

# A CHARACTERIZATION OF $(\mathcal{I}, \mathcal{J})$ -REGULAR MATRICES

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**ABSTRACT.** Let  $\mathcal{I}, \mathcal{J}$  be two ideals on  $\mathbf{N}$  which contain the family  $\text{Fin}$  of finite sets. We provide necessary and sufficient conditions on the entries of an infinite real matrix  $A = (a_{n,k})$  which maps  $\mathcal{I}$ -convergent bounded sequences into  $\mathcal{J}$ -convergent bounded sequences and preserves the corresponding ideal limits. The well-known characterization of regular matrices due to Silverman–Toeplitz corresponds to the case  $\mathcal{I} = \mathcal{J} = \text{Fin}$ .

Lastly, we provide some applications to permutation and diagonal matrices, which extend several known results in the literature.

## 1. INTRODUCTION

Let  $\omega$  be the set of all real sequences indexed by the positive integers  $\mathbf{N}$ , and let  $A = (a_{n,k})$  be an infinite real matrix. A sequence  $x \in \omega$  is said to be  $A$ -summable if the sequence  $Ax = (\sum_k a_{n,k}x_k)$  is well defined and it is convergent, i.e.,  $Ax \in c$ . The matrix  $A$  is called **regular** if every convergent sequence is  $A$ -summable and it preserves the limit, i.e.,  $\lim Ax = \lim x$  for all  $x \in c$ . A classical result of Silverman–Toeplitz characterizes the class of regular matrices:

**Theorem 1.1.** *A matrix  $A$  is regular if and only if:*

- (S1)  $\sup_n \sum_k |a_{n,k}| < \infty$ ;
- (S2)  $\lim_n a_{n,k} = 0$  for each  $k$ ;
- (S3)  $\lim_n \sum_k a_{n,k} = 1$ .

The aim of this work is to generalize Theorem 1.1 in the context of ideal convergence.

Recall that an ideal  $\mathcal{I} \subseteq \mathcal{P}(\mathbf{N})$  is a family of subsets of  $\mathbf{N}$  closed under taking finite unions and subsets. Unless otherwise stated, it is also assumed that  $\mathcal{I}$  is admissible, i.e., it contains the family of finite sets  $\text{Fin}$  and  $\mathcal{I} \neq \mathcal{P}(\mathbf{N})$ . Let  $\mathcal{I}^* = \{A \subseteq \mathbf{N} : A^c \in \mathcal{I}\}$  be its dual filter. An important example of ideal is the family of asymptotic density zero sets, that is,

$$\mathcal{Z} = \left\{ S \subseteq \mathbf{N} : \lim_{n \rightarrow \infty} \frac{1}{n} |S \cap [1, n]| = 0 \right\}.$$

A sequence  $x \in \omega$  is  $\mathcal{I}$ -convergent to  $\eta$ , shortened as  $\mathcal{I}\text{-}\lim x = \eta$ , if  $\{n \in \mathbf{N} : |x_n - \eta| > \varepsilon\} \in \mathcal{I}$  for all  $\varepsilon > 0$ ;  $\mathcal{Z}$ -convergence is often referred to as statistical convergence in the summability theory literature. We let  $c(\mathcal{I})$  be the vector space of  $\mathcal{I}$ -convergent sequences and  $c_0(\mathcal{I})$  be its subspace of sequences with  $\mathcal{I}$ -limit 0. Structural properties of the set of bounded  $\mathcal{I}$ -convergent sequences  $c(\mathcal{I}) \cap \ell_\infty$  and its subspace  $c_0(\mathcal{I}) \cap \ell_\infty$  have been recently studied in the literature, sometimes providing answers to longstanding questions, see e.g. [5, 18, 22, 24].

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2010 *Mathematics Subject Classification.* Primary: 40A35, 40G15. Secondary: 54A20, 40A05.

*Key words and phrases.* Ideal and statistical convergence; summability; regular matrices; convergent sequences.

Given sequence spaces  $X, Y \subseteq \omega$ , we let  $(X, Y)$  be the set of infinite real matrices  $A$  such that  $Ax$  is well defined and belongs to  $Y$  for all  $x \in X$ . Accordingly, a matrix  $A$  is regular if  $A \in (c, c)$  and preserves the (ordinary) limits. The relationship between ideal convergence and matrix summability has been recently studied in [16].

With these premises, we can now state the main definition of this work: Given ideals  $\mathcal{I}, \mathcal{J}$  on  $\mathbf{N}$ , a matrix  $A$  is said to be **( $\mathcal{I}, \mathcal{J}$ )-regular** if  $A \in (c(\mathcal{I}) \cap \ell_\infty, c(\mathcal{J}) \cap \ell_\infty)$  and

$$\forall x \in c(\mathcal{I}) \cap \ell_\infty, \quad \mathcal{I}\text{-}\lim x = \mathcal{J}\text{-}\lim Ax.$$

Considering that  $c = c(\text{Fin}) \cap \ell_\infty$ , Theorem 1.1 amounts to a characterize the class of  $(\text{Fin}, \text{Fin})$ -regular matrices.

**Theorem 1.2.** *Let  $\mathcal{I}, \mathcal{J}$  be ideals on  $\mathbf{N}$ . Then a matrix  $A$  is  $(\mathcal{I}, \mathcal{J})$ -regular provided that*

- (T1)  $\sup_n \sum_k |a_{n,k}| < \infty$ ;
- (T2)  $\mathcal{J}\text{-}\lim_n \sum_k a_{n,k} = 1$ ;
- (T3)  $\mathcal{J}\text{-}\lim_n \sum_{k \in E} |a_{n,k}| = 0$  for all  $E \in \mathcal{I}$ .

*Conversely, if  $A$  is  $(\mathcal{I}, \mathcal{J})$ -regular, then it satisfies (T1) and (T2).*

A nontrivial problem is to characterize when conditions (T1)–(T3), which depend only on the entries of  $A$ , provide a full characterization of  $(\mathcal{I}, \mathcal{J})$ -regularity, that is, equivalently, when an  $(\mathcal{I}, \mathcal{J})$ -regular matrix satisfies condition (T3).

On the positive side, we have the following result.

**Theorem 1.3.** *Let  $\mathcal{I}, \mathcal{J}$  be ideals on  $\mathbf{N}$  such that  $A$  is nonnegative or  $\mathcal{I} = \text{Fin}$  or  $\mathcal{J} = \text{Fin}$ . Then  $A$  is  $(\mathcal{I}, \mathcal{J})$ -regular if and only if it satisfies (T1)–(T3).*

This provides a generalization to [13, Theorem 3.1], which studies the case where  $A$  is nonnegative and, in addition, it belongs to  $(\ell_\infty, \ell_\infty)$ , cf. Theorem 2.1 below. If  $A$  is nonnegative, note that conditions (T2)–(T3) may be simplified together into

$$(T4) \quad \mathcal{J}\text{-}\lim_n \sum_{k \in I} a_{n,k} = 1 \text{ for all } I \in \mathcal{I}^*.$$

On the negative side, it turns out that an  $(\mathcal{I}, \mathcal{J})$ -regular matrix does not necessarily satisfy condition (T3). To the best of authors' knowledge, this is the first result of this type.

**Theorem 1.4.** *Let  $\mathcal{I}$  be an ideal on  $\mathbf{N}$  such that  $\text{Fin} \subsetneq \mathcal{I} \subseteq \mathcal{Z}$ . Then there exists an  $(\mathcal{I}, \mathcal{Z})$ -regular matrix  $A$  which does not satisfy (T3).*

A remarkable example of ideal  $\mathcal{I}$  which satisfies the hypotheses of Theorem 1.4 is the ideal of uniform density zero sets  $\{S \subseteq \mathbf{N} : \lim_{n \rightarrow \infty} \max_{k \geq 0} \frac{1}{n} |S \cap [k+1, k+n]| = 0\}$ , cf. [3].

Lastly, note that Theorem 1.2 is also analogous to [9, Theorem 5.1], which studies the case where both  $\mathcal{I}$  and  $\mathcal{J}$  are assumed to be ideals generated by nonnegative regular matrices. However, it is known by [4, Proposition 13] that, if we regard ideals as subsets of the Cantor space  $\{0, 1\}^{\mathbf{N}}$  endowed with the product topology, then the ideal

$$\mathcal{I}_A := \{S \subseteq \mathbf{N} : \lim_n \sum_{k \in S} a_{n,k} = 0\}$$

generated by a regular matrix  $A$  is necessarily an  $F_{\sigma\delta}$ -ideal, and hence [9, Theorem 5.1] only applies to a restricted class of ideals.

Several other applications for permutations and diagonal matrices have been studied in the literature [11, 12, 14], see Section 3 below.

Related results which characterize matrix classes of the type  $(c(\mathcal{I}) \cap \ell_\infty, c(\mathcal{J}) \cap \ell_\infty)$  (hence, without preserving necessarily their ideal limits) can be found e.g. in [19, 20, 23, 27].

## 2. MAIN PROOFS

Hereafter,  $\ell_\infty$  and any of its subspaces are endowed with the supremum norm. The following result is well known, see e.g. [8, Theorem 2.3.5]:

**Theorem 2.1.**  *$A \in (\ell_\infty, \ell_\infty)$  if and only if  $A \in (c_0, \ell_\infty)$  if and only if  $A$  satisfies (T1).*

The next result which will be the key tool in the proof of our main result. Related results can be found in [15, Proposition 6.3] (for the case of  $F_\sigma$ -ideals  $\mathcal{I}, \mathcal{J}$ ) and in [10, Theorem 3.7] (for the case  $\mathcal{I} = \mathcal{Z}$  and  $\mathcal{J} = \text{Fin}$ ).

**Lemma 2.2.** *Let  $\mathcal{I}, \mathcal{J}$  be ideals on  $\mathbf{N}$ . If a matrix  $A$  satisfies (T1) and (T3) then  $A \in (c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{J}) \cap \ell_\infty)$ , which in turn implies that  $A$  satisfies (T1).*

*Proof.* First, let us prove that if  $A$  satisfies (T1) and (T3) then  $A \in (c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{J}) \cap \ell_\infty)$ . Fix  $x \in c_0(\mathcal{I}) \cap \ell_\infty$ . By condition (T1) and Theorem 2.1, we get  $A \in (\ell_\infty, \ell_\infty)$ , hence  $Ax$  is a well-defined bounded sequence. At this point, fix  $\varepsilon > 0$  and define

$$\delta := \frac{\varepsilon}{2 \max\{1, M, \|x\|\}} > 0,$$

where  $M := \sup_n \sum_k |a_{n,k}|$  and  $\|x\| = \sup_k |x_k|$ . Note that, since  $x \in c_0(\mathcal{I})$  then  $K := \{n \in \mathbf{N} : |x_n| > \delta\} \in \mathcal{I}$ . By condition (T3), it follows that  $S := \{n \in \mathbf{N} : \sum_{k \in K} |a_{n,k}| > \delta\} \in \mathcal{J}$ . Now observe that if  $\sum_k |a_{n,k}x_k| > \varepsilon$  then

$$\sum_{k \in K} |a_{n,k}x_k| > \frac{\varepsilon}{2} \quad \text{or} \quad \sum_{k \notin K} |a_{n,k}x_k| > \frac{\varepsilon}{2}.$$

If it is the case that  $\sum_{k \in K} |a_{n,k}x_k| > \frac{\varepsilon}{2}$  then

$$\frac{\varepsilon}{2} < \sum_{k \in K} |a_{n,k}| |x_k| \leq \|x\| \sum_{k \in K} |a_{n,k}|$$

and hence  $\delta \leq \varepsilon/2 \max\{1, \|x\|\} < \sum_{k \in K} |a_{n,k}|$ , i.e.,  $n \in S$ . In the event that  $\sum_{k \notin K} |a_{n,k}x_k| > \frac{\varepsilon}{2}$ , it follows that

$$\frac{\varepsilon}{2} < \sum_{k \notin K} |a_{n,k}| |x_k| \leq \delta \sum_{k \notin K} |a_{n,k}| \leq \delta M \leq \frac{\varepsilon}{2},$$

which is a contradiction, hence  $\{n \in \mathbf{N} : |\sum_k a_{n,k}x_k| > \varepsilon\} \subseteq S \in \mathcal{J}$ . Since  $\varepsilon$  is arbitrary, we conclude that  $\mathcal{J}\text{-lim } Ax = 0$ .

To prove the second part of the statement, it is sufficient to observe that

$$(c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{J}) \cap \ell_\infty) \subseteq (c_0, \ell_\infty)$$

and to use Theorem 2.1. □

**Lemma 2.3.** *Let  $\mathcal{I}, \mathcal{J}$  be ideals on  $\mathbf{N}$ . Then  $A$  is  $(\mathcal{I}, \mathcal{J})$ -regular if and only if  $A \in (c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{J}) \cap \ell_\infty)$  and (T2) holds.*

*Proof.* IF PART. Suppose that  $A \in (c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{J}) \cap \ell_\infty)$  and satisfies condition (T2). Fix a sequence  $x \in c(\mathcal{I}) \cap \ell_\infty$  such that  $\mathcal{I}\text{-lim } x = \eta$ . It follows that the sequence  $y$  defined by  $y_n := x_n - \eta$  for all  $n$  belongs to  $c_0(\mathcal{I}) \cap \ell_\infty$ , so that  $Ay \in c_0(\mathcal{J}) \cap \ell_\infty$ . Therefore

$$0 = \mathcal{J}\text{-lim } Ay = \mathcal{J}\text{-lim } Ax - \eta \cdot \mathcal{J}\text{-lim}_n \sum_k a_{n,k},$$

which implies that  $Ax \in c(\mathcal{J}) \cap \ell_\infty$  and  $\mathcal{J}\text{-lim } Ax = \eta$ .

**ONLY IF PART.** Suppose that  $A$  is a  $(\mathcal{I}, \mathcal{J})$ -regular matrix. In particular, it is clear that  $A \in (c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{J}) \cap \ell_\infty)$ . Finally, let  $x$  be the constant sequence 1. Then  $\lim_n x_n = 1$ , so that  $Ax = (\sum_k a_{n,k}) \in c(\mathcal{J}) \cap \ell_\infty$  and  $\mathcal{J}\text{-lim}_n \sum_k a_{n,k} = 1$ , i.e., condition **(T2)** holds.  $\square$

*Proof of Theorem 1.2.* It follows putting together Lemma 2.2 and Lemma 2.3.  $\square$

*Proof of Theorem 1.3.* Thanks to Theorem 1.2, it is sufficient to show that, in each case, an  $(\mathcal{I}, \mathcal{J})$ -regular matrix satisfies condition **(T3)**. Note that if  $A$  is  $(\mathcal{I}, \mathcal{J})$ -regular, then  $\mathcal{J}\text{-lim } Ae^k = 0$  by condition **(T2)**, where  $e^k$  is the  $k$ th unit vector of  $\ell_\infty$  for each  $k \in \mathbf{N}$ , i.e.,  $\mathcal{J}\text{-lim}_n |a_{n,k}| = 0$ . This implies that

$$\forall m \in \mathbf{N}, \quad \lim_n \sum_{k \leq m} |a_{n,k}| = 0. \quad (1)$$

First, suppose that  $A$  is a nonnegative  $(\mathcal{I}, \mathcal{J})$ -regular matrix and fix  $E \in \mathcal{I}$ . Since  $\mathbf{1}_E \in c_0(\mathcal{I}) \cap \ell_\infty$ , we obtain that  $\mathcal{J}\text{-lim } A\mathbf{1}_E = \mathcal{J}\text{-lim}_n \sum_{k \in E} a_{n,k} = 0$ .

Second, thanks to (1), condition **(T3)** holds if  $\mathcal{I} = \text{Fin}$ .

Third, let  $A$  be an  $(\mathcal{I}, \text{Fin})$ -regular matrix with  $\mathcal{I} \neq \text{Fin}$ , and hence  $\mathcal{J} = \text{Fin}$  and  $\mathcal{J}\text{-lim}$  is the ordinary limit. Suppose for the sake of contradiction that condition **(T3)** fails. Hence there exists an infinite set  $I \in \mathcal{I}$  such that

$$\limsup_n \sum_{k \in I} |a_{n,k}| \neq 0. \quad (2)$$

Accordingly, let  $(i_n)$  be the increasing enumeration of  $I$ . Taking into account (2) and the fact that  $\sum_{k \in I} |a_{n,k}| \leq \sum_k |a_{n,k}| \leq \sup_i \sum_i |a_{i,k}|$  for all  $n \in \mathbf{N}$ , it follows that the bounded sequence  $(\sum_{k \in I} |a_{n,k}|)$  has at least one accumulation point, let us say  $\kappa \in [0, \sup_n \sum_k |a_{n,k}|]$ , that is,  $\kappa$  verifies  $\{n \in \mathbf{N} : |\sum_{k \in I} |a_{n,k}| - \kappa| < \varepsilon\} \notin \text{Fin}$  for all  $\varepsilon > 0$ . In addition, thanks to (2), we may assume that  $\kappa \neq 0$ . This implies that

$$S := \{n \in \mathbf{N} : \frac{7}{8}\kappa < \sum_{k \in I} |a_{n,k}| < \frac{9}{8}\kappa\} \notin \text{Fin}.$$

With these premises, we are going to construct a sequence  $x \in \{-1, 0, 1\}^{\mathbf{N}}$  supported on  $I$  (hence, in particular,  $x \in c_0(\mathcal{I}) \cap \ell_\infty$ ) such that the sequence  $Ax$  is (well defined and) not convergent to 0. This would provide the desired contradiction.

Thus, we define two increasing sequences  $(s_n)$  and  $(m_n)$  of positive integers as it follows. Fix arbitrarily  $s_1 \in S$  and choose  $m_1 \in \mathbf{N}$  such that  $\sum_{j \leq m_1} |a_{s_1, i_j}| \geq \frac{7}{8}\kappa$ . At this point, suppose that  $s_1, \dots, s_{n-1}$  and  $m_1, \dots, m_{n-1}$  have been defined. Then choose  $s_n$  and  $m_n$  recursively such that:

- (i)  $s_n \in S \setminus [1, s_{n-1}]$  and  $\sum_{j \leq m_{n-1}} |a_{s_n, i_j}| \leq \frac{1}{8}\kappa$  (which is possible, thanks to (1));
- (ii)  $m_n > m_{n-1}$  and  $\sum_{j \leq m_n} |a_{s_n, i_j}| \geq \frac{7}{8}\kappa$ .

Thus, for each  $n \in \mathbf{N}$ , define

$$\alpha_n := \sum_{j \leq m_{n-1}} |a_{s_n, i_j}|, \quad \beta_n := \sum_{m_{n-1} < j \leq m_n} |a_{s_n, i_j}|, \quad \text{and} \quad \gamma_n := \sum_{j > m_n} |a_{s_n, i_j}|.$$

According to points (i) and (ii) above and the definition of  $S$ , we have

$$\forall n \in \mathbf{N}, \quad \alpha_n \leq \frac{1}{8}\kappa, \quad \alpha_n + \beta_n \geq \frac{7}{8}\kappa, \quad \text{and} \quad \frac{7}{8}\kappa < \alpha_n + \beta_n + \gamma_n < \frac{9}{8}\kappa. \quad (3)$$

To conclude, let  $x = (x_n)$  be the sequence supported on  $I$  such that  $x_{i_j} = 1$  if there exists  $n \in \mathbf{N}$  for which  $j \in (m_{n-1}, m_n]$  and  $a_{s_n, i_j} > 0$ , where  $m_0 := 1$ ; otherwise  $x_{i_j} = -1$ . It follows by construction that

$$(Ax)_{s_n} = \sum_{i \in I} a_{s_n, i} x_i = \sum_{j \leq m_{n-1}} a_{s_n, i_j} x_{i_j} + \sum_{m_{n-1} < j \leq m_n} |a_{s_n, i_j}| + \sum_{j > m_n} a_{s_n, i_j} x_{i_j}$$

for all  $n \in \mathbf{N}$ . Hence, thanks to (3), we obtain

$$\begin{aligned} \forall n \in \mathbf{N}, \quad |(Ax)_{s_n}| &\geq \beta_n - \alpha_n - \gamma_n \\ &= 2(\alpha_n + \beta_n) - 2\alpha_n - (\alpha_n + \beta_n + \gamma_n) > \frac{3}{8}\kappa. \end{aligned}$$

Hence  $\{n \in \mathbf{N} : |(Ax)_n| > \frac{3}{8}\kappa\}$  contains the infinite set  $\{s_n : n \in \mathbf{N}\} \notin \mathcal{J} = \text{Fin}$ , which implies that  $\lim Ax \neq 0$ . This contradiction concludes the proof.  $\square$

*Proof of Theorem 1.4.* The proof has two main steps. First we construct a matrix  $A \in (\ell_\infty, c_0(\mathcal{Z}) \cap \ell_\infty)$  which fails (T3), and then, using  $A$ , construct a matrix  $B$  that is  $(\mathcal{I}, \mathcal{Z})$ -regular and fails (T3).

For each  $m \in \mathbf{N}$ , let  $\lambda_m$  be the smallest nonnegative integer  $t$  such that  $2^t \geq m!$ , so that

$$\forall m \in \mathbf{N}, \quad m! \leq 2^{\lambda_m} = 2^{\lceil \log_2 m! \rceil} \leq 2 \cdot m!.$$

Set  $R := \bigcup_m R_m$ , where  $R_m := \{m!, m! + 1, \dots, m! + 2^{\lambda_m} - 1\}$  for all  $m \in \mathbf{N}$ . In particular,  $R_1 = \{1, 2\}$ ,  $R_2 = \{2, 3\}$ ,  $R_3 = \{6, 7, \dots, 13\}$ , etc., so that  $\max R_m < \min R_{m+1}$  for all  $m \geq 2$ . By hypothesis  $\mathcal{I} \neq \text{Fin}$ , hence there exists an infinite set  $I \in \mathcal{I} \setminus \text{Fin}$ . Let  $(i_j)$  be the increasing enumeration of the elements of  $I$  and define

$$\forall m \in \mathbf{N}, \quad Q_m := R_m \times C_m,$$

where  $C_m := \{i_{\alpha_{m-1}+1}, i_{\alpha_{m-1}+2}, \dots, i_{\alpha_m}\}$ ,  $\alpha_m := \sum_{i \leq m} \lambda_i$ , and, by convention,  $\alpha_0 := 0$ .

At this point, let us define the matrix  $A$  such that

$$\forall n, k \in \mathbf{N}, \quad |a_{n,k}| = \begin{cases} 1/m & \text{if there exists } m \in \mathbf{N} \text{ such that } (n, k) \in Q_m, \\ 0 & \text{otherwise,} \end{cases}$$

and, in addition, for all  $m \in \mathbf{N}$ , the vectors of signs of each row of  $Q_m$

$$(\text{sgn}(a_{n, i_{\alpha_{m-1}+1}}), \text{sgn}(a_{n, i_{\alpha_{m-1}+2}}), \dots, \text{sgn}(a_{n, i_{\alpha_m}}))$$

are all distinct (note this is possible; here  $\text{sgn}(z) := 1$  if  $z > 0$  and  $\text{sgn}(z) = -1$  if  $z < 0$ ).

Now, we claim that  $A \in (\ell_\infty, c_0(\mathcal{Z}) \cap \ell_\infty)$ . Let  $x$  be a nonzero bounded sequence. Then  $Ax$  is the bounded sequence such that  $(Ax)_n$  is equal to 0 if  $n \notin R$  and  $\frac{1}{m} \sum_k \text{sgn}(a_{n,k}) x_k$  for all  $n \in R_m$  and  $m \in \mathbf{N}$ . For each  $\varepsilon > 0$ , it follows that

$$\{n \in \mathbf{N} : |(Ax)_n| > \varepsilon\} \subseteq R_1 \cup \bigcup_{m \geq 2} \left\{ n \in R_m : \frac{1}{m} \left| \sum_{k \in C_m} \text{sgn}(a_{n,k}) x_k \right| > \varepsilon \right\}. \quad (4)$$

In addition, by the weak law of large numbers we have that

$$\lim_{m \rightarrow \infty} \frac{1}{2^{\lambda_m}} \left| \left\{ n \in R_m : \frac{1}{m} \left| \sum_{k \in C_m} \text{sgn}(a_{n,k}) x_k \right| > \varepsilon \right\} \right| = 0,$$

cf. e.g. [6, Theorem 6.2]. Hence the upper asymptotic density (which is the function  $\mathbf{d}_g^*$  defined below in (5) with  $g(n) = n$  for all  $n$ ) of the latter union in (4) is at most

$$\limsup_{m \rightarrow \infty} \frac{\max R_{m-1} + o(2^{\lambda_m})}{\min R_m} \leq \limsup_{m \rightarrow \infty} \frac{2 \cdot (m-1)! + o(m!)}{m!} = 0.$$

Therefore  $\mathcal{Z}$ - $\lim Ax = 0$ .

Observe that  $R \notin \mathcal{Z}$ , indeed its upper asymptotic is at least

$$\limsup_{m \rightarrow \infty} \frac{|R \cap [1, \max R_m]|}{\max R_m} \geq \limsup_{m \rightarrow \infty} \frac{|R_m|}{m! + |R_m|} \geq \limsup_{m \rightarrow \infty} \frac{m!}{m! + 2 \cdot m! - 1} = \frac{1}{3}.$$

Therefore  $\mathcal{Z}$ - $\lim_n \sum_{k \in I} |a_{n,k}| \neq 0$ , i.e.,  $A$  does not satisfy (T3). However, with a similar reasoning, the matrix  $A$  does not satisfy (T2) as well. Hence, we need to modify slightly the definition of  $A$  to conclude the proof.

For, define  $B = (b_{n,k}) := A + \text{Id}$ , where  $\text{Id}$  stands for the (infinite) identity matrix. Then  $B \in (\ell_\infty, \ell_\infty)$  since  $\sup_n \sum_k |b_{n,k}| \leq 1 + \sup_n \sum_k |a_{n,k}| < \infty$ . Fix a sequence  $x \in c(\mathcal{I}) \cap \ell_\infty$  with  $\eta := \mathcal{I}$ - $\lim x$ . Thus  $Bx$  is a well-defined bounded sequence. In addition, since  $A \in (\ell_\infty, c_0(\mathcal{Z}) \cap \ell_\infty)$  and  $\mathcal{I} \subseteq \mathcal{Z}$ , we have

$$\mathcal{Z}$$
- $\lim Bx = \mathcal{Z}$ - $\lim Ax + \mathcal{Z}$ - $\lim x = \eta$ .

Therefore  $B$  is  $(\mathcal{I}, \mathcal{Z})$ -regular (in particular, differently from  $A$ , the matrix  $B$  satisfies (T2)). Lastly, note that it follows by construction that

$$\sum_{k \in I} |b_{n,k}| \geq \left(1 - \frac{1}{m}\right) + (m-1) \cdot \frac{1}{m} \geq 1$$

for all integers  $m \geq 2$  and  $n \in R_m$ . Since  $R \notin \mathcal{Z}$  (hence also  $R \setminus R_1 \notin \mathcal{Z}$ ), we conclude that  $\mathcal{Z}$ - $\lim_n \sum_{k \in I} |b_{n,k}| \neq 0$ . Therefore  $B$  does not satisfy (T3).  $\square$

Incidentally, note that matrix  $A$  defined above belongs to  $(\ell_\infty, c_0(\mathcal{Z}) \cap \ell_\infty)$ , hence also to  $(c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{Z}) \cap \ell_\infty)$ . This shows that the reverse implication of the first part of the statement of Lemma 2.2 does not hold as well.

### 3. APPLICATIONS

**3.1. Diagonal matrices.** Given a sequence  $s \in \omega$ , we denote by  $\text{diag}(s)$  the diagonal matrix  $A_s = (a_{n,k})$  defined by  $a_{n,k} = 0$  and  $a_{n,n} = s_n$  for all distinct  $n, k \in \mathbf{N}$ . Accordingly, given sequence space  $X, Y \subseteq \omega$ , we let  $m(X, Y)$  be the set of so-called **multipliers** from  $X$  into  $Y$ , i.e., the set of all sequences  $s \in \omega$  such that  $sx := (s_n x_n)$  is a well-defined sequence in  $Y$  for all  $x \in X$ , cf. [12]. In other words,

$$m(X, Y) := \{s \in \omega : \text{diag}(s) \in (X, Y)\}.$$

**Theorem 3.1.** *Let  $\mathcal{I}, \mathcal{J}$  be ideals on  $\mathbf{N}$ . Then*

$$m(c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{J}) \cap \ell_\infty) = \{s \in \ell_\infty : \mathcal{J}\text{-}\lim s \mathbf{1}_E = 0 \text{ for all } E \in \mathcal{I}\}.$$

*Proof.* First suppose that  $s \in \ell_\infty$  and  $\mathcal{J}$ - $\lim s \mathbf{1}_E = 0$  for all  $E \in \mathcal{I}$ . Then  $s \in \ell_\infty$  yields that  $\text{diag}(s)$  satisfies (T1) and, as  $\mathcal{J}$ - $\lim s \mathbf{1}_E = 0$  implies  $\mathcal{J}$ - $\lim |s| \mathbf{1}_E = 0$ , (T3) is satisfied and hence  $s \in m(c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{J}) \cap \ell_\infty)$  as  $\text{diag}(s) \in (c_0(\mathcal{I}) \cap \ell_\infty, (c_0(\mathcal{J}) \cap \ell_\infty))$ .

Conversely,  $\text{diag}(s) \in (c_0(\mathcal{I}) \cap \ell_\infty, (c_0(\mathcal{J}) \cap \ell_\infty))$  yields  $s \in \ell_\infty$ , by Lemma 2.2, and  $\mathcal{J}$ - $\lim s\mathbf{1}_E = 0$  for all  $E \in \mathcal{I}$  by the definition of a multiplier.  $\square$

At this point, note that, for each  $s \in \omega$ , the sequence  $s\mathbf{1}_E$  is supported on  $E$ . In particular  $\mathcal{J}$ - $\lim s\mathbf{1}_E = 0$  whenever  $E \in \mathcal{J}$ . This implies that:

**Corollary 3.2.**  $m(c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{J}) \cap \ell_\infty) = \ell_\infty$  for all ideals  $\mathcal{I} \subseteq \mathcal{J}$ .

Particular instances of Corollary 3.2 have been already obtained in [14, Theorem 3] where  $\mathcal{I} = \mathcal{J} = \mathcal{I}_A$  for some nonnegative regular matrix  $A$  and [12, Theorem 1 and Theorem 3] where  $\mathcal{I} = \mathcal{J}$  or  $\mathcal{I} = \text{Fin}$ .

**3.2. Permutations.** Given a permutation  $\sigma$  of  $\mathbf{N}$  and a sequence  $x \in \omega$ , we let  $\sigma(x)$  be the sequence defined by  $\sigma(x)_n := x_{\sigma^{-1}(n)}$  for all  $n \in \mathbf{N}$ , cf. e.g. [7, 25]. Equivalently,

$$\forall x \in \omega, \quad \sigma(x) = A_\sigma x,$$

where  $A_\sigma = (a_{n,k})$  is the matrix defined by  $a_{n,k} = 1$  if  $\sigma^{-1}(n) = k$  and  $a_{n,k} = 0$  otherwise. Accordingly, given sequence spaces  $X, Y \subseteq \omega$ , we write  $\sigma \in (X, Y)$  to mean  $A_\sigma \in (X, Y)$  (and similarly for  $(\mathcal{I}, \mathcal{J})$ -regularity). In addition, let  $\widehat{\sigma}$  be the sequence defined by

$$\forall n \in \mathbf{N}, \quad \widehat{\sigma}_n := n/\sigma^{-1}(n).$$

Note that  $\widehat{\sigma}$  already appeared in the literature: indeed, it has been shown in [26] that a permutation  $\sigma$  belongs to the Lévy group, i.e.,  $\sigma$  satisfies  $\lim_n \frac{1}{n} |\{k \in [1, n] : \sigma(k) > n\}| = 0$ , if and only if  $\mathcal{Z}$ - $\lim \widehat{\sigma}^{-1} = 1$ ; see also [26, Theorem 2.3] for a related result.

Before we state our result on permutations, we recall that an ideal  $\mathcal{I}$  is said to be a **simple density ideal** if there exists a function  $g : \mathbf{N} \rightarrow [0, \infty)$  such that  $\lim_n g(n) = +\infty$ ,  $\lim_n n/g(n) \neq 0$ , and  $\mathcal{I} = \mathcal{Z}_g := \{S \subseteq \mathbf{N} : \mathbf{d}_g^*(S) = 0\}$ , where  $\mathbf{d}_g^* : \mathcal{P}(\mathbf{N}) \rightarrow [0, \infty]$  is the function defined by

$$\forall S \subseteq \mathbf{N}, \quad \mathbf{d}_g^*(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap [1, n]|}{g(n)}. \quad (5)$$

In particular,  $\mathcal{Z}$  is the simple density ideal generated by  $g(n) = n$ , cf. [1, 21]. Note that the simple density ideal  $\mathcal{Z}_g$  for which  $n/g(n)$  is bounded does not necessarily coincide with  $\mathcal{Z}$ . Indeed, set  $S_k := [(2k)!, (2k+1)!)$  for all  $k \in \mathbf{N}$  and  $S := \bigcup_k S_{2k}$ . Define  $g(n) = n^2$  if  $n \in S$  and  $g(n) = n$  otherwise. Then it is easy to see that  $S \in \mathcal{Z}_g \setminus \mathcal{Z}$ . Hence  $\mathcal{Z}$  is properly contained in  $\mathcal{Z}_g$ .

Lastly, given an ideal  $\mathcal{I}$  and a sequence  $x \in \omega$ , we say that  $\eta \in \mathbf{R}$  is an  $\mathcal{I}$ -limit point of  $x$  if there exists a subsequence  $(x_{n_k})$  for which  $\lim_k x_{n_k} = \eta$  and  $\{n_k : k \in \mathbf{N}\} \notin \mathcal{I}$ , cf. [17].

The following result has been essentially proved for the case  $\mathcal{I} = \mathcal{J} = \mathcal{Z}$  in [11].

**Theorem 3.3.** *Let  $\sigma$  be a permutation of  $\mathbf{N}$  and let  $\mathcal{I}, \mathcal{J}$  be ideals on  $\mathbf{N}$  such that  $\mathcal{J}$  is not maximal. Then the following are equivalent:*

- (P1)  $\sigma \in (c(\mathcal{I}), c(\mathcal{J}))$  and  $\mathcal{I}$ - $\lim x = \mathcal{J}$ - $\lim \sigma(x)$  for all  $x \in c(\mathcal{I})$ ;
- (P2)  $\sigma \in (c(\mathcal{I}), c(\mathcal{J}))$ ;
- (P3)  $\sigma \in (c_0(\mathcal{I}), c_0(\mathcal{J}))$ ;
- (P4)  $\sigma \in (c_0(\mathcal{I}), c(\mathcal{J}))$ ;
- (P5)  $\sigma(\mathcal{I}) \subseteq \mathcal{J}$ ;

(P6)  $\sigma$  is  $(\mathcal{I}, \mathcal{J})$ -regular;

(P7)  $\sigma \in (c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{J}) \cap \ell_\infty)$ .

In addition, if  $\mathcal{I} = \mathcal{Z}_g$  and  $\mathcal{J} = \mathcal{Z}_h$  are two simple density ideals such that

$$\forall \alpha > 1, \quad \limsup_{n \rightarrow \infty} \frac{n}{g(n)} < \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{g(\lfloor \alpha n \rfloor)}{h(n)} < \infty, \quad (6)$$

then the above conditions are equivalent to:

(P8) 0 is not a  $\mathcal{J}$ -limit point of  $\widehat{\sigma}$ .

*Proof.* (P1)  $\implies$  (P2)  $\implies$  (P4) and (P1)  $\implies$  (P3)  $\implies$  (P4) are obvious.

(P4)  $\implies$  (P5) Suppose for the sake of contradiction that there exists  $S \in \sigma(\mathcal{I}) \setminus \mathcal{J}$ . Then  $\mathbf{1}_{\sigma^{-1}(S)} \in c_0(\mathcal{I})$  and  $\sigma(\mathbf{1}_{\sigma^{-1}(S)}) = \mathbf{1}_S \in c(\mathcal{J})$ , so that  $S \in \mathcal{J}^*$ . Since  $\mathcal{J}$  is not a maximal ideal, there exists a set  $T \notin \mathcal{J} \cup \mathcal{J}^*$ , so that also  $W := T \cap S \notin \mathcal{J} \cup \mathcal{J}^*$ . Then  $\sigma^{-1}(W) \subseteq \sigma^{-1}(S) \in \mathcal{I}$ , which implies  $\mathbf{1}_{\sigma^{-1}(W)} \in c_0(\mathcal{I})$  and  $\sigma(\mathbf{1}_{\sigma^{-1}(W)}) = \mathbf{1}_W \notin c(\mathcal{J})$ .

(P5)  $\implies$  (P1) Fix  $x \in c(\mathcal{I})$  such that  $\mathcal{I}$ - $\lim x = \eta$ . Given  $\varepsilon > 0$ , set  $S := \{n \in \mathbf{N} : |x_n - \eta| > \varepsilon\}$ . Then  $S \in \mathcal{I}$  and  $\{\sigma^{-1}(n) \in \mathbf{N} : |x_{\sigma^{-1}(n)} - \eta| > \varepsilon\} = \sigma(S) \in \mathcal{J}$ . Since  $\varepsilon$  is arbitrary, it follows that  $\sigma(x) \in c(\mathcal{J})$  and  $\mathcal{J}$ - $\lim \sigma(x) = \eta$ .

(P5)  $\iff$  (P6)  $\iff$  (P7) Note that if  $\sigma$  is a permutation of  $\mathbf{N}$  then conditions (T1) and (T2) are trivially verified for the matrix  $A_\sigma$ . Also observe that, for any set  $E \subset \mathbf{N}$  one has that  $A_\sigma(\mathbf{1}_E) = \mathbf{1}_{\sigma(E)}$ . It is clear that (P6) implies (P7). If  $\sigma$  satisfies (P7), then  $E \in \mathcal{I}$  implies that  $\sigma(E) \in \mathcal{J}$ , which, in turn, yields (P5). Next we establish that if (P5) holds, then (T3) holds for  $A_\sigma$ ; this will yield, via Theorem 1.2, that (P6) holds. To see this, fix  $E \in \mathcal{I}$  and observe that  $\sum_{k \in E} a_{n,k} = \sigma(\mathbf{1}_E)_n$  for all  $n \in \mathbf{N}$  and  $\mathbf{1}_{\sigma(E)} \in c_0(\mathcal{J}) \cap \ell_\infty$ , i.e.,  $\mathcal{J}$ - $\lim_n \sum_{k \in E} |a_{n,k}| = 0$ .

For the second part of the statement, assume that  $\mathcal{I}$  is the simple density ideal  $\mathcal{Z}_g$  such that  $n/g(n)$  is bounded sequence which, thanks to [21, Proposition 1], is equivalent to  $\mathcal{Z} \subseteq \mathcal{Z}_g$ .

(P5)  $\implies$  (P8) Suppose that 0 is a  $\mathcal{J}$ -limit point of  $\widehat{\sigma}$ , hence there exists a subsequence  $(\widehat{\sigma}_{n_k})$  such that  $\lim_k \widehat{\sigma}_{n_k} = 0$  and  $S := \{n_k : k \in \mathbf{N}\} \notin \mathcal{J}$ . To conclude, we claim that  $S \in \sigma(\mathcal{I})$ , that is,  $W := \{\sigma^{-1}(n_k) : k \in \mathbf{N}\} \in \mathcal{I}$ . Since  $\lim_k n_k/\sigma^{-1}(n_k) = 0$ , we conclude that for all  $c > 0$  there exists  $k_0$  such that  $\sigma^{-1}(n_k) \geq cn_k$  for all  $k \geq k_0$ . It follows that  $W \in \mathcal{Z} \subseteq \mathcal{I}$ .

(P8)  $\implies$  (P5) It is not difficult to see that 0 is not a  $\mathcal{J}$ -statistical limit point of  $\widehat{\sigma}$  if and only if

$$\lim_{\varepsilon \rightarrow 0^+} \mathbf{d}_h^*(E_\varepsilon) = 0, \quad \text{where} \quad E_\varepsilon := \{n \in \mathbf{N} : \widehat{\sigma}_n < \varepsilon\},$$

cf. [2, Theorem 2.2] or [4, Theorem 18]. At this point, fix  $S \subseteq \mathbf{N}$  such that  $\sigma(S) \notin \mathcal{J}$ . We will show that  $S \notin \mathcal{I}$ . Note that if  $n \in [1, m] \setminus E_\varepsilon$  then  $\sigma^{-1}(n) \leq n/\varepsilon \leq m/\varepsilon$ . Since

$$\forall \varepsilon > 0, \forall m \in \mathbf{N}, \quad \sigma^{-1}(\sigma(S) \cap [1, m] \setminus E_\varepsilon) \subseteq S \cap [1, m/\varepsilon],$$

we obtain that

$$\forall \varepsilon > 0, \forall m \in \mathbf{N}, \quad \frac{|\sigma(S) \cap [1, m] \setminus E_\varepsilon|}{h(m)} \leq \frac{|S \cap [1, m/\varepsilon]|}{g(\lfloor m/\varepsilon \rfloor)} \cdot \frac{g(\lfloor m/\varepsilon \rfloor)}{h(m)},$$

Considering that the function  $\mathbf{d}_h^*$  is monotone and subadditive, we have that

$$\mathbf{d}_h^*(\sigma(S) \setminus E_\varepsilon) \geq \mathbf{d}_h^*(\sigma(S)) - \mathbf{d}_h^*(\sigma(S) \cap E_\varepsilon) \geq \mathbf{d}_h^*(\sigma(S)) - \mathbf{d}_h^*(E_\varepsilon),$$



hence there exists  $\varepsilon_0 > 0$  such that  $\mathbf{d}_h^*(\sigma(S) \setminus E_{\varepsilon_0}) > 0$ . To conclude the proof, since

$$c := \limsup_{m \rightarrow \infty} \frac{g(\lfloor m/\varepsilon_0 \rfloor)}{h(m)} < \infty$$

by the hypothesis (6), we obtain that

$$0 < \mathbf{d}_h^*(\sigma(S) \setminus E_{\varepsilon_0}) \leq c \cdot \limsup_{m \rightarrow \infty} \frac{|S \cap [1, m/\varepsilon_0]|}{g(\lfloor m/\varepsilon_0 \rfloor)} \leq c \cdot \mathbf{d}_g^*(S),$$

which implies that  $S \notin \mathcal{I}$ . □

#### 4. CONCLUDING REMARKS AND OPEN QUESTIONS

It follows by Theorem 3.3 that, if  $\mathcal{J}$  is not a maximal ideal, then a permutation matrix belongs to  $(c_0(\mathcal{I}), c_0(\mathcal{J}))$  if and only if it belongs to  $(c_0(\mathcal{I}) \cap \ell_\infty, c_0(\mathcal{J}) \cap \ell_\infty)$ . However, this does not hold in general. Indeed, the characterization provided in Theorem 2.2 does not hold for its unbounded analogue  $(c_0(\mathcal{I}), c_0(\mathcal{J}))$ , that we leave as an open question for the interested reader. To this aim, consider the following example. Let  $A = (a_{n,k})$  be the matrix defined by  $a_{n,n} = n$  if  $n \in S := \{k^2 : k \in \mathbf{N}\}$ ,  $a_{n,n} = 1$  if  $n \notin S$ , and  $a_{n,k} = 0$  otherwise. Then it is easy to see that  $A \in (c_0(\mathcal{Z}), c_0(\mathcal{Z}))$ . On the other hand, if  $x = \mathbf{1}_S$ , then  $Ax \notin \ell_\infty$ , which proves that  $A \notin (c_0(\mathcal{Z}) \cap \ell_\infty, c_0(\mathcal{Z}) \cap \ell_\infty)$ . This implies that condition (T1) is not necessary for a characterization of the class  $(c_0(\mathcal{I}), c_0(\mathcal{J}))$ .

Lastly, we also leave as an open question to characterize the class of bounded [resp. unbounded]  $(\mathcal{I}, \mathcal{J})$ -conservative matrices, that is, the set of matrices  $A \in (c(\mathcal{I}) \cap \ell_\infty, c(\mathcal{J}) \cap \ell_\infty)$  [resp.,  $A \in (c(\mathcal{I}), c(\mathcal{J}))$ ] which do not necessarily preserve the corresponding ideal limits.

**Acknowledgments.** P.L. is grateful to PRIN 2017 (grant 2017CY2NCA) for financial support.

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