [17] gave a lower bound for their counting function.

In this work, we study the binomial coefficients which are also practical numbers. Our first result, informally, states that for almost all positive integers n

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there is a negligible amount of binomial coefficients $\binom{n}{k}$, with $0 \leq k \leq n$, which are not practical. Precisely, for each positive integer n , define

$$f(n) := \#\left\{0 \leq k \leq n : \binom{n}{k} \text{ is not a practical number}\right\}.$$

Our first result is the following.

Theorem 1.1. *For all $x \geq 3$ and $0 < \gamma < \delta < \log 2$, we have*

$$f(n) < n^{1-(\log 2-\delta)/\log \log n}$$

for all integers $n \in [3, x]$, but at most $O_\gamma(x^{1-(\delta-\gamma)/\log \log x})$ exceptions.

As a consequence, we obtain that as $x \rightarrow +\infty$ almost all binomial coefficients $\binom{n}{k}$, with $0 \leq k \leq n \leq x$, are practical numbers.

Corollary 1.1. *We have*

$$\sum_{n \leq x} f(n) \ll_\varepsilon x^{2-(\frac{1}{2}\log 2-\varepsilon)/\log \log x},$$

for all $x \geq 3$ and $\varepsilon > 0$.

Among the binomial coefficients, the *central binomial coefficients* $\binom{2n}{n}$ are of great interest. In particular, several authors have studied their arithmetic and divisibility properties, see e.g. [1, 2, 14, 16, 18].

In this direction, our second result, again informally, states that almost all central binomial coefficients $\binom{2n}{n}$ are practical numbers.

Theorem 1.2. *For $x \geq 1$, the central binomial coefficient $\binom{2n}{n}$ is a practical number for all positive integers $n \leq x$ but at most $O(x^{0.88097})$ exceptions.*

Probably, there are only finitely many positive integers n such that $\binom{2n}{n}$ is not a practical number. By a computer search, we found only three of them below 10^6 , namely $n = 4, 10, 256$. However, proving the finiteness seems to be out of reach with actual techniques. Indeed, on the one hand, if n is a power of 2 whose base 3 representation contains only the digits 0 and 1, then it can be shown that $\binom{2n}{n}$ is not a practical number (see Proposition 2.1 below). On the other hand, it is an open problem to establish whether there are finitely or infinitely many powers of 2 of this type [4, 6, 8, 12].

We conclude by leaving two open questions. Note that since $\binom{n}{0} = \binom{n}{n} = 1$, we have $0 \leq f(n) \leq n - 1$ for all positive integers n . It is natural to ask when one of the equalities is satisfied.

Question 1.1. What are the positive integers n such that $f(n) = 0$?

Question 1.2. What are the positive integers n such that $f(n) = n - 1$?

Regarding Question 1.1, if $f(n) = 0$ then n must be a power of 2, otherwise there would exist (see Lemma 2.4 below) an odd binomial coefficient $\binom{n}{k}$, with $0 < k < n$, and since 1 is the only odd practical number, we would have $f(n) > 0$. However, this is not a sufficient condition, since $f(8) = 1$. Regarding Question 1.2, if $n = 2^k - 1$, for some positive integer k , then $f(n) = n - 1$, because all the binomial coefficients $\binom{n}{k}$, with $0 < k < n$, are odd (see Lemma 2.4 below) and greater than 1, and consequently they are not practical numbers. However, this is not a necessary condition, since $f(5) = 4$.

Notation. We employ the Landau–Bachmann “Big Oh” notation O and the associated Vinogradov symbol \ll . In particular, any dependence of the implied constants is indicated with subscripts. We write p_i for the i th prime number.

2. PRELIMINARIES

This section is devoted to some preliminary results needed in the later proofs. We begin with some lemmas about practical numbers.

Lemma 2.1. *If m is a practical number and $n \leq 2m$ is a positive integer, then mn is a practical number.*

Proof. See [10, Lemma 4]. □

Lemma 2.2. *If d is a practical number and n is a positive integer divisible by d and having all prime factors not exceeding $2d$, then n is a practical number.*

Proof. By hypothesis, there exist positive integers $q_1, \dots, q_k \leq 2d$ such that $n = dq_1 \cdots q_k$. Then, using Lemma 2.1, it follows by induction that $dq_1 \cdots q_m$ is practical for all $m = 1, \dots, k$. In particular, n is practical. □

Lemma 2.3. *We have that $p_1^{a_1} \cdots p_s^{a_s}$ is a practical number, for all positive integers a_1, \dots, a_s .*

Proof. It follows easily by induction on s , using Lemma 2.1 and Bertrand’s postulate $p_{i+1} < 2p_i$. □

For each prime number p and for each positive integer n , put

$$T_p(n) := \#\left\{0 \leq k \leq n : p \nmid \binom{n}{k}\right\}.$$

We have the following formula for $T_p(n)$.

Lemma 2.4. *Let p be a prime number and let*

$$n = \sum_{j=0}^s d_j p^j, \quad d_0, \dots, d_s \in \{0, \dots, p-1\}, \quad d_s \neq 0,$$

be the representation in base p of the positive integer n . Then we have

$$T_p(n) = \prod_{j=0}^s (d_j + 1).$$

Proof. See [3, Theorem 2]. □

For each prime number p , let us define

$$\omega_p := \frac{\log((p+1)/2)}{\log p}.$$

The quantity ω_p appears in the following upper bound for $T_p(n)$.

Lemma 2.5. *Let p be a prime number and fix $\varepsilon \in (0, 1/2)$. Then, for all $x \geq 1$, we have*

$$T_p(n) < n^{\omega_p + \varepsilon}$$

for all positive integers $n \leq x$ but at most $O(p^3 x^{1-\varepsilon})$ exceptions.

Proof. For $x \geq 1$, let k be the smallest integer such that $x < p^k$. Clearly, we have

$$\begin{aligned} E(x) &:= \#\{n \leq x : T_p(n) \geq n^{\omega_p + \varepsilon}\} \\ &\leq \sum_{j=1}^k \#\{p^{j-1} \leq n < p^j : T_p(n) \geq p^{(j-1)(\omega_p + \varepsilon)}\}. \end{aligned} \quad (1)$$

Moreover, thanks to Lemma 2.4, we have

$$\sum_{p^{j-1} < n \leq p^j} T_p(n) \leq \sum_{0 \leq d_0, \dots, d_{j-1} < p} \prod_{i=0}^{j-1} (d_i + 1) = \left(\sum_{d=0}^{p-1} (d+1) \right)^j = \left(\frac{p(p+1)}{2} \right)^j,$$

and consequently

$$\begin{aligned} \#\{p^{j-1} \leq n < p^j : T_p(n) \geq p^{(j-1)(\omega_p + \varepsilon)}\} &\leq \frac{1}{p^{(j-1)(\omega_p + \varepsilon)}} \sum_{p^{j-1} < n \leq p^j} T_p(n) \\ &\leq \frac{1}{p^{(j-1)(\omega_p + \varepsilon)}} \left(\frac{p(p+1)}{2} \right)^j = \frac{p(p+1)}{2} p^{(1-\varepsilon)(j-1)} < p^{2+(1-\varepsilon)(j-1)}, \end{aligned} \quad (2)$$

for all positive integers j . Therefore, putting together (1) and (2), and using that $\varepsilon < 1/2$, we obtain

$$E(x) < \sum_{j=1}^k p^{2+(1-\varepsilon)(j-1)} \ll p^{2+(1-\varepsilon)k} \leq p^{2+(1-\varepsilon)(\log x / \log p + 1)} < p^3 x^{1-\varepsilon}, \quad (3)$$

as desired. □

Remark 2.1. The constant $1/2$ in the statement of Lemma 2.5 has no particular importance, it is only needed to justify the \ll in (3). Any other real number less than 1 would be fine.

For all $x \geq 1$, let $\kappa(x)$ be the smallest integer $k \geq 1$ such that $p_1 \cdots p_k \geq x$.

Lemma 2.6. *We have*

$$\kappa(x) \sim \frac{\log x}{\log \log x} \quad \text{and} \quad p_{\kappa(x)} \sim \log x,$$

as $x \rightarrow \infty$.

Proof. As a well-known consequence of the Prime Number Theorem, we have

$$\log(p_1 \cdots p_k) \sim p_k \sim k \log k, \tag{4}$$

as $k \rightarrow +\infty$. Since

$$\log(p_1 \cdots p_{\kappa(x)-1}) < \log x \leq \log(p_1 \cdots p_{\kappa(x)}),$$

and $\kappa(x) \rightarrow +\infty$ as $x \rightarrow +\infty$, by (4) we obtain

$$p_{\kappa(x)} \sim \kappa(x) \log \kappa(x) \sim \log x,$$

which in turn implies

$$\kappa(x) \sim \frac{\kappa(x) \log \kappa(x)}{\log \kappa(x) + \log \log \kappa(x)} \sim \frac{\log x}{\log \log x},$$

as desired. □

For every prime number p and every positive integer n , let $\beta_p(n)$ be the p -adic valuation of the central binomial coefficient $\binom{2n}{n}$.

Lemma 2.7. *For each prime p and all positive integers n , we have that $\beta_p(n)$ is equal to the number of digits of n in base p which are greater than $(p-1)/2$.*

Proof. The claim is a straightforward consequence of a theorem of Kummer [7] which says that, for positive integers m, n , the p -adic valuation of $\binom{m+n}{n}$ is equal to the number of carries in the addition $m+n$ done in base p . □

Proposition 2.1. *If n is a power of 2 and if all the digits of n in base 3 are equal to 0 or 1, then $\binom{2n}{n}$ is not a practical number.*

Proof. It follows by Lemma 2.7 that $\beta_2(n) = 1$ and $\beta_3(n) = 0$, that is, $\binom{2n}{n}$ is an integer of the form $12k \pm 2$. However, it is known that, other than 1 and 2, every practical number is divisible by 4 or 6, see [19]. □

We will make use of the following result of probability theory.

Lemma 2.8. *Let X be a random variable following a binomial distribution with j trials and probability of success α . Then*

$$\mathbb{P}[X \leq (\alpha - \varepsilon)j] \leq e^{-2\varepsilon^2 j}$$

for all $\varepsilon > 0$.

Proof. See [13, Theorem 1]. □

For each prime number p , let us define

$$\alpha_p := \frac{1}{p} \left\lceil \frac{p-1}{2} \right\rceil,$$

so that α_p is the probability that a random digit in base p is greater than $(p-1)/2$.

Lemma 2.9. *Let p be a prime number and fix $\varepsilon \in (0, 1/2)$. Then, for all $x \geq 1$, we have*

$$\beta_p(n) > (\alpha_p - \varepsilon) \frac{\log n}{\log p}$$

for all positive integers $n \leq x$ but at most $O(px^{1-2\varepsilon^2/\log p})$ exceptions.

Proof. For $x \geq 1$, let k be the smallest integer such that $x < p^k$. Clearly, we have

$$\begin{aligned} E(x) &:= \#\left\{n \leq x : \beta_p(n) \leq (\alpha_p - \varepsilon) \frac{\log n}{\log p}\right\} \\ &\leq \sum_{j=1}^k \#\{p^{j-1} \leq n < p^j : \beta_p(n) \leq (\alpha_p - \varepsilon)j\} \\ &\leq \sum_{j=1}^k \#\{0 \leq n < p^j : \beta_p(n) \leq (\alpha_p - \varepsilon)j\}. \end{aligned} \quad (5)$$

Given an integer $j \geq 1$, let us for a moment consider n as a random variable uniformly distributed in $\{0, \dots, p^j - 1\}$. Then, the digits of n in base p are j independent random variables uniformly distributed in $\{0, \dots, p-1\}$. Hence, as a consequence of Lemma 2.7, we obtain that $\beta_p(n)$ follows a binomial distribution with j trials and probability of success α_p . In turn, Lemma 2.8 yields

$$\#\{0 \leq n < p^j : \beta_p(n) \leq (\alpha_p - \varepsilon)j\} \leq p^j e^{-2\varepsilon^2 j}. \quad (6)$$

Therefore, putting together (5) and (6), and using that $\varepsilon < 1/2$, we get

$$E(x) \leq \sum_{j=1}^k p^j e^{-2\varepsilon^2 j} \ll (pe^{-2\varepsilon^2})^k \leq (pe^{-2\varepsilon^2})^{\log x / \log p + 1} < px^{1-2\varepsilon^2/\log p}, \quad (7)$$

as desired. □

Remark 2.2. The constant $1/2$ in the statement of Lemma 2.9 has no particular importance, it is only needed to justify the \ll in (7). Any other real number less than $(\frac{1}{2} \log 2)^{1/2}$ would be fine.

3. PROOF OF THEOREM 1.1

Assume $x \geq 3$ sufficiently large, and put

$$\varepsilon := \frac{\delta - \gamma}{\log \log x} + \frac{4 \log \log x}{\log x} \in (0, 1/2).$$

Let n be a positive integer. By Lemma 2.3 and by the definition of $\kappa(n)$, we know that $p_1 \cdots p_{\kappa(n)}$ is a practical number greater than or equal to n . Since all the prime factors of $\binom{n}{k}$ are not exceeding n , Lemma 2.2 tell us that if $p_1 \cdots p_{\kappa(n)}$ divides $\binom{n}{k}$ then $\binom{n}{k}$ is practical. Consequently, we have

$$f(n) \leq \#\left\{0 \leq k \leq n : p_1 \cdots p_{\kappa(n)} \nmid \binom{n}{k}\right\} \leq \sum_{j=1}^{\kappa(n)} T_{p_j}(n).$$

Therefore, it follows from Lemma 2.5 that

$$f(n) < \sum_{j=1}^{\kappa(n)} n^{\omega_{p_j} + \varepsilon}, \quad (8)$$

for all positive integers $n \leq x$ but at most

$$\ll \sum_{j=1}^{\kappa(x)} p_j^3 x^{1-\varepsilon} \ll p_{\kappa(x)}^4 x^{1-\varepsilon} \ll (\log x)^4 x^{1-\varepsilon} = x^{1-(\delta-\gamma)/\log \log x}$$

exceptions, where we also used Lemma 2.6.

Suppose that n satisfies (8). Since ω_p is a monotone increasing function of p , we get that

$$f(n) < \kappa(n) n^{\omega_{p_{\kappa(n)}} + \varepsilon} = n^{\omega_{p_{\kappa(n)}} + \log \kappa(n) / \log n + \varepsilon}. \quad (9)$$

Moreover, for $n \gg_{\gamma} 1$ we have

$$\omega_{p_{\kappa(n)}} < 1 - \frac{\log 2}{\log p_{\kappa(n)}} + \frac{1}{p_{\kappa(n)} \log p_{\kappa(n)}} < 1 - \frac{\log 2 - \gamma/4}{\log \log n}, \quad (10)$$

and

$$\frac{\log \kappa(n)}{\log n} < \frac{\gamma/4}{\log \log n}, \quad (11)$$

where we used Lemma 2.6. Furthermore, since $n \leq x$, we have

$$\varepsilon < \frac{\delta - \gamma/2}{\log \log n}. \quad (12)$$

Consequently, putting together (10), (11), and (12), we obtain

$$\omega_{p_{\kappa(n)}} + \frac{\log \kappa(n)}{\log n} + \varepsilon < 1 - \frac{\log 2 - \delta}{\log \log n},$$

which, inserted into (9), gives

$$f(n) < n^{1-(\log 2 - \delta)/\log \log n}$$

as desired. The proof is complete.

4. PROOF OF COROLLARY 1.1

Obviously, we can assume $\varepsilon < \frac{1}{2} \log 2$. Put $\gamma := 2\varepsilon$ and $\delta := \frac{1}{2} \log 2 + \varepsilon$, so that $0 < \gamma < \delta < \log 2$. For all $x \geq 3$, let $\mathcal{E}(x)$ be the set of exceptional $n \leq x$ of Theorem 1.1. Then we have

$$\begin{aligned} \sum_{n \leq x} f(n) &= \sum_{n \notin \mathcal{E}(x)} f(n) + \sum_{n \in \mathcal{E}(x)} f(n) < \sum_{n \leq x} n^{1-(\log 2 - \delta)/\log \log n} + \#\mathcal{E}(x)x \\ &\ll_{\varepsilon} x^{2-(\log 2 - \delta)/\log \log x} + x^{2-(\delta - \gamma)/\log \log x} \ll x^{2-(\frac{1}{2} \log 2 - \varepsilon)/\log \log x}, \end{aligned}$$

as claimed.

5. PROOF OF THEOREM 1.2

For the sake of notation, put

$$s := 16, \quad \eta_s := \frac{\sum_{i=1}^s \alpha_{p_i} - 1}{\sum_{i=1}^s \sqrt{\log p_i}}, \quad \varepsilon_j := \eta_s \sqrt{\log p_j},$$

for $j = 1, \dots, s$. A computation shows that $\varepsilon_j \in (0, 1/2)$ for $j = 1, \dots, s$.

For $x \geq 1$, it follows from Lemma 2.9 that

$$\sum_{j=1}^s \beta_{p_j}(n) \log p_j > \sum_{j=1}^s (\alpha_{p_j} - \varepsilon_j) \log n = \log n, \quad (13)$$

for all positive integers $n \leq x$, but at most

$$\ll \sum_{j=1}^s p_j x^{1-2\varepsilon_j^2/\log p_j} \ll x^{1-2\eta_s^2} < x^{0.88097}$$

exceptions. Suppose that n is a positive integer satisfying (13). Then,

$$d := \prod_{j=1}^s p_j^{\beta_{p_j}(n)} > n.$$

Also, by Lemma 2.3 we have that d is a practical number, and by the definition of $\beta_{p_j}(n)$ we have that d divides $\binom{2n}{n}$. Moreover, since all the prime factors of

$\binom{2n}{n}$ are not exceeding $2d$, Lemma 2.2 yields that $\binom{2n}{n}$ is practical. The proof is complete.

Remark 5.1. A comment is in order to explain the choice of the parameters ε_j in the proof of Theorem 1.2. Given a positive integer s , one could fix some prime numbers $q_1 < \dots < q_s$ and some real numbers $\varepsilon_1, \dots, \varepsilon_s \in (0, 1/2)$ such that $q_1 \cdots q_s$ is a practical number and $\sum_{j=1}^s (\alpha_{q_j} - \varepsilon_j) \geq 1$. Everything would proceed similarly, with an estimate of the number of exceptions given by

$$O\left(x^{\max\{1-2\varepsilon_1^2/\log q_1, \dots, 1-2\varepsilon_s^2/\log q_s\}}\right).$$

To minimize the exponent of x , the optimal choice for ε_j is

$$\varepsilon_j = \eta_s(q_1, \dots, q_s) \sqrt{\log q_j}, \quad \eta_s(q_1, \dots, q_s) := \frac{\sum_{i=1}^s \alpha_{q_i} - 1}{\sum_{i=1}^s \sqrt{\log q_i}},$$

for $j = 1, \dots, s$, which gives the estimate

$$O\left(x^{1-2\eta_s(q_1, \dots, q_s)^2}\right).$$

Since $\alpha_p = \frac{1}{2} + O(\frac{1}{p})$ for each prime number p , we get that $\eta_s(q_1, \dots, q_s)$ is maximized when $q_j = p_j$, for $j = 1, \dots, s$, and that $\eta_s(p_1, \dots, p_s) \rightarrow 0$ as $s \rightarrow +\infty$. Lastly, some numerical computations verify that the maximum of $\eta_s(p_1, \dots, p_s)$ is reached for $s = 16$.

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GRAZ UNIVERSITY OF TECHNOLOGY, INSTITUTE OF ANALYSIS AND NUMBER THEORY,
GRAZ, AUSTRIA

E-mail address: leonetti.paolo@gmail.com

UNIVERSITÀ DEGLI STUDI DI GENOVA, DEPARTMENT OF MATHEMATICS, GENOVA, ITALY

E-mail address: carlo.sanna.dev@gmail.com