Letter

Cauchy universality and random billiards

Roberto Artuso [1,2,*] and Dario Javier Zamora [1,3,†]
[1] *Center for Nonlinear and Complex Systems and Dipartimento di Scienza e Alta Tecnologia, Via Valleggio 11, 22100 Como, Italy*
[2] *I.N.F.N., Sezione di Milano, Via Celoria 16, 20133 Milano, Italy*
[3] *Instituto de Fisica del Noroeste Argentino, CONICET and Universidad Nacional de Tucuman, Av. Independencia 1800, Tucuman, CP 4000, Argentina*

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Motion in bounded domains represents a paradigm in several settings: from billiard dynamics, to random walks in a finite lattice, with applications to relevant physical, ecological, and biological problems. A remarkable universal property, involving the average of return times to the boundary, has been theoretically proposed and experimentally verified in quite different contexts. We discuss here mechanisms that lead to violations of universality, induced by boundary effects and we also emphasize the role played by replacing straight lines with random walks in this framework. We suggest that our analysis should be relevant where nonhomogeneity appears in the stationary probability distribution in bounded domain.

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What do neutrons and ants have in common? A possible answer, generalizing a geometric result established by Cauchy [1], is that their average residence time $\langle\tau\rangle$ in a d-dimensional bounded domain Ω does not depend on the nature of their dynamics, nor on the shape of the bounded region; it is proportional to the ratio between the volume V_Ω and the surface Σ_Ω of Ω: $\langle\tau\rangle = \eta_d V_\Omega/\Sigma_\Omega = \tau_C$, where η_d is a numerical factor depending only on space dimensionality d (in the case we will consider in detail $d = 2$ and $\eta_2 = \pi$) [2].

More precisely, the Cauchy theorem concerns the case where $\langle\tau\rangle$ is the mean length of randomly distributed chords intersecting a convex body: geometric generalizations of this results have been discussed since then (see [3–5]; in particular the Cauchy theorem has been extended to nonconvex bodies in [6]). From a physical perspective, Cauchy theorem plays an important role in evaluating the residence time of neutrons (assuming they have a constant speed v_0) in a bounded region, provided their density is homogeneous and isotropic, and the medium is nonscattering [7–9].

An unexpected breakthrough came from two independent studies (in wildly different contexts) [10,11]: in [10] it was claimed that the average chord length is preserved even if the bounded medium is scattering (so the trajectories of entering particles are not straight lines but random paths), while the same result was stated in [11], when considering the average time spent by ants injected in a bounded domain before they leave it (see also [12]). This appears very surprising since, intuitively, if we replace straight segments (or arcs) with erratic trajectories, we expect that a twofold mechanism deeply modifies the dynamics (stochastic short returns to the boundary, and long wandering walks in the domain without touching the boundary); see [13]. A considerable effort has been conveyed in checking under which conditions this generalized Cauchy universality holds [14–16], in particular the importance of homogeneous probability distribution inside the domain and of detailed balance in the scattering kernel have been pointed out. Remarkably, Cauchy universality has been recently supported by experimental results, in bacterial motion [17], and light propagation in scattering media [18,19].

On the other side, complex dynamics in bounded domains is sometimes associated with nonhomogeneous distributions, like in the case of active particles (see for instance [20–22]), or in the presence of interactions with the walls [23], so a natural question is whether Cauchy universality is maintained when nonuniform densities are present, or boundary effects are taken into account.

This is the main point we are going to discuss, providing examples where boundary conditions lead to nonuniform stationary probability distribution and violations of Cauchy universality; we will consider both the case of rectilinear trajectories between successive collisions with the boundary (billiard) and the case where instead particles move in the interior of the body in a stochastic fashion (random walk).

In order to provide a simple geometric setting (which is typical of experimental realizations, too [17]) we will take Ω as the unit, $2d$, disk, and consider a particle moving with a constant speed ($v_0 = 1$) inside Ω. To fix the dynamics we have to specify (i) the way the particle moves inside Ω after a collision with the boundary and (ii) the rules prescribing the outgoing angle, once the ingoing angle is known, at a collision. Both prescriptions can be either deterministic or stochastic: the invariance of the average chord length (which in our units is $\tau_C = \pi/2$) has been verified up to now for both

*Contact author: roberto.artuso@uninsubria.it
†Contact author: dariojavier.zamora@uninsubria.it

billiard and random walk cases for elastic (specular reflection) boundary collisions [10,11,14–16]. While considering the distribution of random chords for a disk is a well defined procedure, recasting it in a billiard framework requires some care, since a circular billiard table with elastic reflections defines an integrable system. Before going on let us fix our notation. We will denote by $\mathbf{x}$ a generic point in the interior of Ω, and by ψ the angle between the speed of the particle and a fixed direction; in this way $(\mathbf{x}_t, \psi_t)$ are the coordinates of the point at time t in the continuous time flow. When considering the collision map instead (mapping the position from one collision with the boundary to the next), the relevant coordinates will be denoted by $\mathbf{q} \in \partial\Omega$ and $\theta \in [-\pi/2, \pi/2]$ outgoing angle with respect to the normal vector $\mathbf{n_q}$ (pointing inward) at $\mathbf{q}$ (alternatively we may use the the angle $\phi \in [0, 2\pi]$, of the outgoing speed with respect to the oriented tangent at $\mathbf{q}$ ($\phi = \theta + \pi/2$). In the case of a circular billiard table (with specular reflections), if we start a trajectory at $\mathbf{q}_0$ with outgoing angle θ_0, chords will be all equal to $\tau_{\theta_0} = 2\cos(\theta_0)$ (in our units $R = 1$). This is not due to an unfortunate choice of the billiard table, since all sufficiently smooth strictly convex billiard tables are nonergodic [24] (for general references on billiard dynamics, see [25–27]). We can, however, average over the initial outgoing angle by choosing an appropriate measure

$$\overline{\tau} = \int_{-\pi/2}^{\pi/2} \mu(d\theta_0)\tau_{\theta_0}. \qquad (1)$$

It is easy to check that we can recover the Cauchy result ($\overline{\tau} = \tau_C = \pi/2$) if we choose $\mu(\theta_0) = \rho(\theta_0)d\theta_0 = \frac{1}{2}\cos(\theta_0)d\theta_0$. This is indeed the invariant distribution for the outgoing angle for a chaotic billiard with a uniform stationary distribution for the continuous flow (with specular reflections) [26,28]; it also corresponds to the angular dependence of the flux generated by a uniform and isotropic distribution of particles entering the bounded region from outside. We remark that (1) holds for arbitrary shapes (in particular nonconvex) of the billiard table; indeed $\overline{\tau}$ is recovered as a time average only under ergodicity (to our knowledge the interpretation of the average chord as a Birkhoff sum for a billiard map has been pointed out for the first time in [29]).

An alternative derivation of this result (that provides the first example of our general setting) is to consider a random billiard, where the deterministic rule for the outgoing angle, at a collision point with the boundary, is replaced by a probability distribution density $\mathcal{P}_\mathbf{q}(\theta|\theta_{in})$ (which in principle may depend, or not, on the collision point $\mathbf{q}$ and the ingoing angle). The Knudsen case [30–32] corresponds to the choice (Lambert reflections)

$$\mathcal{P}_\mathbf{q}(\theta|\theta_{in}) = \mathcal{P}_K(\theta) = \tfrac{1}{2}\cos(\theta). \qquad (2)$$

In this framework, checking for Cauchy universality consists of replacing rectilinear motion between collisions with a stochastic process, for instance, in [11,15] a Pearson random walk is considered: the particle travels with a constant speed for a random time t (or distance, since $v = 1$) extracted from an exponential distribution

$$\mathcal{Q}_E(t) = \frac{1}{\lambda}\exp(-t/\lambda). \qquad (3)$$

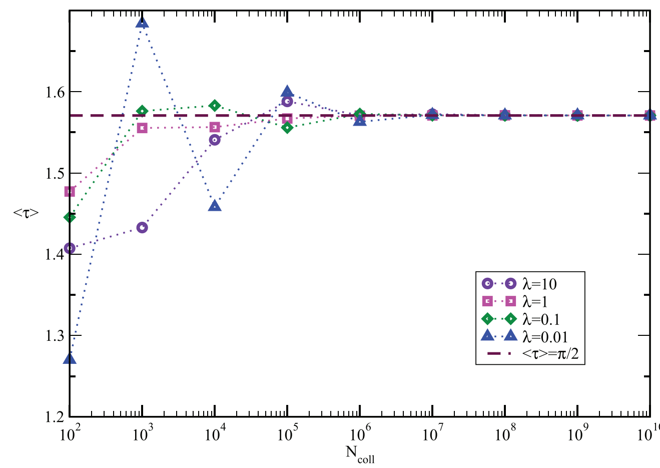

FIG. 1. Time average $\langle\tau\rangle$ of the time between collisions, for increasing number of collisions, computed for different values of the parameter of the exponential distribution (3) λ, and elastic reflections. The dashed line corresponds to τ_C.

At the end of the rectilinear walk (at space point $\mathbf{x}$), the direction of the velocity is randomly reoriented, according to a chosen distribution density. In this paper (following [11,15]) we will employ the random reorientation; the new direction ψ_+ is independent of the old orientation ψ_-: ψ_+ is a random variable uniform in $[0, 2\pi]$ [33]. If the exponential excursion crosses $\partial\Omega$, it is reflected back according to the boundary conditions we are considering.

We first check the case of elastic boundary conditions (specular reflections). In this case we already know Cauchy universality holds, but this numerical experiment is useful to gauge how the asymptotic result is reached by increasing the statistics, see Fig. 1.

There are two physical motivations that suggest considering deviations from specular reflections: the case in which the agent is not properly modeled by a point particle (see for instance [34]), and possible roughness of the boundary [35,36]. Other physical situations are modeled by nonstandard boundary conditions as well [37]. The first example we study is that of a fully random billiard [38] (Evans random billiard), where the particle moves ballistically between collisions, while the outgoing angle θ is a random variable, independent on the ingoing angle, uniformly distributed in $[-\pi/2, \pi/2]$. For this random billiard, the dynamics enjoys strong ergodic properties [38,39]; we mention that such a billiard also has an independent interest in sampling problems of convex sets [40]. The average chord between collisions is easily computed as in (1), where now $\mu_R(d\theta_0) = d\theta_0/\pi$, corresponding to a collision rule

$$\mathcal{P}_\mathbf{q}(\theta|\theta_{in}) = \mathcal{P}_R(\theta) = \frac{1}{\pi}. \qquad (4)$$

As a matter of fact,

$$\overline{\tau} = \tau_R = \int_{-\pi/2}^{\pi/2} \mu_R(d\theta_0)\tau_{\theta_0} = \frac{4}{\pi}, \qquad (5)$$

which is different from the Cauchy value. Actually, the identity (5), together with estimates for other geometric shapes, has been discussed in the framework of integral geometry, see [41,42], where the average chord theorem is discussed, with an emphasis on different notions of "random set of chords." In terms of billiard dynamics, the elastic reflection law corresponds to what has been called μ randomness, while Evans random billiard is referred to as ν randomness [42,43]. Random reflections have also been considered in the problem of mean free path of electrons moving in a wire [44]. Since the Lambert form of the invariant measure (2) is strictly associated to a uniform stationary density for the billiard flow, we expect that Evans random billiards will present a nonuniform stationary probability distribution in space, as we show now. We will denote the invariant probability density for the flow as $\varrho(\mathbf{x}, \phi)$ [45]. In all the cases we will consider rotational invariance is preserved (in biological settings however more complex patterns may arise [46]), so

$$\varrho(\mathbf{x}, \phi) = \frac{1}{2\pi} g(r), \tag{6}$$

where r is the distance of $\mathbf{x}$ from the center of the disk Ω. In this way, $\Phi(r) = 2\pi r g(r)$ is the stationary probability distribution of the distance from the origin: for a uniform spatial probability distribution [$g(r)$ constant], we have $\Phi(r) = 2r$.

Now consider a chord [of length $2\cos(\theta_0)$] corresponding to an outgoing angle θ_0: the values of r along the chord range from $\sin(\theta_0)$ to one, and a uniform distribution along the chord leads to the corresponding r density

$$\mathcal{W}_{\theta_0}(r) = \frac{1}{\cos(\theta_0)} \frac{r}{\sqrt{r^2 - \sin^2(\theta_0)}} \quad r \in [\sin(\theta_0), 1]. \tag{7}$$

Now we evaluate the average of this expression by using the appropriate measure, and by taking into account that the single θ_0 contribution has to be weighted by the ratio of the chord length and the average τ_R:

$$\wp_R(\theta_0) = \frac{2\cos(\theta_0)}{4/\pi}, \tag{8}$$

so

$$\Phi_R(r) = \int_0^{\arcsin(r)} \mu_R(d\theta_0) \wp_R(\theta_0) \mathcal{W}_{\theta_0}(r). \tag{9}$$

By changing variable $r\sin(\theta_0) = \sin(\alpha)$, we get

$$\Phi_R(r) = r \int_0^{\pi/2} d\alpha \, \frac{1}{\sqrt{1 - r^2 \sin^2(\alpha)}} = rK(r^2), \tag{10}$$

where $K(s)$ is the complete elliptic integral of the first kind [47]. The expression Eq. (10) deviates from the linear behavior corresponding to a uniform density (and it has a logarithmic singularity as $r \to 1$, due to the complete elliptic integral), so the random billiard, for which the Cauchy formula does not hold has a nonuniform spatial probability distribution (see Fig. 2, first two lines), peaking close to the boundary.

Though our analytic argument in deriving Eq. (10) is not rigorous, the result can be validated by employing the findings in [38].

A natural question is then to check what happens when we substitute ballistic walks between collisions with an exponential random walk of parameter λ [Eq. (3)], with uniform direction resetting. This is illustrated in Fig. 3. As we see there is no universal behavior by varying λ, while when the average excursion grows, we are close to the random billiard estimate (5), while on the other side for very short average steps the mean chord grows. Our numerical data do not allow to conclude if a well defined value is reached in the limit $\lambda \to 0$.

Finally we consider another kind of boundary conditions, where possible roughness of the boundary is incorporated in a milder way than fully random reflections. The suggestion came from [48], where an ε stochastic perturbation of elastic boundary conditions was shown to lead to ergodic behavior for strictly convex billiard tables. More precisely, fix $\varepsilon \in [0, \pi/2]$, and denote by ϕ_{el} the outgoing angle (measured with respect to the oriented tangent), determined by the specular reflection law. The random elastic perturbed billiard is then defined by making the outgoing angle a random variable, uniformly distributed in the interval $[\phi_{el} - \varepsilon, \phi_{el} + \varepsilon]$, for

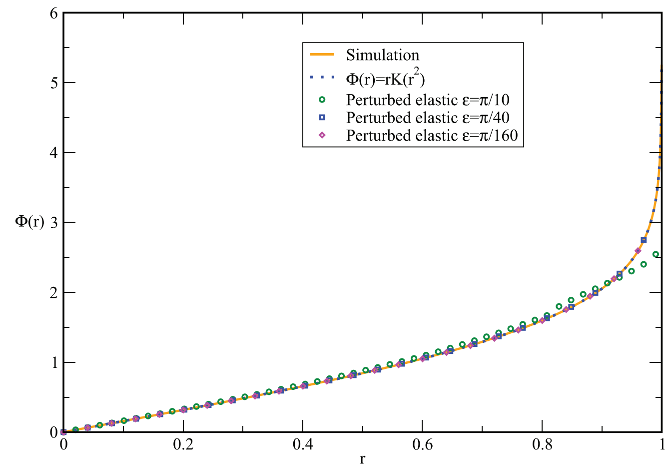

FIG. 2. Radial probability for the random billiard and perturbed elastic reflections (see text). The numerical distributions were obtained by discretizing the trajectories with a 10^{-4} time step, considering 10^8 collisions, and binning into 2500 intervals the r range.

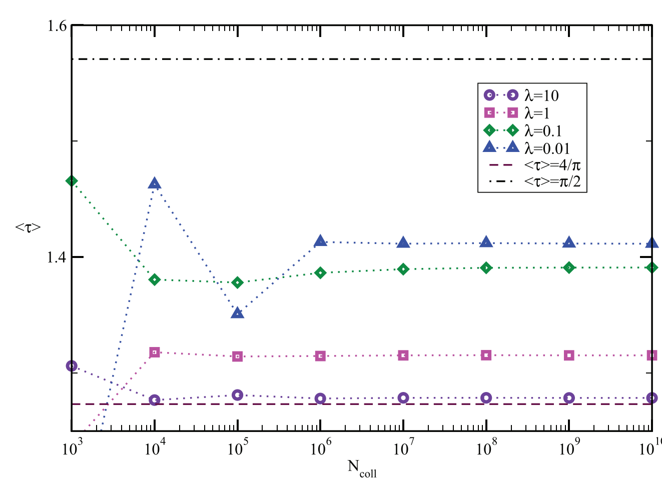

FIG. 3. Time average $\langle\tau\rangle$ of the time between collisions, for increasing number of collisions, computed for different values of the parameter of the exponential distribution (3) λ, for the random billiard. The dashed line corresponds to τ_R, while the dashed dotted line is Cauchy value τ_C.

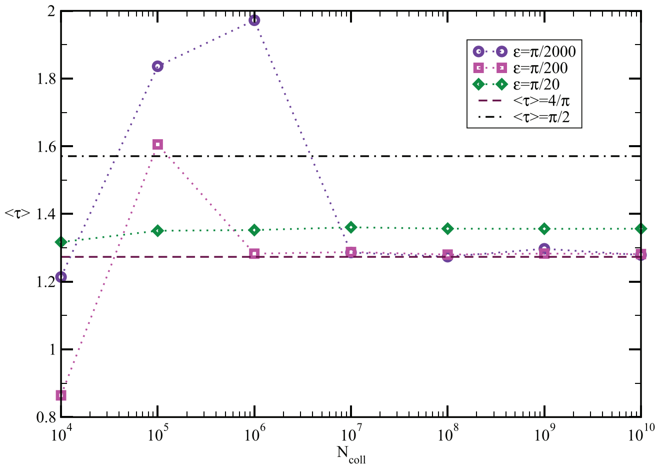

FIG. 4. Time average $\langle\tau\rangle$ of the time between collisions, for increasing number of collisions, computed for a billiard with random elastic perturbation (see text). For very small perturbations the results are very close to the fully random outgoing angle case. The dashed line corresponds to τ_R, while the dashed dotted line corresponds to the Cauchy value τ_C.

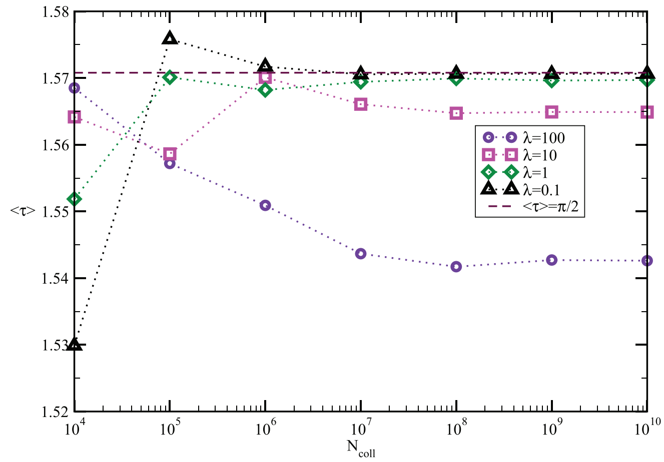

FIG. 5. Time average $\langle\tau\rangle$ of the time between collisions, for increasing number of collisions, computed for weakly stochastically perturbed elastic perturbation, $\varepsilon = \pi/200$ (see text), and an exponential random walk between collisions. For λ not too big, the results are very close to the Cauchy estimate. The dashed line corresponds to τ_C.

$\phi_{el} > \varepsilon$ and $\phi_{el} < \pi - \varepsilon$; the rule must be modified when ϕ_{el} is sufficiently close to the tangent to avoid orbits leaving the region Ω. A possible choice is [48] to reset ϕ_{el} to ε when $\phi_{el} < \varepsilon$, with the new outgoing angle uniformly distributed in $[0, 2\varepsilon]$ (and the analogous prescription when ϕ_{el} is close to π). One expects that for very small ε one should recover the universal behavior, since the elastic reflection law is only slightly perturbed. Numerical experiments, however, suggest that instead the behavior is closer to the fully random billiard (see Fig. 2); the radial distribution displays the same logarithmic (weak) singularity close to the boundary. This somehow surprising result is confirmed by simulations on the average chord, see Fig. 4. When the perturbation is very small the simulations are very close to the random billiard value (5). A theoretical support for such findings comes from considering the reduced discrete dynamics for the outgoing angle $\phi_n \to \phi_{n+1}$. Away from small intervals around 0 and π, the stochastic dynamics is equivalent to a random walk with a uniform and symmetric jump distribution of width ε, so, away from the boundaries we expect a uniform stationary distribution [49].

These features completely change when, while maintaining a stochastic perturbation of elastic reflections at the boundary, we turn from billiards to random walks (we still consider path segments generated by an exponential distribution of parameter λ followed by a uniform random redirection of the velocity direction). A complete analysis is outside the scope of the present paper, we just point out when the mean free path λ is sufficiently small, Cauchy universality is recovered (but the average chord does vary on increasing λ); see Fig. 5.

In conclusion, we have considered the average chord problem for either deterministic or stochastic motion in a bounded domain, for which remarkable universal properties have been proposed. Inspired by possible complex mechanisms altering the specular reflection law for complex agents, we have considered random boundary conditions of different types. Typically under these conditions Cauchy universality is violated, and this is associated to the appearance of nonuniform spatial densities. In general, when random walks are considered instead of billiard ballistic motion between collisions, the picture modifies, and, in particular, when considering slightly stochastically perturbed elastic reflections, in the small average jump regime Cauchy universality is restored. This may be relevant for other physically important problems of motion in confined systems, like the narrow escape problem [50], since enhanced spatial density close to the boundary increases the probability of hitting the target. Random reflections may also be relevant when considering reversible sticking to $\partial\Omega$ (see for instance [51–53]), since when a particles sticks to the boundary, it is natural to consider the release outgoing angle as random, uncorrelated to the incoming direction.

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