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Global vs Blow-Up Solutions and Optimal Threshold for Hyperbolic ODEs with Possibly Singular Nonlinearities

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Abstract

We consider a hyperbolic ordinary differential equation perturbed by a nonlinearity which can be singular at a point and in particular this includes MEMS type equations. We first study qualitative properties of the solution to the stationary problem. Then, for small value of the perturbation parameter as well as initial value, we establish the existence of a global solution by means of the Lyapunov function and we show that the omega limit set consists of a solution to the stationary problem. For strong perturbations or large initial values, we show that the solution blows up. Finally, we discuss the relationship between upper bounds of the perturbation parameter for the existence of time-dependent and stationary solutions, for which we establish an optimal threshold.

Keywords Lyapunov functions · Global solutions · Omega limit set · Blow-up · Dynamical threshold · MEMS

Mathematics Subject Classification 34A12 · 34G20 · 37L25

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1 Introduction

In this paper, we study the following ordinary differential equation:

$$\begin{cases} u_{tt} + \alpha f(u_t) + \beta u = \lambda g(u) \text{ for } t > 0, \\ u(0) = u_0, \\ u_t(0) = v_0, \end{cases} \tag{1}$$

where $\lambda > 0$, $\alpha \geq 0$, $\beta > 0$, $u_0 \in \mathbb{R}$ and $v_0 \in \mathbb{R}$. Under appropriate assumptions on f and g, we discuss the global existence and blow-up of the solutions to (1). For $\alpha \geq 0$ and the initial value $(u_0, v_0) \in \mathbb{R}^2$, we denote by $\lambda^*(u_0, v_0)(\alpha)$ the dynamical threshold for the existence of a global solution of (1). Namely, the solution of (1) exists globally in time for $0 < \lambda < \lambda^*(u_0, v_0)(\alpha)$ and blows up for $\lambda > \lambda^*(u_0, v_0)(\alpha)$. At the same time, let us denote by λ^* the stationary threshold, see Theorem 1 below, for the existence of solutions to

$$\beta\phi = \lambda g(\phi). \tag{2}$$

From the point of view of applications to Micro Electro Mechanical Systems, the value λ^* plays an important role as it is connected to the so-called "pull-in instability", see [1, 2] and references therein.

Assumption 1 We impose the following assumptions on the damping term f:

- $(f1)$ $f \in C^1(\mathbb{R})$;
- $(f2)$ $f(0) = 0$;
- $(f3)$ $f(v)v > 0$ for $v \in \mathbb{R} \setminus \{0\}$;
- $(f4)$ there exist $\eta > 0$ and $\theta \geq 0$ such that $|f(v)| \leq \eta |v|^{\theta+1}$ for $v \in \mathbb{R}$.

Remark 1 Note that under these assumptions one has $f'(0) \geq 0$.

Assumption 2 Let $b \in (0, +\infty]$ and $I = (-\infty, b)$. We impose the following assumptions on the nonlinearity g:

- $(g1)$ $g(u) \in C^2(I)$;
- $(g2)$ $g(u) > 0$ for $u \in I$. Moreover, we normalize $g(u)$ by $g(0) = 1$;
- $(g3)$ $g'(0) \geq 0$;
- $(g4)$ $g''(u) > 0$ for $u \in I$;
- $(g5)$ If $I = \mathbb{R}$, for sufficiently large $R > 0$, there exists $\tau > 0$ such that $g''(u) > \tau$ for $u > R$.
  If $I = (-\infty, b)$, $b < +\infty$, the following hold:

$$\lim_{u \to b^-} g(u) = \lim_{u \to b^-} g'(u) = \lim_{u \to b^-} g''(u) = +\infty,$$

$$\lim_{u \to b^-} \int_0^u g(s)\,\mathrm{d}s = +\infty \quad \text{and} \quad \lim_{u \to b^-} \left(\int_0^u g(s)\,\mathrm{d}s - \frac{u}{2} g(u) \right) < 0.$$

Remark 2 Note that under these assumptions solutions to (2) are positive.

So far, a huge literature has been devoted to (1) and related problems from the theoretical point of view as well as from the point of view of applications, see for instance [6, 9, 12] and references therein.

In [5], Flores studies the following problem

$$\begin{cases} u_{tt} + \alpha u_t + u = \frac{\lambda}{(1-u)^2}, & \text{for } t > 0, \\ u(0) = u_0 \in [0, 1), \\ u_t(0) = v_0. \end{cases}$$

He proves that $0 < \lambda(0,0)(\alpha) < \lambda^*$ for $\alpha \geq 0$ and that $\lim_{\alpha\to+\infty} \lambda(0,0)(\alpha) = \lambda^*$. In [7], Haraux considers the following

$$u_{tt} + c\,|u_t|^{\alpha}\, u_t + |u|^{\beta}\, u = 0, \qquad \text{for } c > 0, \alpha > 0, \beta > 0$$

and studies the existence of sign-changing solutions, number of the zeros and the decay estimates for solutions depending on the value of the parameters c, α, β. Moreover, the results are generalized to

$$u_{tt} + c\,|u_t|^{\alpha}\, u_t + |u|^{\beta}\, u = f(t), \qquad \text{for } c > 0, \alpha > 0, \beta > 0$$

for a continuous function $f(t)$ with some decaying properties. In [11], Souplet studies the following backward equation

$$u_{tt} - |u_t|^{\alpha}\, u_t + |u|^{\beta}\, u = 0 \qquad \text{for} \alpha > 0, \beta > 0$$

and proves the existence of unbounded global solutions, unbounded oscillatory solutions, as well as their blow-up rate and asymptotic behaviour. In [3, 10], the authors consider

$$u_{tt} - |u_t|^{\alpha} + \lambda\,|u|^{\beta} = 0, \qquad \text{for } \lambda > 0, \alpha > 0, \beta > 0.$$

They investigate for which value of parameters one has existence of solutions, derive their asymptotic behaviour and classify the ground state solutions according to the value of parameters.

Here we aim at extending the result in [5] to more general nonlinearities f and g. Henceforth, we consider (1) and (2) under Assumptions 1 and 2 unless otherwise stated. We consider first the stationary problem (2) for which we obtain the bifurcation diagram of the solution set $\{(\lambda, \phi)\}$.

Theorem 1 *There exists a unique* $p > 0$ *such that*

$$g(p) - pg'(p) = 0.$$

Let

$$\lambda^* := \frac{\beta p}{g(p)}.$$

The following hold:

(i) *For any* $\lambda < \lambda^*$, *there are two solutions* $\phi_1 = \phi_1(\lambda)$ *and* $\phi_2 = \phi_2(\lambda)$ *of* (2) *with* $0 < \phi_1 < p < \phi_2 < b$ *and*

$$\lim_{\lambda \searrow 0} (\phi_1, \phi_2) = (0, b) \quad \text{and} \quad \lim_{\lambda \nearrow \lambda^*} (\phi_1, \phi_2) =: \left(\phi_1^*, \phi_2^*\right) = (p, p) .$$

Moreover, $\phi_1(\lambda)$ *and* $\phi_2(\lambda)$ *are respectively increasing and decreasing with respect to* λ*;*

(ii) *For* $\lambda = \lambda^*$, *there exists a unique solution* $\phi_1 = \phi_2 = p$ *of (2);*

(iii) *For* $\lambda > \lambda^*$, *do not exist solutions to* (2).

In order to investigate the dynamical behaviour, we find the solution $(\overline{\lambda}, \overline{\phi_2})$ with the following properties:

Theorem 2 *There exists a unique solution* $(\lambda, \phi_2(\lambda)) = (\overline{\lambda}, \overline{\phi_2})$ *of* (2) *satisfying the following:*

$$0 < \overline{\lambda} < \lambda^*, \quad p < \overline{\phi_2} < b \quad \text{and} \quad \overline{\lambda} = \frac{\beta}{2} \frac{\overline{\phi_2}^2}{\int_0^{\overline{\phi_2}} g(s)\, \mathrm{d}s}.$$

Next, on the one hand we consider the time-dependent equation (1) and derive the conditions for the existence of global bounded solutions. To state the theorem, we define the functional

$$J_\lambda(u) := \frac{\beta}{2} u^2 - \lambda \int_0^u g(s)\, \mathrm{d}s.$$

Theorem 3 *For* $\lambda < \lambda^*$ *and* $\alpha \geq 0$, *let*

$$D_0 := \left\{ (u, v) \in \mathbb{R}^2 \mid l(\lambda) < u < \phi_2(\lambda),\ v^2 < J_\lambda(\phi_2) - J_\lambda(u_0) \right\},$$

where $l(\lambda)$ *is a constant depending only on* λ, β, g *and* u_0. *If* $(u_0, v_0) \in D_0$, *then (1) has a unique global solution* $u \in W^{2,\infty}([0, \infty))$. *Moreover, if* $\alpha > 0$, *then the following holds*

$$\lim_{t \to +\infty} \Big(|u(t) - \phi_1(\lambda)| + |u_t(t)| \Big) = 0. \tag{3}$$

On the other hand, we obtain the following blow-up result for $\lambda > \lambda^*$ or $\phi_2 < u_0 < b$. Let $T_\infty \in (0, +\infty]$, be the maximal time of existence of a solution to (1). Then, we have the following

Theorem 4 *Let* $\lambda > \lambda^*$, $\alpha \geq 0$ *and* $u_0, v_0 \geq 0$. *The solution* (u, u_t) *of* (1) *blows up to* $(+\infty, +\infty)$ *for* $b = +\infty$ *and quenches to* $(b, +\infty)$ *for* $0 < b < +\infty$ *as* $t \to T_\infty$, *where* $T_\infty < +\infty$ *if* $\alpha = 0$ *or* $0 < b < +\infty$.

Theorem 5 *Let* $0 < \lambda \leq \lambda^*$ *and* $\alpha \geq 0$. *For any* $\phi_2(\lambda) < u_0 < b$ *and* $v_0 \geq 0$, *the same conclusion of Theorem 4 holds.*

The main result of this paper is concerned with establishing an optimal dynamical threshold, indeed we have

Theorem 6 *Let* $f(v) = v$. *The function* $\lambda(0,0)(\alpha)$ *is continuous and monotone increasing with respect to* $\alpha \geq 0$ *and satisfies:*

(i) $\overline{\lambda} \leq \lambda(0,0)(\alpha) < \lambda^*$ *for* $\alpha \geq 0$;
(ii) $\lim_{\alpha \to 0^+} \lambda(0,0)(\alpha) = \overline{\lambda}$;
(iii) $\lim_{\alpha \to +\infty} \lambda(0,0)(\alpha) = \lambda^*$.

This paper is organized as follows: In Sect. 2, we consider the stationary problem. Thanks to the assumptions on g, we get at most two positive solutions. In Sect. 3, we settle preliminary lemmas, involving energy and dynamical system, in order to investigate the dynamical behaviour. On the one hand, in Sect. 4, we establish the existence of a global solution and periodic orbit under some appropriate conditions. On the other hand, for large values of the perturbation parameter or large initial values, we prove that the solution blows up. In Sect. 5, we discuss qualitative properties of the orbit such as openness, monotonicity and continuity. In Sect. 6, we prove our main result, namely Theorem 6.

2 The Stationary Problem

Here we study the solution set of the function equation (2) and obtain the upper bound λ^* for the existence of solutions. We regard $\phi = \phi(\lambda)$ as a function of the parameter λ.

Proof of Theorem 1 Since solutions are positive, we consider in (2) $\phi \geq 0$. By Assumption 2,

$$F(u) := \frac{u}{g(u)}$$

is well-defined for $u \geq 0$ and the following holds

$$F'(u) = \frac{1}{g(u)^2}\left(g(u) - ug'(u)\right).$$

From $F(0) = F(b) = 0$ and $F'(0) > 0$, there exists $p > 0$ such that $F'(p) = 0$. Next we show the uniqueness of such p. Set $G(u) := g(u) - ug'(u)$, and consider the sign of $G(u)$. We have $G(0) = 1$,

$$G'(u) = -ug''(u) < 0$$

for all $0 < u < b$, $G'(0) = 0$ and $G'(b) = -\infty$, which implies that $G(u) > 0$ for $u < p$ and $G(u) < 0$ for $u > p$ and thus such p is uniquely determined. The statement follows by drawing the graph of $y = \beta F(u)$ and $y = \lambda$. □

Define

$$H_\lambda(u) := \beta u - \lambda g(u).$$

Then $\beta F(u) = \lambda$ is equivalent to $H_\lambda(u) = 0$ and the sign of $H_\lambda(u)$ is given by the value of λ.

Corollary 1 *For $\lambda < \lambda^*$, there exist ϕ_1, ϕ_2 with $\phi_1 < \phi_2 < b$ such that*

(i) $H_\lambda(\phi_1) = H_\lambda(\phi_2) = 0$*;*
(ii) $H_\lambda(u) < 0$*, for* $u < \phi_1$*,* $\phi_2 < u < b$*;*
(iii) $H_\lambda(u) > 0$*, for* $\phi_1 < u < \phi_2$*.*

Furthermore, $H_\lambda(u)$ is increasing in $(-\infty, q)$ and decreasing in (q, b), where q satisfies $\beta = \lambda g'(q)$ and $p < q < \phi_2 < b$.

Proof Noting that $H_\lambda(\phi_1) = H_\lambda(\phi_2) = 0$, $H'_\lambda(u) = \beta - \lambda g'(u)$ and that $H''_\lambda(u) = -\lambda g''(u) < 0$, we find $q \in (\phi_1, \phi_2)$ uniquely satisfying $H'_\lambda(q) = 0$. Moreover, from

$$H'_\lambda(p) = \beta - \lambda g'(p) = \lambda^* \frac{g(p)}{p} - \lambda \frac{g(p)}{p} = \left(\lambda^* - \lambda\right) \frac{g(p)}{p} > 0,$$

one has $p < q$. □

Remark 3 For $\lambda = \lambda^*$, we have $\phi_1(\lambda^*) = \phi_2(\lambda^*) = p$ and $H_{\lambda^*}(u) < 0$, for $u \neq p$, $H_{\lambda^*}(p) = 0$. Furthermore, $H_{\lambda^*}(u)$ is increasing in $(-\infty, p)$ and decreasing in (p, b).

Corollary 2 *For $\lambda > \lambda^*$, we have*

$$H_\lambda(u) < 0 \quad \text{for } u < b.$$

In particular, there exists $\xi > 0$ such that

$$H_\lambda(u) \le -\xi < 0$$

for all $u \ge 0$, where ξ depends only on λ, β and g.

Proof Set $\xi = \lambda - \lambda^*$, to have

$$\xi = \lambda - \beta F(p) \le \lambda - \beta F(u) = -\frac{1}{g(u)} H_\lambda(u) \le -H_\lambda(u)$$

as $1 = g(0) \le g(u)$ holds for all $u \ge 0$. □

Lemma 1 *For $\lambda < \lambda^*$, one has*

$$H'_\lambda(\phi_1(\lambda)) = \sup_{\phi_1 \le u \le \phi_2} \frac{H_\lambda(u)}{u - \phi_1(\lambda)}.$$

Proof Noting that $H_\lambda(\phi_1) = 0$, we apply the mean value theorem to obtain the conclusion, as $H_\lambda''(u) < 0$ for $\phi_1 \le u \le \phi_2$. □

Proof of Theorem 2 Let us define

$$I(u) := 2\int_0^u g(s)\,\mathrm{d}s - ug(u), \quad 0 \le u < b.$$

We *claim* that $I(u_0) = 0$ for some $p < u_0 < b$. As a consequence, we find a solution $(\lambda, \phi_2) = (\beta u_0/g(u_0), u_0)$ of (2) by Theorem 1 and

$$\lambda = \frac{\beta u_0}{g(u_0)} = \frac{\beta u_0^2}{2\int_0^{u_0} g(s)\,\mathrm{d}s} \in (0, \lambda^*).$$

Let us prove the claim. First we have

$$\begin{aligned} I(p) &= 2\int_0^p g(s)\,\mathrm{d}s - pg(p) \\ &= 2\int_0^p g(s)\,\mathrm{d}s - \frac{\beta}{\lambda^*}p^2 \\ &= \frac{2}{\lambda^*}\int_0^p \left(\lambda^* g(s) - \beta s\right)\,\mathrm{d}s \\ &= -\frac{2}{\lambda^*}\int_0^p H_{\lambda^*}(s)\,\mathrm{d}s > 0 \end{aligned}$$

by Remark 3. Observe that $I'(u) = G(u) < 0$ and that $I''(u) = G'(u) < 0$ for $p < u < b$, where G is defined as in the proof of Theorem 1. By the monotonicity of I, we need only to show that $I(b) < 0$. In the case of $b < +\infty$, $I(b) < 0$ follows from Assumption 2. In the case $b = +\infty$, since $I''(+\infty) = G'(+\infty) = -\infty$ holds, we have $I'(+\infty) = G(+\infty) = -\infty$ and finally $I(+\infty) = -\infty$, which proves the theorem. □

Let us give a few examples where (λ^*, p) and $(\overline{\lambda}, \overline{\phi_2})$ are explicitly known.

Example 1 If $\beta = 1$ and $g(u) = e^u$, Assumption 2 is satisfied with $b = +\infty$. We have

$$\left(\lambda^*, p\right) = \left(\frac{1}{e}, 1\right)$$

and $\left(\overline{\lambda}, \overline{\phi_2}\right)$ satisfies

$$\overline{\phi_2}e^{\overline{\phi_2}} - 2e^{\overline{\phi_2}} + 2 = 0, \quad \overline{\lambda} = \overline{\phi_2}e^{-\overline{\phi_2}}, \quad \overline{\phi_2} \in (1, 2).$$

Example 2 If $\beta = 1$ and $g(u) = 1 + u^{2k}$ for $k \in \mathbb{N}$, Assumption 2 is satisfied with $b = +\infty$. In particular, in the case $k = 1$, we have

$$\left(\lambda^*, p\right) = \left(\frac{1}{2}, 1\right) \quad \text{and} \quad (\overline{\lambda}, \overline{\phi_2}) = \left(\frac{\sqrt{3}}{4}, \sqrt{3}\right).$$

Example 3 If $\beta = 1$ and $g(u) = 1/(1-u)^p$ for $p > 1$, $g(u)$ has a singularity at $u = 1$. However, Assumption 2 is satisfied with $b = 1$. In particular, in the case of $p = 2$ we have

$$\left(\lambda^*, p\right) = \left(\frac{4}{27}, \frac{1}{3}\right) \quad \text{and} \quad (\overline{\lambda}, \overline{\phi_2}) = \left(\frac{1}{8}, \frac{1}{2}\right).$$

3 A Dynamical System

Consider the following energy functional

$$E_\lambda(u, v) := \frac{\beta}{2}u^2 + \frac{1}{2}v^2 - \lambda \int_0^u g(s)\,\mathrm{d}s.$$

Then, $E_\lambda(u(t), u_t(t))$ turns out to be the Lyapunov function for (1). In fact, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\lambda(u(t), u_t(t)) = -\alpha f(u_t(t))u_t(t) \le 0,$$

which yields

$$E_\lambda(u(t), u_t(t)) + \alpha \int_0^t f(u_t(r))u_t(r)\,\mathrm{d}r = E_\lambda(u_0, v_0). \tag{4}$$

Hence, by use of

$$J_\lambda(u) := \frac{\beta}{2}u^2 - \lambda \int_0^u g(s)\,\mathrm{d}s,$$

the energy inequality

$$J_\lambda(u) \le J_\lambda(u_0) + \frac{1}{2}v_0^2 \tag{5}$$

holds by (4). Every local solution satisfies (5) as long as it exists. To extend the solution globally in time, we consider some properties of $J_\lambda(u)$ in the following two lemmas:

Lemma 2 *For $\lambda < \lambda^*$, $J_\lambda(u)$ has a local minimum at $u = \phi_1(\lambda)$ and a local maximum $u = \phi_2(\lambda)$. Moreover, $J_\lambda(\phi_1(\lambda)) < 0$ holds.*

Proof Since $J'_\lambda(u) = \beta u - \lambda g(u) = H_\lambda(u)$ and $J''_\lambda(u) = H'_\lambda(u)$, we have $J'_\lambda(\phi_1) = J'_\lambda(\phi_2) = 0$ and $J''_\lambda(\phi_2) < 0 < J''_\lambda(\phi_1)$ by Corollary 1. Then $J_\lambda(\phi_1(\lambda)) < J_\lambda(0) = 0$ for all $\lambda \in (0, \lambda^*)$, which completes the proof. □

Lemma 3 *The following hold:*

(i) $J_\lambda(\phi_2(\lambda)) > 0$ *for* $0 < \lambda < \overline{\lambda}$;
(ii) $J_\lambda(\phi_2(\lambda)) = 0$ *for* $\lambda = \overline{\lambda}$;
(iii) $J_\lambda(\phi_2(\lambda)) < 0$ *for* $\overline{\lambda} < \lambda < \lambda^*$.

Proof From

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J_\lambda(\phi_2(\lambda)) = H_\lambda(\phi_2(\lambda)) \frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_2(\lambda) - \int_0^{\phi_2(\lambda)} g(s)\,\mathrm{d}s = -\int_0^{\phi_2(\lambda)} g(s)\,\mathrm{d}s < 0$$

by Corollary 1, $J_\lambda(\phi_2(\lambda))$ is monotone decreasing in λ. It follows from Theorem 2 that $J_{\overline{\lambda}}(\phi_2(\overline{\lambda})) = 0$, which yields the conclusions. □

In the next section, we consider dynamical properties of the solution of (1), which can be written in the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ u_t \end{bmatrix} = \begin{bmatrix} u_t \\ -\alpha f(u_t) - H_\lambda(u) \end{bmatrix}. \tag{6}$$

Now under Assumptions 1 and 2, we obtain a local solution. Next, we establish the existence of a global solution exploiting Lemmas 2 and 3. For this purpose, we consider the stability of the equilibrium point. At the equilibrium point

$$(u, u_t) = (\phi_i(\lambda), 0)$$

for $i = 1, 2$, the linearized equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -H'_\lambda(\phi_i) & -\alpha f'(0) \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix}$$

and the eigenvalues $\mu_i^\pm$ of coefficient matrix are given respectively by

$$\mu_i^\pm = \frac{-\alpha f'(0) \pm \sqrt{\{\alpha f'(0)\}^2 - 4H'_\lambda(\phi_i)}}{2}.$$

Along with Corollary 1, we have the following two lemmas:

Lemma 4 *For $\lambda < \lambda^*$, the equilibrium point $(u, u_t) = (\phi_1(\lambda), 0)$ is a stable focus for $0 < \alpha f'(0) < 2\sqrt{H'_\lambda(\phi_1(\lambda))}$, a stable node for $\alpha f'(0) \geq 2\sqrt{H'_\lambda(\phi_1(\lambda))}$ and it is a centre for $\alpha f'(0) = 0$.*

Lemma 5 *For* $\lambda < \lambda^*$, *the equilibrium point* $(u, u_t) = (\phi_2(\lambda), 0)$ *is a saddle for all* $\alpha \geq 0$.

4 The Time-Dependent Problem

In this Section, for small parameters and small initial values, we establish the existence of a global solution by means of the Lyapunov function method. For the dissipative case, that is $\alpha > 0$, we show that the global solution converges to the stationary solution. For the conservative case, namely $\alpha = 0$, we consider the periodic orbit starting at $(0, 0)$. Note that a trivial periodic orbit does not exist by $u_{tt}(0) = \lambda g(0) = \lambda > 0$. Finally, we show that the solution becomes unbounded for large parameters as well as for large initial values.

Proof of Theorem 3 We define by $l(\lambda) \in (-\infty, \phi_1(\lambda))$ the point satisfying $J_\lambda(u) = J_\lambda(\phi_2(\lambda))$. Then, we have $l(\lambda) < 0$, $l(\lambda) = 0$ and $l(\lambda) > 0$ for $\lambda \in (0, \overline{\lambda})$, $\lambda = \overline{\lambda}$ and $\lambda \in (\overline{\lambda}, \lambda^*)$, respectively. In fact, by $J'_\lambda(u) = H_\lambda(u)$ and Corollary 1, we have $J_\lambda(\phi_1) < J_\lambda(u_1) < 0 = J_\lambda(0) < J_\lambda(u_2)$ for $u_2 < 0 < u_1 < \phi_1$. For $\lambda \in (0, \overline{\lambda})$, $J_\lambda(u)$ attains a local maximum $J_\lambda(\phi_2) > 0$ by Lemma 3. Hence we have $l(\lambda) < 0$. Other cases follows easily according to the sign of $J_\lambda(\phi_2)$. Note that $J_\lambda(u_0) < J_\lambda(\phi_2)$ holds for $l(\lambda) < u_0 < \phi_2(\lambda)$ by Lemma 2. Then, for $(u_0, v_0) \in D_0$, (5) yields

$$J_\lambda(u) < \frac{1}{2}\left(J_\lambda(\phi_2) + J_\lambda(u_0)\right) < J_\lambda(\phi_2)$$

and moreover

$$l(\lambda) < u(t) < J_\lambda^{-1}\left(\frac{J_\lambda(\phi_2) + J_\lambda(u_0)}{2}\right) < \phi_2, \tag{7}$$

where J_λ^{-1} is the inverse function of J_λ defined at (ϕ_1, ϕ_2). Finally (4) implies that

$$u_t^2 \leq 2E_\lambda(u_0, v_0) + 2\lambda \int_0^u g(s)\,\mathrm{d}s < 2E_\lambda(u_0, v_0) + 2\lambda g(\phi_2)\phi_2$$

and that $(u(t), u_t(t))$ is uniformly bounded in $\mathbb{R}^2$ for all $t \geq 0$. Therefore, since u_{tt} is also bounded for $t \geq 0$ by (1), we have $u(t) \in W^{2,\infty}([0, \infty))$. If $\alpha > 0$, the Lyapunov function $E_\lambda(u(t), u_t(t))$ is strictly decreasing in t. Thus it follows from Theorem 5.1.8 and Corollary 8.5.1 in [8] that the omega limit set $\omega(u_0, v_0)$ is connected in $\mathbb{R}^2$ and included in the solution set of (2). Since $\overline{\cup_{t \geq 0}(u(t), u_t(t))} \notin \{(\phi_2, 0)\}$ by (7), we obtain (3). □

Remark 4 Since $l(\lambda) < \phi_1$ and $J_\lambda(u) < J_\lambda(\phi_2)$ for all $\lambda < \lambda^*$ and $\phi_1 < u < \phi_2$, we have $(u_0, 0) \in D_0$ with $\phi_1 < u_0 < \phi_2$. In other words, the solution exists globally in time for the initial value (u_0, v_0) with $\phi_1 < u_0 < \phi_2$ and $v_0 = 0$.

Proof of Theorem 4 Let us divide the proof into four cases:

I. The case $\alpha = 0$ and $I = (-\infty, b)$. Integrating (1) and thanks to Corollary 2, we have

$$u_t \geq v_0 + \xi t \geq 0 \quad \text{and} \quad u \geq u_0 + v_0 t + \frac{\xi}{2} t^2 \geq 0, \tag{8}$$

which implies that $u(t)$ reaches b for finite $T_\infty < \infty$. Then

$$u_t^2(T_\infty) = 2E_\lambda(u_0, v_0) + 2\lambda \int_0^b g(s)\,\mathrm{d}s - \beta b^2 = +\infty$$

by (4) and Assumption 2.

II. The case $\alpha = 0$ and $I = \mathbb{R}$. Assume by contradiction that $T_\infty = \infty$. By (8), there exists sufficiently large $T > 0$ such that

$$g''(u(t)) > \tau \tag{9}$$

for all $t > T$, where τ is given in Assumption 2 and T can be taken as $T = \sqrt{2\xi^{-1}R}$. Then, integrating inequality (9) twice, with respect to u, over $[u(T), u]$ we get

$$g(u(t)) \geq g(u(T)) + \frac{\tau}{2}\left(u(t) - u(T)\right)^2$$

as $g'(u) \geq 0$ for $u \geq 0$. Thus we have

$$\begin{aligned} u_{tt}(t) &= \lambda g(u(t)) - \beta u(t) \\ &\geq \lambda \left\{ g(u(T)) + \frac{\tau}{2}\left(u(t) - u(T)\right)^2 \right\} - \beta u(t) \\ &= -H_\lambda(u(T)) + \frac{\tau\lambda}{2}\left(u(t) - u(T)\right)^2 - \beta\left(u(t) - u(T)\right) \\ &> \frac{\tau\lambda}{2}\left(u(t) - u(T)\right)^2 - \beta\left(u(t) - u(T)\right) \end{aligned} \tag{10}$$

for all $t > T$. Since $u'(t) > 0$ holds for all $t > T$ by (8), we have

$$u_t u_{tt} > \frac{\tau\lambda}{2}\left(u(t) - u(T)\right)^2 u_t - \beta\left(u(t) - u(T)\right) u_t$$

and

$$\begin{aligned} \{u_t(t)\}^2 &> \{u_t(T)\}^2 + \frac{\tau\lambda}{3}\left(u(t) - u(T)\right)^3 - \beta\left(u(t) - u(T)\right)^2 \\ &> \left\{ \frac{\tau\lambda}{3}\left(u(t) - u(T)\right) - \beta \right\} \left(u(t) - u(T)\right)^2. \end{aligned}$$

Take $T_1 \in (T, +\infty)$ such that

$$\frac{\tau\lambda}{3}\left(u(t) - u(T)\right) - \beta > \frac{\tau\lambda}{6}\left(u(t) - u(T)\right)$$

holds for all $t > T_1$. For instance, we can take T_1 as follows

$$T_1 = \sqrt{\frac{2}{\xi}}\sqrt{\frac{6\beta}{\tau\lambda} + u(T)} > \sqrt{\frac{2u(T)}{\xi}} \geq \sqrt{\frac{2R}{\xi}} = T.$$

Then we have

$$u_t(t) > \sqrt{\frac{\tau\lambda}{6}}\,(u(t) - u(T))^{\frac{3}{2}}$$

and

$$(u(t) - u(T))^{-\frac{3}{2}}\,u_t(t) > \sqrt{\frac{\tau\lambda}{6}}$$

for all $t > T_1$. Integrating this inequality over $[T_1, t]$, we have

$$0 < \frac{2}{\sqrt{u(t) - u(T)}} < \frac{2}{\sqrt{u(T_1) - u(T)}} - \sqrt{\frac{\tau\lambda}{6}}\,(t - T_1)\,,$$

which implies that $\lim_{t \to T_2} u(t) = +\infty$, where

$$T_2 = T_1 + \sqrt{\frac{6}{\tau\lambda}}\frac{2}{\sqrt{u(T_1) - u(T)}} < +\infty,$$

contradicting the maximality of T_∞ and hence necessarily $T_\infty < +\infty$. Thanks to (4), we also have

$$\xi u + E_\lambda(u_0, v_0) \leq -\int_0^u H_\lambda(s)\,\mathrm{d}s + E_\lambda(u_0, v_0) = \frac{1}{2}u_t^2, \tag{11}$$

which implies that both u and u_t blow up to $+\infty$ as $t \to T_\infty$.
III. The case $\alpha \neq 0$ and $I = (-\infty, b)$. If $v_0 = 0$, we have

$$u_{tt}(0) = -H_\lambda(u_0) \geq \xi > 0. \tag{12}$$

Hence for $v_0 \geq 0$, we have $u_t(t) > 0$ for sufficiently small $t > 0$. If there exists $T_3 \in (0, T_\infty)$ such that $u_t(t) > 0$ for all $t \in (0, T_3)$ and $u_t(T_3) = 0$, then we have $u_{tt}(T_3) > 0$ similarly to (12), which contradicts the positivity of u_t. Hence, if necessary, we retake the initial value as $u_0 = u(T_4)$ and $v_0 = v(T_4)$ for some $T_4 > 0$ so that $u_0 > 0$, $v_0 > 0$, $u(t) > 0$ and $u_t(t) > 0$ hold for all $t \in (0, T_\infty)$. We estimate $u_t(t)$. First if $u_{tt}(0) > 0$ holds, we have $u_t(t) \geq v_0$ for sufficiently small $t > 0$. On the other hand, if $u_{tt}(t) \leq 0$ holds for some $t \geq 0$, we have

$$\alpha\eta u_t^{\theta+1} \geq u_{tt} + \alpha f(u_t) = -H_\lambda(u) \geq \xi > 0$$

and then

$$u_t \geq \left(\frac{\xi}{\alpha\eta}\right)^{\frac{1}{\theta+1}}.$$

Thus we have

$$u_t \geq \min\left\{v_0, \left(\frac{\xi}{\alpha\eta}\right)^{\frac{1}{\theta+1}}\right\} \equiv C_1 > 0 \tag{13}$$

for all $t \in (0, T_\infty)$, which yields

$$u(t) > u_0 + C_1 t \tag{14}$$

for all $t \in (0, T_\infty)$, which brings back to the same situation of case I above.

IV. The case $\alpha \neq 0$ and $I = \mathbb{R}$. By estimates carried out in the case III, we have that (13) and (14) hold. Hence, (u, u_t) is unbounded in $\mathbb{R}^2$ for $t \in (0, T_\infty)$, where $T_\infty \leq +\infty$. In the case of $T_\infty < +\infty$, both u and u_t blow up to $+\infty$ as $t \to T_\infty$ similarly to (11). Next let us consider the case $T_\infty = +\infty$ and let us prove that $u_t(t) \to +\infty$, as $t \to +\infty$. Now suppose that there exists a constant $C_2 > 0$ satisfying

$$C_1 < u_t < C_2$$

for all $t \geq T$. Then, buying the line of (10) we obtain the following differential inequality

$$\begin{aligned} u_{tt} + \alpha\eta u_t^{\theta+1} &\geq u_{tt} + \alpha f(u_t) \\ &> \frac{\tau\lambda}{2}\left(u(t) - u(T)\right)^2 - \beta\left(u(t) - u(T)\right) \\ &= \frac{\tau\lambda}{2}\left(u(t) - u(T) - \frac{\beta}{\tau\lambda}\right)^2 - \frac{\beta^2}{2\tau\lambda} \end{aligned}$$

for all $t \geq T$. Thus for sufficiently large $t > T$, we have

$$u_{tt} \geq \frac{\tau\lambda}{2}\left(C_1 t + u_0 - u(T) - \frac{\beta}{\tau\lambda}\right)^2 - \frac{\beta^2}{2\tau\lambda} - \alpha\eta C_2^{\theta+1},$$

which yields $u_t(t) \to +\infty$, as $t \to +\infty$ by integration and contradicting the boundedness of u_t. Hence, both u and u_t blow up to $+\infty$ as $t \to T_\infty$. □

Proof of Theorem 5 By hypothesis, $\phi_2 < u(t) < b$ holds for sufficiently small $t > 0$. If $v_0 = 0$, we have

$$u_{tt}(0) = -H_\lambda(u_0) > -H_\lambda(\phi_2) = 0$$

by Corollary 1 and Remark 3. Hence we may assume that $\phi_2 < u_0 < u(t) < b$ and $u_t(t) > 0$ holds for all $t \in (0, T_\infty)$. Now set $\xi = -H_\lambda(u_0) > 0$ and proceed as in the proof of Theorem 4. □

Let us denote by $\gamma(t; \lambda, \alpha)$ the orbit of solution of (1) starting at $(0, 0)$. We note that from Lemma 4, one has $\alpha f'(0) = 0$ if and only if $(\phi_1(\lambda), 0)$ is a centre.

Proposition 1 *Let $\alpha = 0$. We have that $0 < \lambda < \overline{\lambda}$ is satisfied if and only if the orbit $\gamma(t; \lambda, 0)$ is periodic. If $\overline{\lambda} < \lambda < \lambda^*$, then the orbit is unbounded.*

Proof In the conservative case $\alpha = 0$, we have

$$E_\lambda(u, u_t) = E_\lambda(0, 0) \Longleftrightarrow J_\lambda(u) + \frac{1}{2}u_t^2 = 0 \tag{15}$$

by (4). $(u_t)_t = -H_\lambda(u)$ is positive either for $u < \phi_1$ or $\phi_2 < u < b$ and negative for $\phi_1 < u < \phi_2$ by Corollary 1. If $0 < \lambda < \overline{\lambda}$, $\gamma(t; \lambda, 0)$ exists globally and never passes $(\phi_2, 0)$ by Theorem 3, (15) and Lemma 3. Hence $\gamma(t; \lambda, 0)$ crosses the u-axis at some point $(u_1, 0)$ for $0 < u_1 < \phi_2$ by Theorem 5 and Remark 4, which implies that the orbit is periodic along with (15). Conversely, if the orbit is periodic, the orbit passes $(u_1, 0)$ for $u_1 \in (\phi_1, \phi_2)$ satisfying (15). Thus $J_\lambda(u)$ has a zero in $u \in (\phi_1, \phi_2)$, which is equivalent to $\lambda \in (0, \overline{\lambda})$ by the monotonicity of $J_\lambda(u)$ with respect to u in Corollary 1, Lemmas 2 and 3. If $\overline{\lambda} < \lambda < \lambda^*$, there does not exist such u_1, which means that $\phi_2 < u < b$ and $u_t > 0$ for sufficiently large t. Finally apply Theorem 5 to conclude. □

Remark 5 Since $\lambda < \overline{\lambda}$ is equivalent to $l(\lambda) < 0$, we have $(0, 0) \in D_0$. Thus by Theorem 3, the orbit $\gamma(t; \lambda, \alpha)$ for $\alpha > 0$ exists globally in time and converges to $(\phi_1(\lambda), 0)$, as $t \to +\infty$. Hence along with Proposition 1, we have $\overline{\lambda} \leq \lambda\,(0, 0)\,(\alpha)$ for $\alpha > 0$ and $\overline{\lambda} = \lambda\,(0, 0)\,(0) < \lambda^*$.

5 Properties of the Dissipative Orbit

In this Section, we study the properties of the orbit starting at the origin for the dissipative case, that is, $\alpha > 0$. Moreover, we also assume that $f(v) = v$. Then, $(\phi_1(\lambda), 0)$ is a hyperbolic sink for all $\lambda \in (\overline{\lambda}, \lambda^*)$. The argument proceeds in the same way as in Sect. 3 of [5].

Let us first define a few sets which will be used in the sequel:

$$\Gamma := \left\{(\lambda, \alpha) \in \mathbb{R}^2 \mid \overline{\lambda} < \lambda < \lambda^*,\ \alpha > 0\right\};$$
$$\Gamma_1 := \{(\lambda, \alpha) \in \Gamma \mid \gamma(t; \lambda, \alpha) \to (\phi_1(\lambda), 0)\,,\ \text{as } t \to +\infty\};$$
$$\Gamma_2 := \{(\lambda, \alpha) \in \Gamma \mid \gamma(t; \lambda, \alpha) \to (\phi_2(\lambda), 0)\,,\ \text{as } t \to +\infty\};$$
$$\Gamma_3 := \{(\lambda, \alpha) \in \Gamma \mid \gamma(t; \lambda, \alpha) \text{ becomes unbounded as } t \to T_\infty\}\,,$$

where $\gamma(t;\lambda,\alpha)$ is the orbit of the solution $(u,v)=(u,u_t)$ of (1) starting at $(0,0)$. In what follows we will also use for convenience the following equivalent notations

$$\gamma(t)=\gamma(t;\lambda,\alpha)=\begin{bmatrix} u(t;\lambda,\alpha) \\ v(t;\lambda,\alpha)\end{bmatrix}=\begin{bmatrix} u(t) \\ v(t)\end{bmatrix}$$

for any fixed $\alpha,\lambda>0$.

Lemma 6 *Γ_1 and Γ_3 are open subsets of Γ.*

Proof If we take $(\lambda_0,\alpha_0)\in\Gamma_1$, then $(u,v)=(\phi_1(\lambda_0),0)$ is a hyperbolic sink by Lemma 4. Hence, there exists $r>0$ depending only on λ_0, α_0, β and g such that we can find an invariant disk $B_r\left(\phi_1(\lambda_0),0\right)$ of radius r centred at $(\phi_1(\lambda_0),0)$ for the dynamical system induced by (6). In other words, there exists $T>0$ such that $\gamma(t;\lambda_0,\alpha_0)\subset B_r\left(\phi_1(\lambda_0),0\right)$ for all $t>T$. If necessary, we can take $r>0$ so small such that

$$B_r\left(\phi_1(\lambda_0),0\right)\subset\left\{(u,v)\in\mathbb{R}^2 \mid u\le p\right\},$$

where p is defined in Theorem 1 and satisfies $\phi_1(\lambda_0)<p<\phi_2(\lambda_0)$. For $(\lambda,\alpha)\in\Gamma$ sufficiently close to (λ_0,α_0), $\gamma(t;\lambda,\alpha)\subset B_r\left(\phi_1(\lambda_0),0\right)$ for sufficiently large t thanks to the continuous dependence on parameters. Since $(\phi_1(\lambda),0)$ is also a hyperbolic sink, $\gamma(t;\lambda,\alpha)$ converges $(\phi_1(\lambda),0)$ as $t\to+\infty$. Hence we have $(\lambda,\alpha)\in\Gamma_1$.
Now take $(\lambda_0,\alpha_0)\in\Gamma_3$ and define

$$\mathcal{U}_{\lambda,\alpha}:=\left\{(u,v)\in\mathbb{R}^2 \mid \phi_2(\lambda)<u<b,\ 0<v<-\frac{1}{\alpha}H_\lambda(u)\right\}.$$

For sufficiently large $t>0$, we have $\gamma(t;\lambda_0,\alpha_0)\subset\mathcal{U}_{\lambda_0,\alpha_0}$ by a phase plane analysis of (6) together with $u_{tt}(0)=\lambda_0>0$ and Remark 4. Again, by continuous dependence, for large $t>0$ we have $\gamma(t;\lambda,\alpha)\subset\mathcal{U}_{\lambda_0,\alpha_0}$ for $(\lambda,\alpha)\in\Gamma$ sufficiently close to (λ_0,α_0). Since we have $\phi_2(\lambda)<u(t;\lambda,\alpha)<b$ and $u_t(t;\lambda,\alpha)>0$ for sufficiently large t, $\gamma(t;\lambda,\alpha)$ becomes unbounded by Theorem 5, which proves that $(\lambda,\alpha)\in\Gamma_3$. □

Proposition 2 *We have $\Gamma=\bigcup_{i=1}^{3}\Gamma_i$.*

Proof Note that every bounded orbit for $\alpha>0$ converges to ϕ_1 or ϕ_2 as $t\to+\infty$ by Corollary 8.5.1 in [8]. If an unbounded orbit $\gamma(t;\lambda,\alpha)$ exists, then $u(t)\ge 0$ and $u_t(t)\ge 0$ hold for sufficiently small $t\ge 0$ by $u_{tt}(0)=\lambda>0$ and the orbit $\gamma(t;\lambda,\alpha)$ enters $\mathcal{U}_{\lambda,\alpha}$ by Remark 4. Then we have $(\lambda,\alpha)\in\Gamma_3$ as in the proof of Theorem 5. □

Let m be a negative constant to be determined later. Let us define the line segments s_i for $i=1,2,3$ and the triangular region $\mathcal{T}$ as follows:

$$s_1:=\{(u,v)=(\phi_2(\lambda),v)\mid m\left(\phi_2(\lambda)-\phi_1(\lambda)\right)\le v\le 0\};$$
$$s_2:=\{(u,v)=(u,0)\mid \phi_1(\lambda)\le u\le\phi_2(\lambda)\};$$

$$s_3 := \{(u, v) = (u, m\,(u - \phi_1\,(\lambda))) \mid \phi_1\,(\lambda) \leq u \leq \phi_2\,(\lambda)\}\,;$$
$$\mathcal{T} := \left\{(u, v) \in \mathbb{R}^2 \mid \phi_1\,(\lambda) \leq u \leq \phi_2\,(\lambda)\,,\ m\,(u - \phi_1\,(\lambda)) \leq v \leq 0\right\}.$$

Proposition 3 *For $\alpha > 2\sqrt{H'_\lambda(\phi_1)}$, there exists a heteroclinic orbit from $(\phi_2, 0)$ to $(\phi_1, 0)$.*

Proof. Since the vector field on s_1 and s_2 defined by (6) points inward $\mathcal{T}$, we can choose $m < 0$ such that the vector field on s_3 also points inward $\mathcal{T}$. Denote by $\boldsymbol{N}$ the normal vector on s_3

$$\boldsymbol{N} = \begin{bmatrix} m \\ -1 \end{bmatrix}$$

and by $\boldsymbol{V}$ the vector on s_3 defined as follows

$$\boldsymbol{V} = \begin{bmatrix} u_t \\ -\alpha u_t - H_\lambda(u) \end{bmatrix} = \begin{bmatrix} m\,(u - \phi_1) \\ -\alpha m\,(u - \phi_1) - H_\lambda(u) \end{bmatrix}$$

for $\phi_1 < u < \phi_2$. Let us compute the inner product between $\boldsymbol{N}$ and $\boldsymbol{V}$ to obtain

$$\begin{aligned} \boldsymbol{N} \cdot \boldsymbol{V} &= m^2\,(u - \phi_1) + \alpha m\,(u - \phi_1) + H_\lambda(u) \\ &= (u - \phi_1)\left(m^2 + \alpha m + \frac{H_\lambda(u)}{u - \phi_1}\right) \\ &\leq (u - \phi_1)\left(m^2 + \alpha m + H'_\lambda(\phi_1)\right) \end{aligned}$$

by Lemma 4. Take m such that $\boldsymbol{N} \cdot \boldsymbol{V} < 0$, that is, $\mu_1^- < m < \mu_1^+$, where $\mu_1^\pm$ are defined in Sect. 3 with $f'(0) = 1$. Now the branch of the unstable manifold of $(\phi_2, 0)$ that points inside the region

$$\left\{(u, v) \in \mathbb{R}^2 \mid u < \phi_2\,(\lambda)\,,\ v < 0\right\}$$

enters $\mathcal{T}$ and does not leave it. Hence this bounded orbit in $\mathcal{T}$ converges to $(\phi_1, 0)$ as $t \to +\infty$. □

Remark 6 A typical example for f is given by $f(v) = |v|^\gamma$, where $\gamma \geq 1$. However, we consider the case of $f(v) = v$. Indeed, if we take $f(v) = |v|^2$, Proposition 3 does not hold because

$$\boldsymbol{N} \cdot \boldsymbol{V} = m^2\,(u - \phi_1) + \alpha m^2\,(u - \phi_1)^2 + H_\lambda(u) > 0$$

for any $m \in \mathbb{R}$ and $u \in (\phi_1, \phi_2)$.

In order to prove the monotonicity of the orbit, let us introduce the following notation

$$\gamma_i(t) = \gamma(t; \lambda_i, \alpha) = \begin{bmatrix} u(t; \lambda_i, \alpha) \\ v(t; \lambda_i, \alpha) \end{bmatrix} = \begin{bmatrix} u_i(t) \\ v_i(t) \end{bmatrix}$$

for any fixed $\alpha > 0$ and $0 < \lambda_i < \lambda^*$, $i = 1, 2$. We next prove the monotonicity of the orbit in λ for fixed α.

Proposition 4 *For fixed $\alpha > 0$, $v(t; \lambda, \alpha)$ is increasing with respect to λ as long as $v(t; \lambda, \alpha) > 0$. Moreover, if there exists $\lambda_0 \in (0, \lambda^*)$ such that $(\lambda_0, \alpha) \in \Gamma_2 \cup \Gamma_3$, then we have $(\lambda, \alpha) \in \Gamma_3$ for all $\lambda > \lambda_0$.*

Proof Since we have $v_i(0) = 0$ and $(v_i)_t(0) = \lambda_i$, $v_1(t) < v_2(t)$ holds for sufficiently small $t > 0$. For the second component of vector field defined in (6), we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(-\alpha v - H_\lambda(u)) = g(u) > 0.$$

Let us prove the second statement. If $(\lambda_0, \alpha) \in \Gamma_3$, the monotonicity yields $(\lambda, \alpha) \in \Gamma_3$ for all $\lambda > \lambda_0$. If $(\lambda_0, \alpha) \in \Gamma_2$, the monotonicity of $\phi_i(\lambda)$ and that of the orbit in λ imply $(\lambda, \alpha) \in \Gamma_3$ by Theorem 5. □

As stated after the proof of Proposition 6 in [5], the stable local manifold of the saddle $(\phi_2(\lambda), 0)$ plays a crucial role in determining the threshold of the parameter λ. Now for fixed $\lambda \in (\overline{\lambda}, \lambda^*)$, we regard the behaviour of the local stable manifold as a function of α. We shall prove that the manifold crosses the positive u-axis for small $\alpha > 0$. Then the orbit $\gamma(t)$ cannot approach the stationary points, which implies that $(\lambda, \alpha) \in \Gamma_3$. On the other hand, for large $\alpha > 0$, we prove that the manifold crosses the negative u-axis. In this case, the solution (u, v) is bounded for all $t \geq 0$ and $(\lambda, \alpha) \notin \Gamma_2$, which means that $(\lambda, \alpha) \in \Gamma_1$. Finally, we uniquely determine $\alpha^*(\lambda) > 0$ such that the manifold crosses the u-axis at $u = 0$. Then we establish the continuity and monotonicity of $\alpha^*(\lambda) > 0$ with respect to λ and define λ^* (0, 0) (α) by the inverse function of $\alpha^*(\lambda)$. In order to analyze the stable manifold of $(\phi_2(\lambda), 0)$, we perform the following change of variables in (1)

$$\begin{cases} t = -s, \\ U(s) = \phi_2(\lambda) - u(t), \\ V(s) = v(t) \end{cases}$$

and consider

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} V \\ \alpha V + H_\lambda\left(\phi_2(\lambda) - U\right) \end{bmatrix} \tag{16}$$

for $s < 0$ with $U(-\infty) = V(-\infty) = 0$. As we have seen in Sect. 3, the eigenvalues $\eta^{\pm}$ of the linearized operator at $(U, V) = (0, 0)$ corresponding to (16) are given by

$$\eta^{\pm} = \eta^{\pm}(\alpha) = \frac{\alpha \pm \sqrt{\alpha^2 - 4H'_{\lambda}(\phi_2)}}{2}.$$

The branch of the local unstable manifold can be expressed by the graph $V = \Phi(U; \lambda, \alpha)$ as long as $V(s) > 0$ since $V(s) = U_s(s) > 0$. First, it is clear that $\Phi(0; \lambda, \alpha) = 0$. Since we concentrate on the unstable manifold, we have $(\mathrm{d}\Phi/\mathrm{d}U)(0; \lambda, \alpha) = \eta^+$ by $H'_{\lambda}(\phi_2) < 0$. Finally, we have

$$\frac{\mathrm{d}\Phi}{\mathrm{d}U}(U; \lambda, \alpha) = \frac{V_s}{U_s} = \alpha + \frac{H_{\lambda}(\phi_2(\lambda) - U)}{\Phi(U; \lambda, \alpha)}. \tag{17}$$

We denote $\Phi_i(U) = \Phi(U; \lambda, \alpha_i)$ for fixed $\lambda \in (\bar{\lambda}, \lambda^*)$ and $\alpha_i > 0$ with $i = 1, 2$ and establish the monotonicity of $\Phi(U; \lambda, \alpha)$ with respect to α.

Proposition 5 *For fixed $\lambda \in (0, \lambda^*)$, let $0 < \alpha_1 < \alpha_2$. The graph $V = \Phi_2(U)$ stays above the graph $V = \Phi_1(U)$. Moreover, the graph $V = \Phi_2(U)$ and $V = \Phi_1(U)$ never intersect each other as long as they are defined.*

Proof Since we have $\Phi_2(0) - \Phi_1(0) = 0$ and $(\mathrm{d}\Phi_2/\mathrm{d}U)(0) - (\mathrm{d}\Phi_1/\mathrm{d}U)(0) = \eta^+(\alpha_2) - \eta^+(\alpha_1) > 0$, we obtain $\Phi_2(U) - \Phi_1(U) > 0$ for sufficiently small $U > 0$. Thanks to (17), $V = \Phi_2(U)$ and $V = \Phi_1(U)$ can not intersect each other. □

For fixed $\lambda \in (\bar{\lambda}, \lambda^*)$, let

$$K(\lambda) := \{\alpha \geq 0 \mid \text{ there exists } P_{\alpha} > 0 \text{ such that } \Phi(P_{\alpha}; \lambda, \alpha) = 0\}.$$

We are interested in the set $L(\lambda)$ of all points P_{α} defined by

$$L(\lambda) := \{P_{\alpha} \mid \alpha \in K(\lambda)\}.$$

$L(\lambda)$ consists of the points where the unstable manifold intersects the positive U-axis. We shall show that $K(\lambda)$ is a non-empty interval and that $L(\lambda)$ is an unbounded interval. For this purpose, let us define two lines parallel to the V-axis as follows:

$$M(\lambda) := \{(\phi_2(\lambda) - \phi_1(\lambda), V) \mid V \geq 0\};$$
$$M_{\Phi}(\lambda) := \{(\phi_2(\lambda) - \phi_1(\lambda), \Phi(\phi_2(\lambda) - \phi_1(\lambda); \lambda, \alpha)) \mid \alpha \geq 0\}.$$

Thanks to the continuity of the intersection point with respect to α proved in [4], $M_{\Phi}(\lambda)$ is an interval in $M(\lambda)$.

Lemma 7 $M_{\Phi}(\lambda) = \{\phi_2(\lambda) - \phi_1(\lambda)\} \times [\Phi(\phi_2(\lambda) - \phi_1(\lambda); \lambda, 0), +\infty)$ *for* $\lambda \in (\bar{\lambda}, \lambda^*)$.

Proof Noting that $H_\lambda(\phi_2 - U) \geq 0$ for $U \in [0, \phi_2 - \phi_1]$, we have

$$\frac{\mathrm{d}\Phi}{\mathrm{d}U}(U; \lambda, \alpha) \geq \alpha$$

for any $\alpha \geq 0$ and $U \in [0, \phi_2 - \phi_1]$ by (17). Hence integrating this inequality over $[0, \phi_2 - \phi_1]$, we obtain

$$\Phi(\phi_2 - \phi_1; \lambda, \alpha) \geq \alpha(\phi_2 - \phi_1) \to +\infty, \text{ as } \alpha \to +\infty.$$

□

Next we establish a few properties of $K(\lambda)$ and $L(\lambda)$, for which we set $A(\lambda) := \sup K(\lambda)$ and $P(\lambda) := \sup L(\lambda)$.

Lemma 8 *$K(\lambda)$ and $L(\lambda)$ are nonempty intervals for $\lambda \in (\overline{\lambda}, \lambda^*)$.*

Proof There exists some $u \in (0, \phi_1)$ such that $J_\lambda(u) = J_\lambda(\phi_2)$ holds. From the arguments of phase plane analysis in Theorem 3, Remark 4 and Proposition 1, $0 \in K(\lambda)$ holds. Hence $K(\lambda) \neq \emptyset$ follows for $\lambda \in (\overline{\lambda}, \lambda^*)$. Finally, as mentioned in Proposition 5, the continuity and monotonicity of Φ yield the conclusion. □

We see that $K(\lambda)$ and $L(\lambda)$ are intervals. Next we prove that the right endpoint of $K(\lambda)$ is open.

Lemma 9 *$K(\lambda) = [0, A(\lambda))$ for $\lambda \in (\overline{\lambda}, \lambda^*)$.*

Proof We may assume $A(\lambda) < +\infty$. Suppose by contradiction that $K(\lambda) = [0, A(\lambda)]$. The orbit $\Phi(U; \lambda, A(\lambda))$ intersects the line $M(\lambda)$. Then, by definition we can find $P_{A(\lambda)} > 0$ such that $\Phi\left(P_{A(\lambda)}; \lambda, A(\lambda)\right) = 0$. Next, the orbit starting at $(P_{A(\lambda)}, 0)$ enters the region $\left\{(U, V) \in \mathbb{R}^2 \mid V < 0\right\}$ with the U-coordinate decreasing. Since the set

$$\{(\phi_2(\lambda) - \phi_1(\lambda), \Phi(\phi_2(\lambda) - \phi_1(\lambda); \lambda, \alpha)) \mid \alpha \in [0, A(\lambda) + \varepsilon]\}$$

for any $\alpha \geq 0$ and sufficiently small $\varepsilon > 0$ describes a closed interval in the line $M(\lambda)$ by the continuity and monotonicity of Φ with respect to α and Lemma 7, the behaviour of the orbit $\Phi(U; \lambda, \alpha)$ is the same as that of $\Phi(U; \lambda, A(\lambda))$ for $\alpha \in (A(\lambda), A(\lambda)+\varepsilon)$ with sufficiently small $\varepsilon > 0$ by the continuous dependence on the parameter and initial value. Hence we obtain P_α satisfying $\Phi(P_\alpha; \lambda, \alpha) = 0$ for $\alpha \in (A(\lambda), A(\lambda) + \varepsilon)$, which contradicts the definition of $A(\lambda)$. □

Finally, we prove that $L(\lambda)$ is an unbounded interval.

Lemma 10 *$P(\lambda) = +\infty$ for $\lambda \in (\overline{\lambda}, \lambda^*)$.*

Proof First, we deal with the case of $A(\lambda) < +\infty$. Assume by contradiction that $P(\lambda) < +\infty$. The orbit $\Phi(U; \lambda, \alpha)$ intersects the line $M(\lambda)$ for any $\alpha \in (0, A(\lambda))$. In addition, due to (17), $(\mathrm{d}\Phi)/(\mathrm{d}U)(U; \lambda, \alpha) < +\infty$ on every finite interval for

any $\alpha \in (0, A(\lambda))$ as long as $\Phi(U; \lambda, \alpha) > 0$. Note that $A(\lambda) \notin K(\lambda)$. Thus the continuous dependence on the parameter yields $\Phi(U; \lambda, A(\lambda)) > 0$ for all $U > 0$. Because of the continuous dependence, for any $\alpha \in (A(\lambda) - \varepsilon, A(\lambda))$ with sufficiently small $\varepsilon > 0$, we have $\Phi(U; \lambda, \alpha) > 0$ for all $U \geq 0$, which contradicts $\alpha \in K(\lambda)$.
Next, we treat the case of $A(\lambda) = +\infty$. For all $\alpha \geq 0$, there exists $P_\alpha \in (\phi_2 - \phi_1, +\infty)$ such that we have $\Phi(P_\alpha; \lambda, \alpha) = 0$. The intersection point of $V = \Phi(U; \lambda, \alpha)$ and $M(\lambda)$ lies in the region $\{(U, V) \mid V' > 0\}$. The graph of $V = \Phi(U; \lambda, \alpha)$ must leave the region at the point (U_α, V_α) satisfying

$$V_\alpha = -\frac{1}{\alpha} H_\lambda (\phi_2 - U_\alpha), \quad \phi_2 - \phi_1 < U_\alpha < P_\alpha \quad \text{and} \quad \Phi(\phi_2 - \phi_1; \lambda, \alpha) < V_\alpha.$$

Then we have

$$H_\lambda (\phi_2 - U_\alpha) < -\alpha \Phi(\phi_2 - \phi_1; \lambda, \alpha) \to -\infty$$

as $\alpha \to +\infty$ by Lemma 7, which implies that $b > p > \phi_1 > \phi_2 - U_\alpha \to -\infty$ as $\alpha \to +\infty$, where p is defined in Theorem 1. Eventually we obtain $P_\alpha > U_\alpha \to \infty$, as $\alpha \to +\infty$. □

6 Proof of Theorem 6

The strategy to prove our main result is the following. For fixed $\lambda \in (\overline{\lambda}, \lambda^*)$, there exists $\alpha^*(\lambda)$ such that the solution of (1) exists globally in time for $\alpha^*(\lambda) < \alpha < +\infty$ and that is unbounded for $0 < \alpha < \alpha^*(\lambda)$. Then, the function $\alpha^*(\lambda)$ turns out to be monotone increasing and continuous with respect to λ. Finally, we have $\alpha^*(\lambda) \to +\infty$, as $\lambda \to \lambda^*$. We denote the inverse function of $\alpha^*(\lambda)$ by $\lambda\,(0, 0)\,(\alpha)$ which we prove to have the desired properties as in Theorem 6.

Lemma 11 *For fixed $\lambda \in (\overline{\lambda}, \lambda^*)$, there exists $\alpha^*(\lambda) > 0$ such that the solution of (1) exists globally in time for $\alpha^*(\lambda) < \alpha < +\infty$ and that becomes unbounded for $0 < \alpha < \alpha^*(\lambda)$.*

Proof For any $\alpha \in (0, A(\lambda))$, we have $P_\alpha > P_0$ such that $\Phi\,(P_\alpha; \lambda, \alpha) = 0$ by Proposition 5. Let

$$x_\alpha = \phi_2 - P_\alpha.$$

In other words, x_α is an intersection point between the stable manifold of the saddle $(\phi_2(\lambda), 0)$ and the u-axis. Since the point P_0 is determined by the orbit of the conservative case, we have $x_0 > 0$, that is, $P_0 < \phi_2$ by Proposition 1 for the case $\lambda \in (\overline{\lambda}, \lambda^*)$. By Lemma 10, the monotonicity and continuity of the intersection point, there exists $\alpha^*(\lambda) > 0$ uniquely determined such that $x_{\alpha^*(\lambda)} = 0$, or $P_{\alpha^*(\lambda)} = \phi_2$. First, we consider the case of $0 < \alpha < \alpha^*(\lambda)$. Since we have

$$\phi_2 - \phi_1 < P_0 < P_\alpha < \phi_2 \Longleftrightarrow 0 < x_\alpha < x_0 < \phi_1,$$

$(0, 0)$ is not in the domain of attraction of $(\phi_1(\lambda), 0)$ for the orbit $\gamma(t; \lambda, \alpha)$. Hence it enters

$$Z_1 := \left\{(u, v) \in \mathbb{R}^2 \mid 0 < u < b,\ v > 0,\ v_t > 0\right\}$$

and

$$Z_2 := \left\{(u, v) \in \mathbb{R}^2 \mid 0 < u < b,\ v > 0,\ v_t < 0\right\}.$$

Then, we can find $T > 0$ such that $\phi_2(\lambda) < u(T) < b$, $v(T) > 0$ and $v_t(T) = 0$ by Proposition 5. Hence the solution blows up by Theorem 5.

Next, we consider the case $\alpha > \alpha^*(\lambda)$. Similarly, we obtain

$$\phi_2 < P_\alpha \Longleftrightarrow x_\alpha < 0.$$

Therefore, $\gamma(t; \lambda, \alpha)$ enters Z_1 and then Z_2. Since $\gamma(t; \lambda, \alpha)$ stays below the stable manifold of $(\phi_2(\lambda), 0)$, it necessarily intersects the u-axis between $\phi_1(\lambda)$ and $\phi_2(\lambda)$. Hence the solution exists globally in time by Remark 4. □

Lemma 12 $\alpha^*(\lambda)$ *is monotone increasing for* $\lambda \in (\overline{\lambda}, \lambda^*)$.

Proof Let $\overline{\lambda} < \lambda_1 < \lambda_2 < \lambda^*$. We have $(\lambda_1, \alpha^*(\lambda_1)) \in \Gamma_2$ by Lemmas 6, 11 and Proposition 2. Thus we have $(\lambda_2, \alpha^*(\lambda_1)) \in \Gamma_3$ by Proposition 4. Hence, from Lemma 11, $\alpha^*(\lambda_1) < \alpha^*(\lambda_2)$ follows. □

Lemma 13 $\alpha^*(\lambda)$ *is continuous for* $\lambda \in (\overline{\lambda}, \lambda^*)$.

Proof Let $\lambda_0 \in (\overline{\lambda}, \lambda^*)$ be fixed. Then we have $(\lambda_0, \alpha^*(\lambda_0) - \varepsilon) \in \Gamma_3$ and $(\lambda_0, \alpha^*(\lambda_0) + \varepsilon) \in \Gamma_1$ for any chosen $\varepsilon \in (0, \alpha^*(\lambda_0))$. By Lemma 6, there exists $\delta > 0$ such that $(\lambda, \alpha^*(\lambda_0) - \varepsilon) \in \Gamma_3$ and $(\lambda, \alpha^*(\lambda_0) + \varepsilon) \in \Gamma_1$ hold for $|\lambda - \lambda_0| < \delta$. Lemma 11 implies that $\alpha^*(\lambda_0) - \varepsilon < \alpha^*(\lambda) < \alpha^*(\lambda_0) + \varepsilon$, and in turn $|\alpha^*(\lambda) - \alpha^*(\lambda_0)| < \varepsilon$. □

Lemma 14 *We have* $\lim_{\lambda \nearrow \lambda^*} \alpha^*(\lambda) = +\infty$.

Proof Assume that for $\lambda = \lambda^*$ there exists $\alpha_0 > 0$ such that $\gamma(t; \lambda^*, \alpha_0) \to (p, 0)$ as $t \to +\infty$, where $\phi_1(\lambda^*) = \phi_2(\lambda^*) = p$. We will show that this assumption leads us to a contradiction. Indeed, in this case we have $(\lambda, \alpha) \in \Gamma_1$ for all $0 < \lambda < \lambda^*$ and $\alpha \geq \alpha_0$ by $\alpha^*(\lambda^*) \leq \alpha_0$ and Lemma 12. Hence $\gamma(t; \lambda^*, \alpha) \to (p, 0)$ as $t \to +\infty$ for all $\alpha \geq \alpha_0$. Now, again through the transformation

$$\begin{cases} t = -s, \\ U(s) = p - u(t), \\ V(s) = v(t), \end{cases}$$

(6) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} V \\ \alpha V + H_{\lambda^*}(p - U) \end{bmatrix}$$

for $s < 0$ with $U(-\infty) = V(-\infty) = 0$. We have one-parameter family of unstable manifold of $(U, V) = (0, 0)$ with a branch which enters

$$W_1 := \left\{(U, V) \in \mathbb{R}^2 \mid U > 0,\ V > \Psi_{\alpha,\lambda^*}(U)\right\},$$
$$W_2 := \left\{(U, V) \in \mathbb{R}^2 \mid U > 0,\ 0 < V < \Psi_{\alpha,\lambda^*}(U)\right\}$$

and approaches $(U, V) = (p, 0)$, where

$$\Psi_{\alpha,\lambda}(U) \equiv -\frac{1}{\alpha} H_\lambda\,(p - U)\,.$$

It is clear that $V_s > 0$ in W_1 and that $V_s < 0$ in W_2. Each of these branches of the unstable manifolds for $\alpha \geq \alpha_0$ is the graph of a function $V = \Phi(U; \lambda^*, \alpha)$ defined for $U \in [0, p]$. Moreover, similarly to (17), we have

$$\frac{\mathrm{d}\Phi}{\mathrm{d}U}(U; \lambda^*, \alpha) = \alpha + \frac{H_{\lambda^*}\,(p - U)}{\Phi(U; \lambda^*, \alpha)}$$

and

$$\frac{\mathrm{d}\Phi}{\mathrm{d}U}(0; \lambda^*, \alpha) = \eta^+(\alpha) = \alpha.$$

As in Proposition 5, the graph $V = \Phi(U; \lambda^*, \alpha_2)$ stays above the graph $V = \Phi(U; \lambda^*, \alpha_1)$ for $\alpha_0 \leq \alpha_1 < \alpha_2$. Let (U_0, V_0) be the intersection point of the branch $V = \Phi(U; \lambda^*, \alpha_0)$ of the unstable manifold with the graph $V = \Psi_{\alpha_0,\lambda^*}(U)$. Then we have $U_0 > 0$, $V_0 > 0$ and

$$V_0 = \Phi(U_0; \lambda^*, \alpha_0).$$

By monotonicity and continuity, there exists $\delta > 0$ such that

$$\frac{1}{2} V_0 = \Phi(U_0 - \delta; \lambda^*, \alpha_0).$$

Let consider the value

$$\alpha > \max\left(\alpha_0, \frac{4\lambda^*}{V_0}\right).$$

Noting that $U_0 < p$ and that $0 > H_{\lambda^*}\,(p - U) \geq H_{\lambda^*}\,(0) = -\lambda^*$ for $U \in [U_0 - \delta, U_0]$, we have

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}U}(U; \lambda^*, \alpha) &= \alpha + \frac{H_{\lambda^*}\,(p - U)}{\Phi(U; \lambda^*, \alpha)}\\ &\geq \alpha + \frac{2H_{\lambda^*}\,(p - U)}{V_0}\end{aligned}$$

$$> \alpha - \frac{2\lambda^*}{V_0}$$
$$> \frac{1}{2}\alpha$$

for $U \in [U_0 - \delta, U_0]$. Apply the mean value theorem to obtain

$$\begin{aligned}\Phi(U;\lambda^*,\alpha) &= \Phi(U;\lambda^*,\alpha) - \Phi(U_0-\delta;\lambda^*,\alpha) + \Phi(U_0-\delta;\lambda^*,\alpha)\\ &> \frac{1}{2}\alpha\left\{U - (U_0-\delta)\right\} + \frac{1}{2}V_0\end{aligned}$$

for $U \in [U_0 - \delta, U_0]$ and in particular

$$\Phi(U_0;\lambda^*,\alpha) > \frac{1}{2}\alpha\delta + \frac{1}{2}V_0.$$

If necessary, we can take α larger and satisfying

$$\frac{1}{2}\alpha\delta + \frac{1}{2}V_0 > \frac{\lambda^*}{\alpha} = \Psi_{\alpha,\lambda^*}(p)$$

so that

$$\Phi(U_0;\lambda^*,\alpha) > \Psi_{\alpha,\lambda^*}(p).$$

Finally we obtain

$$\left\{(U,V)\in\mathbb{R}^2 \mid V = \Phi(U;\lambda^*,\alpha),\ U_0 < U < p\right\} \subset W_1,$$

and thus

$$V_0 < \Phi(p;\lambda^*,\alpha) = 0,$$

which contradicts the fact $V_0 > 0$. □

Proof of Theorem 6 $\alpha^*(\lambda)$ is the desired threshold with respect to α by Lemma 11. Since $\alpha^*(\lambda)$ is strictly increasing and continuous by Lemmas 12 and 13 respectively, the inverse function of $\alpha^*(\lambda)$, denoted by $\lambda\,(0,0)\,(\alpha)$, is well-defined for $\alpha \geq 0$ and inherits the properties of monotonicity and continuity: Remark 5 and Lemma 14 complete the proof. □

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