



Another characterization of meager ideals

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Abstract

We show that an ideal \mathcal{I} on the positive integers is meager if and only if there exists a bounded nonconvergent real sequence x such that the set of subsequences [resp. permutations] of x which preserve the set of \mathcal{I} -limit points is comeager and, in addition, every accumulation point of x is also an \mathcal{I} -limit point (that is, a limit of a subsequence (x_{n_k}) such that $\{n_1, n_2, \dots\} \notin \mathcal{I}$). The analogous characterization holds also for \mathcal{I} -cluster points.

Keywords Ideal limit point · Ideal cluster point · Meager ideal · Subsequences · Permutations.

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1 Introduction

By a known result due to Buck [7], almost every subsequence, in the sense of measure, of a given real sequence x has the same set of ordinary limit points of the original sequence x . Extensions and other measure-related results may be found in [1, 17, 18, 22–24]. The aim of this note is to prove its topological [non]analogue in the context of ideal convergence, following the line of research in [3, 19, 21, 25]. This will allow us to obtain a characterization of meager ideals in Theorem 1.4; the adjective “another” in the title hints at the known characterization of meager ideals due to Talagrand in [26], cf. also [12, 15, 27, 28] for related works.

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We recall briefly the main definitions given in [3]. Let \mathcal{I} be an ideal on the positive integers \mathbf{N} , that is, a proper subset of $\mathcal{P}(\mathbf{N})$ closed under taking finite unions and subsets and containing the family Fin of finite sets. Ideals will be regarded as subsets of the Cantor space $\{0, 1\}^{\mathbf{N}}$ endowed with the product topology. Hence, it makes sense to speak about F_σ -ideals, meager ideals, etc; note that an ideal \mathcal{I} on \mathbf{N} is meager if and only if its dual filter $\{A \subseteq \mathbf{N} : A^c \in \mathcal{I}\}$ is so, via the homeomorphism $A \mapsto A^c$. Let also $x = (x_n)$ be a sequence taking values in a topological space X , which will be always assumed Hausdorff. Then, denote by $\Gamma_x(\mathcal{I})$ the set of its \mathcal{I} -cluster points, that is, the set of all $\eta \in X$ such that $\{n \in \mathbf{N} : x_n \in U\} \notin \mathcal{I}$ for all neighborhoods U of η . Lastly, let $L_x := \Gamma_x(\text{Fin})$ be the set of ordinary accumulation points of x .

Following the same notations as in [3], define

$$\Sigma := \left\{ \sigma \in \mathbf{N}^{\mathbf{N}} : \sigma \text{ is strictly increasing} \right\}.$$

For each $\sigma \in \Sigma$, we denote by $\sigma(x)$ the subsequence $(x_{\sigma(n)})$. We identify each subsequence of (x_{k_n}) of x with the function $\sigma \in \Sigma$ defined by $\sigma(n) = k_n$ for all $n \in \mathbf{N}$. Similarly, define

$$\Pi := \left\{ \pi \in \mathbf{N}^{\mathbf{N}} : \pi \text{ is a bijection} \right\}$$

and write $\pi(x)$ for the rearranged sequence $(x_{\pi(n)})$. Endow both Σ and Π with their relative topology and note, since they are G_δ -subsets of $\mathbf{N}^{\mathbf{N}}$, they are Polish spaces by Alexandrov’s theorem. In particular, they are not meager in themselves. Finally, denote by

$$\Sigma_x(\mathcal{I}) := \left\{ \sigma \in \Sigma : \Gamma_{\sigma(x)}(\mathcal{I}) = \Gamma_x(\mathcal{I}) \right\}$$

the set of \mathcal{I} -subsequences of x which preserve the \mathcal{I} -cluster points of x , and by

$$\Pi_x(\mathcal{I}) := \left\{ \pi \in \Pi : \Gamma_{\pi(x)}(\mathcal{I}) = \Gamma_x(\mathcal{I}) \right\}$$

its permutation analogue.

Following the informal argument that an ideal \mathcal{I} on \mathbf{N} can only be meager if it is regular enough, as in having the Baire Property, the following result has been shown in [3, Theorem 2.2]:

Theorem 1.1 *Let x be a sequence taking values in a first countable space X such that all closed sets are separable and let \mathcal{I} be a meager ideal on \mathbf{N} .*

Then the following are equivalent:

- (C1) $\Sigma_x(\mathcal{I})$ is comeager;
- (C2) $\Sigma_x(\mathcal{I})$ is not meager;
- (C3) $\Pi_x(\mathcal{I})$ is comeager;
- (C4) $\Pi_x(\mathcal{I})$ is not meager;
- (C5) $\Gamma_x(\mathcal{I}) = L_x$.

It is worth to remark that the class of first countable spaces such that all closed sets are separable contains also nonmetrizable spaces, and that there exists a separable first countable space with a nonseparable closed subset, see [3, Examples 2.3 and 2.4]. In addition, there exists a first countable space outside this family which satisfies the statement of Theorem 1.1, see [3, Example 2.5].

Given a sequence $x = (x_n)$ taking values in a topological space X , we denote by $\Lambda_x(\mathcal{I})$ the set of \mathcal{I} -limit points of x , that is, the set of all $\eta \in X$ such that $\lim \sigma(x) = \eta$ for some $\sigma \in \Sigma$ such that $\sigma[\mathbf{N}] \notin \mathcal{I}$. It is well known and easy to show that $\Lambda_x(\mathcal{I}) \subseteq \Gamma_x(\mathcal{I}) \subseteq L_x$,

see e.g. [20, Lemma 3.1]. Further relationships between \mathcal{I} -cluster points and \mathcal{I} -limit points have been studied in [2]. Similarly, we denote by

$$\tilde{\Sigma}_x(\mathcal{I}) := \{ \sigma \in \Sigma : \Lambda_{\sigma(x)}(\mathcal{I}) = \Lambda_x(\mathcal{I}) \}$$

and

$$\tilde{\Pi}_x(\mathcal{I}) := \{ \pi \in \Pi : \Lambda_{\pi(x)}(\mathcal{I}) = \Lambda_x(\mathcal{I}) \}$$

the analogues of $\Sigma_x(\mathcal{I})$ and $\Pi_x(\mathcal{I})$, respectively, for \mathcal{I} -limit points.

Our first main result is the exact analogue of Theorem 1.1 for \mathcal{I} -limit points.

Theorem 1.2 *Let x be a sequence taking values in a first countable space X such that all closed sets are separable and let \mathcal{I} be a meager ideal on \mathbf{N} .*

Then the following are equivalent:

- (L1) $\tilde{\Sigma}_x(\mathcal{I})$ is comeager;
- (L2) $\tilde{\Sigma}_x(\mathcal{I})$ is not meager;
- (L3) $\tilde{\Pi}_x(\mathcal{I})$ is comeager;
- (L4) $\tilde{\Pi}_x(\mathcal{I})$ is not meager;
- (L5) $\Lambda_x(\mathcal{I}) = L_x$.

We remark that Theorem 1.2 provides an affirmative answer to the open question stated at the end of Sect. 2 in [3]. In addition, several special cases of Theorem 1.1 have been already obtained in the literature:

- (i) [21, Theorem 2.3] for the case where $X = \mathbf{R}$, $\mathcal{I} = \mathcal{Z} := \{A \subseteq \mathbf{N} : \lim_n |A \cap [1, n]|/n = 0\}$, and the equivalences (L1) \iff (L2) \iff (L5);
- (ii) [19, Theorem 2.3] for the case where X is chosen as in Theorem 1.2, \mathcal{I} is a generalized density ideal, and the equivalences (L1) \iff (L2) \iff (L5);
- (iii) [3, Theorem 2.9] for the case where X is chosen as in Theorem 1.2 and \mathcal{I} is an analytic P-ideal;
- (iv) [25, Theorem 1] for the case where $X = \mathbf{R}$ with the equivalence (L1) \iff (L5).

At this point, as it has been shown in [3, Example 2.6], it is worth noting that, if \mathcal{I} is maximal (that is, the complement of a free ultrafilter), then there exists a bounded real sequence x which satisfies (C2) but not (C5), cf. Remark 1.6 below. Hence the statement of Theorem 1.1 (and similarly for Theorem 1.2) certainly does not apply to all ideals.

We are going to show below that this is not a coincidence, roughly meaning that the hypothesis of meagerness of the ideal \mathcal{I} is essential for all the above equivalences. In other words, some of the above equivalences turn out to be necessary and sufficient for the meagerness of the ideal \mathcal{I} .

Before we present this characterization, we recall a similar result on series of real numbers. To this aim, we need some additional notation: for each real sequence x , let Sx be the sequence of its partial sums, that is, $Sx = (S_n x)$ with $S_n x := \sum_{i \leq n} x_i$ for all $n \in \mathbf{N}$. The sequence x is said to be \mathcal{I} -bounded if $\{n \in \mathbf{N} : |x_n| > k\} \in \mathcal{I}$ for some $k \in \mathbf{N}$. The vector space of \mathcal{I} -bounded sequences, denoted by $\ell_\infty(\mathcal{I})$, has been studied, e.g., in [6, 10]. Then, define

$$\Sigma_{S,x}(\mathcal{I}) := \{ \sigma \in \Sigma : S\sigma(x) \in \ell_\infty(\mathcal{I}) \}.$$

Accordingly, the following result has been shown in [4, Theorem 4.1].

Theorem 1.3 *Let x be a real sequence such that the series $\sum_n x_n$ is not unconditionally convergent and let \mathcal{I} be a meager ideal on \mathbf{N} . Then:*

(S1) $\Sigma_{S,x}(\mathcal{I})$ is meager.

A related result on series taking values in Banach spaces has been shown in [5, Corollary 3.3], cf. also [4, Theorem 3.4 and Example 2]. It is worth bearing in mind that, if $\liminf_n x_n > 0$, then the series $\sum_n x_n$ is not unconditionally convergent and, for each $\sigma \in \Sigma$ and all ideals \mathcal{I} , the sequence of partial sums $S\sigma(x)$ has limit infinity, hence $\Sigma_{S,x}(\mathcal{I}) = \emptyset$; this example proves that the condition (S1) alone cannot provide a characterization of meager ideals.

For the sake of exposition, let \mathcal{A} be the set of all sequences taking values in some first countable space X such that all closed sets are separable, and \mathcal{N}_1 be its subset of nonconvergent sequences with at least one (ordinary) accumulation point. Lastly, let \mathcal{D} be the set of real sequences x with dense image $\{x_n : n \in \mathbf{N}\}$.

Theorem 1.4 *Let \mathcal{I} be an ideal on \mathbf{N} . Then the following are equivalent:*

- (M1) For all sequences $x \in \mathcal{A}$ we have (C1) \iff (C5);
- (M2) For all sequences $x \in \mathcal{A}$ we have (C3) \iff (C5);
- (M3) For all sequences $x \in \mathcal{A}$ we have (L1) \iff (L5);
- (M4) For all sequences $x \in \mathcal{A}$ we have (L3) \iff (L5);
- (M5) There exists a sequence $x \in \mathcal{N}_1$ such that both (C1) and (C5) hold;
- (M6) There exists a sequence $x \in \mathcal{N}_1$ such that both (C3) and (C5) hold;
- (M7) There exists a sequence $x \in \mathcal{N}_1$ such that both (L1) and (L5) hold;
- (M8) There exists a sequence $x \in \mathcal{N}_1$ such that both (L3) and (L5) hold;
- (M9) There exists a sequence $x \in \mathcal{D}$ such that (S1) holds;
- (M10) \mathcal{I} is meager.

Considering that real bounded nonconvergent sequences belong to \mathcal{N}_1 and the equivalence (M7) \iff (M10) in Theorem 1.4, we obtain the following corollary (which is stated in the abstract):

Corollary 1.5 *An ideal \mathcal{I} on \mathbf{N} is meager if and only if there exists a real bounded nonconvergent sequence x such that $\Lambda_x(\mathcal{I}) = L_x$ and $\tilde{\Sigma}_x(\mathcal{I})$ is comeager.*

Note that the definition of \mathcal{N}_1 imposes that each of its elements has at least one accumulation point. This constraint cannot be removed. Indeed, if $x \in \mathcal{A}$ verifies $L_x = \emptyset$, then $\Gamma_{\sigma(x)}(\mathcal{I}) \subseteq L_{\sigma(x)} \subseteq L_x = \emptyset$ for all $\sigma \in \Sigma$. Hence $\Sigma_x(\mathcal{I}) = \Sigma$, independently of the choice of the ideal \mathcal{I} . This would imply that both (C1) and (C5) hold true, also for nonmeager ideals, and it would provide a counterexample to the equivalence (M5) \iff (M10) in Theorem 1.4.

Remark 1.6 It may be hypothesized that (M10) is equivalent, e.g., also to the following:

- (M11) There exists a sequence $x \in \mathcal{N}_1$ such that both (C2) and (C5) hold.

However we can prove that this is false by the following example. Let \mathcal{I} be a maximal ideal. Since \mathcal{I} is maximal, there exists a unique $A \in \{2\mathbf{N}+1, 2\mathbf{N}+2\}$ such that $A \in \mathcal{I}$. Accordingly, let x be the sequence defined by $x_n = n$ if $n \in A$ and $x_n = 0$ otherwise. Then $x \rightarrow_{\mathcal{I}} 0$ and $\Gamma_x(\mathcal{I}) = L_x = \{0\}$, hence $x \in \mathcal{N}_1$ and (C5) holds. In addition,

$$\Sigma_x(\mathcal{I}) = \{\sigma \in \Sigma : \{n \in \mathbf{N} : x_{\sigma(n)} \geq 1\} \in \mathcal{I}\} = \{\sigma \in \Sigma : \sigma^{-1}[A] \in \mathcal{I}\}.$$

It follows by [3, Example 2.6] that $\Sigma_x(\mathcal{I})$ is not meager, hence also (C2) holds. (Note that the same argument shows that $\Sigma_x(\mathcal{I})$ is not comeager, hence (C1) fails.) This proves that (M11) is verified, while (M10) is not. (The same example works with \mathcal{I} -limit points.)

2 Proof of Theorem 1.2

The following result strenghtens [3, Lemma 3.3].

Lemma 2.1 *Let x be a sequence taking values in a first countable space X and let \mathcal{I} be a meager ideal on \mathbf{N} . Then*

$$\{\sigma \in \Sigma : \eta \in \Lambda_{\sigma(x)}(\mathcal{I})\} \quad \text{and} \quad \{\pi \in \Pi : \eta \in \Lambda_{\pi(x)}(\mathcal{I})\}$$

are comeager for each $\eta \in L_x$.

Proof Assume that $L_x \neq \emptyset$, otherwise there is nothing to prove. Fix $\eta \in L_x$ and let $(U_{\eta,m})$ be a decreasing local base at η . Since \mathcal{I} is a meager ideal, its dual filter $\{A \subseteq \mathbf{N} : A^c \in \mathcal{I}\}$ is meager too. It follows by Talagrand’s characterization of meager filters [26, Theorem 21] that there exists an increasing sequence (ι_n) of positive integers such that $A \notin \mathcal{I}$ whenever $I_n := [\iota_n, \iota_{n+1}) \subseteq A$ for infinitely many n . For each $m \in \mathbf{N}$ define

$$S_m(\eta) := \bigcup_{k \geq m} \{\sigma \in \Sigma : x_{\sigma(n)} \in U_{\eta,m} \text{ for all } n \in I_k\}. \tag{1}$$

Since $\bigcap_m S_m(\eta)$ is contained in $\{\sigma \in \Sigma : \eta \in \Lambda_{\sigma(x)}(\mathcal{I})\}$ by the previous observation, it will be sufficient to show that each $S_m(\eta)$ is comeager. To this aim, fix $m \in \mathbf{N}$ and let $C = \{\sigma \in \Sigma : \sigma(1) = a_1, \dots, \sigma(k) = a_k\}$ be a basic open set, for some positive integers $a_1 < \dots < a_k$. Note that the set $E := \{n \in \mathbf{N} : x_n \in U_{\eta,m}\}$ is infinite since $\eta \in L_x$. At this point, it is enough to see that

$$\{\sigma \in C : \sigma(n) = e_n \text{ for all } n \text{ with } k < n < \iota_{k+m}\},$$

for some increasing values $e_n \in E \setminus [1, a_k]$, is a nonempty open set contained in $C \cap S_m(\eta)$. This proves that $S_m(\eta)$ is comeager, completing the first part. The second part of the proof concerning permutations proceeds verbatim. \square

We are ready for the proof of Theorem 1.2.

Proof of Theorem 1.2 (L1) \implies (L2) It is obvious.

(L2) \implies (L5) Suppose that there exists $\ell \in L_x \setminus \Lambda_x(\mathcal{I})$. Then $\tilde{\Sigma}_x(\mathcal{I})$ is contained in $\Sigma \setminus \{\sigma \in \Sigma : \eta \in \Lambda_{\sigma(x)}(\mathcal{I})\}$, which is meager by Lemma 2.1.

(L5) \implies (L1) Suppose that $L_x \neq \emptyset$, otherwise the claim is trivial. Let \mathcal{L} be a countable dense subset of L_x , and denote by $(U_{\eta,m} : m \geq 1)$ a decreasing local base at each $\eta \in X$. As is shown in the proof of Lemma 2.1, the set $S_{m,\eta}$ defined in (1) is comeager for all $m \in \mathbf{N}$ and $\eta \in L_x$, therefore also $S := \bigcap_{m \geq 1} \bigcap_{\eta \in \mathcal{L}} S_m(\eta)$ is comeager. Note that, equivalently,

$$S = \{\sigma \in \Sigma : \forall \eta \in \mathcal{L}, \exists^\infty k \in \mathbf{N}, \forall n \in I_k, x_{\sigma(n)} \in U_{\eta,m}\}. \tag{2}$$

Since $S \subseteq \{\sigma \in \Sigma : \eta \in \Lambda_{\sigma(x)}(\mathcal{I}) \text{ for all } \eta \in \mathcal{L}\}$, we have that $\mathcal{L} \subseteq \Lambda_{\sigma(x)}(\mathcal{I})$ for all $\sigma \in S$. To conclude the proof, we claim that S is contained in $\tilde{\Sigma}_x(\mathcal{I})$.

To this aim, fix $\sigma \in S$. On the one hand, we have

$$\Lambda_{\sigma(x)}(\mathcal{I}) \subseteq L_{\sigma(x)} \subseteq L_x = \Lambda_x(\mathcal{I}).$$

Conversely, fix $\eta \in L_x$. Since \mathcal{L} is dense, there exists a sequence (η_t) in \mathcal{L} which is convergent to η . Without loss of generality we can assume that $\eta_t \in U_{\eta,t}$ for all $t \in \mathbf{N}$. At this point, for each $t \in \mathbf{N}$, there exists $m_t \in \mathbf{N}$ such that $U_{\eta_t,m_t} \subseteq U_{\eta,t}$. By the explicit expression (2), S is contained in

$$\{\sigma \in \Sigma : \forall t \in \mathbf{N}, \exists^\infty k \in \mathbf{N}, \forall n \in I_k, x_{\sigma(n)} \in U_{\eta_t,m_t}\}.$$

It follows that the subsequence $\sigma(x)$ has a subsubsequence $\hat{\sigma}(\sigma(x))$ which is convergent to η and such that $\hat{\sigma}[\mathbb{N}]$ contains infinitely many intervals I_k , hence η is an \mathcal{I} -limit point of $\sigma(x)$. Therefore

$$\Lambda_x(\mathcal{I}) = L_x \subseteq \Lambda_{\sigma(x)}(\mathcal{I}).$$

This proves that $S \subseteq \tilde{\Sigma}_x(\mathcal{I})$, completing the proof.

The proof of (L3) \implies (L4) \implies (L5) \implies (L3) goes verbatim. □

3 Proof of Theorem 1.4

The following result will be the key tool in the proof of the characterization of meager ideals.

Theorem 3.1 *Let \mathcal{I} be an ideal on \mathbb{N} and let x be a nonconvergent sequence in a topological space X such that there exists $\eta \in X$ for which at least one among*

$$\{\sigma \in \Sigma : \eta \in \Gamma_{\sigma(x)}(\mathcal{I})\} \quad \text{and} \quad \{\pi \in \Pi : \eta \in \Gamma_{\pi(x)}(\mathcal{I})\} \tag{3}$$

is comeager. Then \mathcal{I} is meager.

Proof First, assume that we can fix $\eta \in X$ such that $S := \{\sigma \in \Sigma : \eta \in \Gamma_{\sigma(x)}(\mathcal{I})\}$ is comeager. Hence there exists a decreasing sequence (G_n) of dense open subsets of Σ such that $\bigcap_n G_n \subseteq S$. In addition, since x is not convergent to η , there is a neighborhood U of η such that

$$E := \{n \in \mathbb{N} : x_n \notin U\}$$

is infinite.

At this point, consider the following game defined by Laflamme in [16]: Players I and II choose alternately subsets $C_1, F_1, C_2, F_2, \dots$ of \mathbb{N} , where the sets $C_1 \supseteq C_2 \supseteq \dots$, which are chosen by Player I, are cofinite and the sets $F_k \subseteq C_k$, which are chosen by Player II, are finite. Player II is declared to be the winner if and only if $\bigcup_k F_k \notin \mathcal{I}$. Note that we may suppose without loss of generality that $F_k \cap C_{k+1} = \emptyset$ and $C_k = [c_k, \infty)$ for all $k \in \mathbb{N}$ (hence, the sequence (c_k) corresponds to arbitrary (large enough) choices made by Player I). By [16, Theorem 2.12], Player II has a winning strategy if and only if \mathcal{I} is meager. Hence, the rest of the proof consists in showing that Player II has a winning strategy.

To this aim, we will define recursively, together with the description of the strategy of Player II, also a decreasing sequence of basic open sets $A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \dots$ in Σ (recall that a basic open set in Σ is a cylinder of the type $D = \{\sigma \in \Sigma : \sigma(1) = a_1, \dots, \sigma(n) = a_n\}$ for some positive integers $a_1 < \dots < a_n$, and we set $m(D) := a_n$). Suppose that the sets $C_1, F_1, \dots, C_{k-1}, F_{k-1}, C_k \subseteq \mathbb{N}$ have been already chosen and that the open sets $A_1, B_1, \dots, A_{k-1}, B_{k-1} \subseteq \Sigma$ have already been defined, for some $k \in \mathbb{N}$, where we assume by convention that $B_0 := \Sigma$ and $m(\Sigma) := 0$. Then we define the sets A_k, B_k , and F_k as follows:

- (i) $A_k := \{\sigma \in B_{k-1} : \sigma(n) = e_n \text{ for all } n \text{ with } m(B_{k-1}) < n < c_k\}$ for some increasing values $e_n \in E$ which are bigger than $\sigma(m(B_{k-1}))$ (note that this is possible since E is infinite);
- (ii) B_k is a nonempty basic open set contained in $G_k \cap A_k$ (note that this is possible since G_k is open dense and A_k is nonempty open);

- (iii) $F_k := \{n \in \mathbf{N} : x_{\sigma(n)} \in U\} \cap [c_k, m(B_k)]$ (note that this is a finite set, possibly empty).

We obtain by construction that there exists $\sigma^* \in \Sigma$ such that

$$\sigma^* \in \bigcap_k B_k \subseteq \bigcap_k G_k \subseteq S,$$

so that η is an \mathcal{I} -cluster point of the subsequence $\sigma^*(x)$, which implies that $\{n \in \mathbf{N} : x_{\sigma^*(n)} \in U\} \notin \mathcal{I}$. At the same time, by the definitions above we get

$$\{n \in \mathbf{N} : x_{\sigma^*(n)} \in U\} = \bigcup_k \{n \in [c_k, m(B_k)] : x_{\sigma^*(n)} \in U\} = \bigcup_k F_k.$$

This proves that Player II has a winning strategy. Therefore \mathcal{I} is meager, concluding the first part of the proof.

For the second part, recall that a basic open set in Π is a cylinder of the type $D = \{\pi \in \Pi : \pi(1) = a_1, \dots, \pi(n) = a_n\}$ for some distinct $a_1, \dots, a_n \in \mathbf{N}$, and set $m(D) := \max\{a_1, \dots, a_n\}$. Minor modifications are necessary in the definitions of the corresponding sets A_k and B_k as it follows:

- (i) $A_k := \{\pi \in B_{k-1} : \pi(n) = e_n \text{ for all } n \text{ with } m(B_{k-1}) < n < c_k\}$ for the smallest possible values of $e_n \in E$ which have not been chosen before in the previous steps;
- (ii) B_k is a nonempty basic open set contained in $G_k \cap A_k$ with the additional condition that if $\pi \in B_k$ then $\{\pi(1), \dots, \pi(m(B_k))\}$ coincides with $\{1, \dots, m(B_k)\}$ (this is still possible replacing B_k , if necessary, with a smaller subset which satisfies this condition).

It follows by construction that $\bigcap_k B_k$ contains an element π^* which is a permutation on \mathbf{N} . Finally, the proof of the permutations case follows the same lines as above. □

Corollary 3.2 *Let \mathcal{I} be an ideal on \mathbf{N} and x be a nonconvergent sequence in a topological space X such that there exists $\eta \in X$ for which at least one among*

$$\{\sigma \in \Sigma : \eta \in \Lambda_{\sigma(x)}(\mathcal{I})\} \quad \text{and} \quad \{\pi \in \Pi : \eta \in \Lambda_{\pi(x)}(\mathcal{I})\}$$

is comeager. Then \mathcal{I} is meager.

Proof This follows by Theorem 3.1 and the fact that every \mathcal{I} -limit point is necessarily an \mathcal{I} -cluster point. □

Now, we show that the converse of Theorem 1.3 holds, provided that the sequence x has a dense image.

Theorem 3.3 *Let $x \in \mathcal{D}$ such that $\Sigma_{S,x}(\mathcal{I})$ is meager. Then \mathcal{I} is meager.*

Proof We follow the same lines of proof as in Theorem 3.1. Suppose that $\Sigma \setminus \Sigma_{S,x}(\mathcal{I})$ is comeager, hence it contains $\bigcap_n G_n$, where (G_n) is a decreasing sequence of dense open sets in Σ . Using the same Laflamme game, we modify the construction of the sets A_k and F_k as it follows:

- (i) $A_k := \{\sigma \in B_{k-1} : \sigma(n) = e_n \text{ for all } n \text{ with } m(B_{k-1}) < n < c_k\}$, where the increasing values e_n are chosen such that $|S_n \sigma(x)| < 1$ (note that this is possible since x has a dense image; here, recall that $S_n \sigma(x) = \sum_{i \leq n} x_{\sigma(i)}$);
- (ii) $F_k := \{n \in \mathbf{N} : |S_n \sigma(x)| \geq 1\} \cap [c_k, m(B_k)]$.

Similarly, we obtain that there exists $\sigma^* \in \bigcap_k B_k \subseteq \Sigma \setminus \Sigma_{S,x}(\mathcal{I})$, so that $S\sigma^*(x)$ is not \mathcal{I} -bounded. It follows that

$$\begin{aligned} \bigcup_k F_k &= \bigcup_k \{n \in [c_k, m(B_k)] : |S_n \sigma^*(x)| \geq 1\} \\ &= \{n \in \mathbf{N} : |S_n \sigma^*(x)| \geq 1\} \notin \mathcal{I}, \end{aligned}$$

which proves that player II has a winning strategy. □

We can finally proceed to the proof of Theorem 1.4.

Proof of Theorem 1.4 (M1) \implies (M5) First, note that the ideal \mathcal{I} cannot be maximal: indeed, otherwise, by Remark 1.6, there exists a real sequence x such that (C5) holds and (C1) does not. Since \mathcal{I} has to be not maximal, there exists a set $A \notin \mathcal{I}$ such that $A^c \notin \mathcal{I}$. At this point, it follows that condition (C5) holds for the real sequence x defined by $x_n = 1$ if $n \in A$ and $x_n = 0$ otherwise. Hence (C1) also holds by (M1), proving the implication.

(M5) \implies (M10) Assume that $\Gamma_x(\mathcal{I}) = L_x \neq \emptyset$. Then

$$\Sigma_x(\mathcal{I}) = \{\sigma \in \Sigma : \Gamma_{\sigma(x)}(\mathcal{I}) = L_x\} = \bigcap_{\eta \in L_x} \{\sigma \in \Sigma : \eta \in \Gamma_{\sigma(x)}(\mathcal{I})\}.$$

Hence (3) holds and the implication follows by Theorem 3.1.

(M10) \implies (M1) This follows by Theorem 1.1.

The proof of **(M2) \implies (M6) \implies (M10) \implies (M2)** runs verbatim as above.

The proofs of **(M3) \implies (M7) \implies (M10) \implies (M3)** and **(M4) \implies (M8) \implies (M10) \implies (M4)** go along the same lines, replacing Theorem 1.1 and Theorem 3.1 with Theorem 1.2 and Corollary 3.2, respectively.

The equivalence **(M9) \iff (M10)** follows by Theorem 1.3 and 3.3. □

4 Concluding remarks

Let us identify each $\sigma \in \Sigma$ with the real number $\sum_{n \geq 1} 2^{-\sigma(n)}$. This provides a homeomorphism $h : \Sigma \rightarrow (0, 1]$. Denote by λ the Lebesgue measure on $(0, 1]$ and let $\hat{\lambda} := \lambda \circ h$ be its pushforward on Σ , cf. e.g. [13, 14].

At this point, one may hope for a measure analogue of Theorem 1.4. However, for the subsequences case, this does not seem to be possible as it follows from the next two examples: (i) the ideal $\mathcal{I} := \{A \subseteq \mathbf{N} : \sum_{a \in A} 1/a < \infty\}$ is F_σ and, by [17, Proposition 2.3 and Theorem 3.1], we have $\hat{\lambda}(\Sigma_x(\mathcal{I})) = 1$ for all real sequences x ; (ii) the Fubini sum $\mathcal{P}(\mathbf{N}) \oplus \text{Fin}$ (see e.g. [11, Sect. 1.2]), which can be identified with $\mathcal{J} := \{A \subseteq \mathbf{N} : |A \cap (2\mathbf{N} + 1)| < \infty\}$, is also F_σ and, by [1, Example 2], there exists a real sequence x such that $\hat{\lambda}(\Sigma_x(\mathcal{J})) = 0$.

On the other hand, a [non]analogue for the permutations is more difficult to obtain for lack of a natural candidate for a measure on Π . Indeed, since Π is a Polish group which is not locally compact, we cannot speak about Haar measure. An alternative could be to consider the notion of *prevalent set* as introduced by Christensen in [8]: a set $S \subseteq \Pi$ is called prevalent if it is universally measurable (i.e., measurable with respect to every complete probability measure on Π that measures all Borel subsets of Π) and there exists a (not necessarily invariant) Borel probability measure μ over Π such that $\mu(\zeta S) = 1$ for all permutations $\zeta \in \Pi$, cf. also [9]. We leave as an open question whether $\Pi_x(\mathcal{I})$ and/or $\tilde{\Pi}_x(\mathcal{I})$ are prevalent sets whenever \mathcal{I} is a meager ideal.

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