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Neumann Cut-Offs and Essential Self-adjointness on Complete Riemannian Manifolds with Boundary

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Abstract

We generalize some fundamental results for noncompact Riemannian manfolds without boundary, that only require completeness and no curvature assumptions, to manifolds with boundary: let M be a smooth Riemannian manifold with boundary ∂M and let $\hat{C}_c^\infty(M)$ denote the space of smooth compactly supported cut-off functions with vanishing normal derivative, *Neumann cut-offs*. We show, among other things, that under completeness:

- $\hat{C}_c^\infty(M)$ is dense in $W^{1,p}(\mathring{M})$ for all $p \in (1, \infty)$; this generalizes a classical result by Aubin (Bull. Sci. Math. 100:149–173, 1976) for $\partial M = \emptyset$.
- M admits a sequence of first order cut-off functions in $\hat{C}_c^\infty(M)$; for $\partial M = \emptyset$ this result can be traced back to Gaffney (Ann. Math. (2) 60(1):140–145, 1954).
- the Laplace–Beltrami operator with domain of definition $\hat{C}_c^\infty(M)$ is essentially self-adjoint; this is a generalization of a classical result by Strichartz (J. Funct. Anal. 52(1):48–79, 1983) for $\partial M = \emptyset$.

Keywords Manifolds with boundary · Neumann cut-offs · Neumann Laplacian · Essential self-adjointness

Mathematics Subject Classification 58J05 · 58J50 · 35J25

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1 First Order Sobolev Density Results, and First Order Sequences of Neumann Cut-Off Functions

Let M be a smooth connected Riemannian manifold with boundary with dimension m, where if there is no danger of confusion we will denote the Riemannian metric g on M simply by $(\cdot,\cdot)$. We consider M as a metric space with respect to its geodesic distance $\varrho(x,y) := \inf L(\gamma)$, where the infimum is taken over all piecewise smooth curves $\gamma : [0,1] \to M$ joining x and y, and where $L(\gamma) := \int_a^b |\dot{\gamma}(t)| \mathrm{d}t$ denotes the length of such a curve. The induced open balls are denoted by $B(x,r)$. Note that ϱ induces the original topology on M [1]. The completeness of M will always mean its completeness as a metric space, and by the Hopf-Rinow Theorem for locally compact length spaces, this is equivalent to bounded sets being relatively compact.

Remark 1.1 Assume M' is an open connected smooth subset of a smooth complete Riemannian manifold without boundary. Then $M := \overline{M'}$ is a complete smooth Riemannian manifold with boundary. This follows from the fact that the (intrinsic!) geodesic distance on M induces the topology on M.

We denote by $\mathring{M} = M \backslash \partial M$ the interior of M in the sense of manifolds with boundary. Following the notation of [8] and [13], for any open set Ω in M we set $\mathring{\Omega} = \Omega \cap \mathring{M}$, $\partial\Omega = \partial\mathring{\Omega}$ and let $\partial_0\Omega = \partial\Omega \cap \mathring{M}$ be the Dirichlet boundary of Ω, and $\partial_1\Omega = \bar{\Omega} \cap \partial M$ be the Neumann boundary of Ω, so that $\partial\Omega = \partial_0\Omega \cup \partial_1\Omega$.

The symbol ∂_ν denotes the outward pointing normal derivative on ∂M.

We understand all L^p-norms and L^2-scalar products with respect to the Riemannian volume measure μ given locally by $\mu(\mathrm{d}x) = \sqrt{g}(x)\mathrm{d}x$, where of course $\mu(M \setminus \mathring{M}) = \mu(\partial M) = 0$, as ∂M is assumed to be smooth.

Let ∇ denote the Levi–Civita connection on M, and let Δ denote the (negative definite) Laplace–Beltrami operator, so that we have Green's formula

$$\langle \Delta f_1, f_2 \rangle = -\langle \nabla f_1 \nabla f_2 \rangle + \int_{\partial M} (\partial_\nu f_1) f_2 \, \mathrm{d}\sigma,$$

valid for all $f_1, f_2 \in C^\infty(M)$ one of which having a compact support, where σ is the $(m-1)$-dimensional Hausdorff measure of the metric space M.

We recall that for every smooth Riemannian manifold without boundary N one defines

$$W^{1,p}(N) = \{f \in L^p(N) : \nabla f \in \Gamma_{L^p}(N, TN)\},$$

where expressions of the form Pf with $f \in L^1_{\mathrm{loc}}(N)$ and P a differential operator with smooth coefficients in N, are understood in the usual sense of distribution theory of open manifolds. Then $W^{1,p}(N)$ becomes a Banach space with respect to the norm $\|f\|_p + \|\nabla f\|_p$.

It has been shown in [18, Corollary 2.3], that if M is complete then $C_c^\infty(M)$ is dense in $W^{1,p}(\mathring{M})$.

The Neumann realization of $-\Delta$ in $L^2(M)$ is defined as the nonnegative self-adjoint operator H in $L^2(M)$ associated with the regular, strongly local Dirichlet form

$$\mathscr{E}(f_1, f_2) := \langle \nabla f_1, \nabla f_2 \rangle = \int_M (\nabla f_1, \nabla f_2) \mathrm{d}\mu,$$

with $\mathrm{Dom}(\mathscr{E}) = W^{1,2}(\mathring{M})$. In other words, H is the uniquely determined nonnegative self-adjoint operator in $L^2(M)$ with $\mathrm{Dom}(H) \subset W^{1,2}(\mathring{M})$ and

$$\langle H f_1, f_2 \rangle = \langle \nabla f_1, \nabla f_2 \rangle \quad \text{for all } f_1 \in \mathrm{Dom}(H),\ f_2 \in W^{1,2}(\mathring{M}).$$

In particular, for all $f \in \mathrm{Dom}(H)$ one has $\Delta f \in L^2(M)$ with $Hf = -\Delta f$.

Remark 1.2 It is a standard fact that the intrinsic distance

$$\varrho_{\mathrm{intr}}(x, y) := \sup\{f(x) - f(y) : f \in W^{1,2}_{\mathrm{loc}}(M) \cap C(M), |\nabla f| \leqslant 1\},$$

is equal to ϱ, where $W^{1,2}_{\mathrm{loc}}(M)$ is defined as all μ-equivalence classes of Borel functions f on M such that $f|_U \in W^{1,2}(\mathring{U})$ for all open relatively compact $U \subset M$.

The following space will be in the center of this work:

Definition 1.3 We call

$$\hat{C}^\infty_c(M) := \{f \in C^\infty_c(M) : \partial_\nu f = 0\}$$

the space of *Neumann cut-off functions on M*.

Clearly, $\hat{C}^\infty_c(M)$ is dense in $L^p(M)$ for all $p \in [1, \infty)$. Defining $\hat{W}^{1,p}_0(M)$ to be the closure of $\hat{C}^\infty_c(M)$ in $W^{1,p}(\mathring{M})$, we obtain:

Theorem 1.4 *If M is complete and $p \in (1, \infty)$, then one has $W^{1,p}(\mathring{M}) = \hat{W}^{1,p}_0(M)$.*

Proof As noted above, $C^\infty_c(M)$ is dense in $W^{1,p}(\mathring{M})$. It remains to show that any $f \in C^\infty_c(M)$ can be approximated by a sequence f_n in $\hat{C}^\infty_c(M)$. To this end, we are going to carefully adapt the arguments which lead to the proof of Theorem 7.2.1 in [6] to our geometric setting.
First of all, using Fermi coordinates, every $y \in \partial M$ has a chart (U_y, ϕ_y) with $\phi_y(U_y) = D_y \times [0, \delta_y)$, where D_y is a disc centered at 0 in $\mathbb{R}^{m-1}$, $\phi_y(y) = 0$ and, for every $u \in D_y$, $t \mapsto \phi_y^{-1}(u, t)$ is a normal geodesic issuing from $\phi_y^{-1}(u, 0) \in \partial M$. By compactness, $\mathrm{supp}\, f \cap \partial M$ is covered by finitely many sets $\frac{1}{2}U_i = \phi_{y_i}^{-1}(\frac{1}{2}D_{y_i}) \times [0, \delta_{y_i}/2)$, $i = 1, \ldots, l$. Similarly, for every point x in $\mathrm{supp} f \setminus \cup_{i=1}^l U_i$, exists a chart (U_x, ϕ_x) such that $U_i \cap \partial M = \emptyset$. Again by compactness, there exist finitely many such charts (U_i, ϕ_i), $i = l + 1, \ldots, l'$ such that $U_i \cap \partial M = \emptyset$ and $\{\frac{1}{2}U_i\}$, $i = l + 1, \ldots, l'$ cover $\mathrm{supp} f \setminus \cup_{i=1}^l U_i$. The usual proof (see, e.g. [14, Proposition 2.5]), shows that there exists a partition of unity ρ_i, $i = 1, \ldots, l'$, subordinate to the covering $\{U_i\}$, $i = 1, \ldots, l'$.

Since $f = \sum_{i=1}^{l'} \rho_i f$, it suffices to show that for every $i = i, \ldots, l$ the function $\rho_i f$ can be approximated by functions in $\hat{C}_c^\infty(M)$ in the $W^{1,p}$ norm.
To this end, let f be a C^∞ function with compact support in a chart (U_y, ϕ_y) with the properties listed above. Next, let $h_n : \mathbb{R} \to \mathbb{R}$ be a smooth function with $h_n(s) = 0$ if $s \leqslant 1/n$, $h_n(s) = s$ if $s \geqslant 2/n$ and $0 \leqslant h_n' \leqslant 3$, and define $\tau_n : U_y \to U_y$ by $\tau_n(x) = \phi_y(y, h_n(s))$ if $x = \phi_y(y, s)$ with $(y, s) \in D_y \times [0, \delta_y)$. Finally, define $f_n(x) = f(\tau_n(x))$ and note that by construction f_n is smooth, compactly supported in U_y and

$$f_n(\phi(y, s)) = f_n(\phi(y, 0)) = f(y) \text{ if } y \in D_y,\ s \in [0, 1/n),$$

so that $f_n \in \hat{C}_c^\infty(M)$. Since $|f - f_n|^p + |\nabla(f - f_n)|^p$ is compactly supported, bounded above and converges to zero pointwise, it follows by dominated convergence that

$$\|f - f_n\|_p + \|\nabla(f - f_n)\|_p \to 0 \text{ as } n \to \infty,$$

as required to complete the proof.

□

Next we are going to prove:

Theorem 1.5 (a) *M is complete, if and only if there exists a sequence of* first order cut-off functions on M, *that is, a sequence $(\chi_n) \subset C_c^\infty(M)$, such that*

(i) $0 \leqslant \chi_n \leqslant 1$,
(ii) *for each compact $K \subset M$ there exists $n_K \in \mathbb{N}$ with $\chi_n \equiv 1$ for all $n \geqslant n_K$,*
(iii) $\|\nabla \chi_n\|_\infty = O\left(\frac{1}{n}\right)$ *as $n \to \infty$.*

(b) *If M is complete, then there exists a sequence of* first order Neumann cut-off functions, *that is, a sequence $(\chi_n) \subset C_c^\infty(M)$ of first order cut-off functions such that in addition*

(iv) $\partial_\nu \chi_n = 0$.

Proof (a) Assume M is complete. According to [18, Theorem A] we can realize M as a domain in a complete Riemannian manifold N without boundary. It is well-known that any such N admits sequences of first order cut-off functions: for example, it goes back to [7] that by smoothing the distance function (to a fixed reference point) one obtains a proper smooth function $f : N \to \mathbb{R}$ with bounded gradient. Then $\tilde{\chi}_n(x) := \psi(f(x)/n)$, where $\psi : \mathbb{R} \to [0, 1]$ is smooth with compact support and equal to 1 near 0, does the job for N. Finally $\chi_n := \tilde{\chi}_n|_M$ does the job for N.
If conversely M admits a sequence of first order cut-off functions, one can follow [17, proof of Theorem 2.29] to see that for a fixed $o \in M$ one has $\varrho(x, o) \to \infty$ as $x \to \infty$. We give the simple proof: Pick a compact $K \subset M$ which includes o and pick $n \in \mathbb{N}$ large enough with $\chi_n = 1$ on K and $\|\nabla \chi_n\|_\infty \leqslant 1/n$. It follows that for all $x \in M \setminus \operatorname{supp}(\chi_n)$ and all piecewise smooth curves $\gamma : [0, 1] \to M$ from o to x one has

$$1 = |\chi_n(\gamma(0)) - \chi_n(\gamma(1))| \leqslant L(\gamma)/n,$$

so that taking the infimum over such curves we have shown that for all n there exists a compact K_n such that for all $x \in M \setminus K_n$ one has $\varrho(x, o) \geqslant n$. (b) Let $\{\tilde{\chi}_n\}$ be a sequence of first order cut-off functions in M. We are going to use the idea employed in the previous theorem to modify each $\tilde{\chi}_n$ in such a way that the normal derivative of the modified functions χ_n vanishes and the other properties of cut-offs still hold.
To achieve this, for every n consider a geodesic ball B_n centered at a fixed point $o \in \partial M$ such that $\varrho(\partial_0 B_n, \operatorname{supp} \tilde{\chi}_n) \geq 1$. Note that this implies that $\varrho(\partial M \setminus B_n, \operatorname{supp}, \tilde{\chi}_n) \geqslant 1$. By compactness, there exists $0 < \delta_n \leqslant 1/2$ such that the Fermi coordinates ϕ_n, which map (y, t) to $\gamma_y(t)$, where γ_y is the unit speed normal geodesic issuing from $y \in \partial M \cap B_n$, are a diffeomorphism of $(\partial M \cap B_n) \times [0, \delta_n)$ onto an open nbd of $\partial M \cap B_n$ in M. Moreover, since $|\mathrm{d}\phi_n| = 1$ on $\partial M \cap B_n$, by taking a smaller δ_n we may arrange that $|d\phi_n| \leqslant 2$ on $(\partial M \cap B_n) \times [0, \delta_n)$.
Next, for every n let $h_n : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$h_n(s) = \begin{cases} 0 & \text{if } s \leqslant \delta_n/4 \\ s & \text{if } s \geqslant \delta_n/2, \end{cases} \quad \text{and} \quad 0 \leqslant h_n'(s) \leqslant 3,$$

and define a map $\tau_n : B_n \to M$ by

$$\tau_n(x) = \begin{cases} \phi_n(y, h_n(s)) & \text{if } x = \phi_n(y, s) \in U_n \\ \tau_n(x) = x & \text{otherwise.} \end{cases}$$

Notice that:

(j) $\tau_n(x) = x$ if $\varrho(x, \partial M) \geq \delta_n/2$,
(jj) if $x \in B_n$ is such that $\varrho(x, \partial_0 B_n) > \delta_n\ \partial_0$ and $\varrho(x, \partial M) < \delta_n/2$, then the minimizing geodesic joining x to ∂M lies entirely in B_n,
(jjj) and therefore τ_n is defined and smooth in

$$U_n := \{x \in B_n : \varrho(x, \partial_0 B_n) > \delta_n\},$$

which is an open nbd of supp χ_n,
(jv) $|\mathrm{d}\tau_n| \leqslant 6$.

Now define $\chi_n(x) = \tilde{\chi}_n(\tau_n(x))$ for $x \in U_n$ and extend it with 0 in the complement of U_n. The above considerations show that χ_n is smooth and that

$$\operatorname{supp}(\tilde{\chi}_n - \chi_n) \subseteq \{x \in U_n : \varrho(x, \partial M) < \delta_n/2\},$$

so that χ_n is compactly supported. Moreover,

$$\|\nabla \chi_n\|_\infty = \|\nabla \tilde{\chi}_n \circ \mathrm{d}\tau_n\|_\infty \leqslant 6\, \|\nabla \tilde{\chi}_n\|_\infty \to 0 \text{ as } n \to \infty,$$

and, by construction $\partial_\nu \chi_n = 0$ on ∂M. It remains to show that property (iii) in the statement holds. To this end, let K be a compact set in M and let B_R be a ball of radius R centered at o such that $\overline{B_{R-1}} \supseteq K$. Since $\tilde{\chi}_n$ is a sequence of cut-offs, there exists $n = n_R$ such that $\tilde{\chi}_n = 1$ on $\overline{B_R}$ for every $n \geqslant n_R$.

We claim that for all such n's one has $\chi_n = 1$ on B_{R-1} and therefore on K, as required to complete the proof. Indeed, by (j) above, $\chi_n(x) = \tilde{\chi}_n(x)$ if $x \in U_n$ and $\varrho(x, \partial M) \geqslant \delta_n/2$. It follows that $\chi_n(x) = \tilde{\chi}_n(x) = 1$ if $x \in B_R$ and $\varrho(x, \partial M) \geqslant \delta_n/2$. On the other hand, if $x \in B_{R-1}$ and $\varrho(x, \partial M) \leqslant \delta_n/2$ the unique minimizing geodesic joining x to ∂M lies entirely in B_R and since $\tilde{\chi}_n$ is identically equal to 1 on its image, so is also χ_n. In particular, $\chi_n(x) = 1$ and the claim is proved. ◻

We note that the above construction can be used to give an alternative proof of Theorem 1.4. However, the given proof of Theorem 1.4 seems to be somewhat simpler.

Remark 1.6 By Theorem A in [18], M can be realized as a domain in a smooth Riemannian manifold N without boundary, which can be chosen complete if M is so. Then:

(α) every $f \in \hat{C}_c^\infty(M)$ can be extended to a function in $C_c^\infty(N)$,
(β) every $f_o \in C_c^\infty(\partial M)$ can be extended to a function in $\hat{C}_c^\infty(M)$.

To see (α), given $f \in \hat{C}_c^\infty(M)$, let V be a relatively compact open set in ∂M containing $\operatorname{supp} f \cap \partial M$. By compactness, the Fermi coordinates are a diffeomorphism of $V \times (-r, r)$ onto a tubular neighborhood U_r of V and it follows that there is a chart (U_r, φ) of N such that $\varphi(U_r) = D \times (-r, r)$ and $\varphi(U_r \cap M) = D \times [0, r)$, where D is a disk in $\mathbb{R}^{m-2}$, in such a way that the normal derivative corresponds to differentiation in the last variable. Using a result of Seeley, [19], and multiplying by a function $h(t)$ which is $= 0$ if $t < -r/2$ and $= 1$ if $t > -r/4$ we extend $f \circ \varphi^{-1}$ to a smooth function defined in $\varphi(U_r)$ which lifted to N provides the required extension $\tilde{f}$ of f. Note that Seeley's construction implies that there exists a constant C which depends only on the the geometry of V and on r such that $||\tilde{f}||_{W^{2,2}} \leqslant C||f||_{W^{2,2}}$.
The proof of (β) is similar but easier: $f_o \in C_c^\infty(\partial M)$ and $V \subset \partial M$ is a relatively compact neighborhood of its support, we can extend f_o to a function on N which is compactly supported in U_r by lifting the function defined on $V \times (-r, r)$ by $f(u, t) = f_o(u)h(t)$, where $h(t)$ is a smooth cutoff on $\mathbb{R}$ such that $h(t) = 1$ if $|t| < r/4$ and $h(t) = 0$ if $|t| > r/2$.

We go on to define $\hat{\mathscr{W}}_0^{2,p}(M)$ to be the closure of $\hat{C}_c^\infty(M)$ with respect to the norm $\|f\|_p + \|\Delta f\|_p$.

Remark 1.7 For all $f \in \hat{\mathscr{W}}_0^{2,p}(M)$ one has $f, \Delta f \in L^p(M)$. To see this, note first that clearly $\hat{\mathscr{W}}_0^{2,p}(M)$ is a subspace of $L^p(M)$, as every sequence in $\hat{C}_c^\infty(M)$ which is Cauchy in $\hat{\mathscr{W}}_0^{2,p}(M)$ is also Cauchy in $L^p(M)$. Given $f \in \hat{\mathscr{W}}_0^{2,p}(M)$, $\phi \in C_c^\infty(\mathring{M})$, pick a sequence $f_n \in \hat{C}_c^\infty(M)$ with $f_n \to f$ in $\hat{\mathscr{W}}_0^{2,p}(M)$. Then Δf_n is a Cauchy sequence in $L^p(M)$, and with $h \in L^p(M)$ the limit of Δf_n in $L^p(M)$ we have

$$\langle f, \Delta\phi\rangle = \lim_n \langle f_n, \Delta\phi\rangle = \lim_n \langle \Delta f_n, \phi\rangle = \langle h, \phi\rangle ,$$

and so $\Delta f \in L^p(M)$.

We close this section with the following embedding result:

Theorem 1.8 *Let* $1 < p \leqslant 2$. *Then for all* $f \in \hat{\mathscr{W}}_0^{2,p}(M)$ *one has*

$$\|\nabla f\|_p^2 \leqslant (p-1)^{-1} \|\Delta f\|_p \|f\|_p < \infty. \tag{1.1}$$

In particular, there is a continuous embedding $\hat{\mathscr{W}}_0^{2,p}(M) \subset \hat{W}_0^{1,p}(M)$, *the norm of which depends only on* p.

Proof It is sufficient to prove the asserted estimate for all $f \in \hat{C}_c^\infty(M)$. If the boundary of M is empty and M is complete, then this estimate can be found in [5], and in its ultimate form with an explicit constant in [12]. If the boundary of M is empty and M is not complete, then by Theorem A in [18] one can realize an open relatively compact and smooth neighbourhood of the support of f as a domain in a complete Riemannian manifold without boundary to obtain the result. Finally, in this way, if the boundary of M is nonempty, then the estimate follows from the first part of Remark 1.6. □

2 Essential self-adjointness of the Neumann-Laplacian

The main result of this section is:

Theorem 2.1 *Let* M *be complete. Then* $-\Delta$ *with domain of definition* $\hat{C}_c^\infty(M)$ *is essentially self-adjoint in* $L^2(M)$; *in particular,* H *is the unique self-adjoint extension of this operator.*

To the best of our knowledge, this is the first result that deals with the essential self-adjointness of the Laplacian on a noncompact Riemannian manifold with boundary. The only mildly related result we are aware of is [16], which deals with complex Riemannian manifolds that have a certain symmetry and that satisfy some additional analytic assumptions; there, the authors prove - with completely different methods - the essential self-adjointness of the Laplacian induced by the $\overline{\partial}$ operator.
In the proof of Theorem 2.1 we will use the following lemma, whose content is that the domain of H is invariant under multiplication by functions in $\hat{C}_c^\infty(M)$:

Lemma 2.2 *Let* $f \in \mathrm{Dom}(H)$ *and let* $\chi \in \hat{C}_c^\infty(M)$. *Then* $f\chi \in W^{1,2}(\mathring{M})$, $\Delta(f\chi) = \chi\Delta f + 2(\nabla f, \nabla\chi) + f\Delta\chi \in L^2(M)$ *and, for all* $\phi \in W^{1,2}(\mathring{M})$,

$$\int_M \phi\Delta(\chi f)\,\mathrm{d}\mu = -\int_M (\nabla\phi, \nabla(\chi f))\,\mathrm{d}\mu.$$

It follows that $\chi f \in \mathrm{Dom}(H)$.

Proof of Lemma 2.2 Clearly $\chi f \in W^{1,2}(\mathring{M})$ and a computation shows that, for every $\phi \in C_c^\infty(\mathring{M})$,

$$\begin{aligned}\int_M \chi f \Delta\phi \,\mathrm{d}\mu &= -\int_M (\nabla\phi, \chi\nabla f + f\nabla\chi)\,\mathrm{d}\mu \\ &= -\int_M (\nabla(\chi\phi), \nabla f)\,\mathrm{d}\mu + \int_M \phi(\nabla\chi, \nabla f)\,\mathrm{d}\mu + \int_M \phi\,\mathrm{div}(f\chi)\,\mathrm{d}\mu \\ &= \int_M \phi[\chi\Delta f + 2(\nabla\chi, \nabla f) + f\Delta\chi]\,\mathrm{d}\mu,\end{aligned}$$

where we have used that $\chi\phi \in W^{1,2}(\mathring{M})$, $f \in \mathrm{Dom}(H)$ and that $f\nabla\chi$ is a $W^{1,2}$ vector field with compact support in $\mathring{M}$. Thus

$$\Delta(\chi f) = \chi\Delta f + 2(\nabla\chi, \nabla f) + f\Delta\chi \in L^2(M).$$

Next let $\phi \in W^{1,2}(\mathring{M})$; using the fact that $\chi f \in W^{1,2}(\mathring{M})$, $f\phi \in W^{1,1}(\mathring{M})$ and that $\partial_\nu\chi = 0$ we obtain

$$\begin{aligned}\int_M \phi\Delta(\chi f)\,\mathrm{d}\mu &= \int_M [(\chi\phi)\Delta f + 2(\nabla f, \phi\nabla\chi) + f\phi\Delta\chi]\,\mathrm{d}\mu \\ &= \int_M [-(\nabla f, \nabla(\chi\phi)) + 2(\nabla f, \phi\nabla\chi) - (\nabla(f\phi), \nabla\chi)]\,\mathrm{d}\mu \\ &\quad + \int_{\partial M} f\phi\partial_\nu\chi\,\mathrm{d}\mu \\ &= -\int_M (\nabla\phi, \nabla(\chi f))\,\mathrm{d}\mu,\end{aligned}$$

as required to conclude the proof of the lemma. □

Proof of Theorem 2.1 We are going to show that $\hat{C}_c^\infty(M)$ is an operator core for H. To this end, our proof will follow Chernoff's philosophy [3] of using the wave equation for proving essential self-adjointness (see also [9, 10]). However, in the presence of boundary, some additional care has to be taken in order obtain enough regularity for functions in the intersection of the spaces $\mathrm{Dom}(H^k)$.

Step 1: For all $t > 0$, $f \in L^2(M)$ one has $\mathrm{e}^{-tH} f \in C^\infty(M)$ with $\partial_\nu(\mathrm{e}^{-tH} f) = 0$.

Proof of step 1: By the spectral calculus we have

$$h := \mathrm{e}^{-tH} f \in \bigcap_{k=0}^{\infty} \mathrm{Dom}(H^k) \subset \{f \in W^{1,2}(\mathring{M}) : \Delta^k f \in L^2(M) \quad \text{for all } k\},$$

and it satisfies the weak Neumann boundary condition

$$\langle \Delta h, \phi\rangle = \langle \nabla h, \nabla\phi\rangle \ ,$$

for all $\phi \in W^{1,2}(\mathring{M})$.

Clearly h is smooth in $\mathring{M}$ due to local interior elliptic regularity and Sobolev embedding. We need to prove that h is smooth up to the boundary and that $\partial_\nu h = 0$. So we fix an arbitrary $x \in \partial M$ and pick a relatively compact coordinate neighbourhood $N' \subset M$ of x such that $N := \overline{N'}$ is a smooth manifold with boundary. Let also $V \Subset N'$ be a neighborhood of x and let $\chi, \rho \in C^\infty(M)$ by such that $\chi = 1$ on V, $\rho = 1$ on $\operatorname{supp}\chi$ and $\operatorname{supp}\rho \Subset N'$. Since $h \in \operatorname{Dom}(H)$, by Lemma 2.2,

$$h_0 := \chi h \in W^{1,2}(\mathring{M}), \quad \Delta h_0 =: h_1 \in L^2(M),$$

and the weak Neumann boundary condition holds

$$\langle \Delta h_0, \phi\rangle = \langle \nabla h_0, \nabla\phi\rangle \quad \text{for all } \phi \in W^{1,2}(\mathring{M}).$$

Thus, given $\phi_0 \in W^{1,2}(\mathring{N})$, $\rho\phi_0 \in W^{1,2}(\mathring{M})$ and we deduce that

$$\int_N \phi_0 h_1 \,\mathrm{d}\mu = \int_M \Delta h_0(\rho\phi_0)\,\mathrm{d}\mu = -\int_M (\nabla h_0, \nabla(\rho\phi_0))\,\mathrm{d}\mu = \int_N (\nabla h_0, \nabla\phi_0)\,\mathrm{d}\mu.$$

with $h_1 = \Delta h_0 \in L^2(N)$.
It follows from [15, Theorem 5.9] that $h_0 \in W^{2,2}(\mathring{N})$. Bootstrapping this argument leads to $h_0 \in W^{k,2}(\mathring{N})$ for all k, and so $h_0 \in C^\infty(\mathring{N})$ by Sobolev embedding and so $h \in C^\infty(M)$. Finally, the weak Neumann boundary condition and Green's formula in combination with the second part of Remark 1.6 give $\partial_\nu h = 0$.

Step 2: The space

$$\operatorname{Dom}_c(H) := \{f \in \operatorname{Dom}(H) : f \text{ has a compact support}\}$$

is an operator core for H, that is, $\operatorname{Dom}_c(H)$ is dense in the norm $\|f\|_H := \|f\| + \|Hf\|$ in $\operatorname{Dom}(H)$.

Proof of step 2: Note first that by Remark 1.2, Theorems A.1 and A.3 we get that for all $U_1, U_2 \subset M$ open and disjoint, $f_1, f_2 \in L^2(M)$ with $\operatorname{supp}(f_i) \subset U_i$, $i = 1, 2$, and all $0 < s < \varrho(U_1, U_2)$, one has

$$\langle \cos(s\sqrt{H}) f_1, f_2\rangle = 0. \tag{2.1}$$

The latter fact implies that if $\operatorname{supp}(f) \subset B(x_0, r)$ for some $x_0 \in M$, $r > 0$, then for any $t > 0$ one has

$$\operatorname{supp}(\cos(t\sqrt{H})f) \subset \overline{B(x, r+t)}, \tag{2.2}$$

and this closed ball is compact due to the completeness of M (since completeness on M is equivalent to the Heine–Borel property of M [18]). Now the claim follows from an abstract functional analytic fact (cf. Theorem A.2).

Step 3: $\hat{C}_c^\infty(M)$ is dense in $\operatorname{Dom}_c(H)$ in the norm $\|\cdot\|_H$.

Proof of step 3: let $f \in \mathrm{Dom}_c(H)$. Pick a function $\chi \in C_c^\infty(M)$ with $\partial_\nu \chi = 0$ and $\chi = 1$ on $\mathrm{supp}(f)$. By step 1 we have $\chi e^{-tH} f \in \hat{C}_c^\infty(M)$ and

$$\left\| \chi e^{-tH} f - f \right\| = \left\| \chi \left(e^{-tH} f - f \right) \right\| \to 0$$

as $t \to 0+$. Moreover

$$\begin{aligned} H(\chi e^{-tH} f - f) &= H \left[\chi \left(e^{-tH} f - f \right) \right] \\ &= \chi \left(H e^{-tH} f - Hf \right) - 2(\nabla \chi, \nabla (e^{-tH} f - f)) \\ &\quad - (H\chi)(e^{tH} f - f), \end{aligned}$$

which shows that $\left\| H(\chi e^{-tH} f - f) \right\| \to 0$ as $t \to 0+$, completing the proof. □

Example 2.3 Let us show that one cannot drop completeness in Theorem 2.1. Pick $M := [1, \infty)$, $\psi := 1$, and fix the Riemannian metric on M as $g(x) := 1/x^4$. Then, the metric distance between two points p, q is given by $\varrho(p, q) = \mathrm{sgn}(q-p)(1/p - 1/q)$ and M is not complete (the sequence $p_n = n$ is Cauchy but does not converge to any point in M). The Laplace–Beltrami operator is given in this case by

$$-\Delta f = -x^2 \frac{\mathrm{d}}{\mathrm{d}x} \left(x^2 f \right).$$

Chose c_1 and c_2 such that

$$\hat{f}(x) := c_1 \cosh(1/x) + c_2 \sinh(1/x)$$

satisfies $\frac{\mathrm{d}\hat{f}}{\mathrm{d}x}_{|x=1} = 0$. It is straightforward to check that $-\Delta \hat{f} = -\hat{f}$. By the boundedness of $\hat{f}$, it follows that $\hat{f}, \Delta \hat{f} \in L^2(\mathring{M})$, and by the Neumann boundary condition at $x = 1$ it follows that

$$\langle -\Delta h, \hat{f} \rangle = \langle h, -\Delta \hat{f} \rangle \quad \text{for every } h \in \hat{C}_c^\infty(M).$$

That is, for H_0 defined as $-\Delta$ with domain of definition $\hat{C}_c^\infty(M)$ one has $\hat{f} \in \mathrm{Dom}(H_0^*)$ and $H_0^* \hat{f} = -\Delta \hat{f}$. As a consequence, $(H_0 + 1)^* \hat{f} = 0$. Therefore, $(H_0 + 1)^* = H_0^* + 1$ has a nontrivial kernel and we conclude that H_0 is not essentially self-adjoint in $L^2(M)$.

Appendix A

Let M be locally compact separable metrizable space with μ a Radon measure on M with full support.

Theorem A.1 (Theorem 3.4 in [4]) *Let H be a self-adjoint nonnegative operator in $L^2(M,\mu)$. Then the following two conditions are equivalent for every distance ϱ on M which induces the original topology:*

(i) *There are constants $C \geqslant 1$, $a > 0$ such that for all $t > 0$, all open $U_1, U_2 \subset M$, and all $f_1, f_2 \in L^2(M,\mu)$ with $\mathrm{supp}(f_i) \subset U_i$, $i = 1, 2$, one has*

$$\left|\left\langle \mathrm{e}^{-tH} f_1, f_2 \right\rangle\right| \leqslant C \mathrm{e}^{at} \mathrm{e}^{-\varrho(U_1,U_2)^2/(4t)} \|f_1\| \|f_2\| . \tag{A.1}$$

(ii) *For all open $U_1, U_2 \subset M$, and all $f_1, f_2 \in L^2(M,\mu)$ with $\mathrm{supp}(f_i) \subset U_i$, $i = 1, 2$, one has*

$$\left\langle \cos\left(s\sqrt{H}\right) f_1, f_2 \right\rangle = 0 \text{ for all } 0 < s < \varrho(U_1, U_2).$$

Note that property (i) above is usually referred to as *Davies-Gaffney bound*, while (ii) is referred to as *finite wave propagation speed.*

Before we state the next theorem, we recall given a self-adjoint operator $H \geqslant 0$ in $L^2(M,\mu)$ and $t \in \mathbb{R}$ one defines the unitary operator $\cos(t\sqrt{H})$ via spectral calculus. Then $\cos(t\sqrt{H})$ preserves $\mathrm{Dom}(H)$, and for all $f \in \mathrm{Dom}(H)$ the function $f(t) := \cos(t\sqrt{H})$ solves the abstract wave equation $f''(t) = -Hf(t)$.

Theorem A.2 (Theorem 3 in [9]) *Let $H \geqslant 0$ be a self-adjoint operator in $L^2(M,\mu)$, and assume furthermore that the compactly supported elements $\mathrm{Dom}_c(H)$ of $\mathrm{Dom}(H)$ are dense in $L^2(M,\mu)$ and that for all $t > 0$ one has the mapping property*

$$\cos(t\sqrt{H}) : \mathrm{Dom}_c(H) \longrightarrow \mathrm{Dom}_c(H).$$

Then $\mathrm{Dom}_c(H)$ is an operator core for H.

Theorem A.3 (Theorem 0.1 in [21]) *Let $\mathscr{E}$ be a strongly local and regular Dirichlet form in $L^2(M,\mu)$, with $H \geqslant 0$ the associated self-adjoint operator, and with Γ the associated energy measure. Assume also that the intrinsic distance*

$$\varrho_{\mathrm{intr}}(x, y) := \sup\{f(x) - f(y) : f \in \mathrm{Dom}_{\mathrm{loc}}(\mathscr{E}) \cap C(M), \mathrm{d}\Gamma(f) \leqslant \mathrm{d}\mu\}$$

induces the original topology on M. Then H satisfies (A.1) with $C = 1$ and $a = 0$.

Note that, in general, ϱ_{intr} is only a pseudo metric, that is, it can attain the value $+\infty$ or may be degenerate (but nevertheless induces a topology).

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