Università degli Studi dell'Insubria, Como, DiSAT

Clifford algebra automorphisms in theoretical physics:

A discussion on the role of spin connection in gravity and weak interactions

Angelo Eduardo da Silva Hartmann
PhD Candidate

Thesis submitted to the DiSAT Program in Physics and Astrophysics, XXXV Cycle, as a partial fulfillment of the requirements for the degree of Doctor of Philosophy.

PhD Director: Prof. Giuliano Benenti.
Advisor: Prof. Sergio Luigi Cacciatori.
Examiners: Prof. Roberto Onofrio, Prof. José Abdalla Helayël-Neto.

Como, 2023

The scientific theorist is not to be envied. For Nature, or more precisely experiment,
is an inexorable and not very friendly judge of his work. It never says “Yes” to a theory.
In the most favorable cases it says “Maybe”, and in the great majority of cases simply “No”.
... Probably every theory will some day experience its “No”.
Most theories, soon after conception.
A. Einstein *apud* J. Agassi [7].

Abstract

The present investigation is a return to an old, and yet open, discussion about the relation between gravitational and weak interactions. Pauli's conjecture on the possibility of new physics being related to the square root of the gravitational constant is addressed. A particular framework were gravity is induced by four-fermions is formulated. A general analysis of the internal consistency of spin connections in Riemannian geometry is considered, with special attention to the arbitrariness entailed by the Fock connection. In particular, the Fock connection is a non-Abelian gauge field of the Clifford bundle. We propose, as a possible synthesis of this analysis, that new degrees of freedom may arise from the group of automorphisms of the associated Clifford bundle. If this theory results internally consistent, a new framework embracing the electroweak model and sterile neutrinos coupled to gravity is envisaged, without changing the (external) properties of the physical spacetime.

Keywords: algebra automorphisms, enlarged internal groups, gravitational four-fermions, Pauli square root conjecture, spin connection.

Acknowledgments

The last three years have been challenging, to say the least. The world is changing, and changing quickly. Living in Europe is an experience that have unavoidably transformed the way how I think. If history still plays a role in our lifes, that should be to not forget the intrinsic value of the free, and open debate. The more the confrontation of ideas and arguments, the less we will see bodies falling at the front lines. It seems that, even in science, we are sometimes on the verge of suppressing our willingness to listen to alternative proposals in the name of paradigms' authority. Infallibility does not apply to science though.

Prof. Sergio L. Cacciatori, to whom I am deeply grateful, gave me the opportunity to exercise the argument and search for a coherent view of the open problems discussed during the previous stages of this thesis.

I also thanks Prof. Mario Novello for his support during the first half of my PhD research.

On November 7 2019, when I arrived at DiSAT to attend the first doctoral seminar of the XXXV Cycle, I was welcomed by Prof. Alberto Parola. His cordiality belongs to the kind of memories that time does not erase. On Prof. Parola's behalf, I salute all the DiSAT members I had the opportunity to dialogue with during these years.

During the vacations of 2020, I spent two weeks as a volunteer at Ca'Mariuccia, a project for biological, ethical agriculture and ecotourism in the surroundings of Asti, Piemonte. There, I basically learnt from zero, by trial and error, to speak in Italian with Giovanni, Angela and Andrea, Gesù, Paola, Stefano, Umberto (*in memoriam*), Asham and Mohammed. I happily accepted the invitation to return for another three weeks period on August 2021. The world needs more communities like Ca'Mariuccia, thinking and acting for a renewed relation with our Home.

I also remind with esteem the way how all attendees were very welcomed at Domodossola by Andrea and his group, during the 2022 edition of the DomoSchool. That one week gave me a renewed perspective on my research. It also allowed me to meet one of my PhD colleagues, Gaia, who became a dear friend since then. On behalf of Gaia and her family, I wish a bright future and fruitful life to all my colleagues at Insubria. The invitation remains open to come and visit the Iguaçu Falls.

A special thanks to Mrs. Giovanna Colombo, the tireless DiSAT librarian who helped me to find copies of references not easily available. I am particularly indebted for the following ones: [527–529] (Biblioteca Nazionale Centrale di Roma), [176, 177, 276] (Biblioteca di Scienze dell'Università degli Studi di Firenze), [213, 277] (Biblioteca delle Scienze dell'Università di Pavia), [28] (Academia delle Scienze di Torino), [306, 307] (Biblioteca dell'INAF, Osservatorio Astronomico di Arcetri), [457–461] (Hemeroteca Digital da Fundação Biblioteca Nacional), [150] (Virginia Military Institute Library), and [271] (Princeton University Library).

To the Niels Bohr Archive, for kindly providing me access to Bohr Scientific Correspondence with Dirac, Gamow, Klein, and Pauli. This material helped me to enlighten Pauli's square root conjecture (Sec. 2.2), whose historical and epistemological relevance is beyond the scope of the present work and are envisaged for further research.

A warm thank you to Prof. Benenti, for his attentive communication throughout these almost four years, including the pandemic, my recovery, the Politecnico residence, the final year seminars and reports, refunds, and the thesis process.

I also want to express my sincere gratitude to the reviewers, Prof. R. Onofrio and Prof. Helayël-Neto. The present version of this work is mainly due to their encouragement, generous remarks and fruitful insights.

And now a word to all my people. It wouldn't make sense to get here if it weren't for you. Thank you for not letting me lose heart in these strange times. Thank you for making me believe in the future, even when I couldn't.

Contents

1. Introduction

One should always guard against getting too attached to one particular line of thought.
P. A. M. Dirac in [11], p.135.

Few topics, if any, were more systematically scrutinized in theoretical physics than the theory of fermions in Clifford algebras. Immediately after its introduction in physics by the seminal papers of Dirac, "The quantum theory of the electron" (1928) [122, 123], it became a subject of analysis in Lorentzian manifolds [24, 98, 174, 175, 179, 242, 259, 349, 392, 419, 425, 446, 447, 463, 469, 545–547, 554, 560, 567], as well as in higher dimensional [60, 126, 270, 281, 390, 391, 399, 400, 464], Euclidean [74, 385], projective [465–468, 491, 492, 519, 520], and quaternionic [91, 119, 297–300] spaces[1].

In the 1950s, new directions for inquiring nature were unveiled with the detection of the electron neutrino [429–431], and the violation of spatial reflection in the β-decay [561] and μ-decay [188, 195] processes. Those experiments[2] demarcated a new stage for the problem of parity conservation in weak interactions, under discussion since, at least, the work by Bargmann and Wigner [26], followed by Yang and Tiomno [496, 565], Zharkov [578], Berestetskiĭ [28, 29], Schwinger [474], Caianiello [69], Wick, Wightman and Wigner [552], Lüders and B. Zumino [328–330] (see, in particular, the note 2 in [330]), Zeldovich [569, 570, 574, 575], Gulmanelli [213], T. D. Lee and Yang [312], Salam [442], and Watanabe [530], to name but a few.

If the neutrino is massive or not, which translates into the question if the neutrino has a Dirac or a Majorana spinor representation, is a key problem within parity violation of weak interactions that received two divergent interpretations: T. D. Lee and Yang [312], Landau [303], and Salam [442] suggested the two-component theory of longitudinal neutrinos; in dissonance, following Gell-Mann and Pais [200] idea of $K^o \rightleftarrows \bar{K}^o$ oscillations, Bruno Pontecorvo [420] leaded the avenue of mixed neutral particles, violating the transitions $\nu_L \rightleftarrows \bar{\nu}_R$ forbidden by the two-component theory [41]. Despite the experimental success of the first, the second was not excluded, and this divergence remains open up to date [40].

Consequently, a review of Fermi's contact interaction [163, 164],

$$\mathcal{L}_{\mathrm{F}} = -\frac{G_F}{\sqrt{2}}\left[\overline{\Psi}_p(x)\gamma_\mu\Psi_n(x)\,\overline{\Psi}_e(x)\gamma^\mu\Psi_\nu(x)\right] + \text{ h.c.} \tag{1.1}$$

was advanced[3] by Sudarshan and Marshak [489, 490], and followed by Feynman and Gell-Mann [168,

[1]For an historical appraisal of the development of Clifford algebras, see Dieudonné's review in [84], and Lounesto in [432].

[2]Some authors, including Wigner himself, suggested that Madame Wu's results could had been anticipated by Cox, McIlwraith and Kurrelmeyer [92] in 1928. See the discussion session after Yang's contribution [562] to the Colloque International sur l'Histoire de la Physique des Particules (Paris, 1982).

[3]In contradiction with the current empirical data at the time [10], according to which "the beta interaction is scalar",

199], Sakurai [441], and J. Leite Lopes [314], in terms of the weak $V - A$ current interaction [579],

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} J^\dagger_\mu(x)\, J^\mu(x) \ + \ \text{h.c.} \tag{1.2}$$

where

$$J^\mu(x) = \overline{\Psi}_e(x)\gamma^\mu(1-\gamma_5)\Psi_\nu(x) = 2\overline{e}_L(x)\gamma^\mu\nu_R(x). \tag{1.3}$$

The first synthesis[4] of an intense decade of debate came in 1961 when Glashow [206] achieved the electro-weak unification. The standard model of particle physics became self-contained in 1967 with the introduction of the Higgs boson by Salam [443], and Weinberg [533]. The extension to include quarks came in 1970, by Glashow, Iliopoulos, and Maiani [207]. This period is widely known and extensively documented [1, 39, 64, 75, 82, 296, 311, 315, 354, 373, 424, 481, 487, 522, 536, 540, 563, 580]. Currently, Fermi's theory (1.2) is assumed to correspond to the *effective* (tree level) low energy limit of the intermediate vector boson theory, where the weak current $J^\mu(x)$ in (1.2) is decomposed into the leptonic and hadronic sectors [82, 413, 579].

What seems less emphasized in the literature is the way how parity violation in weak interactions became the seed for a renewed interest in to understand the *language* connecting neutrinos to the gravitational interaction, and, in particular, the algebra of spinors in curved spaces. Why gravity? Because it was believed, in the 1950s, that the neutrino of the electron, the only known type of neutrino at the time[5], was massless, electromagnetically neutral, and weakly universal.

Dicke [115], and by Brill and Wheeler [62] figure among the pioneer papers to address this situation soon after the refutation of parity conservation by neutrinos and muons. We shall recall that Wheeler was a fervent supporter of the geometrization program of all fundamental interactions [341, 548–550], an interpretation that became predominant in the gravitational physics of the second-half of the 20th century. For the neutrino-gravity coupling in particular, Wheeler stated the problem of reducing the neutrino to the metric field, known in the literature as the Rainich problem for neutrinos [243, 294]. The authors of [62] conclude with the typical puzzle posed by spinor fields to the geometrization program:

> What is there about the description of the geometry of space which is not already adequately covered by ordinary scalars, vectors, and tensors of standard tensor analysis? To this question

Zeldovich and Gershtein [575] discussed in 1955, with "no practical significance but only methodological interest", how the conservation of weak vector and electromagnetic currents might be related. In 1955, Landau [304] discuss the limits of QED at high energies, recognizing the nonrenormalizability of the pseudovector coupling.

[4] According to J. Tiomno in [12], the idea of unify electromagnetism and weak interactions was proposed in 1958 by J. Leite Lopes [314]. Back in 1938, O. Klein [281] suggested a 5D theory of gravity where the neutrino and the electron constitute the same family, both described by massless Dirac spinors (but one should recall that there was no distinction, at that point, between strong and weak interactions, cf. [212]). See also Salam and Ward comments in [445], Weinberg Nobel lecture [535], and Leite Lopes lectures on weak interactions [316].

[5] The distinction between the electron neutrino ν_e and the muon neutrino ν_μ was introduced only in 1962 with the Brookhaven neutrino experiment [95]; the third generation of neutrinos, namely the tau neutrino ν_τ, was announced in 2000 by the DONUT Collaboration [282]; see also [39].

> the mathematics of spinor fields gives a well known answer: spinors allow one to describe rotations at one point in space completely independently of rotations at all other points in space−rotations that have nothing to do with the coordinate transformations that are treated in the usual tensor analysis. Fully to see at work this machinery of independent rotations at each point in space, we do best to consider the spinor field in a general curved space, as in this paper. But the deeper part of such rotations in the description of nature is still mysterious.
> Brill & Wheeler [62], p.479.

The explanation why spinors do not behave like vectors, and how they can be defined in (pseudo) Riemannian manifolds, only came to light after the meeting of topology with differential geometry, and the subsequent introduction of Ehresmann's connections and the theory of fibrations in theoretical physics, cf. Rudolph and Schmidt [436]; we also refer to Dieudonné [118, Part 3, Chp. III], and Kay [261].

Besides, parity has an additional deviation from the classical theory of transformations, as inaugurated in physics by H. Weyl [452, 545, 546]: while Weyl identified gauge symmetry with the group of continuous transformations that leaves invariant the physics described by the theory, parity $P \in \mathrm{Aut}(Cl(V, \mathrm{q}))$ is, in the case of fermions, a Clifford algebra automorphism, and belongs to the set of discrete automorphisms $\mathcal{A} = \{C, P, T\}$ along with charge conjugation C (Dirac, 1931 [125]), and time inversion T (Wigner, 1932 [555]). Together, CPT constitutes a fundamental symmetry to be satisfied by any local Lorentz invariant theory of fermions [50, 104, 265, 579].

In this way, the puzzle on spinors and curved space, neutrinos and gravity, discrete and continuous transformations, seems to be inextricably linked to the dialectic inquiry into the realms of algebra and geometry. In the 1950s, a variety of alternative views to tackle the neutrino-gravity coupling problem emerged. Some seeking for the geometrization of spinors, some dealing with the spinor-algebraic characterization of spacetimes. However lay the emphasis on algebraic or geometrical methods, the decade post-1957 portrays a vivid and fecund chapter in the history of theoretical physics, as we shall evoke some examples in the following.

1956a Mario Schönberg [455, 456, 462] was among the first to explore the properties of Graßmann and Clifford algebras, as well as its extensions to symplectic spinors, towards an unified approach to field theories; subsequently, in the Part IV of a series of five communications [457–461], Schönberg addresses the possibility of the Lee-Yang theory be related to the intrinsic orientability of spacetime:

> The recent discoveries about the breakdown of reflection invariance in the interactions involving neutrinos show that the space-like sections of the space-time are endowed with a screw-orientation, an intrinsic orientation not depending on the embedding into the four-dimensional space-time.
> M. Schönberg [460], p. 276.

A partial account of Schönberg's ideas is presented in [186]. About a decade later, Hestenes [237, 238] eventually started to pursue a similar path towards an algebraic definition of spinors [347].

Symplectic spinors were treated by Kostant [286], while symplectic Dirac operators were defined only in 1995 by Katharina Habermann, cf. [215, 216, 336].

1956b Oskar Klein [273, 276] and Bertel Laurent [305] review Bargmann's paper [24] on the introduction of Dirac's theory in curved spacetime, insofar as to discuss a generalized Dirac adjoint that preserves unitarity; Laurent [306–308] further analysed Klein's covariant formulation by relaxing the unitarity of the Jacobian; a perspective from the Palatini variational principle is carried on by Klein [277–280] so as to embrace a generalized principle of equivalence in the unified picture including gravity (see also Dirac in [121]);

1959 Louis Witten [558] establishes the first correspondence between the invariants of GTR and spinors, by using Petrov's classification of Einstein spaces [105, 414]; in 1960, Penrose [404] rephrases Petrov's method in a spinor approach to GTR; a new variant, self-contained version of the spinor method was proposed by Newman and Penrose [358] in terms of double-null tetrades; the spinor method was systematically extended to the theory of twistors by Penrose and Rindler [406], where the spacetime is replaced by the space of light rays (see also [334, 418]).

1960 Louis de Broglie [99] revives his iconic program of Double Solution, with a critical appraisal of his original ideas (1925-1927), Pauli's objections at the 1927 Solvay Congress, Dirac's theory (1928), and further developments by Bohm, Vigier, and Petiau (1952-1954); de Broglie inherits Einstein's classical approach to physics, without ignoring the dense discussion in the 1930's and 1940's on the role of Planck's constant in the quantum approach to gravity [88, 98]; and yet pursues, in a rather independent path, a defense of objective particles's trajectories aiming at a dynamical synthesis of gravity and matter:

> The goal to be achieved would be to represent every type of particle (*including the photon*) as a singular region in a u-wave field properly incorporated into the structure of space-time. (...) Einstein has called these fields containing strong local condensations, which he thinks must be the true representation of particles, "bunch-like fields". (...) the u-wave theory may perhaps one day help to achieve a magnificent synthesis of General Relativity and Quantum Theory.
> de Broglie [99], pp.291-292.

Currently, a modified version of de Broglie's program is referred to as the de Broglie-Bohm theory, and has unique implications in cosmology, where the Copenhagen interpretation unequivocally fails due to the inexistence of an external observer in a classical domain [387, 415].

1961a C. Møller [343–345] evaluate gravity as described by all the 16 degrees of freedom of tetrad fields, which are required to be invariant under constant tetrad rotations; the consequences and further extension to the coupling with neutrinos were investigated by Pellegrini and Plebański [403]; there, the torsion appears to be naturally identified with the antisymmetric part of the energy-momentum tensor of Dirac's theory; independently of Møller's hypothesis, Finkelstein [171], Rodičev [434], and Ivanenko [246] arrived at the neutrino-gravity coupling via torsion;

1961b Kibble [264] applies the Palatini variational principle to infer a non-symmetric contribution to the Levi-Civita connection of GTR; the interpretation given by the author is that fermions in a gravitational potential, in the gauge approach of Utiyama [513], give rise to a repulsive axial-axial coupling term $\sim \kappa(\overline{\psi}\gamma_\mu\gamma_5\psi)^2$ due to the torsion of spacetime structure; since then, the Einstein-Cartan theory has been taken as an alternative avenue to address the strong regime field of gravity in high energy physics [32, 52, 71, 101, 117, 158, 159, 196, 226, 235, 244, 331, 504, 516];

1963a E. Lubkin [327] approaches, apparently for the first time, gauge invariance in the fibre bundle language, introducing a new perspective on Utiyama's approach [513] to the analogy between the Yang-Mills theory and GTR;

1963b in parallel, the definitions of spin manifolds and Clifford modules appeared for the first time in the literature with the works by Milnor [340], and Atiyah, Bott and Shapiro [19], respectively;

1964-70 Lichnerowicz [318–320], Trautman [502, 503], Penrose [405], Geroch [202, 203], and Hermann [236], among others, push forward a systematic development of field theory in the fibre bundle language; in this period, it is formulated our modern view of matter and gauge fields [350, 436];

After the theoretical completion of the Salam-Glashow-Weinberg model, and its acknowledge at the 1978 Tokyo conference as the standard model of electro-weak interactions, the neutrino-gravity coupling, although still present in the literature [101, 107–109, 162, 197, 198, 204, 294, 482, 559], started to lose its appeal as the road to unification. In its place, it was believed that strong gravity could play a distinctive role in the hadronic sector [102, 226, 244, 482].

Besides, the Bargmann-Wigner formalism [26] of Dirac's equation received a plethora of modifications so as to include explicit dependence on the chiral element of the relying Clifford algebra [214, 254, 356, 531, 532]. Also, the Fierz-Pauli formalism [170] of generalized spin became an attractive point of discussion [253], specially with respect to the half-integral spin of Rarita-Schwinger theory [428], that would eventually lead to the so-called theory of supergravity [112, 134, 184, 185, 334].

Furthermore, a variety of extensions to the SGW-model brought to light the properties of Clifford algebras and spin structures, as the Spin(10) model, proposed independently by Georgi and Glashow [201], and Fritzsch and Minkowski [189]; the Pati-Salam model [342, 388, 389]; and more recently, the division algebras in the lines of Dixon [129], Furey [190–192] and Singh [480], as well as the spinor-algebraic approaches developed by Trayling and Baylis [506], Castro [78, 79], Lopes and da Rocha [325], Hoff da Silva *et al* [239, 240], Prinz [423], Todorov [497–500], Krasnov [292, 293], Varlamov [518], to mention a few. Partial accounts are given by Krasnov and Percacci [291], and Chester, Marrani and Rios [83].

Unification might be aimless, though, if no better understanding of each interaction, gravity and weak interactions, is achieved previously. Explanation, rather than reduction, should guide our inquiry, even if the latter is desirable as an emergent[6] synthesis of our theories in its maturity. While unification aims at

[6]This is Pauli's view of unification, as presented in the Sec. 2.2. Later, I also found in Popper [421, pp.290-295] a very similar analysis.

the mathematical structure of our theories, explanation inevitably entails bold physical conjectures. New physics, or new synthesis as I prefer, is more likely to emerge from new conjectures, a feature that hardly will be achieved in terms of mathematical resets. Hence, focusing on Clifford algebras *per se* could be a pseudodirection.

At this point, an unavoidable question arises: is it still possible to look at parity violation of weak interactions, and its relation with gravity, from a renewed perspective? Are there elements claiming for a critical revision of the standard structures of theoretical physics? The present work's proposition pursue an affirmative answer not only as possible, but as inexorable.

For a unnoticed conjecture, that goes back to Bohr, Pauli, and Wataghin, on the relation between gravity and weak interactions will be retaken into consideration. Although the Planck scale has an eminent position in quantum approaches to gravity, its relation to Fermi's coupling scarcely play a key role in the deductive structure of physical theories. To bring back this situation to current research is the next chapter's challenge (Chapter 2).

The following steps correspond to a preliminary implementation of Pauli's conjecture (Chapter 3), and to the critical revision of its internal consistency (Chapter 4). Each section of the following chapters is interpreted by the author as an open line of research. The challenge is how to make of them a consistent, and coherent, research program of long term. To our view, a potential synthesis should retain, from epistemology, the role of deducibility in the construction of new *testable* physical theories; from theoretical physics, field thery and the weight of bold conjectures, as exemplified by Pauli's square root conjecture; and from mathematical physics, the theory of fiber bundles as the natural language to deal with Clifford algebra automorphisms associated to Riemannian geometry.

2. Gravity and weak interactions

> Whenever the need for a new theory is felt in some field of factual science, both the theory builder and the metascientist are confronted with the problem of choosing the *kind* of theory that should be tried next. Shall the next endeavor be in the direction of increasing detail and depth (growth of the population of theoretical entities)? Or shall it eschew speculation on what goes on in the innermost recesses of reality and focus, on the other hand, on data fitting, with the sole help of fairly directly observable variables? In other words, shall *the* future theory be representational or phenomenological, shall it be conceived as a more faithful picture of reality or only as a more effective tool for summarizing and predicting observations?
>
> M. Bunge [65], p.234.

The problem-situation posed by Mario Bunge in the excerpt above gives us a glimpse about the status of field theories since Maxwell electrodynamics. From a logical point of view, one expect all physical theories[1] to share some degree of deducibility, that is to say one expect to infer its results from a minimal set of initial hypotheses, as much as possible without relying upon eventual additional inputs to deal with the difficulties appearing along the road. But the relevance of how much deductive a theory is only becomes explicit when testability[2] is taken into account. By increasing the degree of deducibility of a theory, one also increases its degree of refutability. If so, then effective, or phenomenological, approaches can be viewed as the method of *weakening* the deducibility of physical theories by recognizing, for instance, that perturbative renormalizability is a too sharp razor to select the good candidates to admissible (quantum) field theories.

Gravity is the perturbatively non-renormalizable interaction that experienced the transition from a representational to a phenomenological picture to the fullest. The highest level of deducibility of the General Theory of Relativity (GTR) is achieved with the vacuum solutions. The introduction of matter into the Einstein-Hilbert action is the first step towards ambiguity. The further introduction of an unknown

[1]By theory, we shall refer, with a certain degree of ambiguity, to a not necessarily deduced (infinite) set of statements τ containing a minimal (finite) subset $\varkappa \subset \tau$ of premises (principles, postulates, or axioms; these terms are often interchangeably in the physics literature). We shall call $\varkappa$ the constituting set of a theory. In general, the subset of statements ς fixing the physical boundary of a new theory τ' in terms of the already tested one may be classified into auxiliary, *ad hoc*, and constitutive hypotheses. Besides, it is a rather non trivial aspect that the same theory may have different degrees of deducibility, as is the case of GTR (Section 2.1).

[2]We follow the testability criteria introduced by Popper [422] to distinguish between auxiliary and *ad hoc* hypotheses. An additional hypothesis that can be tested independently from the constituting set $\varkappa$ is auxiliary, rather than *ad hoc*. While *ad hoc* hypotheses are typically invoked in order to avoid the theory's refutation, the auxiliary hypotheses, on the contrary, increases the degree of testability of the theory by fixing its limits of validity with respect to the already tested ones. Moreover, it is clear that an *ad hoc* hypothesis may become auxiliary, as it happened, for instance, with Planck's relation $E = h\nu$, and Pauli's neutrino hypothesis. The Higgs boson and the Maldacena conjecture, on the other hand, seems to remain *ad hoc* up to date. By 'constitutive hypothesis', we shall refer to those statements in $\varkappa$ that play the role of fixing the physical boundary of the theory. Since a mathematical hypothesis may not necessarily carry physical content, it is natural to split the three categories of statements (auxiliary, *ad hoc*, and constitutive) into mathematical and physical ones. In this way, gauge invariance can be seen as a selection rule of those mathematical statements within the theory that are also physical.

type of matter as an additional source to the Einstein's field equations aimed to save the data breaks its refutability[3]. Thus, deducibility provide for a better elucidation when dealing with effective approaches in theoretical physics. In short, testability should not be taken for granted.

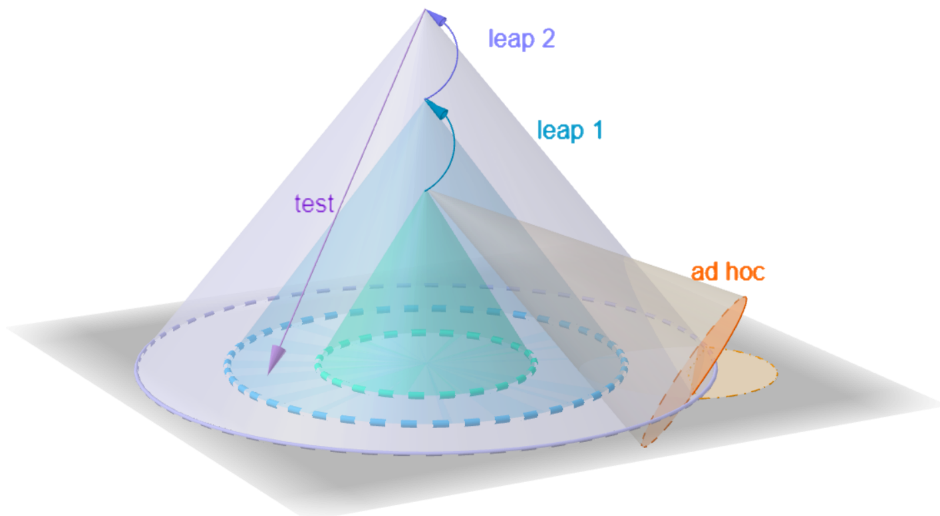

Figure 2.1: Diagramatic view of deductive systems in theoretical (mathematical) physics: every leap is guided by a physical (mathematical) conjecture. Only those tests that can potentially refute the theory are admissible as to support it. Hence, *ad hoc* hypotheses are mainly responsible for precluding the theory's testability. Adapted from [138].

That GTR has to be modified was recognized in 1916 by Einstein [141] himself[4]. Nowadays, the question is how to pursue a theory of gravity that can be made compatible with the picture of Quantum Field Theories (QFTs). Such a candidate is expected to exhibit its typical features at the Planck scale as a potential lower bound for all known interactions [2, 5, 103, 271, 272, 274, 275, 304, 381–383]. The search for a 'quantum gravity', as it is refered to in the literature [3, 46, 47, 67, 90, 132, 167, 241, 265, 409, 494, 524, 576], is an open discussion, and a breakthrough in the road is still lacking[5]. It might be helpful to get some insights from the most notable alternative programs to the standard picture composed by GTR and the SGW-model. For the length scale problem may gives us a clue not only about the difficulties involved, as well as how deducibility and testability are implicated by any new development (Section 2.1).

[3]The only empirical tests that are admissible as to support (or to corroborate) the theory are the ones that can potentially refute it. According to Popper, irrefutability rules out the theory's corroboration, and, therefore, its status as scientific. That is why *ad hocness* should be avoided as far as possible. On the claim [346] that it is possible to falsify particular subcases of GTR+ΛCDM, one might recall that unless the *ad hoc* hypothesis become auxiliary (testable independently from the model), the multiplicity of 'theoretical entities' is innocuous. That is another reason to take deducibility into consideration, once it allows one to measure the contrast between representational and phenomenological approaches without relying exclusively upon refutability. In addition, theories can not be refuted by theories, as suggested by one of the reviewers of [367].

[4]See also Kiefer [265, pp.26-27].

[5]The leading difficulty with the program of quantum gravity is the lack of an objective problem to start with, which depends, by its turn, on the yet open problem of how to describe the gravitational field. If gravity (as a spin 2 field) is expected to be quantized as matter fields are, then GTR hardly is the proper framework for that path. If instead gravity is kept at the semiclassical level and only matter fields are quantized, then one faces again inconsistent Einstein's field equations. One shall recall Feynman's remark on the quantization of gravity: "It's clear that the problem we are working on is not the correct problem; the correct problem is: What determines the size of gravitation?" [576, p.77]. Then one is led, once again, to the claim of the Planck scale as a lower bound for all known interactions. However, what explains the introduction of the quantum of action $\hbar$ into the framework of GTR?

Coincidently or not, the weak interactions of old also are perturbatively non-renormalizable, bounded by a well defined length scale, and the most universal[6] among the currently known interactions−aside from gravity. Due to dimensional reasons, these features were not unnoticed by Pauli, who identified in the hierarchy problem between gravity and weak interactions the possibility of a distinct path in understanding the quantum effects of gravity. An attempt to rescue this unfashionable discussion is presented in the Section 2.2.

The most striking development in the search for the coupling between gravity and weak interactions was achieved even before the electroweak unification in terms of the Einstein-Cartan theory, which is seem as an effective approach in particle physics when gravity is taken into account [117, 337]. By effective, we shall refer to any theory that cannot be built without *ad hocness*. This criteria seems to cover not only all QFTs, including the so called Effective Field Theories (EFTs), as well as modified theories of gravity, as is the case of ECT. This is the scope of Section 2.3.

2.1 The length scale problem

Constitutive, *ad hoc*, and auxiliary hypotheses − Deducibility in GTR − Extensions of GTR (Weyl, Kaluza, Klein) − Discretization of spacetime − Born-Infeld theory − Heisenberg's program.

There is hardly a more transversal problem in theoretical physics than the one of a characteristic, fundamental, or minimal *length* (*energy*) *scale* associated to each one of the four interactions. This is a modern way of stating the problem of how to demarcate the limits of validity of a physical theory in terms of the typical range of the interaction under description. It is also a way of to quest for the truly existence of four distinct, irreducible, albeit not completely independent, physical interactions. It contains the Cluster Decomposition Principle [68, 536] as auxiliary hypothesis.

Some of the bold conjectures on gravity, electromagnetic, and weak interactions during the 1915-1939 period were able to present a well posed length scale, and hence a well defined structure of deducibility for the respective physical theories. For this reason, the response to the length scale problem is often given by a constitutive hypothesis, which shapes the new theory by including the old one as a limiting case[7].

Schwarzschild solution. To illustrate this point, let us consider the spherically symmetric solutions of GTR, namely [87, 418, 485]

$$ds^2 = \mathrm{e}^{2\nu} dt^2 \,-\, \mathrm{e}^{-2\nu} dr^2 \,-\, r^2 \left(d\theta + \sin^2\theta d\varphi^2\right), \tag{2.1}$$

where

$$g_{oo} = \mathrm{e}^{2\nu} = 1 + \frac{C}{r} + \frac{1}{3}\Lambda r^2 + \frac{e^2}{r^2}. \tag{2.2}$$

[6]Only gluons, among the known elementary particles, do not interact weakly.
[7]That is precisely what constitutive is supposed to mean. See also [422, § 79].

The exact solution of Einstein's vacuum equations,

$$R_{\mu\nu} = 0, \tag{2.3}$$

was found independently by K. Schwarzschild [472] and J. Droste [133], and it corresponds to the particular case[8] of (2.2) for $\Lambda = e = 0$. In the weak field regime, the GTR is expected to reproduce Newtonian gravity, which is fixed in the Schwarzschild solution by identifying the constant of integration C in (2.2) with the Newtonian potential ϕ [4, 87, 534],

$$g_{oo} \cong 1 + \frac{2\phi}{c^2} \cong 1 - \frac{2G_{\rm N}M}{c^2 r}, \qquad \phi = -\frac{G_{\rm N}M}{r}, \quad r \ll M \quad (M > 0). \tag{2.4}$$

Hence, the Schwarzschild spacetime is a static[9], oriented, 4-dimensional product manifold of $\mathbb{R}^3 \cap \{r > a > 2m\} \cong S^2 \times \mathbb{R}_+$ by $\mathbb{R}$, endowed with a metric (2.1) with auxiliary condition (2.4). The notation $2m \equiv 2G_{\rm N}M/c^2$ is oftenly used in order to absorb the gravitational constant $G_{\rm N}$ and the velocity of light c.

Furthermore, the Schwarzschild spacetime is asymptotically flat: without any further assumption, as $r \to \infty$, the solution of Einstein's equations in vacuum reaches the Minkowski spacetime in spherical coordinates. Also, the validity of (2.4) is restricted to the exterior region $r > 2m$,

$$2m = \frac{2G_{\rm N}M}{c^2} \approx 3\left(\frac{M}{M_\odot}\right)\text{km}, \tag{2.5}$$

where $M_\odot = 2 \cdot 10^{33}$g is the mass of Sun [526]. The Schwarzschild spacetime has a spurious singularity at $r = 2m$, and a physical singularity at $r = 0$. The region $r < 2m$ is only mathematically complementary to $r > 2m$, where the constitutive (physical) part of the Schwarzschild spacetime holds. Notwithstanding, the region $r < 2m$ is supplemented with auxiliary hypotheses (like the equation of state) in order to build the so-called interior solutions [153, 418, 526].

The formal structure of the Schwarzschild spacetime is of particular interest once it shows why GTR is the prototype of a deductive theory in physics: the three classical tests of GTR, namely the light bending, the gravitational red shift, and the perihelion precession, are tests of the Schwarzschild-Droste solution. It also exhibits how the degree of deducibility varies within GTR[10], and the universal role of Einstein's

[8] The other cases are: $C = e = 0$ (De Sitter, 1916), $\Lambda = 0$ (Reissner, 1916; Nordström, 1918), $e = 0$ (Kottler, 1918). More general solutions were found by Cahen and Defrise (1968), and by Kinnersley (1969), cf. [418]. Higher dimensional extensions of spherical solutions are examined in [21].

[9] A result found independently by J. T. Jebsen (1921) and G. D. Birkhoff (1923), and usually refered as Birkhoff's theorem. See also the footnote on p. 174 of [418].

[10] "(...) energy-momentum tensors, however, must be regarded as purely temporary and more or less phenomenological devices for representing the structure of matter, and their entry into the equations makes it impossible to determine how far the results obtained are independent of the particular assumption made concerning the constitution of matter. Actually, the only equations of gravitation which follow without ambiguity from the fundamental assumptions of the general theory of relativity are the equations for empty space (...)." Einstein, Infeld, and Hoffmann [152, p.65]. According to Pauli [396], "this tensor [of energy and momentum], as well as the constant of gravitation, remains the phenomenological constituents of the general theory of relativity". Nonetheless, there are other elements that makes this claim explicit, as the fact that not all classes of exact solutions of GTR are asymptotically flat, cf. [485].

Equivalence Principle (EEP) in a theory of gravitation, where the minimal coupling 'principle'[11] is a particular subcase.

Weyl geometry. In his first attempt[12] at a unified theory of gravity and electromagnetism, Weyl [543, 544] introduced an extension of GTR through a length connection $\phi = \phi_\mu dx^\mu$ satisfying the *Eichtransformations* [454]

$$\phi' = \phi - d(\log \Omega)\,, \qquad g' = \Omega^2 g. \tag{2.6}$$

The equivalence class of pairs (g, ϕ) defines a Weyl metric. In Weyl geometry (M, g, ϕ), the compatible affine connection $\Gamma = \Gamma(g, \phi)$ is uniquely determined by (g, ϕ), and transfers the scale invariance to the Riemann curvature as well as to the geodesics of (M, g, ϕ). Under the (auxiliary) integrability condition $d\phi = 0$, Weyl geometry is asymptotically Riemannian[13]. Despite its inability to describe the atomic spectrum correctly due to the appearance of a second clock effect [4, 322], Weyl geometry remains an active topic of research in conformal theories of gravity [453, 551].

Kaluza, Klein, and the fifth dimension. A similar pattern to Weyl's geometry was followed, in 1921, by Theodor Kaluza [260] in terms of a 5-dimensional[14] relativistic extension to GTR as an alternative solution to the problem of unification of gravity with electromagnetism. The line element of Kaluza's theory is given by

$$d\sigma = \sqrt{(dx^o + \beta\phi_\mu dx^\mu)^2 + g_{\mu\nu}dx^\mu dx^\nu}\,, \qquad \mu, \nu = 1, 2, 3, 4 \tag{2.7}$$

where $\{x^1, x^2, x^3, x^4\}$ are the coordinates of the 4-dimensional Minkowski spacetime, ϕ_μ is the 4-potential of Maxwell's equations, and x^o is the 5th coordinate. In natural units, the parameter β has dimension of length. The attempts to interpret the fifth dimension physically came five years later by H. Mandel [332, 333], V. Fock [173], Einstein [143], and O. Klein [267–269], independently. Klein called the attention to the fact that, if β has the value

$$\beta = \sqrt{2\kappa_E}\,, \tag{2.8}$$

[11] The role of the minimal coupling is more like a razor, or a constraint, inferred from EEP and imposed to the field equations so as to prevent the energy-momentum tensor from representing the back-reaction of the gravitational field on matter [176, 177]. This partially explains why Einstein was assertive in regarding the energy-momentum tensors as provisional representations of matter [148, 152].

[12] The second one [545, 546] was made in 1929 as a criticism to Einstein's letters [144] on teleparallelism. There, Weyl gives a different interpretation to the *vierbeins* so as to describe Dirac's theory in a 4-dimensional curved space. A third development comes in 1950 when Weyl [547] once again addresses the coupling of an electron to the gravitational field. See also [106, 208, 209, 451].

[13] This is an example of additional hypothesis that *increases* the refutability of the theory. See, for instance, [422, §20].

[14] In 1914, Gunnar Nordström[359] anticipated the idea of a 5-dimensional scalar unified field theory, where the 4-dimensional spacetime of the Special Theory of Relativity is a surface embedded in a 5-dimensional geometry. Yet the works by Kaluza and O. Klein are not directly linked to Nordström's. Notwithstanding, it is possible that O. Klein started his interest in the 5D formulation after his contact with P. Ehrenfest, who also worked in collaboration with Nordström , cf. [221].

where κ_E is the Einstein constant, and if p^o, the 5th component of the momentum of a particle, is proportional to the electric charge,

$$p^o = \frac{e}{\beta c} \tag{2.9}$$

which also corresponds to an integer (positive or negative) multiple of a least quantum of action h,

$$p^o = \frac{Nh}{l} \tag{2.10}$$

then "any electric particle will in the 5-dimensional representation be periodic functions of x^o, the periodicity introducing a fundamental length"

$$l_{\text{Klein}} := \frac{hc\sqrt{2\kappa_E}}{e} \approx 0.8 \cdot 10^{-30}\text{cm}. \tag{2.11}$$

Klein sought to provide a less artificial explanation for introducing an extra dimension to the Minkowski spacetime in terms of a boundary condition to the physics described in the 4-dimensional sector.

> The small value of this length together with the periodicity in the fifth dimension may perhaps be taken as a support of the theory of Kaluza in the sense that they may explain the non-appearance of the fifth dimension in ordinary experiments as the result of averaging over the fifth dimension. (...) Although incomplete, this result, together with the considerations given here, suggests that the origin of Planck's quantum may be sought just in this periodicity in the fifth dimension.
> O. Klein in [268].

Einstein and Bergmann [151] came in with an auxiliary topological condition to support Klein's interpretation. If one considers the fifth dimension as a long thin tube, then locally it looks like an extended two-dimensional tube, while from very long distances it is just a one-dimensional string. Remarkably, what remains physically invariant is its length. As conceived by Einstein and Bergmann, the extra dimension would be controlled by a running universal parameter, the typical radius b of the cylinder, that allows one to reach the limiting classical fields as $b \to 0$, a clear resemblance of Born-Infeld theory as we shall see below.

Notwithstanding, Einstein and Bergmann breaks the 5-dimensional symmetry by introducing an (*ad hoc*) geodesic postulate in order to clamp down on the appearance of a massless scalar field (a dilaton in modern theories) from the fluctuations in the length of the fifth dimension, which would lead to a scalar-tensor theory of gravity, cf. [557]. Subsequently, Einstein, V. Bargmann, and P. Bergmann [150] reduce Klein's topological postulate to a circle.

It is particularly intriguing how Kaluza-Klein theory allowed such a wide range of variations, including structural modifications in Theoretical Physics: a varying gravitational constant appears in Jordan's 5D cosmology [257]; Cartan's exterior calculus is applied in the study of the global aspects of 5D relativistic theories already in the 1940s, cf. [321]; in quantum optics, Yurii B. Rumer [439] ascribe to the 5th coordinate of a Riemannian space the quantum action with fundamental periodicity h; the first fibre bundle treatment appears in 1968 by R. Kerner [262], where the 5D perspective is generalized for non-abelian theories. Since then, Kaluza-Klein theories were developed to attack a wide variety of problems: from the existence of monopoles, chiral fermions, and scale hierarchy to quantum effects in gravity and cosmology [18, 208, 380, 410, 418].

For some reason, Klein's further developments[15] [270–281] of his 5D research program did not receive the attention as the earlier papers [267, 268]. In the 1950s, Klein suggests that a 5D Dirac theory allows one to extend the unitarity of QED so as to embrace a generalized equivalence principle. The seeds of that view are in Bargmann's seminal paper of 1932 [24], although restricted to a 4-dimensional spacetime. We will reconsider this interpretation in Chapter 4.

Discretization of spacetime. All field theories discussed above share the premise according to which there is a continuous spacetime evolving dynamically, that allows one to describe matter by means of a fixed background. Also, it is taken for granted the existence of points constituting the continuous fabric of spacetime.

Motivated by Klein's insights [267–269] on the existence of a fundamental length l, the idea that protons and electrons could ocuppy only lattice points in a 3-dimensional cubic volume was proposed in 1930 by Ambarzumian and Iwanenko [14], and pushed forward by Arthur March [335], Heisenberg [230], Snyder [483], Rosen [435], and Schild [449, 450], among others. While the discrete structure is ruled by integral Lorentz transformations, the continuous background of special relativity is reached as $l \to 0$.

The discrete perspective in quantum approaches to gravity is mainly driven by spin-foam models, where the path-integral is constructed in terms of spin networks in time [265, 409, 494].

Born-Infeld theory. The introduction of the length scale problem in electrodynamics came when, in the early 1930's, Max Born and Leopold Infeld [54, 55] tried to tackle the electron self-energy problem in terms of a non-linear extension of Maxwell's electrodynamics [38, 394, 473]. One shall recall that, by dimensional analysis, no characteristic length in terms of the electric charge e, the Planck's constant h, and the velocity of light c can be constructed [194, 304].

In straight analogy[16] with the Special Theory of Relativity (STR), where the velocity v of a particle of mass m, in any inertial frame, is limited by the velocity of light c,

$$\mathcal{L} = -m^2 c^2 \sqrt{1 - \frac{v^2}{c^2}}\,, \tag{2.12}$$

[15] The very exceptional case is the paper [281], prepared for the Conference New Theories in Physics (Warsaw, 1938). There, Klein suggests that Fermi's β-decay theory, in analogy with Maxwell's theory, should be mediated by charged vector bosons, which would indicate an unification of electromagnetism with Yukawa's theory of nuclear mesons. See also [212].

[16] To compare with [140], and [147, p.67].

Born and Infeld conjectured that the classical radiation field must reach a maximal intensity b,

$$\mathcal{L}_{BI} = b^2c^2 \left[1 - \sqrt{1 + \frac{F}{2b^2} - \frac{G^2}{16b^4}}\right], \tag{2.13}$$

where F and G are denoting the Lorentz electromagnetic invariants,

$$F \equiv F^{\mu\nu}F_{\mu\nu} = 2(\mathcal{E}^2/c^2 - \mathcal{B}^2), \qquad G \equiv \overset{*}{F}{}^{\mu\nu}F_{\mu\nu} = -4(\mathcal{E}\cdot\mathcal{B}). \tag{2.14}$$

Without extracting from the theory a prediction for its value, the authors assumed that the upper bound parameter b has the dimension of the critical magnetic field $\mathcal{B}_{crit}$. In SI units, it reads

$$[b^2] = [\mathcal{B}^2_{crit}] = \mathrm{L}^{-4}\mathrm{MT}. \tag{2.15}$$

In this way, (2.13) states that $\mathcal{L}_{BI} \to \mathcal{L}_{Max}$ as $b \to \infty$: the Maxwell theory is the weak regime of Born-Infeld electrodynamics.

Since the motion of a free particle described by STR can be deduced from the least action principle, Born and Infeld showed that the expression within the square root in (2.13) is the determinant of the quantity

$$g^{\text{eff}}_{\mu\nu} = g_{\mu\nu} + \frac{1}{b}F_{\mu\nu}. \tag{2.16}$$

Another key feature of $\mathcal{L}_{BI}$ is that the self-energy of the classical electron is finite, possibly indicating that the incompleteness of Maxwell's theory relies typically upon its linearity, rather than upon its classical level. The ultra limiting case of $\mathcal{L}_{BI}$, when $b \to 0$, exhibits an infinite hierarchy of conservation laws [36, 37].

Despite its interesting achievements, the quantization of $\mathcal{L}_{BI}$ appeared to be a partial obstruction to further developments of the theory [127]. The algebraic properties of a non-linear field theory constructed from a Lagrangian of the form $\mathcal{L} = \mathcal{L}(F, G)$ were studied by Plebański [417], and Boillat [51]. In 1985, the idea of an upper limit on the strength of the electromagnetic field was rescued in the context of brane theory by Fradkin and Tseytlin [182, 183]. Another variation of BI models was proposed by Lisa Randall and Sundrum [426, 427] in terms of a non-factorizable metric. Currently, light scattering is approached by hadronic, dispersion relations and lattice models [48, 72]. For an extension of the hypercharge sector of the electroweak model, see [100]. For a survey on BI modifications of gravity, see [252].

The relevant point here is that, by introducing b as an upper bound limit to the electromagnetic field, the Born-Infeld Lagrangian translated the problem of a characteristic length scale to the context of classical radiation. With no parallel in field theory, $\mathcal{L}_{BI}$ is an attempt to understand the departure from the classical standpoint: How to discern between nonlinear effects at the critical value (if any) of the classical radiation field, and the non-linear effects of the quantized Maxwell field? Regarding this aspect, the Born-Infeld scenario plays a straight analogy with GTR, once quantization is not necessary to introduce non-linear effects (in the sense of self-interacting fields) into the theory [524].

Euler-Heisenberg Lagrangian. The typical example of BI electrodynamics is the photon-photon system, also refered to as the light by light scattering [249, 473]. In spinor QED, it appears as a quantum effect at the fourth order of perturbation, as suggested by O. Halpern [220], Debye, and Heisenberg [228]. Heisenberg's students, Euler and Kockel [157], computed the leading quantum correction $O(\alpha^2)$ to Maxwell's Lagrangian due to vacuum polarization [413],

$$\mathcal{L}_{EH} = \mathcal{L}_{Max} + \frac{\alpha^2}{m_e^4}\,\left[C_1\,F^2 + C_2\,G^2\right]\,, \tag{2.17}$$

where α is the fine structure constant, the energy scale is fixed by the electron mass m_e, and

$$C_1 = \frac{1}{90}\,, \qquad C_2 = \frac{7}{90}. \tag{2.18}$$

Usually refered to as the Heisenberg-Euler model, $\mathcal{L}_{EH}$ is interpreted as the low energy of light scattering [135, 136, 413], and constitutes an independent field theory from $\mathcal{L}_{BI}$ [181].

Heisenberg's program. The question posed by Born and Infeld reflects the premise of Heisenberg's paper "Die Selbstenergie des Elektrons" [227], published in 1930. The common ground seems to be how causality makes quantum mechanics and special relativity incompatible, at the same time that it is at the origin of the singularities arisen after the quantization of relativistic field theories. In Heisenberg's lines, it should be possible to demarcate the limit of validity of quantum physics in terms of a characteristic length[17], as an universal invariant of spacetime. The price to pay, though, would be the breakdown of Lorentz invariance. Quoting Heisenberg,

> In particular, the statement that a minimum length exists is no longer relativistically invariant, and one sees no way to reconcile the requirement of relativistic invariance with the fundamental introduction of a minimum length. For the time being it seems more correct *not* to introduce

[17]The claim [241, p.7] that "Heisenberg was very worried about the non-renormalizability of Fermi's theory of β-decay" as early as 1936 is not only anachronistic, as mistaken. Renormalization as a method in field theory only arised in the late 1940s [76]. Yet, Heisenberg (as Dirac, Schrödinger, Wheeler) never took renormalizability as a serious requirement to any fundamental field theory, otherwise it hardly would make sense Heisenberg dedicate his last three decades to a perturbatively non-renormalizable spinor theory. Regarding the quantization of the nonlinear spinor field (see next section, Eq. (2.25)), Heisenberg stated in 1955 [49, Series B-1, p.537]: "... we can divide all possible interactions in two types: one type can be renormalized and shows what can be called weak interaction; the other type has what we may call strong interaction, and for strong interactions this process does not work. This interaction here [$l^2\psi(\psi^+\psi)$, $\psi^+ \equiv \psi^\dagger\gamma^o$], however, belongs to the strong interaction-type, and regardless of what kind of nonlinear wave equation we would write for spinor waves, we would always get the strong-interaction type, which cannot be renormalized. Therefore, we have to invent a new scheme of quantization". As also explained by Dürr [49, Series A-3, pp.136-137], the correct argument is more likely to occur in the opposite direction: at the limits of validity of quantum mechanics, whose breakdown would be measured by a characteristic length scale, a new theory would face strong interactions type, and would be nonrenormalizable by definition (see also [154, pp.315-317]). Unlike Pauli, Heisenberg saw renormalizable theories as the low energy, hence phenomenological, limiting case of (unknown) fundamental ones containing a universal length parameter. This might give some clue on Pauli's reluctance in associate the square root of the gravitational constant with Fermi constant, cf. addressed in the next section.

> the length r_o into the fundamentals of the theory, but to stick to the relativistic invariance.[18]
> W. Heisenberg [227], p. 5; my translation.

Eight years later, Heisenberg [229] eventually changed his orientation, and the discussion about the fundamental length became the background for his program of a unified field theory, cf. Dürr in [49, Series A-2, pp. 133-141]. From 1950 until 1976, Heisenberg worked on the constituting hypothesis that all mass spectrum of the elementary particles should be obtained from a fundamental equation with one coupling parameter. While the basic elements of Einstein's attempts at a unified field theory were of tensorial character, Heisenberg agreed with de Broglie [98, 99], Dirac [124], Pauli [394], Jordan [255, 256], Born [56], Wataghin [527–529], and Stückelberg [488] that a fundamental representation of matter should rest on a spinor field[19]. This is the point where Pauli's conjecture on the connection between gravity and weak interactions comes in.

2.2 Pauli's square root conjecture

> Strengthening the degree of deducibility in Theoretical Physics – The attractive nature of the gravitational constant – Phenomenological theories in Pauli's view.

On coupling constants in theoretical physics. Although unfashionable nowadays, the problem of how fundamental are the physical constants, and why their corresponding fields are attractive or repulsive, belongs to the oldest, and still open, discussions in theoretical physics. Until 1948, the nature of the electric charge, for instance, was, according to Pauli (and Einstein), one of the key problems left untouched whether by relativity theory or quantum mechanics. According to Enz [154, p.255], "the question of the atomicity of the electric charge ... was central to Pauli's thinking". In rather different historical moments (1921, 1949 and 1958), Pauli's criticism remained the same regarding the role of the coupling constants in the structure of physical theories. It might be enlightening to quote directly from his works, starting with the *Handbook article*:

> It is the aim of all continuum theories to derive the atomic nature of electricity from the property that the differential equations expressing the physical laws have only a discrete number of solutions which are everywhere regular, static, and spherically symmetric...
> Furthermore, ... the continuum theories are forced to introduce special forces which keep the Coulomb repulsive forces in the interior of the electrical elementary particles in equilibrium. If we assume that these forces are *electrical in nature*, we have to assign an absolute meaning to the four-vector potential... The other alternative, that the electrical elementary particles

[18] "Insbesondere ist die Aussage, daß eine kleinste Länge existiert, nicht mehr relativistisch invariant und man sieht keinen Weg, die Forderung der relativistischen Invarianz mit der grundsätzlichen Einführung einer kleinsten Länge in Einklang zu bringen. Es erscheint also einstweilen richtiger, die Länge r_o *nicht* in die Grundlagen der Theorie einzuführen, sondern an der relativistischen Invarianz festzuhalten."

[19] See also Darrigol's contribution to [217, pp.53-72].

are held together by *gravitational forces*, is however countered by a very weighty, empirical, argument. For one would expect, in such a case, that a simple numerical relation would exist between the gravitational mass of the electron and its charge. Actually, the relevant dimensionless number $e/m\sqrt{\kappa}$) (κ =ordinary gravitational constant) is of the order of 10^{20}!
Pauli (1921) in [397, p.205].

The continuum theories refers to Einstein, Mie, and Weyl developments on the connection between gravity and electromagnetism. The same problem is retaken by Pauli almost three decades later, in his contribution to the provocative volume *Albert Einstein, Philosopher-Scientist*, which can be read as a continued review of the relativity theory, now including 'Einstein's contributions to quantum physics':

Inside physics in the proper sense we are well aware that the present form of quantum mechanics is far from anything final, but, on the contrary, leaves problems open which Einstein considered long ago. In his previously cited paper of 1909, he stresses the importance of Jean's remark that the elementary electric charge e, with the help of the velocity of light c, determines the constant e^2/c of the same dimension as the quantum of action h (thus aiming at the now well known fine structure constant $2\pi e^2/hc$). He recalled "that the elementary quantum of electricity e is a stranger in Maxwell-Lorentz' electrodynamics" and expressed the hope that "the same modification of the theory which will contain the elementary quantum e as a consequence, will also have as a consequence the quantum theory of radiation". The reverse certainly turned out to be not true, since the new quantum theory of matter and radiation does not have the value of the elementary electric charge as a consequence, which is still a stranger in quantum mechanics, too.
Pauli (1949) in [395], p.158.

The road to unity as envisaged by Pauli, though, goes vertically opposite to Einstein's:

The theoretical determination of the fine structure constant is certainly the most important of the unsolved problems of modern physics. We believe that any regression to the ideas of classical physics (as, for instance, to the use of the classical field concept) cannot bring us nearer to this goal. To reach it, we shall, presumably, have to pay with further revolutionary changes of the fundamental concepts of physics with a still further digression from the concepts of the classical theories.
Pauli, *ibid*.

The synthesis given by Pauli reads: any physical theory should be capable of entailing its coupling constants, rather than being supplemented by it. Only then one would be able to explain its attractive or repulsive nature, as well as its scale of range, and limits of validity hence. (One may recall Feynman's remark on the quantization of gravity: "It's clear that the problem we are working on is not the correct problem; the correct problem is: What determines the size of gravitation?" [576, p.77].) The claim that the analysis given by Pauli is not restricted to the electric charge, but encompasses any coupling constant, may be supported by two further quotations. While the empirical content of Einstein's constant is crucial for Pauli since the *Handbook article* of 1921, Fermi's constant will draw Pauli's attention specially after the fall of parity in 1957.

> The general theory of relativity, therefore, does not provide a physical interpretation for the sign (gravitational attraction, and not repulsion) and numerical value of the gravitational constant, but takes these data from experiment.
> Pauli (1921) [397, p.163].

For an emergent unifying scheme would be more likely to achieve such a disposition to a fundamental relation between the coupling constants.

> A point now appears to have been reached when the physics of the neutrino merges with the more general physics of elementary particles. Nowadays we still describe each of these particles by its own field and each type of interaction by its own coupling constants.
> What, for example, is the significance of the small numerical value of the constant of the *Fermi* interaction, of the dimension of a cross-section, compared with other cross-sections? The next step, the suppression of the phenomenological physics of individual fields and coupling constants in favour of a unified conception is likely to be much more difficult than what has so far been achieved.
> Pauli (1958) in [155, p.217].

Among 'what has so far been achieved', one may mention Klein [268] interpretation, back in 1926, of the 5-dimensional Kaluza's extension of GTR.Pauli [390, 391] developed Klein's interpretation by setting the projective spinor as $\Psi = \psi\, F^l$, where ψ denotes the ordinary Dirac spinor, F a real scalar, and l a purely imaginary phase given by (Eq. (56) in [391])

$$l_{\text{Pauli}} = \frac{ie}{\hbar}\frac{1}{\sqrt{\kappa}}\frac{1}{r} = \frac{2\pi i}{l_{\text{Klein}}}\frac{\sqrt{2}}{r}. \tag{2.19}$$

In Pauli's notation [397, n.320 on p.163], κ is related to Newton's constant G_{N} by $\kappa = 8\pi\, G_{\text{N}}/c^4$. The number r is set by the redefined electromagnetic fields in the 4-dimensional spacetime (Eq. (44) in [390]),

$$X_{ij} = r f_{ij} = r\sqrt{\kappa}F_{ij}. \tag{2.20}$$

Note that the square root of Einstein's constant appears in the description of the electromagnetic field as a factor of 'geometrization' of the classical electron in the presence of gravity, an interpretation claimed by Weyl since 1918 and never shared by Einstein, cf. [313]. But the point here is that Pauli arrived at a

Dirac equation with a gravitational magnetic-moment coupling (Eq. (58) in [393]), where the anomalous magnetic moment is given by

$$\mu = i\frac{r}{8}\sqrt{\kappa}. \tag{2.21}$$

For a detailed account on Pauli's scrutiny of Klein's early formulation, we shall refer to [154, pp.260-270]. While the Born-Infeld theory was received by Pauli as rather unsatisfactory, once it left undetermined a new parameter b characterizing the maximal strength of the electromagnetic field, the Kaluza-Klein theory was seen as 'a *general* method' for 'a *logical unification* of the foundations of natural law' [391, p.337]. Once again, a general method for, not a realization of, as stressed by Pauli himself in the supplementary note 23 to [397].

But no attempt at a unified picture would make Pauli's view more explicit than the nonlinear spinor theory (NST), which raised structural changes in quantum mechanics. The indefinite metric problem aside, Pauli and Heisenberg discussed qualitatively in the Unpublished Preprint of 1958 [234] three key problems that would shape the construction of the NST:

(P_1) How to quantize nonlinear (self-interacting) fields?

(P_2) How to give mass to the elementary particles?

(P_3) How to deduce the coupling constants?

All the three points are detailed by Dürr in [49, Series B-1, pp.325-334], and Enz [154, pp.523-533]. What is relevant here is to extract Pauli's view on the formulation of a fundamental theory, as claimed in the Introduction. The quantization of NST was answered with the group of canonical transformations found by Pauli-Gürsey (Eqs. (I) in [234])

$$\psi' = a\psi + b\gamma_5\psi^c = a\psi + b\gamma_5 C^{-1}\overline{\psi}\,, \quad \overline{\psi}' = a^*\overline{\psi} - b^*\overline{(\psi^c)}\gamma_5 = a^*\overline{\psi} + b^*\psi C\gamma_5\,, \tag{2.22}$$

with $|a|^2 + |b|^2 = 1$, and Touschek (Eqs. (II) in [234])

$$\psi' = e^{i\alpha\gamma_5}\psi \quad \overline{\psi}' = \overline{\psi}e^{i\alpha\gamma_5}. \tag{2.23}$$

In the authors notation, $\psi^c = C^{-1}\overline{\psi}$, $\overline{\psi} \equiv \psi^*\gamma_4$ and C is a unitary matrix set by (Eq. (6) in [234])

$$C\gamma_\mu C^{-1} = -{\gamma_\mu}^T, \quad C\gamma_5 C^{-1} = {\gamma_5}^T, \quad C^T = -C. \tag{2.24}$$

From the full group constituted by (I) and (II), it follows the obstruction of a mass term in the Lagragian; the selection, among all five Lorentz invariants of Dirac theory, of the axial-axial current coupling; and the doubling of the Dirac adjoint. The Lagrangian has the form (Eq.(11) in [234])

$$L = \overline{\psi}\gamma^\nu\partial_\nu\psi \pm l^2(\overline{\psi}\gamma_\mu\gamma_5\psi)^2. \tag{2.25}$$

Since L is now invariant under groups (I) and (II), a quantum theory based upon (2.25) should be able to explain the existence of the two quantum numbers for [electric] charge and baryonic charge, and of the isotopic spin.

In constructing this quantum theory it will be possible to give a precise mathematical meaning to the three approximations, which are usually distinguished by the terms strong, electromagnetic and weak interactions. It will, however, not be possible to introduce any arbitrary constants into the theory.

Heisenberg and Pauli (1958) in [234], p.339.

In the rest of the preprint, Heisenberg and Pauli elaborate how the three interactions could follow from the NST, how the masses of the particles would be related to a nontrivial vacuum, and how to calculate an approximate value for the fine structure constant. As well known, Pauli withdrew his participation as a co-author of the publication, being the reasons mainly attributed to his rejection of the vacuum degeneracy, not to mention his personal relation with Heisenberg. A substantially reviewed version of [234] was further elaborated Heisenberg in collaboration with Dürr, Mitter, Schlieder, and Yamazaki [137]. On the impacts of Heisenberg's program in the formulation of the SGW model and supersymmetry, see, for instance, Weinberg [535], Nambu [354] and Shifman [478].

This brief account on Pauli's conception of the coupling constants in physics might be helpful to put its square root conjecture into a better perspective.

Pauli's conjecture. In November 1934, Pauli [393] gave an appraisal on the current situation of theoretical physics to the Philosophical Society in Zürich, by examining "the role of three universal constants of nature − c the velocity of light *in vacuo*, κ the constant of gravitation and h Planck's quantum of action". Special emphasis is given to the conservation law of the electric charge, which "has not yet found its appropriate place beside the constants c, h and κ."

It seems worth mentioning that a similar dimensional analysis of the relation between e, c, h, and χ (the now called Newton's constant, G_{N} in our notation) was made in 1928 by Gamow, Ivanenko and Landau [194], where the impossibility of a well posed length scale in terms of e, c, and h appears to be linked to the fact that QED, as well as "a nonquantum electron in the general theory of relativity" are incomplete systems. In other words, the dimensional analysis as discussed in the early days of QED was not restricted to the necessity of introducing regulators when dealing with UV divergences entailed by the theory [241]. Rather, it was seen as related to the own shaping of every physical theory, being it classical or quantum, continuous or discrete, statistical or not. In the absence of such a scale entailed by the theory, as is the case of Dirac's theory, its limits of validity would be fixed by an *ad hoc* length characterizing, ideally, the transition from the continuous to the discrete domain of elementary particles. Dirac[20], Bohr[21], and Heisenberg [227] agreed that a new departure from quantum mechanics in the late 1929s would be quite

[20]Niels Bohr Archive, BSC-DIR-291209t.

[21]Niels Bohr Archive, BSC-DIR-292312f.

premature in a context where even physical principles like Lorentz invariance and energy conservation were in dispute.

> In particular, the statement that a minimum length exists is no longer relativistically invariant, and one sees no way to reconcile the requirement of relativistic invariance with the fundamental introduction of a minimum length. For the time being it seems more correct *not* to introduce the length r_o into the fundamentals of the theory, but to stick to the relativistic invariance.
> Heisenberg (1930) in [227, p.5]; my translation.

Although Bohr's objections to the law of energy conservation has been extensively examined in the literature [250, 289], its impact on theoretical physics is far from being exhausted. According to Pauli [155, p.202], "it was not until 1936 that [Bohr] completely accepted the validity of the energy law in beta decay and the neutrino". Two years earlier, though, Bohr wrote to Pauli:

> I was also pleased that you understood the basic attitude of my concluding remarks about energy conservation. Since then, however, I have become more skeptical with respect to the implicit intention of these remarks, namely to use the theory of gravitation for a corresponding derivation of the law of β-decay. The idea was that a neutrino, for which one assumes a zero rest mass, can be nothing else than a gravitational wave with suitable quantization. However, I have convinced myself that the gravitational constant is far too small to be able to justify such a view, and [I am] therefore fully prepared to accept that we have here a really new atomic fact before us, which could be equivalent to the real existence of the neutrino.[22]
> Bohr to Pauli (15 March 1934), Doc. [366] in [525]; translation with the help of DeepL[23].

To what extent Bohr's objection to the neutrino's existence was the seed of a new *gedanken* for Pauli's sharp intuition is left to further research. The relevant fact is that Pauli (and perhaps Fermi[24]) objectively considered the possibility, raised by Bohr, of a deeper connection between the β-decay nuclear processes and gravity.

Back to the 1934 lecture, Pauli concluded his appraisal sketching out the present difficulties in to explain both the sign of κ within GTR and the β-decay processes, whose description was requiring a new coupling constant:

---

[22] Det glædede mig ogsaa, at Du forstod Grundstemningen i mine Slutbemærkninger om Energibevarelsen. Jeg er dog siden blevet mere skeptisk med Hensyn til det Haab, som implicit ligger i disse Bemærkninger, nemlig at benytte Gravitationsteorien til en Korrespondensudledning af Loven for β-Straaleemissionen. Tanken var den, at en Neutrino, for hvilken man antager en Hvilemasse 0, vel ikke kan være andet end en Gravitationsbølge med passende Kvantisering. Jeg har imidlertid overbevist mig om, at Gravitationskonstanten er altfor lille til at kunne berettige en saadan Opfattelse og [er] derfor fuldt forberedt paa, at vi her virkelig har et nyt Atomartræ for os, der kunde være ensbetydende med Neutrinoens reale Eksistens.

[23] I am thankful to Prof. Erhard Scholz for this tip.

[24] See the comments on Docs. [348] and [351] in [525]. Also, Kragh [288, pp.30-31].

> We cannot here enter into a general discussion of the unsolved problems of nuclear physics; we may however add one further remark in this connection. There are several indications that the phenomenon of so-called β-radioactivity i.e., the spontaneous emission of electrons by atomic nuclei – as well as the related only recently discovered phenomenon of artificially induced positron radioactivity – bears witness to a deeper level, so to speak, of physical reality, than the other empirically known phenomena of nuclear physics. For these phenomena appear, according to recent theories, to be governed by a further constant of nature, which cannot be directly reduced to the usual constants of atomic physics. In this connection it is of interest to point out that present-day classical field theories, including the relativistic theory of gravitation, do not give a satisfactory interpretation of the essentially *positive* character of the constant κ, which is responsible for the fact that gravitation manifests itself as an attraction and not a repulsion of gravitating masses. Such an interpretation could consist only in the reduction of the constant κ to the *square* of another constant of nature. This suggests looking for phenomena in which the square root of the constant κ plays a part. While hitherto it has been regarded as almost certain that gravitational phenomena play practically no part in nuclear physics, it now appears that the possibility that the phenomena of β-radioactivity might be connected with the square root of κ can no longer be rejected out of hand. It must however be left to the future to decide whether or not this hypothesis is appropriate.
> Pauli in [393], pp.104-105; italics as in the original.

This excerpt vividly contrasts to Pauli's rigorous scrutiny of any physical theory. Notwithstanding its loose conception, what makes Pauli's argument very unique is the situation in which the attractive character of the gravitational constant is outlined: at the frontiers of nuclear physics. The "another constant of nature" is the key piece of Pauli's reasoning, and was interpreted in a brief communication by Gleb Wataghin [528], regarding the neutrino theory of light and the quantization of gravity, as corresponding to Fermi's constant:

> It seems that one could achieve a remarkable simplification of the theory of gravitation by accepting that the same neutrinos are responsible for the gravitational action. Indeed, it is possible to apply the theory of Jordan (that allows one to obtain Bose's statistics for the quanta of gravitation from Fermi's statistics for the neutrinos) to substitute, in every case, the action of a gravitational quantum by a pair of neutrinos. In this way, 'the sea of neutrinos with negative energy' would constitute a new kind of ether that determines the geodetics of the universe, and allows one to distinguish locally between inertial and accelerated systems. This point of view agrees with the idea expressed by W. Pauli on the existence of a relation between the square root of the gravitational constant k and the new constant g introduced by Fermi in the theory of β rays.
> G. Wataghin (1936) [528]; my translation.

For a further overview of Wataghin's contributions, we refer to Hagar [218] and Rocci [433]. The relevant point here is: the unknown coupling interaction expected to be proportional to the square root of the gravitational constant was, according to Pauli, *not* necessarily the Fermi's one. In fact, Pauli considered already in 1936, from an analysis of the Fermi length $l^2_{\text{Fermi}} = G_{\text{F}}/\hbar c$ in a lattice description of the field, that "one would consider Fermi theory as being refuted" [154, pp.315-316].

A couple of days after receiving the results of Madame Wu's experiments on parity violation by β-decay processes, Pauli recalls, in a letter to Victor Weisskopf, his early hypothesis:

> Incidentally, I have published a remark in 1936 (in a somewhat hidden place) that perhaps the constant of the Fermi interactions could be proportional to the *square root of the gravitational constant*. No method exists to confirm or disprove such a conjecture. However, I think that one should *also* keep in mind the possibility that *a still unknown field* plays a role here. That this is just the case for the weak interactions may have its special reasons which ought to be connected with the unknown physical nature of the fields. Many questions, no answers!
> Pauli to Weisskopf (Zürich, 27/28 January 1957), Doc. [2476] in [525]; my translation, italics in original.

Beside, Bohr's early objections were still fresh to Pauli:

> I am now fain to apply *Bohr*'s warning, mentioned earlier, that in the case of weak interactions (as they are called nowadays) one must "be prepared for surprises", to the violation of C- and P-symmetries separately. While his special idea, which he abandoned later, of a violation of the energy law in these interactions would have concerned the *continuous* group of translations in space and time (contained in the inhomogeneous Lorentz group); our actual surprise, however, is with reference to the lowering of symmetry in the *discrete* groups of reflections in the case of weak interactions.
> Pauli (1957/8) in [155], p.212.

These remarks help to situate Pauli's bold conjecture into a better perspective. It is not by chance that the square root conjecture appears in Pauli's writings when two episodes were challenging the foundations of theoretical physics. Few, if any, could be able to express this situation better than Wheeler. In 1963, during the discussion session after Heisenberg's contribution to the Conference of Commemoration of the Fiftieth Anniversary of Niels Bohr's First Papers on Atomic Constitutions, Wheeler described the "very large numbers of nucleons" and its role on the stability of the sun as the problem that "poses issues that I at any rate don't have the faintest idea how we are going to approach" (see also [62, 549]). After a brief digression on the theme, Wheeler concludes:

> Therefore I would suppose that we don't have really to go to the realm of unbelievable energies, or unbelievable extensions in space and issues of cosmology: really on the very modest scale

> of a star we are at the border line of quite a new issue in physics, where we shall have to be prepared for quite new concepts. I wonder whether you have any comments on this border line area between elementary particle physics and gravitation physics.
> Wheeler (1963) [49, Series B-1, pp.629-630].

Heisenberg was assertive:

> No real opinion of my own. I remember, I should say with pleasure, the idea that gravitation could possibly be connected with the weak interactions in a similar way as on the other hand weak interactions may be connected with the electromagnetic ones, so that the square of the electromagnetic coupling constant could roughly be the weak coupling constant and the square of the weak coupling constant would be the gravitational coupling constant. I remember that there have been some papers in the early years; have you, Dirac, not written about these problems once? Ah, it was Gamow and Teller. Yes, well, but I am not very familiar with these ideas, and I find it simply too early in the present state to think about these problems; but some day these problems will certainly come up again.
> Heisenberg in [49], Series B-1, p.630.

One shall notice that Pauli's argument appears inverted in Heisenberg's reply. Nonetheless, evidence for such a glimpse (suggested by Heisenberg or someone else from the audience, among which Casimir, Dirac, Kronig, Pais, Rabi, and Weisskopf directly intervened in the discussion) on the Gamow-Teller part in the square root conjecture is still lacking. Yet, it is not totally unfeasible that Gamow may have taken part on this topic. Curiously, the square root of Newton's constant appears, in Gamow's paper co-authored with Ivanenko and Landau [194], as proportional to the ratio between the charge and the mass of the electron (in CGS units) for purely dimensional reasons. Although few information is given by Okun's review [375] on the availability of the GIL-paper outside the Russian community, there is evidence[25] of Gamow's exchanges with Bohr and Dirac as early as 1929.

The growing acceptance of perturbative renormalizability as a razor for new field theories only became consensual in the 1970s [70]. The quotations above suggest that Pauli had a remarkably different perspective than widespread on the structure and development of physical theories: that deducibility, rather than renormalizability, was a desirable property of new models, regardless its phenomenological or fundamental aspect. In Pauli's view, any description of a particular field would be nothing but phenomenological, meaning not only partial or provisional, as any physical theory is, but also incapable of deducing its coupling constant – and that precisely is the reason why unification echoed without barriers.

However, what characterized every program of unification was the role of gravity in the orchestration of matter. While Einstein [142, 144], Weyl [543, 545, 546], Fock [174, 175], Schrödinger [469], Bargmann [24] and O. Klein [270–273, 276] suggested many attempts towards a compatible structure between the

[25] Niels Bohr Archive, BSC-DIR-291126t

electric charge and the gravitational field, Dirac [120] and Heisenberg [231] aimed at a fundamental picture of elementary particles without gravity. Bohr envisaged, in a more radical way, matter as quantum actions of the gravitational field, and resisted (until 1936) to the existence of the neutrino by claiming the violation of energy conservation. But Pauli, quite distinctively and in the opposite direction of Einstein's [142], pointed at gravity as an emergent phenomenon of matter fields as early as 1934, three decades before Sakharov [440]. Alternative theories 'beyond Einstein' so as to include high energy physics only started to draw attention in the middle 1980s, cf. Will [556, pp.105-106]. This brief background might be helpful in order to bring Pauli's conjecture to current researches[26] at the frontiers of gravity. We start in the next section with the still phenomenological context restricted to gravity and weak interactions.

2.3 Fermi's coupling in effective scenarios

Weakening the deducibility's degree in Theoretical Physics – Planck and Fermi scales – gravitational four-fermions.

Breve intermezzo. The way how Effective Field Theories[27] (EFTs) are currently described in the literature, the way how EFTs were conceived by Heisenberg, Schwinger, and Weinberg, and the way how field theories were viewed[28] by the founders of GTR and QFT are significantly different. For let us illustrate this assertion with some quotations.

> The philosophy of effective field theories valid up to a certain energy scale Λ seems so obvious by now that it is almost difficult to imagine that at one time many eminent physicists demanded much more of quantum field theory: that it be fundamental up to arbitrarily high energy scales. Indeed, we now regard all quantum field theories as effective field theor[ies].
> A. Zee [568], pp.456-7.

Perhaps the claimed historical contrast between QFTs and EFTs, insofar as it refers to energy scale validity, is somewhat artificial. Here is a brief excerpt from a discussion between two of those eminent physicists, as mentioned by Zee:

[26] One example is the possibility of reproducing gravity, at the classical [80, 116] and quantum [33, 34, 57, 66, 348] levels, as the double copy of non-Abelian gauge fields.

[27] Initially, the present section was thought as a technical appraisal of effective methods applied to gravitational four-fermions, which are expected to play an important role in a maturate level of the problem posed in Chapter 4. It happened that this stage is yet to be developed. Hence the present section will contain a reduced version of its original proposal, dealing mainly with a qualitative analysis of the physical arguments that motivate the next two Chapters.

[28] Even at the classical level, one may recall the contrast between Einstein's and Weyl's interpretations of GTR [313] and its coupling to Maxwell electrodynamics [144, 545–547], which become unequivocally incompatible once Dirac's theory is taken into account [310], as well as the divergent viewpoints of Einstein [142], Dirac (followed by Heisenberg [232]), Klein [273, 274, 277, 279], Fock [176–178], Ivanenko [245–247], Feynman [167], Wheeler [548–550], Deser [110, 111], Thirring [495], Sakharov [440], Zeldovich [571, 572, 576, 577], and Treder [508–511, 524] on the role of gravity in particle physics. Once more, theoretical physics was anything but paradigmatic.

> Dirac: (...) It always seems to me that the S-matrix does not apply to the whole of physics but just to high energy physics; it has its limitation somewhere for lower energy physics. I would like to have your view of that.
>
> Heisenberg: Well, I could not agree more. I have never really liked the idea that one should explain everything by means of the S-matrix; but one had to emphasize this point perhaps to some extent for a time in order to gain some freedom from the old Hamiltonian scheme. But certainly the S-matrix does not contain everything.
>
> Discussion after Heisenberg's talk in the *Commemoration of the Fiftieth Anniversary of Niels Bohr's First Papers on Atomic Constitution*, held in Copenhagen on July 8-15, 1963.
> Reprinted in [49], Series B-1, p.623.

If the sections 2.1 and 2.2 above were successful in to communicate its main idea, it should be clear by now that Dirac and Heisenberg were not alone on this topic. It is mostly a prevailing view of our times that QFT and GTR are solid, fundamental theories, and that EFTs are a step further. Such a picture may be taken, for instance, from Weinberg:

> The essential point in using effective field theory is you're not allowed to make any assumption of simplicity about the Lagrangian. Certainly you're not allowed to assume renormalizability. Such assumptions might be appropriate if you were dealing with a fundamental theory, but not for an effective field theory, where you must include all possible terms that are consistent with the symmetry. The thing that makes this procedure useful is that although the more complicated terms are not excluded because they're non-renormalizable, their effect is suppressed by factors of the ratio of the energy to some fundamental energy scale of the theory. Of course, as you go to higher and higher energies, you have more and more of these suppressed terms that you have to worry about.
> S. Weinberg [539], p. 9.

The statement above contains all the key elements under discussion in the present chapter. First, the possibility of to interfere in the structure of the theory by claiming simplicity, which usually is said to be enough to allow *ad hoc* modifications. Second, the role of renormalizability in the classification of field theories. At this point, it might be useful to recall that, in the perturbation approach, a quantum field theory is composed of an action (dynamics), a vacuum, and a *regulator*, according to which the Lagrangian is said to be [68, 536]

- *non-renormalizable*, if $k_{D,n} > 0$;
- *superrenormalizable*, if $k_{D,n} < 0$;
- *renormalizable*, if $k_{D,n} = 0$,

where the superficial degree of divergence of a graph Γ reads

$$\deg_D(\Gamma) = k_{D,n}\, L + \frac{2(n-E)}{n-2}\,, \quad k_{D,n} \equiv \frac{D(n-2)-2n}{n-2}\,, \tag{2.26}$$

for E external legs, L loops, D spacetime dimensions and n valency. For a D-dimensional scalar QED, say, the vertex coupling constant g_n is related to the renormalizability condition by

$$[g_n] = -\frac{n-2}{2}\, k_{D,n}\,, \tag{2.27}$$

which satisfies the Dyson condition $[g_j] \geqslant 0$ if $n \leqslant 2D/(D-2)$. In the quotation above, Weinberg is refering to non-renormalizable theories in the Dyson power-counting sense. However, the status of renormalizability as a criteria to select physical theories is not the same anymore as it was in the 1960s, when the Salam-Glashow-Weinberg model was under construction.[29] The turning point seems to be its irrelevance in to treat the ultraviolet divergences [536, I, p.499], which led to the re-evaluation of Yang-Mills theory and Einstein gravity by Veltman and t'Hooft, among many others; see, for instance, [522]; we also refer to Cao and Schweber [70] for an analysis of the status of renormalizability from the late 1940s until 1996. More recent accounts are provided by Nambu [354], Pittau [416], and Weinberg [537, 538].

Planck and Fermi scales. It seems that Gleb Wataghin's excerpt quoted in the previous section is the only explicit record of Pauli's conjecture reported in the literature, beside Mario Novello's recollections [362, 367] of his private conversations, in the early seventies, with Josef-Maria Jauch and Ernst Stückelberg, both connected to Pauli at the ETH Zürich (the first as student in the 1933-1938 period, the second as *Privatdozent* as of late 1933, cf. Wanders in [217]; see also Enz [154]).

According to Novello, the argument as reconstructed by Jauch and Stückelberg relies upon the physical and dimensional features of the relation between the Planck and Fermi scales:

i. the Fermi length l_{Fermi} is like the square root of the Planck length l_{Planck},

$$l_{\text{Fermi}} \approx \sqrt{l_{\text{Planck}}} \approx 10^{-16}\text{cm}. \tag{2.28}$$

ii. Fermi's G_{F} and Newton's G_{N} constants share the highest degree of universality[30] among the coupling constants of physics;

iii. in natural units ($c = \hbar = 1$), G_{F} and G_{N} have dimension of length squared,

$$[G_{\text{F}}] = [G_{\text{N}}] = \text{L}^2. \tag{2.29}$$

While properties (i.) and (ii.) are markedly physical, property (iii.) is directly implicated by the dynamical variables of Einstein's and Fermi's Lagrangians. For Pauli [396], the gravitational constant,

[29] See, for instance, Salam [443]. Moreover, it is interesting to notice that Heisenberg was alone on this topic in the 1950's, as commented in the Section 2.1.

[30] To date, only gluons do not couple weakly.

as well as the energy-momentum tensor, are admittedly the 'phenomenological constituents of the GTR' (rather than fundamental, as sometimes stated). Curiously, T. D. Lee [311] asserts that $G_{\rm F}$ was inspired by Newton's constant. What is far from trivial, according to the square root conjecture, is the reason why, among all field theories, the only ones carrying $G_{\rm F}$ and $G_{\rm N}$ allows one to define length scales such that the first is like the square root of the second, and how it explains the positive (attractive) character of gravitating masses.

Although the Planck scale has an eminent position in quantum approaches to gravity, its relation to Fermi scale scarcely play any role in the deductive structure of new models. The Fermi length $l_F \equiv \sqrt{G_{\rm F}/\hbar c}$ was introduced by Ivanenko and Sokolow [248] in 1936 as a typical radius of heavy particles (protons and neutrons). Its *ad hoc* status is slightly different, though, from Planck scale; for it contains the three constants – c, $\hbar$, and $G_{\rm F}$ – present in Fermi's theory, while no consistently testable theory relating c, $\hbar$, and $G_{\rm N}$ is yet known.

More recently, the relation between Newton's and Fermi's constant has been revived [367, 377–379]. In particular, let us consider the Fermi scale defined by

$$l_F := \left(\frac{G_{\rm F}}{\hbar c}\right)^{1/2}, \quad m_F = \left(\frac{\hbar^3}{c G_{\rm F}}\right)^{1/2}, \quad t_F = \left(\frac{G_{\rm F}}{\hbar c^3}\right)^{1/2}, \quad \theta_F = \left(\frac{\hbar^3 c^3}{G_{\rm F} k_B^2}\right)^{1/2}. \tag{2.30}$$

Then, one shall note that the ratio $s_{\rm Fermi}/s_{\rm Planck}$ generates an adimensional characteristic value given by

$$\frac{l_F}{l_P} = \frac{m_P}{m_F} = \frac{t_F}{t_P} = \frac{\theta_P}{\theta_F} = \frac{c}{\hbar}\left(\frac{G_{\rm F}}{G_{\rm N}}\right)^{1/2} =: \sqrt{\xi} \simeq \sqrt{1.738 \cdot 10^{33}}\,. \tag{2.31}$$

Then, following the suggestion made by Prof. R. Onofrio [377], the renormalization of $G_{\rm N}$ by reabsorbing ξ into $\tilde{G}_N$, so that

$$\tilde{G}_N := \sqrt{2}\,\xi\, G_{\rm N} = 2.458 \cdot 10^{33} G_{\rm N} \tag{2.32}$$

would imply that, at subnuclear distances, $l_P \approx_{\rm eff} l_F$. Qualitatively, it makes explicit the phenomenological aspect of gravity (in Pauli's sense) as described by GTR. Among the possibilities to make further advances on this argument, there is one of special interest, where the physical spacetime is kept a 4-dimensional, Lorentzian manifold as in GTR, while the internal group of symmetries is enlarged.

Gravitational spin connections. As stressed in the Introduction, a renewed interest in the 1950s on the construction of Dirac's theory in curved spacetime was brought forward by Klein [273, 276], and Laurent [305], were a more general Dirac adjoint preserving unitarity was reviewed in the light of Bargmann's seminal paper [24] of 1932. It is interesting to note that Klein's 5-dimensional relativistic formalism seems to have paved the way for another type of unifying scenarios, where the physical spacetime is kept 4-dimensional and the internal space is extended via generalized spin connections (GSC) [93, 161, 210, 211, 323, 324, 362–364, 367, 371, 401, 402, 507].

This possibility is mainly related to the well known arbitrariness of the Riemannian compatibility constraint imposed to the Dirac basis of the Clifford bundle. A similar situation occurs from the point of view of soldering forms (globally defined frame fields) in classical unified models (of Einstein-Cartan- and GraviGUT-type) with enlarged internal spaces, cf. Krasnov and Percacci [291]. What distinguish the GSC approaches from the models reviewed in [291] is the absence, in the GSC approaches, of frame fields (whether locally, or globally defined) in the construction of the Dirac operator. This aspect draw some attention to the internal consistency of the GSC approaches, once fermions can not be described in curved spacetime without tetrads. We return to this point in the Section 4.1.

Consistency aside, the GSC formulations contain an interesting line of thought, which explore the possibility of introducing new degrees of freedom into Dirac's theory via spin connection. Recently, a suggestion made by Donoghue [131] also consider the same path, but from a different perspective. The argument presented by Donoghue relies upon the arbitrary choice of imposing the metric compatibility as an additional constraint to the spin connection. His motivation is stressed by the fact that the spin connection, seen as non-Abelian gauge fields, give negative β functions,

$$\beta(g) = -\frac{22}{3}\frac{g^3}{16\pi^2} \tag{2.33}$$

For a recent discussion of Donoghue's proposal, we refer to Alexander and Manton [8]. The relevant point to be highlighted is that the metricity compatibility does not preclude the spin connection from having independent degrees of freedom. Notwithstanding, that is not the only possibility to introduce internal degrees of freedom in a theory of fermions in curved spacetime. This point also is continued in Chapter 4.

Comment on the Born-Infeld running scale. From the works of Sauter [448], Halpern [220], Euler and Heisenberg [156, 233], and Schwinger [474] among others [473], the departure between Maxwell electrodynamics and non-linear quantum effects, predicted by pure QED, is characterized by the *critical electric field* $\mathcal{E}_{crit}$, given by

$$\mathcal{E}_{crit}\Big|_{\text{QED}} = \frac{m_e^2 c^3}{e\hbar}\,. \tag{2.34}$$

By noting that $[\mathcal{E}_{crit}] = \mathrm{L}^{-1}\mathrm{M}^{1/2}\mathrm{T}^{1/2}$, one can infer that the Born-Infeld parameter b_{crit} reads

$$b_{\text{QED}} \equiv b_{crit}\Big|_{\text{QED}} = \frac{\mathcal{E}_{crit}}{c}\Big|_{\text{QED}} = \mathcal{B}_{crit}\Big|_{\text{QED}} = \frac{m_e^2 c^2}{e\hbar}. \tag{2.35}$$

If we consider the possibility[31] of to construct an electromagnetic scale in terms of $\hbar, c, e$ *and* b, then we are tempted to state a *Born-Infeld running scale* s_{BI}, namely

$$l_{BI} := \left(\frac{\hbar^3 c^4}{b e^5}\right)^{1/2}, \quad m_{BI} := \left(\frac{b e^5}{\hbar c^6}\right)^{1/2}, \quad t_{BI} := \left(\frac{\hbar^3 c^2}{b e^5}\right)^{1/2}, \quad \theta_{BI} := \left(\frac{b e^5}{\hbar c^2 k_B^2}\right)^{1/2}. \tag{2.36}$$

At the critical value b_{QED} of the electromagnetic field predicted by QED (eq. 2.35), the Born-Infeld scale corresponds to

$$\lim_{b\to b_{\text{QED}}} l_{BI} = \frac{\hbar^2 c}{m_e e^2}, \quad \lim_{b\to b_{\text{QED}}} m_{BI} = \frac{m_e e^2}{\hbar c^2}, \quad \lim_{b\to b_{\text{QED}}} t_{BI} = \frac{\hbar^2}{m_e e^2}, \quad \lim_{b\to b_{\text{QED}}} \theta_{BI} = \frac{m_e e^2}{\hbar k_B}. \tag{2.37}$$

[31] This idea was suggested by Prof. M. Novello in private communication.

It also follows that s_{BI} reaches the Planck scale s_P,

$$\lim_{b\to b_{max}} l_{BI} = l_P\,, \qquad \lim_{b\to b_{max}} m_{BI} = m_P\,, \qquad \lim_{b\to b_{max}} t_{BI} = t_P\,, \qquad \lim_{b\to b_{max}} \theta_{BI} = \theta_P\,, \tag{2.38}$$

at the critical (maximal) value b_{max} determined by

$$b_{max} := \left(\frac{\hbar c}{e^2}\right)^2 \frac{c^5}{eG_{\rm N}} \approx 2.14\,409\cdot 10^{105}\,\text{GeV}\,\text{m}^{-4}\text{kg}^{-1/2}\text{s}^{5/2}. \tag{2.39}$$

We shall also note that the ratios s_P/s_{BI}, and s_F/s_{BI} produce, respectively, the adimensional running parameters given by

$$\frac{l_P}{l_{BI}} = \frac{m_{BI}}{m_P} = \frac{t_P}{t_{BI}} = \frac{\theta_{BI}}{\theta_P} = \left(\frac{be^5 G_{\rm N}}{\hbar^2 c^7}\right)^{1/2}, \tag{2.40}$$

$$\frac{l_F}{l_{BI}} = \frac{m_{BI}}{m_F} = \frac{t_F}{t_{BI}} = \frac{\theta_{BI}}{\theta_F} = \left(\frac{be^5 G_{\rm F}}{\hbar^4 c^5}\right)^{1/2}. \tag{2.41}$$

One may claim that the electromagnetic field reaches the Planck and Fermi scales, respectively, as $b \longrightarrow b_P$ and $b \longrightarrow b_F$, for

$$b_P := \lim_{l_{BI}\to l_P} b = \frac{\hbar^2 c^7}{e^5 G_{\rm N}}, \tag{2.42}$$

$$b_F := \lim_{l_{BI}\to l_F} b = \frac{\hbar^4 c^5}{e^5 G_{\rm F}}. \tag{2.43}$$

The strongest magnetic fields detected in Astrophysics comes from neutron stars; we refer to [338, 437].

2.4 Outlook

Historically, the statement of energy scale problem associated to each interaction shapes, and sometimes even predate, the own physical theory, and allows one to rethink the way how theories are fabricated since the 1970s. While the generation of field theories in the 1915-1939 period were mainly bold, constitutive hypotheses with increasing degree of deducibility, the period post-1970 admittedly portraits an increasing adhesion in the effective approaches to model-building aimed at fitting the empirical data, even if they are openly *ad hoc*.

If that is the case, we do not see EFTs as a departure from QFTs: How could quantum field theories be unequivocally classified as effective, or phenomenological, and non-effective, or non-phenomenological? Born-Infeld (1934), Euler-Heisenberg (1935-1936), Heisenberg's unified field theory (1947-1959) are all typically nonrenormalizable theories, proposed as early as the foundational papers of QED. The novelty, perhaps, is in to admit that perturbative renormalizability is not enough to eliminate theories from our bundle of frameworks.

However, the crucial question seems to be this: What is expected from the new theories in physics after a decisive test like parity violation? While the non inclusion of the empirical evidence would be a nonsense, any test of the future theory can not rely upon its ability of to describe parity violation. Corroboration can not exclusively follow from the reproducibility of a previous refutation. Any further development post-1957 must be able to present new logical consequences, including parity violation, from its constitutive and auxiliary hypotheses. For a new bold conjecture, rather than a mathematical reset, is necessary, even if its results are more likely to require a long term research program. Oriented by the Pauli conjecture, the next chapters are small steps in this open direction.

3. Grasping an effective approach to gravity

Any physical theory, any physical notion, is, as a matter of fact, an approximation.
Each great progress of physical science is related not only to the creation of new notions,
but also to the critical revision of old ones.
V. A. Fock in [160], Paper 36-1.

Current researches connecting cosmology and neutrino physics have been confronting over the last five decades the Salam-Glashow-Weinberg model limitations to explain the massive [39, 42, 331] and sterile [58] neutrinos, its Dirac or Majorana nature [40], and how its parity violation could be indicating a deeper connection with external symmetries of spacetime [9, 71, 97, 198, 205, 263, 357, 376–379, 522].

In cosmology, the lackness of matter and energy content in the universe within the inflationary model combined to recent Planck data seems to indicate a complete exhaustion of the present theoretical scheme, as well as the necessity of a new departure, or yet *new physics* [59, 61, 114]. Meanwhile, alternative perspectives [20, 71, 117, 309, 355, 365, 370, 386, 411, 412, 515, 517, 521, 542] do not discard the achievements of the standard model, but keep open the path in searching for a *new synthesis*.

We intend to address the second road by starting with a brute force exercise: the implementation[1] of non-linear spinor fields inducing an effective spacetime [367–369]. In this effective approach, also called spinor theory of gravity[2] (STG), the following statements are assumed as the orientation for this program:

I. The construction of the STG relies upon Pauli's square root conjecture that there is a deeper relation between gravity and weak interactions (Sec. 2.2). There are at least three common aspects pointing to that direction: their high level of universality, the same dimensionality of their coupling constants (in the natural unit system), and their manifest external symmetries (parity violation, for instance).

II. The mathematical framework of STG is a Clifford algebra $Cl(V_4, \eta)$ associated to a 4-dimensional Minkowski space (V_4, η), with an extention of the spin connection by a scalar field H. In effect, this prevents us from introducing the antisymmetric part of the Riemannian connection, as suggested by the Einstein-Cartan theory. As a key outcome, the four-fermion coupling to gravity (in the sense of GR) does not induces torsion.

III. The self-interaction of the gravitational field is implemented by a Heisenberg spinor in (V_4, η) (Sec. 3.1). That is, gravity is locally described by a Fermi's contact interaction.

[1] The results of this chapter were developed in collaboration with M. Novello, and E. Bittencourt.
[2] Although 'spinor approach to gravity' would be a better name, once the proposal's testability is still lacking.

IV. In the lines of Sakharov's conjecture, the physical spacetime (M, g) emerges as an effective process induced by weak $(V-A)$-currents (Sec. 3.2). By construction, gravity is in the large scale an effective weak interaction.

We argument that gravity as an effective manifestation of internal degrees of freedom of the spacetime structure is closer to Einstein's ideas than regarding it to a metric representation. Indeed, Einstein's *apriorism* relies, not upon the (pseudo) Riemannian structure of (M, g) as usually is asserted, but instead on the belief that the gravitational field is as *intrinsic* to the spacetime structure as the curvature is intrinsic to a (pseudo) Riemannian manifold. Hence, unification and geometrization are two distinct programs [313]. Einstein followed the first, while Weyl pursued their identification. If that is the case, then geometry might be sufficient, albeit not necessary, to describe gravity.

That allow us to infer the possibility of introducing a metric structure in M satisfying the metricity condition, without imposing a dynamics over g. Einstein equations, in that case, are tautologically satisfied in compatibility with the Bianchi identities. The evolution of g comes from the hypothesis that g inherits its dynamics from non-linear spinor fields. That is the content of what we call by "effective" here.

Let us point out that, in the GTR, Einstein's equivalence principle (EEP) is assumed to be valid in the (external) physical spacetime. From the point of view of STG, one is led to consider as a key feature of the theory that the universal coupling of matter with the non-linear spinor fields satisfies the EEP in the effective spacetime, while internally, matter is intrinsically carrying gravitational content. This seems to be another point of contact between Heisenberg's program and Einstein's gravitational physics [142] (Sec. 3.3).

3.1 The action

The spin connection. Let $Cl(V_4\,, \eta)$ be a Clifford algebra associated to a 4-dimensional Minkowski space $(V_4\,, \eta)$ with signature $+\,-\,-\,-$, and ideal given by

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\,\eta^{\mu\nu}(x)\,\mathbb{I}. \tag{3.1}$$

Since η is a bilinear in arbitrary coordinates, we require that it satisfies a (pseudo-)Riemannian structure, so that

$$\nabla_{\rho}\,\eta_{\mu\nu} = \partial_{\rho}\eta_{\mu\nu} - \Gamma^{\sigma}{}_{\rho\mu}\,\eta_{\sigma\nu} - \Gamma^{\sigma}{}_{\rho\nu}\,\eta_{\mu\sigma} = 0, \tag{3.2}$$

where $\Gamma^{\sigma}{}_{\rho\nu} = \Gamma^{\sigma}{}_{\nu\rho}$ is the Levi-Civita connection. The Dirac basis defining the ideal (3.1) is made compatible with (3.2) under the further constraint given by the covariant derivative[3]

$$\nabla_{\mu}\,\gamma_{\nu} = \partial_{\mu}\,\gamma_{\nu} - \Gamma^{\sigma}{}_{\rho\mu}\,\gamma_{\sigma} - \Gamma_{\mu}\,\gamma_{\nu} + \gamma_{\nu}\,\Gamma_{\mu} = 0, \tag{3.3}$$

[3] This expression corresponds to the Eq. (36) in Fock (1929) [174], Eq. (8) in Schrödinger (1932) [469], and Eq. (18) in Bargmann (1932) [24], and Eq. (2) in Brill and Wheeler (1957) [62]. A discussion is presented in Chapter 4.

where, for any vector field $H_\mu \in \mathfrak{X}(V_4)$, the spin connection $\Gamma_\mu = \Gamma^{FI}_\mu + H_\mu$ is linear in $\gamma_\mu(x) \in Cl(V_4\,,\eta)$. Note that the Fock-Ivanenko connection is a recurrence relation. Contracting (3.3) with γ^ν on the left and on the right, separately, and taking the difference, one gets

$$\gamma^\nu \partial_\mu \gamma_\nu - \partial_\mu \gamma_\nu \gamma^\nu - \Gamma^\sigma{}_{\rho\mu} \left(\gamma^\nu \gamma_\sigma \,-\, \gamma_\sigma \gamma^\nu\right) \,+\, 8\Gamma^{FI}_\mu \,-\, 2\gamma^\nu \Gamma^{FI}_\mu \gamma_\nu \,= 0. \tag{3.4}$$

In particular, the Fock-Ivanenko connection satisfies[4]

$$\Gamma^{FI}_\mu := -\frac{1}{8}\left[\gamma^\kappa\, \partial_\mu\, \gamma_\kappa - \partial_\mu \gamma_\kappa\, \gamma^\kappa - \Gamma^\lambda{}_{\mu\kappa}(\gamma^\kappa \gamma_\lambda - \gamma_\lambda \gamma^\kappa)\right]. \tag{3.5}$$

Setting (3.5) is equivalent to fixing an orthonormal frame bundle for V_4, where $\Gamma^{FI}_\mu = \omega_\mu$ is the spin connection [265, 526]. Moreover, let $H(x)$ be a real Klein-Gordon field, such that

$$H_\mu(x) := \frac{1}{4}\, \gamma_\mu\, \gamma^\kappa\, \partial_\kappa H(x). \tag{3.6}$$

One may note that (3.3) is linear in γ_μ, since

$$[H_\mu\,, \gamma_\nu] = \frac{1}{2}(H_{,\mu} \gamma_\nu - \eta_{\mu\nu}\, \gamma^\kappa H_{,\kappa}). \tag{3.7}$$

To ensure that the $H(x)$ does not carry any dependence on the spinor fields, its dynamics is constrained by

$$\mathcal{S}_o[H] = \int_{V_4} \sqrt{-\eta}\, \left[\mathcal{L}_o + b^{\mu\nu}(\nabla_\mu\, \gamma_\nu + 2\eta_{\mu\nu}\gamma^\kappa\, \partial_\kappa H - 2\gamma_\nu\, \partial_\mu H)\right] d^4x, \tag{3.8}$$

where the Lagrange multipliers $b_{\mu\nu} \in Cl(V_4\,,\eta)$ are functions of the elements of the Clifford algebra. In this way, the variation of $\mathcal{S}_o[H]$ with respect to H reduces to the massless Klein-Gordon equations,

$$\delta_H\, \mathcal{S}_o[H] = 0 : \qquad \Box\, H(x) = 0\,, \qquad b_{\mu\nu} = \gamma_\mu \gamma_\nu\,. \tag{3.9}$$

In consequence, the action of the Lorentz covariant derivative over the spinor fields $\{\overline{\Psi}, \Psi\}$ is given by[5]

$$\nabla_\mu \Psi = \partial_\mu \Psi - \Gamma_\mu \Psi\,, \tag{3.10}$$

$$\overline{\nabla}_\mu \overline{\Psi} = \partial_\mu \overline{\Psi} + \overline{\Psi}\, \Gamma_\mu\,, \tag{3.11}$$

where $\overline{\Psi} := \Psi^\dagger \gamma^o$ is the ordinary Dirac adjoint.

We shall see that the *ad hoc* introduction of H is meant to save the stability of the static and spherically symmetric solutions of STG. In the context of QED, the Heisenberg equation allows one to interpret Ψ and $\overline{\Psi}$ as bounded states, self-interacting via a scalar field [35]. In the present formulation, the scalar field plays no other role than adjusting the stability of the static sphere configurations (Sec. 3.4).

[4] Eq. (10-6.29) in Anderson [15].

[5] This convention follows Brill and Wheeler [62].

The Fierz identities. Let $\Gamma \in Cl(V_4, \eta)$ be the set of 16 independent elements of the Clifford algebra associated to the Minkowski space. Then Γ satisfies the Fierz(-Pauli-Kofink) identities[6],

$$(\overline{\Psi}\,\Gamma\gamma_\sigma\,\Psi)\,\gamma^\sigma\,\Psi \;=\; (\overline{\Psi}\,\Gamma\,\Psi)\;\Psi - (\overline{\Psi}\,\Gamma\gamma_5\,\Psi)\,\gamma_5\,\Psi\,, \qquad \forall\,\Gamma \in Cl(V_4,\eta). \tag{3.12}$$

Denoting by

$$A := \overline{\Psi}\,\Psi, \quad B := i\overline{\Psi}\gamma_5\Psi, \quad J^\mu := \overline{\Psi}\gamma^\mu\Psi, \quad I^\mu := \overline{\Psi}\gamma^\mu\gamma_5\Psi \tag{3.13}$$

the scalar, pseudoscalar, vector and axial currents, respectively, it holds the following relations:

$$J_\mu\,\gamma^\mu\,\Psi \;=\; (A + iB\gamma_5)\,\Psi \tag{3.14}$$

$$J_\mu\,\gamma^\mu\gamma_5\Psi \;=\; -(A + iB\gamma_5)\gamma_5\Psi \tag{3.15}$$

$$I_\mu\gamma^\mu\,\Psi \;=\; (A + iB\gamma_5)\gamma_5\Psi \tag{3.16}$$

$$I_\mu\gamma^\mu\gamma_5\,\Psi \;=\; -(A + iB\gamma_5)\,\Psi\,. \tag{3.17}$$

In terms of the projector operators $P_\pm$,

$$P_\pm := \frac{1}{2}(\mathbb{I} \pm \gamma_5), \tag{3.18}$$

one may define the chiral states $\{\overline{\chi}, \chi\}$, such that

$$\chi := P_+\,\Psi, \qquad \overline{\chi} := \overline{(P_+\Psi)} = \overline{\Psi}\,P_-\,. \tag{3.19}$$

Since $P_\pm^2 = P_\pm$ and $\gamma_5 P_\pm = \pm P_\pm$, the relations between the projectors $P_\pm$ and the 16 independent elements Γ of the Clifford algebra are summarized by[7]

$$\Gamma = \left\{\mathbb{I}\,,\gamma_5\,,\gamma_\mu\gamma_\nu\right\}\;: \qquad P_\mp\Gamma P_\pm = 0\,, \tag{3.20}$$

$$\Gamma = \left\{\gamma_\mu\,,\gamma_\mu\,\gamma_5\right\}\;: \qquad P_\mp\Gamma P_\pm = \pm P_\mp\gamma_\mu P_\pm\,. \tag{3.21}$$

Hence, the Fierz identities for the chiral states reduces to

$$(\overline{\chi}\,\Gamma\gamma_\sigma\,\chi)\,\gamma^\sigma\,\chi \;=\; 0, \qquad \Gamma = \left\{\mathbb{I}\,,\gamma_5\,,\gamma_\mu\gamma_\nu\right\} \tag{3.22}$$

$$(\overline{\chi}\,\Gamma\gamma_\sigma\,\chi)\,\gamma^\sigma\,\chi \;=\; (\overline{\chi}\,\Gamma\,\chi)\;\chi - (\overline{\chi}\,\Gamma\gamma_5\,\chi)\,\gamma_5\,\chi\,, \qquad \Gamma = \left\{\gamma_\mu\,,\gamma_\mu\,\gamma_5\right\}. \tag{3.23}$$

In summary, only vector and axial currents of chiral states survives. This is a curious property indeed, once it reduces the arbitrariness of which kind of spinors can condensate into Einstein-Bose states. In other words, a four-fermion interaction can not be responsible for inducing the mass of particles without breaking parity.

---

[6] Cf. Fierz [169], Fierz and Pauli [170], and Kofink [283–285]; see also the Chapter II.8 in Castellani, D'Auria and Fré [77].
[7] See also Feynman and Gell-Mann [168].

The Heisenberg fields. By construction, $\{\overline{\Psi}, \Psi\}$ are Dirac spinors with a Fermi's contact interaction, constrained by the Action

$$\mathcal{S}[\overline{\Psi}, \Psi] = \frac{i}{2} \int_{V_4} \sqrt{-\eta} \Big(\overline{\Psi}\, \overline{\nabla} \Psi - \overline{\overline{\nabla}\Psi}\, \Psi + s\, J^{\mu} J_{\mu} \Big) d^4 x \,. \tag{3.24}$$

The coupling parameter $s \in \mathbb{R}$ has dimension of length squared. The dynamics for Ψ is

$$\delta_{\overline{\Psi}} \mathcal{S}[\overline{\Psi}, \Psi] = 0 : \qquad \gamma^{\mu}(\partial_{\mu} - \Gamma^{FI}_{\mu} - H_{\mu})\Psi - 2s(A + iB\gamma_5)\Psi = 0. \tag{3.25}$$

We shall note that the Heisenberg fields [137, 193, 231, 234, 517] correspond to the particular case when $\nabla_{\mu} \rightarrow \partial_{\mu}$. Moreover, the Heisenberg dynamics reduces to Dirac equation for $\Psi = \gamma_5\, \Psi$, cf. [243].

3.2 The effective spacetime

Einstein's *apriorism* in the formulation of the GTR relies, not upon the pseudo-Riemannian structure (M, g) itself, but instead upon the belief that the gravitational field is as intrinsic to the spacetime structure as the Riemannian curvature is to (M, g). That is the root for requiring general covariance in GTR: once gravity is, by hypothesis, intrinsic to the structure of spacetime, the theoretical system describing gravity should not be dependent on the coordinate system. Physical reality, according to Einstein, cannot depend on the nets choose to catch it.

That Einstein did not see the GTR as a geometrization of gravity is well documented in the literature, cf. Lehmkuhl [313]; see also Kiefer [265, p. 351], and Darrigol [96, §9.6]. The standard interpretation present in any textbook on the subject is due to Hermann Weyl [543, 545–547], and his successor John A. Wheeler [548–550]. The metric was seen by Einstein as a secondary element of the theory, while the affine connection is what characterize the physical content of GTR. One of the few to enlighten this distinction was Schrödinger:

> As early as 1918, H. Weyl drew attention to the fact that in Einstein's relativistic theory of gravitation of 1915, gravitation was based not directly on the metric $g_{\mu\nu}$ but on the affine connection
>
> $$\Gamma^{\alpha}{}_{\mu\nu} = \left\{ {}_{\mu}{}^{\alpha}{}_{\nu} \right\} \qquad (\ldots).$$
>
> Schrödinger [470], p.147.

Einstein's Equivalence Principle (EEP) between inertial and gravitational effects is directly dependent upon the Christoffel symbols, which define a family of geodesics characterized by its arc length s,

$$\frac{d^2 x^{\rho}}{ds^2} + \Gamma^{\rho}{}_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} = 0. \tag{3.26}$$

The metric plays no direct role on parallel displacements.

Trautman [505] defined a “weak equivalence principle”, according to which the reproducibility of measurements on a given physical system is assured by the same affinity preserving the covariance of the system, if the back-reaction of measurements on the source of the gravitational field is neglectable. In the lines of Anderson [15, 16], one may state that EEP introduces a selection, and very restrictive indeed, rule to the coupling of gravity with matter in GR. The fact that the Levi-Civita connection is unique and compatible with the metric has no other physical implication than preserving lengths under parallel transport, the key property lost in Weyl’s geometry [4].

One may also be led to notice that Schrödinger’s comment is coherent with how Einstein’s severe self-criticism predominantly reflects on modifications of GTR via the affine connection, as is the case of his collaborations with Élie Cartan, Eddington, Mayer, Schrödinger, Straus, and Bruria Kaufman, cf. [25, 96, 209, 291, 471, 501]. The exception is the bitensor theory, developed with V. Bargmann [149]. There is more: Einstein also argued that gravity should play a key role not only in the stability of matter [142], but also in the creation of the masses of particles [146, p.675]. That is precisely what we mean by saying that gravity, according to Einstein, is intrinsic to the spacetime structure[8]. According to Einstein, GTR is nothing else than a provisional step towards a relativistic theory of gravity.

In these lines, Pauli [222] and Sakharov [440] seems to have understood Einstein’s physics in deep when he proposed gravity as an emergent phenomenon of matter fields. That was the seed to think gravity as an emergent, analogue or yet effective, process [13, 22, 23, 46, 81, 130, 523, 576].

In the STG, we call *effective spacetime* a 4-dimensional, oriented manifold (M, g^{eff}) equipped with an effective metric g, as defined by

$$g^{\text{eff}}_{\mu\nu} = \eta_{\mu\nu}(x) - \kappa h_{\mu\nu}(\overline{\Psi}, \Psi), \tag{3.27}$$

where the modifications $h_{\mu\nu}(\overline{\Psi}, \Psi)$ of the Minkowski background are constructed in terms of the $V - A$ currents[9]

$$h_{\mu\nu}(\overline{\Psi}, \Psi) = l_\mu \, l_\nu, \qquad l_\mu := \left(\frac{g_w}{J}\right)^{1/4} (J_\mu - I_\mu) \, , \qquad J \equiv \eta_{\mu\nu} J^\mu J^\nu. \tag{3.29}$$

We also require that g^{eff} satisfies a pesudo-Riemannian structure,

$$\nabla_\rho \, g^{\text{eff}}_{\mu\nu} = 0. \tag{3.30}$$

As a consequence of the Fierz identities, the vector current J^μ is timelike, the axial current I^μ is spacelike, and the l_μ are null. In addition, it holds that

$$\eta_{\mu\nu} J^\mu J^\nu = -\eta_{\mu\nu} I^\mu I^\nu = -A^2 - B^2 \, , \tag{3.31}$$

[8] Dewar [113] argues that diffeomorsphisms should be understood as internal, more than external, automorphisms in the context of GTR.

[9] We shall note that in Heisenberg’s QED [49, Series B-1, p.663], the projection operator for a Dirac spin 1 state is proportional to

$$\left(\frac{kl}{2\pi}\right)^2 \left(\eta_{\mu\nu} - \frac{J_\mu J_\nu}{J}\right) q_1(\lambda) \, , \qquad J \equiv \eta_{\mu\nu} J^\mu J^\nu . \tag{3.28}$$

A further elucidation of this parallel with the effective metric is needed.

and consequently,

$$\eta^{\mu\nu} h_{\mu\nu}(\overline{\Psi}, \Psi) = 0\,, \qquad h_{\mu\kappa}\,\eta^{\kappa\lambda}\,h_{\lambda_\nu} = 0\,, \qquad \det g^{\text{eff}}_{\mu\nu}(x) = \det \eta_{\mu\nu}(x). \tag{3.32}$$

That is another relevant property related to the spinor representation: the existence of one spinor field does not affect the total volume of the physical spacetime (compare, for instance, with Nambu [353, p. 402]). These properties do not hold for contact interactions of two distinct spinor fields.

3.3 Universal coupling with matter

One of the fundamental problems present in Einstein's formulation of GTR is how to interpret the energy-momentum tensor [141, 251, 302]. In *The meaning of Relativity*, Einstein asserts:

> We have seen, indeed, that in a more complete analysis the energy tensor can be regarded only as a provisional means of representing matter. In reality, matter consists of electrically charged particles, and is to be regarded itself as a part, in fact, the principal part, of the electromagnetic field. It is only the circumstance that we have no sufficient knowledge of the electromagnetic field of concentrated charges that compels us, provisionally, to leave undetermined, in presenting the theory, the true form of this tensor.
> Einstein [148], p.84-85.

As in GR, the action of the free matter $\mathcal{S}_m$ is stated by

$$\mathcal{S}_m = \int \sqrt{-g}\;\mathcal{L}_m\, d^4x\,, \tag{3.33}$$

with energy-momentum distribution defined by

$$\delta\,(\sqrt{-g}\;\mathcal{L}_m) := \frac{1}{2}\sqrt{-g}\,T_{\mu\nu}\;\delta g^{\mu\nu}. \tag{3.34}$$

However, the metric in STG is not a dynamical field, since it is nothing but a representation the action of the Heisenberg field Ψ upon the Minkowski space. Recalling the notation in [367],

$$\delta_{\overline{\Psi}}\, g^{\mu\nu} = 2\kappa\, l^{\mu}\,\delta_{\overline{\Psi}}\, l^{\nu} \equiv 2\kappa\, Q^{\mu\nu}\Psi, \tag{3.35}$$

with

$$Q^{\mu\nu} \equiv \left(\frac{g_w}{J}\right)^{1/4} l^{\mu}\,\gamma^{\nu}\,(\mathbb{I} - \gamma_5) - \frac{1}{2J}\,h^{\mu\nu}\,(A + iB\gamma_5). \tag{3.36}$$

Thus, the physical content of the variational relation (3.34) for $T_{\mu\nu}$ is given by

$$\delta_{\overline{\Psi}}\,\mathcal{L}_m = \kappa\,T_{\mu\nu}\;Q^{\mu\nu}\Psi. \tag{3.37}$$

Note that the energy tensor is written without distinction between the Minkowski background and the effective spacetime. This is due to the fact that the Heisenberg field do not affect the effective metric,

$g_{\mu\nu}\, l^\nu = \eta_{\mu\nu}\, l^\nu$. In other words, matter couples to $Q^{\mu\nu}$ as if it were in the background spacetime. Otherwise, instead of an exact expression for the inverse of g, we would have an infinite series, as it is the case of field theoretic formulation of gravity[10].

Hence, the universal coupling of matter $\mathcal{S}_m$ with the effective metric induced by the Heisenberg spinor Ψ implies the non-minimal coupling of matter with the Heisenberg field,

$$i\nabla\!\!\!/\, \Psi(x) = -\kappa\, T_{\mu\nu}\, Q^{\mu\nu}\Psi(x)\,. \tag{3.38}$$

3.4 The exact solutions of STG

In this section, we start by considering the two observed solutions[11] of GTR: the Schwarzschild metric [472] and the Friedman universe [187]. We shall see that, differently from Einstein's gravity, the STG is not constrained to the Birkhoff statement, which allows us to explore two distinct situations: with and without a Heisenberg potential. Thereafter, we indicate how gravitational waves can be defined in STG following the Kundt's criteria [295]. Finally, we start the discussion about the weak field regime of the STG.

Static and spherically symmetric solution I. In the spherical coordinate system, the Minkowski line element

$$ds^2 \;=\; dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \tag{3.39}$$

is associated to the following non trivial Christoffel symbols:

$$\Gamma^1{}_{22} = -r\,, \quad \Gamma^1{}_{33} = -r\sin^2\theta\,, \quad \Gamma^2{}_{21} = \Gamma^3{}_{31} = \frac{1}{r}\,, \Gamma^2{}_{33} = -\sin\theta\cos\theta\,, \quad \Gamma^3{}_{32} = \cot\theta\,, \tag{3.40}$$

and the Lorentz volume of $\eta_{\mu\nu}$ reduces to $\sqrt{-\eta} = r^2\sin\theta$. Thus, the γ's reproduces the metric elements as follows:

$$\gamma_o = \widetilde{\gamma}_o\,, \quad \gamma_1 = \widetilde{\gamma}_1\,, \quad \gamma_2 = r\,\widetilde{\gamma}_2\,, \quad \gamma_3 = r\sin\theta\,\widetilde{\gamma}_3\,, \tag{3.41}$$

where the constant basis $\widetilde{\gamma}_\mu$ corresponds to the Dirac representation of $Cl(V_4, \eta)$,

$$\widetilde{\gamma}_o \;=\; \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \qquad \widetilde{\gamma}_j \;=\; \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \qquad \widetilde{\gamma}_5 \;=\; \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}. \tag{3.42}$$

The inverse components of (3.41) are defined by

$$\gamma^\mu := \eta^{\mu\nu}\gamma_\nu\,, \qquad \eta_{\mu\nu}\gamma^\mu\gamma^\nu = 4. \tag{3.43}$$

Explicitly, we have

$$\gamma^o = \widetilde{\gamma}_o\,, \quad \gamma^1 = -\widetilde{\gamma}_1\,, \quad \gamma^2 = -\frac{1}{r}\,\widetilde{\gamma}_2\,, \quad \gamma^3 = -\frac{1}{r\sin\theta}\,\widetilde{\gamma}_3.$$

[10] See, for instance Feynman in [167], and Prinz [423].

[11] See also [63, 89].

The components of the Fock-Ivanenko connection (3.5) are given by

$$\Gamma_2^{FI} = -\frac{1}{2}\widetilde{\gamma}_1\,\widetilde{\gamma}_2\,, \qquad \Gamma_3^{FI} = -\frac{1}{2}\big(\sin\theta\widetilde{\gamma}_1 + \cos\theta\widetilde{\gamma}_2\big)\widetilde{\gamma}_3. \tag{3.44}$$

In addition, the choice $H = \ln\det\gamma_3 = \ln\sqrt{r\sin\theta}$ is a solution of (3.9). In these coordinates, the Heisenberg field satisfies the equations of motion given by

$$\left[\left(-\partial_r - \frac{1}{r} + \frac{4\varepsilon}{r}\right)\widetilde{\gamma}_1 + \left(-\frac{1}{r}\partial_\theta - \frac{\cot\theta}{2r} + \frac{2\varepsilon\cot\theta}{r}\right)\widetilde{\gamma}_2\right]\Psi(r,\theta) = 0. \tag{3.45}$$

Setting

$$\Psi = f(r)\,B(\theta)\,\Psi^o\,, \tag{3.46}$$

we obtain that

$$\left[\widetilde{\gamma}_1\left(\frac{f'}{f} + \frac{1}{2r}\right) + \widetilde{\gamma}_2\frac{1}{rB}\frac{dB}{d\theta}\right]\Psi^o = 0\,. \tag{3.47}$$

Hence the Heisenberg field Ψ only depends on the radial part,

$$\Psi(r) = \frac{1}{\sqrt{r}}\Psi^o\,, \qquad \Psi^o := \begin{pmatrix}\zeta\\ \eta\end{pmatrix}. \tag{3.48}$$

We are looking for a solution such that

$$h_{22} = h_{33} = 0\,. \tag{3.49}$$

From the spatial components of J^μ and I^μ,

$$J_i - I_i = \frac{a}{r}\,(\zeta^\dagger - \eta^\dagger)\,\sigma_i\,(\eta - \zeta)\,, \tag{3.50}$$

we infer that a sufficient condition to guarantee (3.49) is by setting

$$(\zeta^\dagger - \eta^\dagger)\,\sigma_2\,(\zeta - \eta) = 0\,, \tag{3.51}$$

$$(\zeta^\dagger - \eta^\dagger)\,\sigma_3\,(\zeta - \eta) = 0\,. \tag{3.52}$$

Furthermore, the components J_o and J_1 are, respectively,

$$J_o = \frac{1}{r}\,(\zeta^\dagger\zeta + \eta^\dagger\eta) \tag{3.53}$$

$$J_1 = \frac{1}{r}\,(\zeta^\dagger\sigma_1\eta + \eta^\dagger\sigma_1\zeta)\,, \tag{3.54}$$

and the normalization factor is

$$J^{1/4} := (\eta_{\mu\nu}J^\mu J^\nu)^{1/4} = (J_oJ^o + J_1J^1)^{1/4} = \frac{1}{\sqrt{r}}\big[(\zeta^\dagger\zeta + \eta^\dagger\eta)^2 - (\zeta^\dagger\sigma_1\eta + \eta^\dagger\sigma_1\zeta)^2\big]^{1/4}. \tag{3.55}$$

It follows that the components l_o and l_1 are given by

$$l_o := \left(\frac{g_w}{J}\right)^{1/4}(J_o - I_o) = g_w^{1/4}\frac{1}{\sqrt{r}}\frac{(\zeta^\dagger - \eta^\dagger)(\zeta - \eta)}{\left[(\zeta^\dagger\zeta + \eta^\dagger\eta)^2 - (\zeta^\dagger\sigma_1\eta + \eta^\dagger\sigma_1\zeta)^2\right]^{1/4}}, \tag{3.56}$$

$$l_1 := \left(\frac{g_w}{J}\right)^{1/4}(J_1 - I_1) = g_w^{1/4}\frac{1}{\sqrt{r}}\frac{(\zeta^\dagger - \eta^\dagger)\,\sigma_1\,(\eta - \zeta)}{\left[(\zeta^\dagger\zeta + \eta^\dagger\eta)^2 - (\zeta^\dagger\sigma_1\eta + \eta^\dagger\sigma_1\zeta)^2\right]^{1/4}}. \tag{3.57}$$

Collecting the results, the non-vanishing components of $h_{\mu\nu}(\overline{\Psi}, \Psi)$ are

$$h_{oo} = \frac{g_w^{1/2}}{r}\frac{\left[(\zeta^\dagger - \eta^\dagger)(\zeta - \eta)\right]^2}{\left[(\zeta^\dagger\zeta + \eta^\dagger\eta)^2 - (\zeta^\dagger\sigma_1\eta + \eta^\dagger\sigma_1\zeta)^2\right]^{1/2}} \equiv \frac{\alpha\, g_w^{1/2}}{r}, \tag{3.58}$$

$$h_{o1} = \frac{g_w^{1/2}}{r}\frac{(\zeta^\dagger - \eta^\dagger)(\zeta - \eta).(\zeta^\dagger - \eta^\dagger)\,\sigma_1\,(\eta - \zeta)}{\left[(\zeta^\dagger\zeta + \eta^\dagger\eta)^2 - (\zeta^\dagger\sigma_1\eta + \eta^\dagger\sigma_1\zeta)^2\right]^{1/2}} \equiv \frac{\beta\, g_w^{1/2}}{r}, \tag{3.59}$$

$$h_{11} = \frac{g_w^{1/2}}{r}\frac{\left[(\zeta^\dagger - \eta^\dagger)\,\sigma_1\,(\eta - \zeta)\right]^2}{\left[(\zeta^\dagger\zeta + \eta^\dagger\eta)^2 - (\zeta^\dagger\sigma_1\eta + \eta^\dagger\sigma_1\zeta)^2\right]^{1/2}} \equiv \frac{\gamma\, g_w^{1/2}}{r}, \tag{3.60}$$

for α, β, and γ constants. Consequently, the effective line element reads

$$ds^2_{eff} = \left(1 - \frac{\alpha\, g_w^{1/2}}{r}\right)dt^2 - \left(1 + \frac{\gamma\, g_w^{1/2}}{r}\right)dr^2 - \frac{2\beta\, g_w^{1/2}}{r}\,dtdr - r^2\, d\Omega. \tag{3.61}$$

In order to eliminate the cross term, we set $\beta = \alpha$, and make the coordinate transformation

$$dt = dT + \frac{\alpha g_w^{1/2}/r}{1 - \alpha g_w^{1/2}/r}\,dr\,. \tag{3.62}$$

The line element (3.61) reduces to

$$ds^2_{eff} = \left(1 - \frac{\alpha\, g_w^{1/2}}{r}\right)dT^2 - \left(1 + \frac{\gamma\, g_w^{1/2}}{r} - \frac{(\alpha\, g_w^{1/2}\,/r)^2}{1 - \alpha\, g_w^{1/2}\,/r}\right)dr^2 - r^2\, d\Omega. \tag{3.63}$$

If $\gamma = \alpha$, the relations (3.56) and (3.57) for l_o and l_1 are compatible with the Schwarzschild solution for $r_H \equiv \alpha\, g_w^{1/2}$,

$$ds^2_{eff} = \left(1 - \frac{r_H}{r}\right)dT^2 - \frac{1}{1 - r_H\,/r}dr^2 - r^2\, d\Omega\,. \tag{3.64}$$

Static and spherically symmetric solution II. Let us consider the case in which the Heisenberg field satisfies the dynamics given by

$$i\nabla\!\!\!\!/\,\Psi - 2s\,(A + i\,B\,\gamma_5\,)\,\Psi = 0\,. \tag{3.65}$$

From the solution I without self-interaction, we know that Ψ carries only radial dependence,

$$\Psi(r) = f(r)\,\Psi^o\,, \qquad \Psi^o := \begin{pmatrix}\zeta\\ \eta\end{pmatrix}. \tag{3.66}$$

Looking at the constant sector of the effective metric,

$$J_\mu^{(o)} - I_\mu^{(o)} = \overline{\Psi}^o \gamma_\mu (1-\gamma_5) \Psi^o , \tag{3.67}$$

we have that

$$J_o^{(o)} - I_o^{(o)} = (\zeta^\dagger - \eta^\dagger)(\zeta - \eta), \tag{3.68}$$

$$J_k^{(o)} - I_k^{(o)} = -(\zeta^\dagger - \eta^\dagger)\sigma_k (\zeta - \eta) . \tag{3.69}$$

Here we need to introduce the criterion that gives the compatibility of $h_{\mu\nu}$ with the spherical symmetry. For it is sufficient to assume ζ and η as eigenstates of the Pauli matrix σ_1,

$$\sigma_1 \zeta = \varepsilon \zeta , \quad \sigma_1 \eta = \varepsilon \eta \qquad (\varepsilon^2 = 1) . \tag{3.70}$$

Then

$$(\zeta^\dagger - \eta^\dagger)\sigma_2 (\zeta - \eta) = 0 , \tag{3.71}$$

$$(\zeta^\dagger - \eta^\dagger)\sigma_3 (\zeta - \eta) = 0 . \tag{3.72}$$

By stating that ζ and η are eigenstates of σ_1, we reduce its components to the bond relations

$$\zeta_2 = \varepsilon \zeta_1 , \qquad \eta_2 = \varepsilon \eta_1 , \tag{3.73}$$

for ζ and η written as

$$\zeta \equiv \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} , \qquad \eta \equiv \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} . \tag{3.74}$$

Hence, the expressions (3.68) and (3.69) reduces to

$$J_o^{(o)} - I_o^{(o)} = 2 (\zeta_1^\dagger - \eta_1^\dagger)(\zeta_1 - \eta_1) , \tag{3.75}$$

$$J_1^{(o)} - I_1^{(o)} = -2\varepsilon (\zeta_1^\dagger - \eta_1^\dagger)(\zeta_1 - \eta_1) . \tag{3.76}$$

In particular, we shall note that the scalar A and the pseudo-scalar B do not depend on ε:

$$A := \overline{\Psi}\Psi = 2 f^* f (\zeta_1^\dagger \zeta_1 - \eta_1^\dagger \eta_1) \equiv 2 f^* f M, \tag{3.77}$$

$$B := i\overline{\Psi}\gamma_5 \Psi = 2i f^* f (\zeta_1^\dagger \eta_1 - \eta_1^\dagger \zeta_1) \equiv 2i f^* f N. \tag{3.78}$$

Collecting the terms, the Heisenberg equation (3.65) returns

$$-i\varepsilon\left(f' + \frac{f}{2r}\right) \begin{pmatrix} \eta \\ -\zeta \end{pmatrix} - 2s f f^* \left[M \begin{pmatrix} \zeta \\ \eta \end{pmatrix} - N \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \right] = 0 . \tag{3.79}$$

Setting $\varepsilon = +1$, and

$$\alpha = \zeta_1 = m + i n , \qquad \beta = \eta_1 = p + i q ,$$

the system yields

$$-i\left(f' + \frac{f}{2r}\right)\beta - 4s\, f f^* f\left(M\,\alpha - N\,\beta\right) = 0\,, \tag{3.80}$$

$$i\left(f' + \frac{f}{2r}\right)\alpha - 4s\, f f^* f\left(M\,\beta - N\,\alpha\right) = 0\,, \tag{3.81}$$

which is of Bernoulli's type,

$$f' + \frac{f}{2r} - \lambda f^3 = 0\,. \tag{3.82}$$

Multiplying by $-2/f^3$ and isolating f on the left side,

$$-\frac{2}{f^3}\frac{df}{dr} - \frac{1}{r f^2} = -2\lambda\,.$$

Defining $u(r) \equiv 1/f^2$, the equation above becomes

$$\frac{du}{dr} - \frac{u}{r} = -2\lambda\,.$$

Taking the multiplicative factor $\mu(r) = e^{\int -\frac{1}{r}\,dr} = 1/r$, and applying the reverse Leibniz rule for products,

$$\frac{d}{dr}\left(\frac{u}{r}\right) = -\frac{2\lambda}{r}\,. \tag{3.83}$$

By the anti-derivative, we have

$$\frac{u(r)}{r} = a_o - 2\lambda\log(r)\,,$$

that is,

$$f(r) = \pm\frac{1}{\sqrt{r(a_o - 2\lambda\log r)}}\,. \tag{3.84}$$

Therefore, the Heisenberg spinor is a solution for

$$\Psi(r) = \frac{1}{\sqrt{r(a_o - 2\lambda\log r)}}\,\Psi^o\,. \tag{3.85}$$

Recalling the definition (3.67), the contribution to $h_{\mu\nu}$ is

$$h_{\mu\nu} = g_w^{1/2}\,\frac{\left(J_\mu^{(o)} - I_\mu^{(o)}\right)\left(J_\nu^{(o)} - I_\nu^{(o)}\right)}{\left[\eta^{\mu\nu}J_\mu^{(o)}J_\nu^{(o)}\right]^{1/2}}\,f^2(r). \tag{3.86}$$

From the equations (3.75) – (3.76) and defining $F(r) \equiv 1/f^2(r)$, it follows that the effective geometry has the form

$$ds^2 = \left(1 - \frac{\xi}{F(r)}\right)dt^2 - \left(1 + \frac{\xi}{F(r)}\right)dr^2 + \frac{2\xi}{F(r)}\,dt\,dr - r^2\,d\theta^2 - r^2\sin^2\theta\, d\phi^2\,, \tag{3.87}$$

where

$$\xi \;\equiv\; \frac{2\,\kappa\, g_w^{1/2}\,(\alpha\;-\;\beta^*)\,(\alpha\;-\;\beta\,)}{\left[\,(2\alpha\beta\,)^2\;-\;(\alpha^2\;+\;\beta^*\,\beta\,)^2\,\right]^{1/2}} \tag{3.88}$$

Under a change of coordinates, say

$$dt = d\tilde{T} - \frac{\xi/F(r)}{1-1/F(r)}\,dr\,, \tag{3.89}$$

we obtain the effective line element,

$$ds^2_{eff} \;=\; \left(1-\frac{\xi}{F}\right)d\tilde{T}^2 \;-\; \left(1-\frac{\xi}{F}\right)^{-1} dr^2 \;-\; r^2\,d\Omega\,. \tag{3.90}$$

In the limit $\lambda \longrightarrow 0$, we must recover the Schwarzschild geometry of solution I, which imposes $a_o \equiv 1/r_H$. We also note that the general solution of (3.82) with $\lambda \;=\; a + ib$ admits the expansion

$$F(r\,,\,\lambda) \;\equiv\; \frac{1}{f^2} \;=\; r\,\left(\frac{1}{r_H}\;-\;2\lambda\log r\right) \;=\; r\,\left[\left(\frac{1}{r_H}\right)^2 \;+\;4\,(a^2\;-\;b^2)\,\log^2 r\;-\;4\frac{a}{r_H}\log r\right]^{1/2}.$$

In particular, for $\lambda \in \mathbb{R}$,

$$F \;\;=\; r\,\left(\frac{1}{r_H}\;\;-\;2a\;\log r\right)\,. \tag{3.91}$$

Let us set $\xi = 1$. The horizon occurs for values at $r \equiv R_H$. Considering the real case, that is $\lambda = a$, we have

$$\log R_H \;=\; \frac{1}{2a}\left(\frac{1}{r_H}\;-\;\frac{1}{R_H}\right)\,. \tag{3.92}$$

The solution is given in terms of the Lambert function $W(z)$,

$$R_H \;=\; -\frac{1}{2a}\,\frac{1}{W(z)}\,, \tag{3.93}$$

with

$$z \;=\; -\frac{1}{2a}\,e^{-1/2ar_H}\,. \tag{3.94}$$

The Lambert function can be written as the infinite series

$$W(z) \;=\; \sum_{n=1}^{\infty}\frac{(-n)^{\,n-1}}{n!}\,z^n\,. \tag{3.95}$$

At first order, $W(z) \approx z$, and (3.93) becomes

$$R_H \approx e^{-1/2ar_H} \;\approx\; 1\;+\;\frac{1}{2a\,r_H}\,. \tag{3.96}$$

That is,

$$R_H - r_H \;=\; 1\;+\;\frac{1}{2a\,r_H}\;-\;r_H \;=\; \frac{1}{2ar_H}\left(1\;+\;2ar_H\;-2ar_H^2\right)\,. \tag{3.97}$$

For

$$0 < r_H < \frac{1}{2} + \sqrt{1\;+\;\frac{2}{a}}\,, \qquad R_H \;>\; r_H\,, \tag{3.98}$$

$$r_H > \frac{1}{2} + \sqrt{1\;+\;\frac{2}{a}}\,, \qquad R_H \;<\; r_H\,. \tag{3.99}$$

Weak field regime. Let us consider the motion of a body $x^\alpha(s)$ in the metric (3.90). Its corresponding velocity $(\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi})$ is provided by

$$\frac{d}{ds}(g_{\kappa\lambda}\,\dot{x}^\kappa) - \frac{1}{2}\,g_{\mu\nu,\lambda}\,\dot{x}^\mu\,\dot{x}^\nu = 0. \tag{3.100}$$

The equations for $x^2 = \theta$ and $x^3 = \varphi$ implies that the angle θ is a constant of motion, say $\theta = \pi/2$. The remaining variables t and φ satisfies, respectively, the following Euler-Lagrange equations [4]

$$\dot{\varphi} = \frac{h}{r^2} \qquad \dot{t}\left(1 - \frac{1}{S}\right) = k. \tag{3.101}$$

for arbitrary constants h and k. Let us define the variable $u = 1/r$, and instead of searching for the evolution du/ds it is convenient to look for the equation of u as a function of the angle coordinate φ. Using the auxiliary condition

$$v^\mu\,v^\nu\,g_{\mu\nu} = 1\,, \tag{3.102}$$

we obtain

$$(u')^2 + (1 - \frac{1}{S})\left(u^2 + \frac{1}{h^2}\right) = \frac{l^2}{h^2}. \tag{3.103}$$

Moreover, the line element of STG for a non-circular orbit is

$$u'' + (1 - \frac{1}{S})\,u + \frac{1}{2h^2}\,\frac{1}{S^2}\,\frac{dS}{du}\,(1 + h^2\,u^2) = 0. \tag{3.104}$$

If such geometry is observable, one should expect to reproduce the Newtonian potential in the weak field regime of STG, when $\lambda \to 0$, so that it holds the correspondence

$$\frac{1}{2M^*} = \frac{1}{2M} - 2\,\lambda\,log(r/r_0)\,, \tag{3.105}$$

where M^* is the Schwarzschild mass. In other words, the value of the Schwarzschild mass that interpret the solution of Spinor Theory of Gravity in terms of General Relativity depends on the distance to the body that generates the field. In that sense, one may assume that

$$M^* \approx \frac{M}{1 - 4M\lambda(r/r_0 - 1)}\,, \tag{3.106}$$

and the effective gravitational potential has the form [369]

$$\Phi_{\rm STG}(r) = -\frac{g_N\,M}{r\left[1 - 4g_N\,M\lambda\,\ln\left(\frac{r}{r_0}\right)\right]}. \tag{3.107}$$

The key property in (3.107) is the logaritmic term depending on r. If these assumptions hold true, it might be of interest to examine how the galaxy rotation curves can eventually be obtained through the virial theorem, say

$$V^2_{\rm STG} = r\frac{d\Phi_{\rm STG}}{dr}. \tag{3.108}$$

A preliminary discussion in comparison with the Navarro, Frenk and White (NFW) profile is started in [369].

Friedman universe with stiff matter. Let us consider the background Minkowski space in flat coordinates,

$$ds^2 \;=\; dt^2 - dx^2 - dy^2 - dz^2. \tag{3.109}$$

The auxiliary field H must satisfy:

- $\Box H(x) = 0$;
- $\partial_\mu \widetilde{\gamma}_\nu = [H_\mu \,, \widetilde{\gamma}_\nu]$, where $4\, H_\mu = \sigma\, \widetilde{\gamma}_\mu\, \widetilde{\gamma}^\lambda \partial_\lambda H$.

Recalling the identity (3.7), we see that $H(t) = mt$ is a solution of the system. In particular,

$$0 = \partial_\mu \widetilde{\gamma}_\nu = [H_\mu \,, \widetilde{\gamma}_\nu] = \frac{1}{2}(\partial_\mu H\, \widetilde{\gamma}_\nu - \eta_{\mu\nu}\, \widetilde{\gamma}^\kappa \partial_\kappa H) \tag{3.110}$$

is identically fulfilled by

$$\partial_o(mt)\, \widetilde{\gamma}_o = \eta_{oo}\, \widetilde{\gamma}^o\, \partial_o(mt). \tag{3.111}$$

In consequence, the dynamics for Ψ is

$$(\widetilde{\gamma}^\mu\, \partial_\mu - \lambda \widetilde{\gamma}^o)\, \Psi = 0\,, \qquad \lambda \equiv \sigma\, m. \tag{3.112}$$

Let us consider the solution given by

$$\Psi(t) \;=\; e^{\lambda t}\, \Psi^{\,o}\,, \qquad \Psi^{\,o} = \begin{pmatrix} \phi \\ \eta \end{pmatrix}. \tag{3.113}$$

The arbitrariness on Ψ^0 allow us to set $(\phi^\dagger\, \sigma_j - \eta^\dagger\, \sigma_j)\; (\eta - \phi)$ to have the same value for $j = (1, 2, 3)$, say

$$l_o = l\,, \qquad l_1 = l_2 = l_3 = l/\sqrt{3} = b e^{\lambda t}\,. \tag{3.114}$$

Hence, the effective line element has the form

$$\begin{aligned} ds^2_{\text{eff}} = (1 - l^2)\; dt^2 - (1 + \frac{l^2}{3})\, (dx^2 + dy^2 + dz^2) - \frac{2}{\sqrt{3}}\, l^2\, dt\, (dx + dy + dz) \\ - \frac{2}{3}\, l^2\, (dx\, dy + dx\, dz + dy\, dz). \end{aligned} \tag{3.115}$$

By a coordinate transformation from (t, x, y, z) to (T, u, v, q), such that we set

$$dt = SdT + du + dv + dq\,, \qquad dx = dy = dz = F dT\,, \tag{3.116}$$

we obtain

$$ds^2_{\text{eff}'} = (1 - l^2)\left[-\frac{S^2}{l^4} dT^2 + du^2 + dv^2 + dq^2 + 2(dudv + dudq + dvdq)\right] \tag{3.117}$$

A second change of coordinate system is required, now from (T, u, v, q) to (T, X, Y, Z):

$$dX = \beta(p_1 du + p_2 dv + p_3 dq)\,, \tag{3.118}$$

$$dY = \beta(p_2 du + p_3 dv + p_1 dq)\,, \tag{3.119}$$

$$dZ = \beta(p_3 du + p_1 dv + p_2 dq)\,, \tag{3.120}$$

where the constant coefficients p_j are constrained by

$$\sum_{j=1}^{3} p_j = \frac{\sqrt{3}}{\beta}\,, \qquad \sum_{j=1}^{3} p_j^2 = \frac{1}{\beta^2}\,. \tag{3.121}$$

Then (3.117) reads

$$ds^2_{\text{eff}''} = (1 - l^2)\left[-\frac{S^2}{l^4} dT^2 + dX^2 + dY^2 + dZ^2\right]. \tag{3.122}$$

Setting $S^2 = l^4$, and

$$(e^{2\lambda T} - 1)\, dT^2 =: d\tau^2\,, \tag{3.123}$$

we obtain a Friedman-type universe in Gaussian global time $\tau := (a(T) - \arctan a(T))$,

$$ds^2_{\text{eff}'''} = d\tau^2 - a^2(\tau)(dX^2 + dY^2 + dZ^2)\,. \tag{3.124}$$

Effective gravitational waves. Following Einstein's path [141], we seek for a solution of the STG that is Ricci-flat [368],

$$R^{\text{eff}}_{\mu\nu}(x) = 0. \tag{3.125}$$

It is interesting to note that the Fock-Ivanenko connection (3.5) satisfies identically Einstein's vacuum equations [529]. So in order to deal with a definition of effective gravitational waves induced by Ψ, we can start with the simplest possible situation, that is when the background is flat (3.109) and the only remaining contribution coming from the spin connection is given by H_μ, cf. (3.7). Moreover, the standard definition in GTR requires that the perturbations of the background are transverse and traceless in the harmonic coordinate system [178, 534],

$$\Box_g\, h^{TT}_{\mu\nu}(x) = 0\,, \qquad \Gamma^{\sigma}{}_{\mu\nu} g^{\mu\nu} = 0. \tag{3.126}$$

In the STG, the effective metric is naturally transverse and traceless,

$$\nabla_\mu\, h^{\mu\nu}(\overline{\Psi}, \Psi) = 0\,, \qquad g^{\mu\nu}_{\text{eff}}\, h_{\mu\nu}(\overline{\Psi}, \Psi) = 0\,, \tag{3.127}$$

for weak currents of the form

$$l_\mu = \exp H\, m_\mu\,, \qquad \eta^{\mu\nu}\, m_\mu m_\nu = 0, \tag{3.128}$$

with H_μ orthogonal to l^μ, say

$$H(x) = \exp\left(m_\mu x^\mu\right) . \tag{3.129}$$

Note that $\nabla_\mu l_\nu = \partial_\mu l_\nu = H_\mu l_\nu$. Hence, the Levi-Civita connection in the effective spacetime fulfills the harmonic gauge (3.126), where

$$\Gamma^\sigma{}_{\mu\nu} = -\frac{\kappa}{2} g^{\sigma\rho}\left(\partial_\mu(l_\nu l_\rho) + \partial_\nu(l_\mu l_\rho) - \partial_\rho(l_\mu l_\nu)\right) . \tag{3.130}$$

The main outcome of the construction above is the parallel that STG exhibit, under certain restrictions, with Kundt's definition of gravitational waves [295, 566], according to which the criteria for the existence of gravitational waves is an isotropic vector field, κ^μ say, that satisfies

$$\nabla_{[\mu}\kappa_{\nu]} = 0 , \qquad \nabla_{(\mu}\kappa_{\nu)} \nabla^\mu \kappa^\nu = 0 \qquad \nabla_\mu \kappa^\mu = 0. \tag{3.131}$$

Therefore, STG fulfills Kundt's criteria for $\kappa_\mu = H_\mu$.

3.5 Further perspectives

The current structure in STG. Let us consider the Gordon decomposition of Heisenberg equation without matter (3.25). Comparing with Dirac's theory, a non-linear spinor field implies a replacement of the Compton's wave length by

$$\lambda_e = \frac{h}{m_e c} \quad \longrightarrow \quad \lambda_g \equiv \frac{h/c}{s\,|J|} , \quad J \equiv J_\mu J^\mu , \tag{3.132}$$

where s is the coupling parameter of Heisenberg's potential, with dimension of length squared (in natural units). In the framework of STG, there is room to consider particle production as induced by gravitational processes. In the present formulation, the symmetries of a (not yet defined) vacuum state for STG is hardly achieved without the scalar field H introduced in the Sec.3.1. A physical interpretation of the role played by H within STG is still lacking.

The vacuum state in STG. One of the crucial questions related to any relativistic field theory is how to characterize its fundamental state. The vacuum state of GTR is identified with the reduction of the energy-momentum tensor to a scalar, Λ say,

$$T_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{3.133}$$

The role of a positive cosmological constant in the vacuum state of the relativistic theory of the electron was examined by Dirac in [126]. In the case of STG, the field equations (3.38) reduce to

$$i\nabla\!\!\!/\, \Psi = -\kappa\, \Lambda Q \Psi , \tag{3.134}$$

where Q is given by

$$Q = g_{\mu\nu} Q^{\mu\nu} = \left(\frac{g_w}{J}\right)^{1/4} (l \cdot \gamma)\, (\mathbb{I} - \gamma_5). \tag{3.135}$$

By contradiction, requiring chiral states would imply that, for a fixed g_w, $Q \to 0$ only if $l \cdot \gamma \to 0$; but then $J = A^2 + B^2 \to 0$, and hence $Q \to \infty$. This situation seems to suggest that the Fermi coupling should have the character of a running parameter, rather than a fixed constant [413]. These properties remains open for further developments.

The generalized spin connection. Some previous versions[12] of the STG suggest the introduction of the weak currents as an extension of the Fock-Ivanenko connection, say (Eq. (15) in [363])

$$\Gamma_\mu = -i(aJ_\mu + bI_\mu)(\mathbb{I} + \gamma^5). \tag{3.136}$$

On the one hand, it allows one to deduce Heisenberg's equation (3.25) in a very natural way. It also gives a curious interpretation to the Fierz identities (3.14-3.17): while the spin connection carries a weak $V - A$ interaction, the Action principle (3.24) results to be bosonic, or Heisenberg-type. From that perspective, the spin connection and the dynamics of the spinor field differ by a contraction with the Clifford basis. Notwithstanding, this approach is not internally consistent, and requires a modification of the relying Clifford algebra, as we shall discuss in the next chapter.

[12] Cf. Novello [362, 363], Novello and Maria Borba [53, 364], Formiga [180], and Fernandes [166]; see also the works on spin connection by Novello and Bittencourt [43–45].

4. In search of Clifford algebra automorphims

The growth of the use of transformation theory, as applied first to relativity and later to quantum theory, is the essence of the new method in theoretical physics. Further progress lies in the direction of making our equations invariant under wider and wider transformations.
Dirac in [128], Preface to the First edition.

Therefore it would be most beautiful, if one were to succeed in expanding the group once more, analogous to the step which led from special relativity to general relativity.
Einstein in [145], p.91.

Élie Cartan seems to have been the first to realize the key role of affine connections in GTR. In a series of four communications during 1922, Cartan [73] discussed the arbitrariness of affinities allowed by the metricity condition

$$\nabla g = 0. \tag{4.1}$$

The fact that (4.1) does not determine univocally the set of affine connections on an oriented, 4-dimensional manifold (M, g) is perhaps the most relevant property that underlies Einstein's attempts at a unified field theory [208, 209], with a few exceptions being the Einstein-Schrödinger [471, 501] and the Einstein-Bargmann [149] theories.

Similarly, Valentine Bargmann [24] and Schrödinger [469] were among the first to recognize, independently, the possibility of extending Dirac's theory to a more general law of transformations preserving unitarity[1]. For Bargmann and Schrödinger posed the problem of how to obtain a generalized Dirac adjoint that brings the relativistic quantum description into GTR. Both addressed this problem by exploring the arbitrariness entailed by the covariant derivative of the Dirac basis, which is the defining relation of the Fock coefficients,

$$\partial_\mu \gamma_\nu - \Gamma^\sigma{}_{\mu\nu}\gamma_\sigma = [\Gamma_\mu, \gamma_\nu], \qquad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{4.2}$$

Many authors consider this as an open path for generalized spin connections, were the Fock coefficients span the independent subspaces of the associated Clifford algebra. It is not automatic, though, that these approaches are internally consistent with the fiber bundle generated by the Dirac basis. Roughly speaking, modified spin connections may not preserve, for instance, the ordinary Dirac basis as a Lorentz 4-vector even if we restrict the analysis to a Minkowski spacetime[2].

The present chapter is an attempt at formulating this problem and it could be addressed in terms of Clifford algebra automorphisms induced by the spin connection.

[1]According to Enz [154], Pauli asked to Bargmann to revise the Einstein-Mayer theory.
[2]This remark was pointed to me by Prof. S. Cacciatori on December 23 2022.

4.1 Situating the problem

Let (M, g) be a Lorentz manifold, with signature $+ - --$, and $\Gamma^{\sigma}{}_{\rho\mu} \in \Gamma(M, LM)$ its Levi-Civita connection. The general situation under discussion relies upon the relation between the following elements:

- The ideal of a Clifford algebra Cl associated to the 4-dimensional spacetime (M, g),

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{I}$$

- The parallel transport of the Lorentz basis,

$$\nabla_\mu v_\nu{}^a \equiv \vartheta_{\mu\nu}{}^a. \tag{II}$$

- The metric compatibility condition of the physical spacetime (M, g),

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^{\sigma}{}_{\rho\mu} g_{\sigma\nu} - \Gamma^{\sigma}{}_{\rho\nu} g_{\mu\sigma} = 0. \tag{III}$$

- The equivalence condition of $(M = \mathbb{R}^{1,3}, \eta)$,

$$\nabla_\rho \eta_{ab} = 0. \tag{IV}$$

The Clifford algebra associated to $(M = \mathbb{R}^{1,3}, \eta)$ will be denoted by $Cl_{1,3}$, cf. [436, p. 5.1].

The different choices between assumptions (I-IV) is what determines the formulations of general relativity and its variations. To give some familiarity with the present notation, let us collect some of these approaches:

∗ Case 1. Fock coefficients: the covariant derivative is constructed as acting directly on the Clifford mapping γ_μ; the expression for (II),

$$\vartheta_{\mu\nu} \equiv \nabla_\mu \gamma_\nu = \partial_\mu \gamma_\nu - \Gamma^{\sigma}{}_{\mu\nu}\gamma_\sigma - \Gamma_\mu \gamma_\nu + \gamma_\nu \Gamma_\mu = 0, \quad \Gamma_\mu \in \Omega^1(Cl(E)\,, Lie(G) \subset \mathrm{End}(Cl)) \tag{4.3}$$

with $Cl(E) := \mathcal{P}_{\mathrm{Fock}} \times_\rho Cl$, $G \subset \mathrm{Aut}(Cl)$, and $\rho : GL(4, \mathbb{C}) \to \mathrm{Aut}(Cl)$, follows from combining (I) and (III). In particular, if Γ_μ is identified to the spin connection ω of Case 2 below, then $\mathcal{P}_{\mathrm{Fock}} = O_+$ and $\rho : SO(1,3) \to \mathrm{Aut}(Cl)$. We retake this point below.

∗ Case 2. Vierbein formulation: this is the canonical formulation of fermions in general relativity, cf. Wald [526] (see also the interesting report by Krasnov and Percacci [291], and [290]).

$$\vartheta_{\mu\nu}{}^a := \partial_\mu e_\nu{}^a - \Gamma^{\sigma}{}_{\mu\nu} e_\sigma{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0, \quad \omega \in \Omega^1(\mathcal{P}, Lie(SO(1,3))) \tag{4.4}$$

- Case 3. Einstein-Cartan theory: the antisymmetric contribution to the Levi-Civita connection is introduced,

$$\widetilde{\vartheta}^a := \widetilde{\nabla}_\mu e_\nu{}^a dx^\mu \wedge dx^\nu = \mathrm{d}e^a + \omega^a{}_b \wedge e^b - T^a = 0, \quad T^a \in \Omega^2(M, \mathrm{T}M) \tag{4.5}$$

with $2T^a := \Gamma^\sigma{}_{[\mu\nu]} e_\sigma{}^a dx^\mu \wedge dx^\nu$ and $e_\mu{}^a \in \Gamma(M, \mathrm{T}^* M \otimes LM)$. We refer to Tecchiolli [493] for an account on the structure of ECT; also, [436, p. 2.1].

- Case 4. Spin connection as a Yang-Mills field: lifting the metricity constraint (III) on the spin connection allows one to approach it as a non-Abelian[3] gauge field $A_\mu{}^a{}_b$ [8, 131],

$$\vartheta_{\mu\nu}{}^a e_\nu{}^a := \partial_\mu e_\nu{}^a - \Gamma^\sigma{}_{\mu\nu} e_\sigma{}^a + A_\mu{}^a{}_b e_\nu{}^b. \tag{4.6}$$

- Case 5. Clifford algebra automorphisms: the relativistic invariance properties are ascribed to the action of a subgroup of the group of automorphisms of the Clifford algebra [224],

$$\boldsymbol{\vartheta}_{\mu\nu}{}^a := \bigoplus_A \left[\partial_\mu \boldsymbol{e}_\nu{}^a - \Gamma^\sigma{}_{\mu\nu} \boldsymbol{e}_\sigma{}^a + \boldsymbol{\omega}^{(A)}{}_\mu{}^a{}_b \boldsymbol{e}_\nu{}^b \right], \quad \boldsymbol{\omega}^{(A)} \in \Omega^1(\boldsymbol{P}, Lie(G) \subset \mathrm{Aut}(Cl)). \tag{4.7}$$

Consistency between $\boldsymbol{\omega}^{(A)}{}_\mu{}^a{}_b$ and $\boldsymbol{e}_\sigma{}^a$ implies that the 'vierbeins' in (4.7) also must be gradings of the Dirac basis. This type of Lorentz basis seems to be new in the literature.

The problem entailed by Fock connection. The formulation of STG in Chapter 3 is based on the covariant derivative of the Dirac basis itself, without introducing frame fields. Historically, it corresponds to the first attempt at a description of Dirac's theory in general relativity, introduced by Fock[4] in 1929 [174, 175], and reviewed, for instance, by Schrödinger [469], Bargmann [24], Wataghin [529], Klein [270, 276, 277], and Gulmanelli [213]. To date, a fiber bundle approach to the Fock connection seems to be lacking in the literature[5]. This case contains the elements for the formulation of our main problem under discussion in this Chapter.

Following Gulmanelli [213], let us start by noting that Eq. (I) is invariant under a nonsingular matrix S transformation,

$$\gamma'_\mu = S^{-1} \gamma_\mu S. \tag{4.8}$$

Under infinitesimal transformations

$$S = 1 + \varepsilon\Lambda, \quad \gamma'_\mu = \gamma_\mu + \varepsilon\eta_\mu, \tag{4.9}$$

[3] For an account of the Yang-Mills connection, see [436, p. 6.2].

[4] Also known in the literature as the Fock-Ivanenko connection [179], or Fock-Ivanenko coefficients. This formulation also appears in the subsequent works by Rumer [438, 439], Brill and Wheeler [62], Green [210, 211], Fletcher [172], Kimura [266], Nakamura and Toyoda [351], Rodichev [434], Peres [407, 408], Loos [323], Pagels [384], Ogievetskiĭ and Toyoda [374], Anderson [15], Loos and Treat [324, 507], Novello [360–363, 367, 371], Fairchild Jr. [161], Chilsholm and Farwell [85, 86], Weldon [541], and Crawford [93, 94], to mention a few. For a historical overview of the earlier works until Brill and Wheeler [62], see [261, 451].

[5] So far, only a few comments are made by Kay [261], as indicated below.

the covariant derivative of the Dirac basis

$$\vartheta_{\mu\nu} \equiv \nabla_\mu \gamma_\nu \tag{4.10}$$

satisfying Eqs (I) and (III) gives

$$\begin{aligned}(\nabla_\rho \gamma'_\mu)\gamma'_\nu + \gamma'_\mu(\nabla_\rho \gamma'_\nu) + (\mu \leftrightarrow \nu) \longrightarrow\ & (\nabla_\rho \gamma_\mu)\gamma_\nu + \gamma_\mu(\nabla_\rho \gamma_\nu) + (\mu \leftrightarrow \nu) \\ & + \varepsilon[(\nabla_\rho \gamma_\mu)\eta_\nu + (\nabla_\rho \eta_\mu)\gamma_\nu + \gamma_\mu(\nabla_\rho \eta_\nu) + \eta_\mu(\nabla_\rho \gamma_\nu) + (\mu \leftrightarrow \nu)] \\ & + O(\varepsilon^2).\end{aligned} \tag{4.11}$$

$$\begin{aligned}(\partial_\rho \gamma'_\mu)\gamma'_\nu + \gamma'_\mu(\partial_\rho \gamma'_\nu) + (\mu \leftrightarrow \nu) \longrightarrow\ & (\partial_\rho \gamma_\mu)\gamma_\nu + \gamma_\mu(\partial_\rho \gamma_\nu) + (\mu \leftrightarrow \nu) \\ & + \varepsilon[(\partial_\rho \gamma_\mu)\eta_\nu + (\partial_\rho \eta_\mu)\gamma_\nu + \gamma_\mu(\partial_\rho \eta_\nu) + \eta_\mu(\partial_\rho \gamma_\nu) + (\mu \leftrightarrow \nu)] + O(\varepsilon^2).\end{aligned} \tag{4.12}$$

$$\Gamma^\sigma{}_{\rho\mu}[\gamma'_\sigma \gamma'_\nu + (\nu \leftrightarrow \sigma)] \longrightarrow \Gamma^\sigma{}_{\rho\mu}[\gamma_\sigma \gamma_\nu + \varepsilon(\gamma_\sigma \eta_\nu + \eta_\sigma \gamma_\nu) + (\sigma \leftrightarrow \nu)] + O(\varepsilon^2). \tag{4.13}$$

$$\Gamma^\sigma{}_{\rho\nu}[\gamma'_\mu \gamma'_\sigma + (\mu \leftrightarrow \sigma)] \longrightarrow \Gamma^\sigma{}_{\rho\nu}[\gamma_\mu \gamma_\sigma + \varepsilon(\gamma_\mu \eta_\sigma + \eta_\mu \gamma_\sigma) + (\mu \leftrightarrow \sigma)] + O(\varepsilon^2). \tag{4.14}$$

From (4.8) and (4.9), it holds $\eta_\mu = \gamma_\mu \Lambda - \Lambda \gamma_\mu$, and the first order contribution in ε of expressions (4.11) and (4.12) generate 16 terms each, 4 canceling pairs of (covariant and partial, resp.) derivatives of Λ, and 4 pairs of (covariant and partial, resp.) derivatives of the Dirac matrices, which gives

$$\varepsilon \sim [(\nabla_\rho \gamma_\mu)\gamma_\nu + \gamma_\mu(\nabla_\rho \gamma_\nu) + (\mu \leftrightarrow \nu)]\Lambda - \Lambda[(\nabla_\rho \gamma_\mu)\gamma_\nu + \gamma_\mu(\nabla_\rho \gamma_\nu) + (\mu \leftrightarrow \nu)]. \tag{4.15}$$

$$\varepsilon \sim [(\partial_\rho \gamma_\mu)\gamma_\nu + \gamma_\mu(\partial_\rho \gamma_\nu) + (\mu \leftrightarrow \nu)]\Lambda - \Lambda[(\partial_\rho \gamma_\mu)\gamma_\nu + \gamma_\mu(\partial_\rho \gamma_\nu) + (\mu \leftrightarrow \nu)]. \tag{4.16}$$

The terms containing the Levi-Civita connection generate 8 terms each, giving zero contribution as expected,

$$\varepsilon \sim \Gamma^\sigma{}_{\rho\mu}(\{\gamma_\sigma, \gamma_\nu\}\Lambda - \Lambda\{\gamma_\sigma, \gamma_\nu\}) = 2\Gamma^\sigma{}_{\rho\mu}(g_{\sigma\nu}\Lambda - \Lambda g_{\sigma\nu}) = 0. \tag{4.17}$$

It results that Λ is totally independent from $\Gamma^\sigma{}_{\rho\mu}$,

$$\begin{aligned}&[\nabla_\rho \gamma_\mu \gamma_\nu + \gamma_\mu \nabla_\rho \gamma_\nu + (\mu \leftrightarrow \nu)]\Lambda - \Lambda[\nabla_\rho \gamma_\mu \gamma_\nu + \gamma_\mu \nabla_\rho \gamma_\nu + (\mu \leftrightarrow \nu)] \\ &= [\partial_\rho \gamma_\mu \gamma_\nu + \gamma_\mu \partial_\rho \gamma_\nu + (\mu \leftrightarrow \nu)]\Lambda - \Lambda[\partial_\rho \gamma_\mu \gamma_\nu + \gamma_\mu \partial_\rho \gamma_\nu + (\mu \leftrightarrow \nu)],\end{aligned} \tag{4.18}$$

and there are 4 Λ's, denoted henceforth by Γ_μ, that satisfy (4.18), namely (Fock, 1929)

$$\partial_\rho \gamma_\mu - \Gamma^\sigma{}_{\rho\mu}\gamma_\sigma = \Gamma_\rho \gamma_\mu - \gamma_\mu \Gamma_\rho. \tag{Fock-1}$$

Combining (4.18) and (Fock-1), it follows that

$$(\Gamma_\rho \gamma_\mu - \gamma_\mu \Gamma_\rho)\gamma_\nu + \gamma_\mu(\Gamma_\rho \gamma_\nu - \gamma_\nu \Gamma_\rho) + (\mu \leftrightarrow \nu) = \Gamma_\rho\{\gamma_\mu, \gamma_\nu\} - \{\gamma_\mu, \gamma_\nu\}\Gamma_\rho \overset{\text{Eq.(I)}}{=} 0. \tag{Fock-2}$$

Moreover, (Fock-1) with (4.8) implies that Γ_μ transforms as (also, Eqs. (19-20) in [24])

$$\partial_\mu \gamma_\nu - \Gamma^{\sigma}{}_{\mu\nu}\gamma_\sigma = -S(\partial_\mu S^{-1})\gamma_\nu - \gamma_\nu(\partial_\mu S)S^{-1} + S\Gamma_\mu S^{-1}\gamma_\nu - \gamma_\nu S\Gamma_\mu S^{-1} = \Gamma_\rho \gamma_\mu - \gamma_\mu \Gamma_\rho, \tag{4.19}$$

that is,

$$\Gamma'_\mu = S\Gamma_\mu S^{-1} - S^{-1}\partial_\mu S, \qquad SS^{-1} = 1. \tag{Fock-3}$$

At this point, some remarks are in order. First, Eq. (Fock-3) follows from infinitesimal Lorentz transformations (4.8) of the γ's. One shall recall that non-Abelian gauge fields are identified as a connection of the spin bundle due to the Lorentz invariance of the Dirac operator,

$$\Psi \longrightarrow \tilde{\Psi} = S\Psi\,, \tag{4.20}$$

$$\nabla_\mu \Psi \longrightarrow \tilde{\nabla}_\mu \tilde{\Psi} = S(\nabla_\mu \Psi)\,. \tag{4.21}$$

If the field strength is non-Abelian, $S^{-1}F_{\mu\nu}S \neq F_{\mu\nu}$. In the case of Fock connection, the internal curvature[6] associated to (Fock-1), given by

$$[\nabla_\mu, \nabla_\nu] = \mathcal{R}_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu - [\Gamma_\mu, \Gamma_\nu], \tag{4.22}$$

acquires a non-Abelian character from the anticommutative properties of the Clifford algebra. If so, then Clifford bundles are responsible for introducing non-Abelian[7] gauge fields in Theoretical Physics.

Secondly, Eq. (Fock-1) is consistent with, albeit *not fixed by*, the metric compatibility condition (III). Three consequences follow from these two aspects:

A. If Γ_μ in (Fock-1) is seen as a connection on the Clifford bundle, with fibers generated by the Dirac basis[8] associated at each spacetime point[9], then the arbitrariness of Γ_μ in (Fock-2) might be interpreted as the possibility of introducing new degrees of freedom, allowed by the group of automorphisms $\mathrm{Aut}(E)$ of the Clifford bundle.

B. Such arbitrariness seems *prima facie* to allow an undefined extension of the Fock coefficients in terms of the Clifford basis itself, which *is* consistent with (Fock-2), and therefore with (I) and (III),

[6] Cf. Wataghin [529], Eqs. (I) and (Fock-1) implies Einstein's vacuum equations.

[7] It is interesting to note that, according to Prof. Straumann [486], Pauli tried to extend the Kaluza-Klein to a 6-dimensional structure $M \times S^2$, where the metric contains the spacetime metric g, the metric on the 2-sphere, and three Killing fields A^a_μ (although not recognized by the author as such). Also, Prof. Straumann explains that Pauli "determines the transformation behavior of A^a_μ under the group $[(x, y) \longrightarrow x, R(x) \cdot y]$ and finds in matrix notation what he [Pauli] calls the 'generalization of the gauge group'" (Eq. (5) in [486]):

$$A_\mu \longrightarrow R^{-1}A_\mu R + R^{-1}\partial_\mu R.$$

I would like to remark that, as it is shown above, the Fock connection (which is present in Gulmanelli's seminar notes [213], based on his correspondence with Pauli) transforms as a Yang-Mills field solely from infinitesimal Lorentz transformations of the γ's.

[8] This nomenclature here stands for gamma matrices satisfying the ideal (I), without any relation to Dirac's theory.

[9] This fiber bundle interpretation of (Fock-1) is also suggested by Kay [261].

but *not* with (Fock-1). For it is sufficient and necessary to take the trace of both sides of (Fock-1) once a generalization of this type is introduced[10], say

$$\widetilde{\Gamma}_\mu = \bigoplus_A \widetilde{\Gamma}^{(A)}_\mu, \tag{4.23}$$

for A independent components of Cl. It follows that

$$\mathrm{Tr}[\partial_\rho \gamma_\mu - \Gamma^\sigma{}_{\rho\mu}\gamma_\sigma] \neq \bigoplus_A \mathrm{Tr}[\widetilde{\Gamma}^{(A)}_\rho \gamma_\mu - \gamma_\mu \widetilde{\Gamma}^{(A)}_\rho], \tag{4.24}$$

unless an extension of the Dirac basis is carried on. While (Fock-2) allows an even higher arbitrariness for the Fock coefficients, (Fock-1) together with (I) reduce the possibilities of (4.23) to some particular cases. We will continue this discussion in the following Section (4.2).

C. If a new synthesis from (A.) and (B.) above is indeed consistent, then a spanning of $\widetilde{\Gamma}_\mu$, as in (4.23), would mean that the double cover of the spin structure (which is required for a consistent definition of the Dirac operator in curved spacetime) appears here as the twisting of the Clifford bundle. If so, the question then is how to make twisted Clifford bundles compatible with the double coverings of the spin structure.

The third point is, despite (Fock-1) being consistent with (I) and (III), an explicit expression for Γ_μ requires the introduction of (local or global) frame fields.

4.2 Extended Clifford bases: a preliminary discussion

General overview. Let us, provisionally, call "extended Clifford bases" the quantities $\{\boldsymbol{\gamma}_\mu\}_{\mu=o}^3$ as stated by

$$\boldsymbol{\gamma}_\mu(x) := \bigoplus_{A=0}^{4} \boldsymbol{e}^{(A)}{}_\mu = \bigoplus_{A=0}^{4} \boldsymbol{e}^{(A)}{}_{\hat\mu}\xi^{(A)}{}_\mu{}^{\hat\mu}, \tag{4.25}$$

where the elements of $\left\{\xi^{(A)}{}_\mu{}^{\hat\mu}\right\}_{A=0}^4$ are, in general, arbitrary non-holonomic multi-vielbeins[11] in $GL(\mathbb{R},4)$, $\left\{\boldsymbol{e}^{(A)}{}_{\hat\mu}\right\}_{A=0}^4$ is a set of linear frames, constructed with the local frames

$$\left\{\boldsymbol{e}^{(o)}{}_{\hat\mu} = e_{\hat\mu}\,\mathbb{I} \;:\; e_{\hat\mu}\, e_{\hat\nu} = \eta_{\hat\mu\hat\nu}\,\mathbb{I}\right\}_p \in (\hat V_4, \hat\eta)\,, \tag{4.26}$$

and the 16 independent components of a Clifford algebra $Cl(\hat V_4, \hat\eta)$ associated to a 4-dimensional Minkowski space $(\hat V_4, \hat\eta)$, denoted by[12]

$$\left\{\boldsymbol{e}^{(A)}{}_{\hat\mu}\right\}_{A=0}^4 = \left\{e_{\hat\mu}\mathbb{I}\,,\, \gamma_{\hat\mu}\,, ie_{\hat\mu}\gamma_{\hat 5}\,, i\gamma_{\hat\mu}\gamma_{\hat 5}\,, \gamma_{\hat\mu}\gamma_{\hat\kappa}\right\}. \tag{4.27}$$

[10]The first to suggest this route was Green [210, 211]. This is the main motivation for the subsequent works mentioned in the note [4] above.

[11]Introduced by Einstein in a series of three letters in 1929, cf. [512].

[12]We shall denote by Greek letters the indices running from 0 to 3. Letters with hat are denoting Lorentz indices, while Greek letters without hat refers to the (external) spacetime. Moreover, we assume that all metrics are equipped with Lorentzian structure of signature $+---$.

The $\gamma_{\hat{5}}$ matrix is defined by

$$\gamma_{\hat{5}} := \frac{i}{4!\sqrt{-\hat{\eta}}}\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\gamma_{\hat{\mu}}\gamma_{\hat{\nu}}\gamma_{\hat{\rho}}\gamma_{\hat{\sigma}} = i\gamma_{\hat{0}}\gamma_{\hat{1}}\gamma_{\hat{2}}\gamma_{\hat{3}}\,, \tag{4.28}$$

and $\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$ is the Levi-Civita tensor in $(\hat{V}_4, \hat{\eta})$. The question to be discussed is under which conditions, if any, the objects $\{\boldsymbol{\gamma}_\mu(x)\}_{\mu=o}^4$ can be bounded by an inner product of the form

$$\{\boldsymbol{\gamma}_\mu, \boldsymbol{\gamma}_\nu\} = \bigoplus_{A,B=0}^{4}\left\{\boldsymbol{e}^{(A)}{}_\mu,\, \boldsymbol{e}^{(B)}{}_\nu\right\} \stackrel{?}{=} 2\bigoplus_{A,B=0}^{4} g\left(\boldsymbol{e}^{(A)}{}_\mu,\, \boldsymbol{e}^{(B)}{}_\nu\right). \tag{4.29}$$

Roughly speaking, is it possible to generate an ideal of the tensor algebra from the direct sum of $k(\leq 5)$ quadratic spaces $(V^{(A)}, g^{(A)})_{A=0}^4$, where the basis of every vector space $V^{(A)}$ corresponds, respectively, to the spanning set (4.27) of independent elements of a Clifford algebra $\hat{C}l := \hat{C}l(\hat{V}_4, \hat{\eta})$ associated to a 4-dimensional Minkowski space?

To elucidate the problem, let us consider our provisional definition (4.25) in a slightly simplified notation, say

$$\begin{aligned}
\boldsymbol{e}^{(o)}{}_\mu &= e^{(o)}{}_\mu\,\mathbb{I}\,,\\
\boldsymbol{e}^{(1)}{}_\mu &= e^{(1)}{}_\mu{}^{\hat{\mu}}\,\gamma_{\hat{\mu}}\,,\\
\boldsymbol{e}^{(2)}{}_\mu &= e^{(2)}{}_\mu\, i\gamma_{\hat{5}}\,,\\
\boldsymbol{e}^{(3)}{}_\mu &= e^{(3)}{}_\mu{}^{\hat{\mu}}\, i\gamma_{\hat{\mu}}\gamma_{\hat{5}}\,,\\
\boldsymbol{e}^{(4)}{}_\mu &= e^{(4)}{}_\mu{}^{\hat{\mu}\hat{\nu}}\,\gamma_{\hat{\mu}}\gamma_{\hat{\nu}}.
\end{aligned}$$

After some arrangements, one gets that

$$\begin{aligned}
\frac{1}{2}\{\boldsymbol{\gamma}_\mu, \boldsymbol{\gamma}_\nu\} = \big[&e^{(o)}{}_\mu\, e^{(o)}{}_\nu + e^{(1)}{}_\mu{}^{\hat{\mu}}\, e^{(1)}{}_\nu{}^{\hat{\nu}}\eta_{\hat{\mu}\hat{\nu}} - e^{(2)}{}_\mu\, e^{(2)}{}_\nu + e^{(3)}{}_\mu{}^{\hat{\mu}}\, e^{(3)}{}_\nu{}^{\hat{\nu}}\eta_{\hat{\mu}\hat{\nu}}\\
&- 2e^{(4)}{}_\mu{}^{\hat{\mu}\hat{\nu}}\, e^{(4)}{}_\nu{}^{\hat{\kappa}\hat{\lambda}}\eta_{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}}\,\big]\cdot\mathbb{I}\\
&\oplus \big[e^{(o)}{}_\mu e^{(1)}{}_\nu{}^{\hat{\lambda}} + e^{(1)}{}_\mu{}^{\hat{\lambda}}\, e^{(o)}{}_\nu + (e^{(3)}{}_\mu{}^{\hat{\kappa}} e^{(4)}{}_\nu{}^{\hat{\mu}\hat{\nu}} + e^{(4)}{}_\mu{}^{\hat{\mu}\hat{\nu}} e^{(3)}{}_\nu{}^{\hat{\kappa}})\varepsilon_{\hat{\mu}\hat{\nu}\hat{\kappa}}{}^{\hat{\lambda}}\big]\,\gamma_{\hat{\lambda}}\\
&\oplus \big[e^{(o)}{}_\mu\, e^{(2)}{}_\nu + e^{(2)}{}_\mu\, e^{(o)}{}_\nu + 2e^{(4)}{}_\mu{}^{\hat{\mu}\hat{\nu}}\, e^{(4)}{}_\nu{}^{\hat{\kappa}\hat{\lambda}}\varepsilon_{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}}\big]\, i\gamma_{\hat{5}}\\
&\oplus \big[e^{(o)}{}_\mu e^{(3)}{}_\nu{}^{\hat{\lambda}} + e^{(3)}{}_\mu{}^{\hat{\lambda}}\, e^{(o)}{}_\nu - 2(e^{(1)}{}_\mu{}^{\hat{\kappa}} e^{(4)}{}_\nu{}^{\hat{\mu}\hat{\nu}} + e^{(4)}{}_\mu{}^{\hat{\mu}\hat{\nu}} e^{(1)}{}_\nu{}^{\hat{\kappa}})\varepsilon_{\hat{\mu}\hat{\nu}\hat{\kappa}}{}^{\hat{\lambda}}\big]\, i\gamma_{\hat{\lambda}}\gamma_{\hat{5}}\\
&\oplus \big[e^{(o)}{}_\mu\, e^{(4)}{}_\nu{}^{\hat{\kappa}\hat{\lambda}} + e^{(4)}{}_\mu{}^{\hat{\kappa}\hat{\lambda}}\, e^{(o)}{}_\nu - \frac{1}{2}(e^{(1)}{}_\mu{}^{\hat{\mu}}\, e^{(3)}{}_\nu{}^{\hat{\nu}} - e^{(3)}{}_\mu{}^{\hat{\mu}}\, e^{(1)}{}_\nu{}^{\hat{\nu}}\\
&\quad + e^{(2)}{}_\mu\, e^{(4)}{}_b{}^{\hat{\mu}\hat{\nu}} + e^{(4)}{}_\mu{}^{\hat{\mu}\hat{\nu}}\, e^{(2)}{}_\nu)\varepsilon_{\hat{\mu}\hat{\nu}}{}^{\hat{\kappa}\hat{\lambda}}\big]\, 2i\sigma_{\hat{\kappa}\hat{\lambda}}\,.
\end{aligned} \tag{4.30}$$

The expression above contains the Levi-Civita tensor $\varepsilon_{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}}$ in $(\hat{V}_4, \hat{\eta})$, and the antisymmetric tensor

$$\eta_{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}} \equiv \eta_{\hat{\mu}\hat{\kappa}}\eta_{\hat{\nu}\hat{\lambda}} - \eta_{\hat{\mu}\hat{\lambda}}\eta_{\hat{\nu}\hat{\kappa}}. \tag{4.31}$$

The first interesting outcome of (4.30) is the fact that, apart from $\left\{\boldsymbol{e}^{(4)}{}_{\mu}\right\}_{\mu=0}^{3}$, the diagonal elements of (4.25) return the sum of the internal product associated to every basis $\left\{\boldsymbol{e}^{(j)}{}_{\hat{\mu}}\right\}_{j\in A=0}^{3}$ multiplied by the identity. In particular, one may infer that the reduction of (4.25) to

$$\{\boldsymbol{\gamma}_{\mu}, \boldsymbol{\gamma}_{\nu}\} = \bigoplus_{A=0}^{3} \left\{\boldsymbol{e}^{(A)}{}_{\mu}, \boldsymbol{e}^{(A)}{}_{\nu}\right\} = 2 \bigoplus_{A=0}^{3} g\left(\boldsymbol{e}^{(A)}{}_{\mu}, \boldsymbol{e}^{(A)}{}_{\nu}\right)_{p\in\hat{V}_4} \tag{4.32}$$

generates an ideal bounded by a bilinear form multiplied by the identity. In addition, the expression (4.30) seems to have, at first glance, a sort of coupling between the vector space associated to the identity $\mathbb{I}$ and the, say, axial space defined by $\gamma_{\hat{5}}$, due to the $\left\{\boldsymbol{e}^{(4)}{}_{\mu}\right\}$ objects. Another pattern occurs between the $\gamma_{\hat{\mu}}-$ and the $i\gamma_{\hat{\mu}}\gamma_{\hat{5}}-$spaces. This suggests us to identify three distinct sectors in (4.29), namely

- the scalar-pseudoscalar sector, spanned by $\left\{\boldsymbol{e}^{(0)}{}_{\mu}\right\}$ and $\left\{e^{(2)}{}_{\mu}\right\}$;
- the vector-axial sector, spanned by $\left\{\boldsymbol{e}^{(1)}{}_{\mu}\right\}$ and $\left\{e^{(3)}{}_{\mu}\right\}$;
- and the bivector sector, generated by $\left\{\boldsymbol{e}^{(4)}{}_{\mu}\right\}$.

Our main focus will be on the scalar-pseudoscalar and vector-axial couplings. From the beginning, we shall note two particular situations: when

- $A = B = 0$, which corresponds to an orthonormal frame bundle $E^{(o)} := O_{+}(M^{(o)})$ of M with structure group $SO(1,3)$;
- $A = B = 1$, which gives a Clifford bundle $E^{(1)} := Cl(E^{(o)})$ of $E^{(o)}$.

Let us briefly address these two subcases.

Subcase $A = B = 0$**.** Setting back our previous notation, let $V^{(o)} = M$ be a 4-dimensional manifold with a set of 4-frames $\{e^{(o)}{}_{\hat{\mu}}\}_p$ on M. Let us denote by $\{\xi^{(o)}{}_{\mu}{}^{\hat{\mu}}\} \in GL(4,\mathbb{R})$ the set of *vierbeins* locally defined at a point $p \in M$, with right action on $L(M)$,

$$R_{\xi^{(o)}} : L(M) \times GL(4,\mathbb{R}), \qquad R_{\xi} \circ e^{(o)}{}_{\hat{\mu}} = e^{(o)}{}_{\hat{\mu}}\, \xi^{(o)}{}_{\mu}{}^{\hat{\mu}} . \tag{4.33}$$

The orbit space of R_{ξ} is M, and the canonical projection $\pi : L(M) \longrightarrow M$ assigns to each $(e^{(o)}{}_{\hat{\mu}})_p$ at $p \in M$. If $(U, \phi)_p$ is a local chart of M at p, then every basis vector $e^{(o)}{}_{\mu} \in \pi^{-1}(U)$ can be represented by

$$e^{(o)}{}_{\mu} = e^{(o)}{}_{\hat{\mu}}\, \xi^{(o)}{}_{\mu}{}^{\hat{\mu}}, \qquad \xi^{(o)}{}_{\mu}{}^{\hat{\mu}} = \left(\frac{\partial \overline{x}^{\hat{\mu}}}{\partial x^{\mu}}\right)_p . \tag{4.34}$$

The mapping

$$\chi : \pi^{-1}(U) \longrightarrow U \times GL(4, \mathbb{R}), \qquad \chi(e^{(o)}{}_{\mu}) := (\pi(e^{(o)}{}_{\hat{\mu}}), \xi^{(o)}) \tag{4.35}$$

is a projection such that $\mathrm{pr}_U \circ \chi = \pi$. If we equipp $L(M)$ with a differential structure, by assuming that (4.35) are diffeomorphisms, the 5-tuple $(L(M), GL(4, \mathbb{R}), M, R_{\xi^{(o)}}, \pi)$ is a principal fibre bundle $\mathcal{P}^{(o)}$ with local trivializations (4.35). Let $E^{(o)} \to M$ be a $\mathbb{R}$-vector bundle of rank 4, with set of bases $L_p^{(o)}$ in the fibre $E_p^{(o)}$. Then, the frame bundle

$$L(E^{(o)}) := \bigcup_{p \in M} L_p^{(o)} \tag{4.36}$$

carries the structure of $\mathcal{P}^{(o)}$ over M with G-structure of $GL(4, \mathbb{R})$. Finally, let $E^{(o)}$ be endowed with a fibre metric $g^{(o)}$, given by

$$g^{(o)}{}_{\mu\nu} := g^{(o)}(e^{(o)}{}_{\mu}, e^{(o)}{}_{\nu})_p = \eta_{\hat{\mu}\hat{\nu}} \xi^{(o)}{}_{\mu}{}^{\hat{\mu}} \xi^{(o)}{}_{\mu}{}^{\hat{\mu}}, \qquad \eta^{(o)}{}_{\hat{\mu}\hat{\nu}} = e^{(o)}{}_{\hat{\mu}} e^{(o)}{}_{\hat{\nu}}. \tag{4.37}$$

Once we have $(E^{(o)}, g^{(o)})$, the set of bases $L_p^{(o)}$ can be made $g^{(o)}$-orthonormal at the fibre $E_p^{(o)}$,

$$O(E^{(o)}) := \bigcup_{p \in M} O_p^{(o)}. \tag{4.38}$$

Requesting that M is oriented, and identifying $E^{(o)}$ with the tangent bundle $(\mathrm{T}M, M, \pi)$ of M, one may restrict the orthonormal frame bundle to the subset $O_+(M) \subset O(M)$ of ordered orthonormal frames with group structure $SO(1, 3) \subset O(1, 3)$.

Subcase $A = B = 1$. In general lines,

$$(V^{(1)}, g^{(1)}) = (M, g), \qquad (\hat{V}^{(1)}, \hat{\eta}^{(1)}) \cong Cl(\hat{V}_4, \hat{\eta}), \tag{4.39}$$

such that

$$\{\boldsymbol{e}^{(1)}{}_{\mu}, \boldsymbol{e}^{(1)}{}_{\nu}\} = 2g(\boldsymbol{e}^{(1)}{}_{\mu}, \boldsymbol{e}^{(1)}{}_{\nu})_{\tilde{p}} =: 2g^{(1)}{}_{\mu\nu}, \qquad \tilde{p} \in \hat{V}^{(1)}. \tag{4.40}$$

In particular, if

$$\boldsymbol{e}^{(1)}{}_{\mu} = \xi_{\mu}{}^{\hat{\mu}} \gamma_{\hat{\mu}}, \qquad \gamma_{\hat{\mu}} \in Cl(\hat{V}_4, \hat{\eta}), \tag{4.41}$$

with holonomic vierbeins $\{\xi_{\mu}{}^{\hat{\mu}} = \delta_{\mu}{}^{\hat{\mu}}\}_{p \in \hat{V}_4}$, then $V^{(1)}$ is isomorphic to the usual Clifford algebra associated to a 4-dimensional Minkowski space,

$$V^{(1)} \cong Cl^{(1)} = Cl(\hat{V}_4, \hat{\eta}). \tag{4.42}$$

Complexification of $\left\{\boldsymbol{e}^{(1)}{}_{\mu}\right\}$. Let us consider the pre-defining relation (4.25) restricted to the case in which $A = 1, 3$. Explicitly,

$$\boldsymbol{\gamma}_{\mu}(x) := \frac{1}{2}\bigoplus_{A=1,3}\boldsymbol{e}^{(A)}{}_{\mu} = \frac{1}{2}\bigoplus_{A=1,3}\boldsymbol{e}^{(A)}{}_{\hat{\mu}}\xi^{(A)}{}_{\mu}{}^{\hat{\mu}}. \tag{4.43}$$

In particular, for $\left\{\xi^{(A)}{}_{\mu}{}^{\hat{\mu}}\right\}$ holonomic, one has

$$\{\boldsymbol{\gamma}_{\mu}, \boldsymbol{\gamma}_{\nu}\} = \frac{1}{4}\bigoplus_{A,B=1,3}\left\{\boldsymbol{e}^{(A)}{}_{\mu}, \boldsymbol{e}^{(B)}{}_{\nu}\right\} = \frac{1}{4}\bigoplus_{A,B=1,3}\left\{e^{(A)}{}_{\hat{\mu}}, e^{(B)}{}_{\hat{\nu}}\right\}\xi^{(A)}{}_{\mu}{}^{\hat{\mu}}\,\xi^{(B)}{}_{\nu}{}^{\hat{\nu}} \tag{4.44}$$

$$= \left(\xi^{(1)}{}_{\mu}{}^{\hat{\mu}}\xi^{(1)}{}_{\nu}{}^{\hat{\nu}} + \xi^{(3)}{}_{\mu}{}^{\hat{\mu}}\xi^{(3)}{}_{\nu}{}^{\hat{\nu}}\right)\eta_{\hat{\mu}\hat{\nu}} = 2\,g_{\mu\nu}\,. \tag{4.45}$$

We recall that

$$\bigoplus_{A,B=1,3}\left\{e^{(A)}{}_{\hat{\mu}}, e^{(B)}{}_{\hat{\nu}}\right\} = 4\,\eta_{\hat{\mu}\hat{\nu}}\,, \tag{4.46}$$

where

$$e^{(1)}{}_{\hat{\mu}} = \gamma_{\hat{\mu}}\,, \qquad e^{(3)}{}_{\hat{\mu}} = i\gamma_{\hat{\mu}}\gamma_{\hat{5}}\,. \tag{4.47}$$

A line element in (M, g) implies that

$$\begin{aligned} ds^2 = g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} &= \frac{1}{8}\bigoplus_{A,B=1,3}\left\{e^{(A)}{}_{\mu}, e^{(B)}{}_{\nu}\right\}dx^{\mu}\,dx^{\nu} \\ &= \frac{1}{8}\bigoplus_{A,B=1,3}\left\{e^{(A)}{}_{\hat{\mu}}, e^{(B)}{}_{\hat{\nu}}\right\}\xi^{(A)}{}_{\mu}{}^{\hat{\mu}}\,\xi^{(B)}{}_{\mu}{}^{\hat{\mu}}\,dx^{\mu}\,dx^{\nu} \\ &= \frac{1}{2}\bigoplus_{A,B=1,3}\xi^{(A)\hat{\mu}}\xi^{(B)\hat{\nu}}\,\eta_{\hat{\mu}\hat{\nu}}\,, \end{aligned} \tag{4.48}$$

where we have set the dual multi-bases

$$\xi^{(A)\hat{\mu}} \equiv dx^{\mu}\xi^{(A)}{}_{\mu}{}^{\hat{\mu}}. \tag{4.49}$$

Given the fact that our candidates to the frame $\left\{e^{(A)}{}_{\hat{\mu}}\right\}_{A=1,3}$ are not Lorentz covariant 4-vectors, but rather elements of a subspace of $Cl(\hat{V}_4, \hat{\eta})$, one has to deal with the spin bundle of $Cl(M, g)$ associated to the oriented Riemannian manifold (M, g) in order to seek a proper definition for what we (provisionally) are calling "extended Dirac bases". This problem is yet to be confronted. For now, we discuss a more suitable combination of the vector-axial vierbeins. Much of this notation is inspired in the seminal work by Petrov [414], from which the Newman-Penrose formalism was developed, cf. [406, 485, 526]. A motivation of this formalism is presented in the App.C.

We construct the set of complex graded vierbeins $Z := \left\{\boldsymbol{\zeta}_{\mu}{}^{\hat{\mu}}, \overline{\boldsymbol{\zeta}}_{\mu}{}^{\hat{\mu}}\right\}$ in the subspace of $Cl(\hat{V}_4, \hat{\eta})$ spanned by $\left\{\mathbb{I}, \gamma_{\hat{5}}\right\}$, as the direct sum of the vector-axial vierbeins $\left\{\xi^{(1)}{}_{\mu}{}^{\hat{\mu}}, \xi^{(3)}{}_{\mu}{}^{\hat{\mu}}\right\}$, that is,

$$\boldsymbol{\zeta}_{\mu}{}^{\hat{\mu}} := \frac{1}{\sqrt{2}}\left(\xi^{(1)}{}_{\mu}{}^{\hat{\mu}} - i\gamma_{\hat{5}}\xi^{(3)}{}_{\mu}{}^{\hat{\mu}}\right), \qquad \overline{\boldsymbol{\zeta}}_{\mu}{}^{\hat{\mu}} := \frac{1}{\sqrt{2}}\left(\xi^{(1)}{}_{\mu}{}^{\hat{\mu}} + i\gamma_{\hat{5}}\xi^{(3)}{}_{\mu}{}^{\hat{\mu}}\right). \tag{4.50}$$

It follows that (4.43) can be rewritten as

$$\boldsymbol{\gamma}_{\mu} = \gamma_{\hat{\mu}} \boldsymbol{\zeta}_{\mu}{}^{\hat{\mu}} \in Cl_{1,3} \times_{\rho} GL(4, \mathbb{R}), \tag{4.51}$$

such that

$$\left\{\boldsymbol{\gamma}_{\mu}, \boldsymbol{\gamma}_{\nu}\right\} = \left\{\gamma_{\hat{\mu}} \boldsymbol{\zeta}_{\mu}{}^{\hat{\mu}}, \gamma_{\hat{\nu}} \boldsymbol{\zeta}_{\nu}{}^{\hat{\nu}}\right\} = \left\{\gamma_{\hat{\mu}}, \gamma_{\hat{\nu}}\right\} \boldsymbol{\zeta}_{\mu}{}^{\hat{\mu}} \overline{\boldsymbol{\zeta}}_{\nu}{}^{\hat{\nu}} = 2\eta_{\hat{\mu}\hat{\nu}} \boldsymbol{\zeta}_{\mu}{}^{\hat{\mu}} \overline{\boldsymbol{\zeta}}_{\nu}{}^{\hat{\nu}} = 2g_{\mu\nu}. \tag{4.52}$$

Hence the covariant metric and its inverse are given by

$$g_{\mu\nu} = \eta_{\hat{\mu}\hat{\nu}}\, \boldsymbol{\zeta}_{\mu}{}^{\hat{\mu}}\, \overline{\boldsymbol{\zeta}}_{\nu}{}^{\hat{\nu}}, \qquad g_{\mu\nu}\, \boldsymbol{\zeta}_{\mu}{}^{\hat{\mu}}\, \overline{\boldsymbol{\zeta}}_{\nu}{}^{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}, \qquad g^{\mu\nu} = \eta^{\hat{\mu}\hat{\nu}}\, \boldsymbol{\zeta}^{\mu}{}_{\hat{\mu}}\, \overline{\boldsymbol{\zeta}}^{\nu}{}_{\hat{\nu}}, \tag{4.53}$$

where the inverse of Z is defined as

$$\boldsymbol{\zeta}^{\mu}{}_{\hat{\mu}} := \frac{1}{\sqrt{2}}\left(\xi^{(1)\mu}{}_{\hat{\mu}} - i\gamma_{\hat{5}}\xi^{(3)\mu}{}_{\hat{\mu}}\right), \qquad \overline{\boldsymbol{\zeta}}^{\mu}{}_{\hat{\mu}} := \frac{1}{\sqrt{2}}\left(\xi^{(1)\mu}{}_{\hat{\mu}} + i\gamma_{\hat{5}}\xi^{(3)\mu}{}_{\hat{\mu}}\right). \tag{4.54}$$

The consistency of Z with $Cl(M, g)$ can be tested by the following relations:

$$\begin{aligned} \overline{\boldsymbol{\zeta}}^{\nu}{}_{\hat{\lambda}}\, \boldsymbol{\zeta}_{\nu}{}^{\hat{\kappa}} &= \frac{1}{2}\left(\xi^{(1)\nu}{}_{\hat{\lambda}} + i\xi^{(3)\nu}{}_{\hat{\lambda}}\, \gamma_{\hat{5}}\right)\left(\xi^{(1)}{}_{\nu}{}^{\hat{\kappa}} - i\xi^{(3)}{}_{\nu}{}^{\hat{\kappa}}\, \gamma_{\hat{5}}\right) \\ &= \frac{1}{2}\left(\xi^{(1)\nu}{}_{\hat{\lambda}}\xi^{(1)}{}_{\nu}{}^{\hat{\kappa}} + \xi^{(3)\nu}{}_{\hat{\lambda}}\xi^{(3)}{}_{\nu}{}^{\hat{\kappa}}\right) = \frac{1}{2}\left(\delta_{\hat{\lambda}}{}^{\hat{\kappa}} + \delta_{\hat{\lambda}}{}^{\hat{\kappa}}\right) = \delta_{\hat{\lambda}}{}^{\hat{\kappa}}\, \mathbb{I}. \end{aligned} \tag{4.55}$$

$$g^{\mu\kappa}\, g_{\kappa\nu} = \boldsymbol{\zeta}^{\mu}{}_{\hat{\mu}}\, \overline{\boldsymbol{\zeta}}^{\kappa}{}_{\hat{\kappa}}\, \boldsymbol{\zeta}_{\kappa}{}^{\hat{\kappa}}\, \overline{\boldsymbol{\zeta}}_{\nu}{}^{\hat{\nu}}\, \eta_{\hat{\kappa}\hat{\nu}}\, \eta^{\hat{\mu}\hat{\kappa}} = \boldsymbol{\zeta}^{\mu}{}_{\hat{\mu}}\, \delta^{\kappa}{}_{\kappa}\, \overline{\boldsymbol{\zeta}}_{\nu}{}^{\hat{\nu}}\, \delta_{\hat{\nu}}{}^{\hat{\mu}} = 4\boldsymbol{\zeta}^{\mu}{}_{\hat{\mu}}\, \overline{\boldsymbol{\zeta}}_{\nu}{}^{\hat{\mu}} = 4\delta^{\mu}{}_{\nu}\, \mathbb{I}. \tag{4.56}$$

$$g^{\rho\mu}\boldsymbol{\zeta}_{\mu}{}^{\hat{\mu}}\eta_{\hat{\mu}\hat{\kappa}} = \boldsymbol{\zeta}^{\rho}{}_{\hat{\rho}}\, \overline{\boldsymbol{\zeta}}^{\mu}{}_{\hat{\sigma}}\boldsymbol{\zeta}_{\mu}{}^{\hat{\mu}}\, \eta_{\hat{\mu}\hat{\kappa}}\, \eta^{\hat{\rho}\hat{\sigma}} = \boldsymbol{\zeta}^{\rho}{}_{\hat{\rho}}\, \delta_{\hat{\sigma}}{}^{\hat{\mu}}\, \eta_{\hat{\mu}\hat{\kappa}}\, \eta^{\hat{\rho}\hat{\sigma}} = \boldsymbol{\zeta}^{\rho}{}_{\hat{\rho}}\, \eta_{\hat{\sigma}\hat{\kappa}}\, \eta^{\hat{\rho}\hat{\sigma}} = \boldsymbol{\zeta}^{\rho}{}_{\hat{\rho}}\, \delta_{\hat{\kappa}}{}^{\hat{\rho}} = \boldsymbol{\zeta}^{\rho}{}_{\hat{\kappa}}. \tag{4.57}$$

Of special relevance is the subset of parity relations:

$$\boldsymbol{\zeta}_{\mu}{}^{\hat{\kappa}}\, \boldsymbol{\zeta}^{\nu}{}_{\hat{\kappa}} = -i\gamma_{\hat{5}}\, \delta_{\mu}{}^{\nu}, \tag{4.58}$$

$$\overline{\boldsymbol{\zeta}}_{\mu}{}^{\hat{\kappa}}\, \overline{\boldsymbol{\zeta}}^{\nu}{}_{\hat{\kappa}} = +i\gamma_{\hat{5}}\, \delta_{\mu}{}^{\nu}, \tag{4.59}$$

$$\boldsymbol{\zeta}_{\mu}{}^{\hat{\mu}}\, \boldsymbol{\zeta}_{\nu}{}^{\hat{\nu}} = -i\gamma_{\hat{5}}\, \delta_{\mu}{}^{\hat{\mu}}\, \delta_{\nu}{}^{\hat{\nu}}, \tag{4.60}$$

$$\overline{\boldsymbol{\zeta}}_{\mu}{}^{\hat{\mu}}\, \overline{\boldsymbol{\zeta}}_{\nu}{}^{\hat{\nu}} = +i\gamma_{\hat{5}}\, \delta_{\mu}{}^{\hat{\mu}}\, \delta_{\nu}{}^{\hat{\nu}}. \tag{4.61}$$

Moreover, the complex vierbeins allows one to define the candidates to contravariant extended Dirac bases $\left\{\boldsymbol{\gamma}^{\mu}(x)\right\}_{\mu=0}^{3}$, for

$$\boldsymbol{\gamma}^{\mu} := g^{\mu\nu}\boldsymbol{\gamma}_{\nu} = g^{\mu\nu}\, \boldsymbol{\zeta}_{\nu}{}^{\hat{\kappa}}\, \gamma_{\hat{\kappa}} = \boldsymbol{\zeta}^{\mu}{}_{\hat{\mu}}\, \overline{\boldsymbol{\zeta}}^{\nu}{}_{\hat{\nu}}\, \boldsymbol{\zeta}_{\nu}{}^{\hat{\kappa}}\, \gamma_{\hat{\kappa}}\eta^{\hat{\mu}\hat{\nu}} = \boldsymbol{\zeta}^{\mu}{}_{\hat{\mu}}\, \gamma_{\hat{\kappa}}\, \eta^{\hat{\mu}\hat{\kappa}} = \boldsymbol{\zeta}^{\mu}{}_{\hat{\mu}}\, \gamma^{\hat{\mu}}. \tag{4.62}$$

A key element of any Clifford algebra is the matrix of chirality $\boldsymbol{\gamma}_5 \in Cl$. In the holonomic case, it reduces to $\gamma_{\hat{5}} \in \hat{C}l$,

$$\boldsymbol{\gamma}^5 = \gamma^{\hat{5}}. \tag{4.63}$$

By direct inspection,

$$
\begin{aligned}
\boldsymbol{\gamma}^5 &= -\frac{i}{4!}\,\varepsilon_{\mu\nu\kappa\lambda}\,\boldsymbol{\gamma}^\mu\,\boldsymbol{\gamma}^\nu\,\boldsymbol{\gamma}^\kappa\,\boldsymbol{\gamma}^\lambda = \frac{i}{4!}\,\varepsilon_{\mu\nu\kappa\lambda}\,\boldsymbol{\zeta}^\mu{}_{\hat\mu}\,\overline{\boldsymbol{\zeta}}^\nu{}_{\hat\nu}\,\boldsymbol{\zeta}^\kappa{}_{\hat\kappa}\,\overline{\boldsymbol{\zeta}}^\lambda{}_{\hat\lambda}\,\gamma^{\hat\mu}\,\gamma^{\hat\nu}\,\gamma^{\hat\kappa}\,\gamma^{\hat\lambda} \\
&= -\frac{i}{4!}\,\varepsilon_{\mu\nu\kappa\lambda}\left(\frac{1}{\sqrt{2}}\right)^4\left(\xi^{(1)\mu}{}_{\hat\mu}\ominus i\xi^{(3)\mu}{}_{\hat\mu}\,\gamma_{\hat 5}\right)\left(\xi^{(1)\nu}{}_{\hat\nu}\oplus i\xi^{(3)\nu}{}_{\hat\nu}\,\gamma_{\hat 5}\right)\left(\xi^{(1)\kappa}{}_{\hat\kappa}\ominus i\xi^{(3)\kappa}{}_{\hat\kappa}\,\gamma_{\hat 5}\right)\left(\xi^{(1)\lambda}{}_{\hat\lambda}\oplus i\xi^{(3)\lambda}{}_{\hat\lambda}\,\gamma_{\hat 5}\right) \\
&\qquad\cdot\,\gamma^{\hat\mu}\,\gamma^{\hat\nu}\,\gamma^{\hat\kappa}\,\gamma^{\hat\lambda} \\
&= -\frac{i}{4!}\,\varepsilon_{\mu\nu\kappa\lambda}\,\frac{1}{4}\left(\xi^{(1)\mu}{}_{\hat\mu}\xi^{(1)\nu}{}_{\hat\nu}+\xi^{(3)\mu}{}_{\hat\mu}\xi^{(3)\nu}{}_{\hat\nu}\right)\left(\xi^{(1)\kappa}{}_{\hat\kappa}\xi^{(1)\lambda}{}_{\hat\lambda}+\xi^{(3)\kappa}{}_{\hat\kappa}\xi^{(3)\lambda}{}_{\hat\lambda}\right)\gamma^{\hat\mu}\,\gamma^{\hat\nu}\,\gamma^{\hat\kappa}\,\gamma^{\hat\lambda} \\
&= -\frac{i}{4!}\,\varepsilon_{\mu\nu\kappa\lambda}\delta^\mu{}_{\hat\mu}\,\delta^\nu{}_{\hat\nu}\,\delta^\kappa{}_{\hat\kappa}\,\delta^\lambda{}_{\hat\lambda}\,\gamma^{\hat\mu}\,\gamma^{\hat\nu}\,\gamma^{\hat\kappa}\,\gamma^{\hat\lambda} \\
&= -\frac{i}{4!}\,\varepsilon_{\hat\mu\hat\nu\hat\kappa\hat\lambda}\,\gamma^{\hat\mu}\,\gamma^{\hat\nu}\,\gamma^{\hat\kappa}\,\gamma^{\hat\lambda} = \gamma^{\hat 5}.
\end{aligned}
$$

Thus, $\boldsymbol{\gamma}^5$ preserves the following properties:

$$
\{\boldsymbol{\gamma}^5\,,\boldsymbol{\gamma}^\mu\} = \boldsymbol{\zeta}^\mu{}_{\hat\mu}\,\{\gamma^{\hat 5}\,,\gamma^{\hat\mu}\} = 0\,, \qquad (\boldsymbol{\gamma}^5)^2 = \mathbb{I}\,, \qquad [\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^\nu\,,\boldsymbol{\gamma}_5\,] = [\gamma^{\hat\mu}\gamma^{\hat\nu}\,,\gamma_{\hat 5}\,] = 0. \tag{4.64}
$$

With that, the projection operators $P_\pm \in Cl$ may act indistinctly in $\hat{Cl}$ as well,

$$
P_\pm := \frac{1}{2}\,(1 \pm \boldsymbol{\gamma}^5) = \hat P_\pm. \tag{4.65}
$$

with $P_+{}^2 = P_+$, $P_-{}^2 = P_-$, and $P_+\,P_- = 0$.

Extended Dirac adjoint. An enlarged Dirac adjoint is one of the main motivations for developing this formalism. The adjoint $\overline{\Psi} \equiv \Psi^*\gamma^o$ of a Dirac field is the conjugate that preserves Lorentz covariance. For it requires a Dirac operator and its respective spin structure, whose complexification also is usually required [350, 436]. For that reason, we proceed with a qualitative discussion of the Dirac field while these structures are still to be developed.

In the present formulation, the grading of the Clifford algebra is expected to induce a consistent grading of both, the connection $\boldsymbol{\omega}^A$ and the bases set $\left\{\boldsymbol{\zeta}_\mu{}^{\hat\mu}\,,\overline{\boldsymbol{\zeta}}_\mu{}^{\hat\mu}\right\}$, while the base manifold (M, g) remains a 4-dimensional Lorentzian structure, as in general relativity.

For let us start with the elements (I-IV) of the previous section. The graded spin connections are expected to satisfy local Lorentz invariance (IV), hence

$$
\boldsymbol{\nabla}_\rho\eta_{ab} = 0 \Longrightarrow \bigoplus_{A=1,3}\left[\boldsymbol{\omega}^{(A)}{}_{\rho ab} + \boldsymbol{\omega}^{(A)}{}_{\rho ba}\right] = 0, \quad \boldsymbol{\omega}^{(A)} \in \Omega^1(\boldsymbol{P}, Lie(G) \subset \mathrm{Aut}(Cl)). \tag{4.66}
$$

From the metric compatibility (III), it must hold

$$
\boldsymbol{\nabla}_\rho g_{\mu\nu} = 0 \Longrightarrow \eta_{ab}(\overline{\boldsymbol{\vartheta}}_{\rho\mu}{}^a\,\boldsymbol{\zeta}_\nu{}^b + \overline{\boldsymbol{\zeta}}_\mu{}^a\boldsymbol{\vartheta}_{\rho\nu}{}^b) = 0. \tag{4.67}
$$

For now, we set the particular choice for (II) given by

$$
\overline{\boldsymbol{\vartheta}}_{\rho\mu}{}^a \equiv \boldsymbol{\nabla}_\rho\overline{\boldsymbol{\zeta}}_\mu{}^a = \bigoplus_A\left[\partial_\mu\overline{\boldsymbol{\zeta}}_\nu{}^a - \Gamma^\sigma{}_{\mu\nu}\overline{\boldsymbol{\zeta}}_\sigma{}^a + \boldsymbol{\omega}^{(A)}{}_\mu{}^a{}_b\overline{\boldsymbol{\zeta}}_\nu{}^b\right] = 0\,, \tag{4.68}
$$

$$\boldsymbol{\vartheta}_{\rho\mu}{}^{a} \equiv \boldsymbol{\nabla}_{\rho}\boldsymbol{\zeta}_{\mu}{}^{a} = \bigoplus_{A}\left[\partial_{\mu}\boldsymbol{\zeta}_{\mu}{}^{a} - \Gamma^{\sigma}{}_{\mu\nu}\boldsymbol{\zeta}_{\sigma}{}^{a} + \boldsymbol{\omega}^{(A)}{}_{\mu}{}^{a}{}_{b}\boldsymbol{\zeta}_{\nu}{}^{b}\right] = 0. \tag{4.69}$$

The ideal of the Clifford algebra (I) is fulfilled by

$$\{\boldsymbol{\gamma}_{\mu}, \boldsymbol{\gamma}_{\nu}\} = 2g_{\mu\nu}, \quad g_{\mu\nu} = \eta_{ab}\overline{\boldsymbol{\zeta}}_{\mu}{}^{a}\boldsymbol{\zeta}_{\nu}{}^{b}. \tag{4.70}$$

Let us consider the Dirac fields, $\boldsymbol{\Psi} \in \boldsymbol{\Gamma}(M, \boldsymbol{B})$ and $\widetilde{\boldsymbol{\Psi}} \in \boldsymbol{\Gamma}(M, \widetilde{\boldsymbol{B}})$, as sections of the vector bundles, $\boldsymbol{B} = \boldsymbol{P}\times_{\rho} V$ and $\widetilde{\boldsymbol{B}} = \boldsymbol{P}\times_{\rho}\widetilde{V}$, associated to the principal G-bundle $(\boldsymbol{P}, G, M, , \pi)$. The relativistic invariance is expected to hold for

$$\boldsymbol{\Psi}'(x') = \boldsymbol{S}(\omega)\boldsymbol{\Psi} \quad \widetilde{\boldsymbol{\Psi}}'(x') = \widetilde{\boldsymbol{S}(\omega)\boldsymbol{\Psi}} = \widetilde{\boldsymbol{\Psi}}\boldsymbol{S}^{-1} \tag{4.71}$$

$$\boldsymbol{S}^{-1}(\omega)\boldsymbol{\gamma}^{\mu}\boldsymbol{S}(\omega) = \Lambda^{\mu}{}_{\nu}\boldsymbol{\gamma}^{\nu} \tag{4.72}$$

Note that (4.72) implies the relation

$$\boldsymbol{\omega}^{\kappa\lambda}[\boldsymbol{\mu}, \boldsymbol{\Omega}_{\kappa\lambda}] = -2i\omega^{\mu}{}_{\nu}\gamma^{\nu}, \quad \boldsymbol{S}(\omega) = \exp\left(\frac{i}{2}\boldsymbol{\omega}^{\kappa\lambda}\boldsymbol{\Omega}_{\kappa\lambda}\right). \tag{4.73}$$

Since the commutator in the left hand side of (4.73) is proportional to $\boldsymbol{\gamma}$, we set $\boldsymbol{\Omega}_{\kappa\lambda} := \eta[\boldsymbol{\gamma}_{\mu}, \boldsymbol{\gamma}_{\nu}]$. Eq. (4.73) implies $4i\xi = 1$, that is, the generators of the group structure are given by

$$\boldsymbol{\Omega}_{\kappa\lambda} := \frac{1}{4i}[\boldsymbol{\gamma}_{\mu}, \boldsymbol{\gamma}_{\nu}]. \tag{4.74}$$

The dynamics is expected to follow from the (bare) action, written as a purely imaginary scalar,

$$\mathcal{S}[\boldsymbol{\Psi}, \widetilde{\boldsymbol{\Psi}}] = \int_{M}\sqrt{-g}\,\left(\boldsymbol{\Psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\boldsymbol{\Psi} - \widetilde{\boldsymbol{\nabla}_{\mu}\boldsymbol{\Psi}}\boldsymbol{\gamma}^{\mu}\boldsymbol{\Psi}\right)d^{4}x. \tag{4.75}$$

The bitensor sector. Despite being beyond the scope of the present work, it is worth mentioning that the 6-dimensional real vector space X_6 associated to $\left\{\boldsymbol{e}^{(4)}{}_{\mu}\right\}_{\tilde{p}}$ at a point $\tilde{p} \in \hat{V}_4$ seems to be naturally inducing a metric for $V^{(4)} := X_6$, with $\dim V^{(4)} = n(n-1)/2 = 6$, where $n = \dim \hat{V}_4 = 4$.

To illustrate that possibility, let us open $\left\{\boldsymbol{e}^{(4)}{}_{\mu}\right\}_{\tilde{p}}$ in a more objective notation. To avoid any confusion, a translation, for this particular case, of the Greek indices into Latin letters when refering to X_6, say $a, b = 1, ..., 6$, is in order. Let $\boldsymbol{F}$ be a skew-symmetric bitensor at $\tilde{p} \in V^{(4)}$, satisfying the transformation relations

$$F^{\mu\nu} \longmapsto F^{\mu'\nu'} = A^{\mu'}{}_{\mu}A^{\nu'}{}_{\nu}F^{\mu\nu} = 2A^{\mu'}{}_{[\mu}A^{\nu'}{}_{\nu]}F^{\mu\nu} \qquad (\mu, \nu = 0, ..., 3). \tag{4.76}$$

In the lines of Petrov's method [219, 414], one may map each skew-symmetric pair of indices $\mu\nu$ into a collective index a. Then (4.76) reads

$$F^{a} \longmapsto F^{a'} = A^{a'}{}_{a}F^{a} \qquad (a = 1, ..., 6), \tag{4.77}$$

and, at a point $\tilde{p} \in V^{(4)}$, the correspondence[13]

$$A^{a'}{}_a = 2A^{\mu'}{}_{[\mu}A^{\nu'}{}_{\nu]} = A^{[\mu'}{}_{[\mu}A^{\nu']}{}_{\nu]} \tag{4.78}$$

defines a centro-affine space E_6 if the set of bivectors in $V^{(4)}$ induces a Klein geometry[14] in X_6, for which

$$F^{a'} = A^{a'}{}_a F^a, \qquad F^a = A^a{}_{a'} F^{a'}, \qquad \left| A^{a'}{}_a \right| \neq 0, \qquad A^a{}_{b'} A^{b'}{}_c = \delta^a_c. \tag{4.79}$$

Returning to our claim. For $A = 4$ in (4.27), we set

$$\boldsymbol{e}^{(4)}{}_a = \tilde{\xi}_a{}^{\hat{\mu}\hat{\kappa}}\, \boldsymbol{e}_{\hat{\mu}\hat{\kappa}}, \qquad \boldsymbol{e}_{\hat{\mu}\hat{\kappa}} = \gamma_{\hat{\mu}}\, \gamma_{\hat{\kappa}} = 2iS_{\hat{\mu}\hat{\kappa}} \in Cl^o(\hat{V}_4, \eta). \tag{4.80}$$

Recall that the subalgebra generated by $\left\{ \boldsymbol{e}^{(4)}{}_\mu \right\}_{\tilde{p}}$ is the even algebra Cl^0.

$$\{\boldsymbol{e}^{(4)}{}_a, \boldsymbol{e}^{(4)}{}_b\} = 2^4\, \tilde{\xi}_a{}^{\hat{\mu}\hat{\nu}}\, \tilde{\xi}_b{}^{\hat{\kappa}\hat{\lambda}}\, (\eta_{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}} - i\varepsilon_{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}}\, \gamma_{\hat{5}}). \tag{4.81}$$

In particular, if we restrict the affine connections to[15]

$$\tilde{\xi}_a{}^{\hat{\mu}[\hat{\nu}}\, \tilde{\xi}_b{}^{\hat{\kappa}\hat{\lambda}]}\, \varepsilon_{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}} = 0\,. \tag{4.82}$$

If that is the case, the expression 4.83 is meant to be read as a mapping from a Clifford algebra associated to a 4-dimensional Minkowski space into a 6-dimensional bivector space R_6, with metric

$$g^{(4)}{}_{ab} = g(\boldsymbol{e}^{(4)}{}_a, \boldsymbol{e}^{(4)}{}_b)_{\tilde{p}} = \eta_{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}}, \qquad (\mu, \nu = 1, ..., 6)\,. \tag{4.83}$$

4.3 Outline of the ongoing research

Summary. The present chapter findings may be outlined in two categories [224].

A Any modifications are introduced in Eq. (Fock-1); then

(A-1) *Without fermion coupling*, the Fock connection Γ_μ *is* a non-Abelian gauge field naturally induced by the Clifford bundle;

(A-2) *With fermion coupling*,

(A-2a) Γ_μ may be identified with the spin connection ω_μ (as described in Case 2, Eq.(4.4));

(A-2b) Γ_μ may not correspond to the spin connection ω_μ;

Two further possibilities follow from (A-2a):

(A-2a') $\Gamma_\mu \sim \omega_\mu$ is further constrained by the metric compatibility (III);

[13] Note that this map is isomorphic relative to addition, subtraction, and multiplication, but not contracted multiplication.
[14] A homogeneous space X_6 with a transitive action on X_6 by a Lie group.
[15] That would be a natural condition for bivectors satisfying the Bianchi identities of first type, for instance.

(A-2a") $\Gamma_\mu \sim \omega_\mu$ is not further constrained by the metric compatibility (III), and retains its non-Abelian character (here, Donoughe's proposal [8, 131] appears as a particular subcase).

B Modifications are introduced in Eq. (Fock-1); then

(B-1) the Fock connection Γ_μ allows a grading involution the Clifford bundle, whose internal consistency with the Cl-mappings require a grade involution of the γ's as well (as approached in Section 4.2),

$$\{\boldsymbol{\gamma}_\mu, \boldsymbol{\gamma}_\nu\} = 2g_{\mu\nu}1_4\,, \quad \boldsymbol{\gamma}_\mu = \bigoplus_A \boldsymbol{\gamma}^{(A)}{}_\mu. \tag{4.84}$$

(B-2) the Cl-mappings are $N \times N$ matrices satisfying the ideal[16]

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}1_N\,, \tag{4.85}$$

associated to the physical spacetime (M, g). Then the Clifford bundle is set by $Cl(E) = \mathcal{P}_{\text{Fock}} \times_\rho Cl$, which contains all the reducible representations $\rho : GL(4, \mathbb{C}) \to \text{Aut}(Cl)$ of the group of automorphisms of Cl. (This alternative is in line with the works by Prof. Pavšič [401, 402], and Prof. Shirokov [479].)

On the compatibility with Riemannian connections. Recall that, for an oriented Riemannian vector bundle $E \to M$ of rank n, the parity automorphism of Clifford algebras over M induces a vertical bundle automorphism of $Cl(E) := O_+(E) \times_\rho Cl$, so that [436]

$$Cl(E) = Cl^0(E) \oplus Cl^1(E)\,. \tag{4.86}$$

A similar grading induced by (4.7) is expected in a further development of situation (B-1). To date, the main difficulty is the construction of the quaternionic frame bundle as a direct sum of the complexified tetrad bundles, which is required in order to obtain a complex spin structure for (M, g). This may also encourage an investigation of a possible relying structure that unifies the twisting of the subspaces of $Cl(E)$. Since sheaves take values on vector spaces[17], it is not totally unfeasible to look at them as possible candidates for a further improvement of the Clifford bundle grade involutions.

On the physical interpretation of $\text{Aut}(Cl)$**.** The properties of the complex tetrads in the limiting case of holonomic coordinates preserves the Dirac operator and the vector-axial currents, while it modifies the other spinor bilinears of the theory. A discussion of these properties is in progress. In particular, we intend to review the role of the axial current in QED, and its link with weak interactions [6, 352]. The new Dirac adjoint is expected to provide some clue on the class of neutrinos that does not interact with the SGW-model of electroweak interactions, also called sterile neutrinos, cf. [58].

[16]This possibility was suggested by Prof. S. Cacciatori, on October 18 2023.

[17]I am thankful to Josh Wrigley, Prof. Simone Noja, and Prof. Olivia Caramello for conversations on this path, at the DomoSchool 2022.

On second class constraints. One of the key features of introducing the extended spin connection is to be able to describe fermions in curved spacetime without introducing torsion. The Einstein-Cartan theory introduces 24 momentum variables in addition to the 16 variables of GTR in the ADM theory; while the later does not entail second class constraints, the ECT does, cf. [291]. Future analysis of the Clifford algebra automorphisms should consider whether or not only first class constraints are entailed by the theory, and under which conditions second class constraints could be absorbed by $\mathrm{Aut}(Cl)$, in case they also appear.

Fermion doubling problem. The definition of chiral fermions in lattice formulations of QFTs is made consistent by doubling the spectrum of fermionic excitations, whose right sector appears as a mirror of the left one [27]. In loop quantum gravity, the fermion doubling is responsible for canceling the chiral anomaly. Further analysis of the enlarged spin connection via Clifford algebra automorphisms might give a new clue, at the quantum level [317], on the problem of the fermion coupling to gravity, and its discrete description hence. For it may prevent the doubling of fermionic spectrum once the extended Dirac basis carries, in a certain sense, the 'doubling' (grading property of Cl) of the Clifford algebra itself. Qualitatively, one may use this formalism to carry the entire grading of the subspaces spanned by $1, \gamma_5$ until the point where the vacuum expectation value select the left and right sectors, with their respective structure groups; see also [291, 357].

5. Open directions

It appears as if general relativity contained within itself
the seeds of its own conceptual destruction ...
P. G. Bergmann [30], p.514.

Criticism is perhaps the attitude that most characterises the growth of scientific knowledge. So it happened in 1905, when Einstein eliminated the contradiction between Newtonian mechanics and Maxwell's theory with the formulation of the STR; and in 1915, when Einstein once again removed the incompatibility between the STR and Newtonian gravity with a new synthesis, the GTR. Then in 1928, when Dirac connected the STR and Quantum Mechanics; and in 1953-54, when Pauli [486], Shaw [477], and Yang and Mills [564] independently fixed the connection of non-Abelian gauge field theories. Since then throughout the last seven decades, a paramount effort has been made in order to tackle the problem of compatibility between the GTR and QFT.

Although a deeper dig is required in order to understand the reasons why it took nearly three decades to accept the fall of parity (1928-1958), there is no prejudice in to state that every time a strong belief replaces scepticism in science, a long period of irrationalism is followed.

Nowadays, it seems almost unthinkable that an alternative direction is still possible without changing the most unshaken concepts since the advent of the leading theories, that is to say, without disclaiming gravity as a metric (spin 2) field; without exploring higher dimensional spacetimes (Nordström, Kaluza, Klein), or changing the natural topological properties of a Riemannian manifold (in terms of Weyl geometry, de Sitter or Anti-de Siter spacetimes, torsion, or even noncommutative geometry); without giving up of a field-theoretical description of reality (twistors, strings). And yet, it appears that there is still room to seek for a new synthesis in the interior of the current conceptual framework (where gravity is an effective spin 2 field, the spacetime is a standard oriented, 4-dimensional Lorentzian manifold, and all anticommutative properties are naturally entailed by Dirac's theory of fermions).

Unlike the attempts adduced above, the quest for a more general Lorentz invariance, for possible violations of EEP, as well as the search for physics beyond Planck scale, belong, among other problems, to the most ordinary dynamics of how scientific knowledge evolve. The distinction between structural modifications of our best theories in physics and its intrinsic limitations is a key element to formulate our research program. That is the relevance of having a clear and objective distinction between auxiliary, *ad hoc*, and constitutive hypotheses. To our view, what makes the problem of a characteristic length scale intrinsic to any theoretical model in physics is the need to demarcate its limits of testability. Our scope always is to increase the degree of testability of the theory by increasing its degree of universality. Excessive generalization, though, might be as inconsequent as remaining at the phenomenological level.

Clifford algebra automorphisms. Gauge symmetry remains as the foremost direction of research in physics. In the framework of GTR, the Lorentz invariance is assumed to be locally valid at every inertial frame of reference. Our point at issue is that a more realistic description of gravity would imply, not necessarily the breakdown of the Lorentz symmetry of local physics [90, 227], but instead a deviation to inertiality due to a spin connection induced by parity automorphisms that locally *preserves* Lorentz invariance. From the point of view of the logical structure of the theory, it implies an enlargement of internal structure group. Clifford algebra automorsphisms comes in as a source to an extended spin connection in curved spacetime. If so, then the resulting framework is expected to embrace the electroweak scenario and the class of neutrinos [58] that, from the viewpoint of SGW-model, only couple to gravity.

Gravity and weak interactions. What describes gravity according to Einstein is infered by measuring the "gravitational intensity" from the acceleration that a classical body may suffer:

> If (...) the acceleration is to be independent of the nature and the condition of the body and always the same for a given gravitational field, then the ratio of the gravitational to the inertial mass must likewise be the same for all bodies. By a suitable choice of units we can thus make this ratio equal to unity. We then have the following law: The *gravitational* mass of a body is equal to its *inertial* mass.
> A. Einstein [147], p.67.

We choose this excerpt among all precisely because of the expression "ratio equal to unit". We retain this as the most universal aspect of EEP[1].

As aforementioned, the GTR was taken by Einstein just as a provisional step towards a relativistic theory of gravity. According to him, a satisfactory explanation of the inertial properties of matter was still lacking [476]. So far as we know, Einstein never identified the gravitational field with the metric or any other element of the theory [145, 146, 313]. That is, even in the original conception of GTR, geometry *might* be one path, but not the only one, to describe gravitational physics. A different viewpoint leading to the same conclusion is presented by Anderson [17]. Recently, an EFT perspective of the minimal coupling prescription within GTR has been under discussion [251].

Hence, we interpret the GTR as an effective theory, and the equivalence principle as a 'first order aproximation' of the universal coupling of matter with gravity, or yet a selection rule for interactions that preserves the local physics as described by the STR. What prevents us from moving to a 'second order aproximation' of the EEP? *Universality* seems to be the answer. That is one of the reasons to retake Pauli's conjecture into consideration, even though weak interactions are not universal in the same sense as gravity is.

[1] See also the discussion by Treder [510, 511].

Classical spacetime. To Sakharov, it should be possible to describe gravity by measuring the quantum fluctuations of the vacuum as a result of the variation of the curvature of space [440],

$$G_N = \frac{k' c^3 l^2}{\hbar}, \qquad l = \frac{\hbar}{p_o}. \tag{5.1}$$

Zel'dovich extended Sakharov's idea to electrodynamics and weak interactions [572]; see also [576, §2.5]. Independently, Oskar Klein [276, 278–280] proposed a similar interpretation of the role of vacuum in the GTR.

There is a yet unexplained channel of interaction in the neutrino's realm [58] that allows one to propose another "ratio equal to unity": the ratio between the Planck's and Fermi's scales,

$$\frac{l_P}{l_F} = \frac{m_F}{m_P} = \frac{t_P}{t_F} = \frac{\theta_F}{\theta_P} = \frac{\hbar}{c}\left(\frac{G_N}{G_F}\right)^{1/2} =: \sqrt{\xi} \simeq \sqrt{1.738 \cdot 10^{33}}\,. \tag{2.31}$$

Besides, the coupling parameters G_N and G_F (and only them, among the four known interactions) have the dimension of length squared,

$$[\mathring{G}_N] = [\mathring{G}_F] = L^2 \quad (= [\mathring{b}_{BI}]^{-1}). \tag{5.2}$$

The fact that both Einstein's and Fermi's theories are perturbatively non-renormalizable leads us to pursue not only an effective approach, but also a common framework for gravity and weak interactions. Besides, as pointed by Pauli [528], the Planck scale is like the square root of the Fermi scale in natural units, indicating a subcase of the hierarchy problem.

If we interpret the introduction of a second order correction to the equivalence principle as a violation of the universality of free fall, which is just one particular realization of EEP, then we should expect that the fundamental coupling constants vary with time. Both, parity and the equivalence principle, violations are expected to be observed in equal footprint [514, 515].

Spin connection. Once parity automorphisms induces, in our approach, an extension of the spin connection, one might consider its physical interpretation as a generalization of the equivalence principle, rather than its violation. Crawford [93] discussed a somewhat similar idea, but its local automorphisms refers to the *drehbeins* ("spin legs") preserving the spinor metric. In our case, the extended Dirac basis corresponds to graded complex vierbeins carrying the chiral element of the Clifford algebra, while the gravitational spin connection induces local Clifford automorphisms with respect to the (external) metric of spacetime.

Cosmological vacuum state. Connection aside, the Weyl tensor is perhaps the most intriguing element within GTR. It completely characterizes a Ricci flat manifold, carrying all the symmetries of the Riemann curvature tensor and having all of its traces zero. Given that the equations of massless fields are conformally invariant, it is said that the Weyl tensor $C_{\alpha\beta\mu\nu}$, also called the conformal tensor, contains the non-Newtonian effects described by GTR [87]. Equivalently, $C_{\alpha\beta\mu\nu}$ contains the gravitational physics that is not determined locally by matter (in the classical sense of GTR) [225].

These properties are interpreted by some authors as indicating a direct analogy with the electromagnetic field in vacuum, where the Bianchi identities of type II are reduced to

$$\nabla_{[\rho}C_{\alpha\beta]\mu\nu} = 0. \tag{5.3}$$

The double projection of the Weyl tensor onto timelike inertial observers, and of its dual onto null directions, are referred in the literature, respectively, as the "electric" and "magnetic" components of the Weyl tensor [225, 258].

It is also curious to note that the symmetries of $C_{\alpha\beta\mu\nu}$ allows one to define a connection for the vacuum state of GTR, as showed by C. Lanczos [301]. It becomes tempting to describe the deviation of the magnetic gravitational monopoles to inertiality in terms of the Lanczos potential. In [366], we propose the simple expression

$$a_\mu = L_{\mu\rho\sigma}\, v^\rho\, v^\sigma\,, \qquad g_{\mu\nu}v^\mu v^\nu = 1, \tag{5.4}$$

and indicate how it gives a direct interpretation in the Schwarzschild solution.

The difficulty of to confront this formulation with observations remains open, mainly due to the absence of a parameter that characterizes the scale of energy at which the acceleration of the gravitational monopoles is expected to occur. In this sense, we see the expression aforesaid as a first order approximation of a more realistic scenario. Moreover, it would be interesting to examine the relation between the Lanczos tensor and the spin connection induced by parity automorphisms. For it would lead to a revision of the analogy between GTR and Maxwell's electrodynamics, were weak interactions are now responsible for inducing the repulsive properties ascribed to the independent components of $C_{\alpha\beta\mu\nu}$.

A. Gleb Wataghin, 'Sulle forze d'inerzia secondo la teoria quantistica della gravitazione' (1936)

Reproduction of:

Wataghin, G. Sulle forze d'inerzia secondo la teoria quantistica della gravitazione [528]
La Ricerca Scientifica, Serie II, Anno VII, Vol. 2, n. 5-6, p. 341.

Sulle forze d'inerzia secondo la teoria quantistica della gravitazione

L'analogia tra le leggi di Newton e di Coulomb ed il fatto che le caratteristiche e le bicaratteristiche delle equazioni gravitazionali di Einstein coincidono con quelle della propagazione della luce inducono ad applicare alla gravitazione il metodo della quantizzazione usato per la radiazione e per le onde elettroniche. Notiamo anche che, per fissare in un punto dello spazio-tempo un sistema di referimento, occorre disporre di regoli rigidi e di orologi capaci di dare un significato invariante ad un quadrivettore $\Sigma x^i \vec{e}_i$ ove $\vec{e}_1\ \vec{e}_2\ \vec{e}_3\ \vec{e}_4$ formano un sistema di « vettori fondamentali ». La indeterminazione caratteristica di ogni nostra misura rende incerti non solo i valori delle x bensì anche i vettori $\vec{e}_i$ ed i coefficienti delle trasformazioni di Lorentz e di conseguenza anche i potenziali $g_{ik} = \vec{e}_i \times \vec{e}_k$.

Perciò diversi Autori hanno proposto e discusso i procedimenti della quantizzazione del campo gravitazionale. Ammesso che le azioni gravitazionali sono trasmesse da quanti di gravitazione (con statistica di Bose) o da neutrini, possiamo dedurne alcune interessanti conseguenze riferentesi alla idea di Mach sulla dipendenza delle forze d'inerzia dalla generale distribuzione di masse nell'Universo. Evidentemente le forze d'inerzia sono dovute ad azioni trasmesse con velocità finita e possono essere attribuite ad azioni dei quanti di gravitazione o dei neutrini.

In questa nota vogliamo esaminare alcuni argomenti che ci inducono a preferire di applicare la teoria di Jordan (dei neutrini) ai quanti di gravitazione invece che ai fotoni (1). Osserviamo che la teoria dei positroni di Dirac ci dà l'esempio di una nuova concezione sulle proprietà dello spazio-tempo: tutto lo spazio è sede di una distribuzione di elettroni negli stati con energia negativa, e questa distribuzione risulta dipendente dal sistema di riferimento usato.

Basandoci sulla teoria dei raggi β di Fermi e sui suoi recenti brillanti sviluppi, dobbiamo anche ammettere l'esistenza di un'analoga distribuzione spaziale dei neutrini con energia negativa.

Ci sembra che si arriverebbe ad una notevole semplificazione della teoria della gravitazione ammettendo che gli stessi neutrini sono responsabili delle azioni gravitazionali. Infatti è possibile applicare la teoria di Jordan (che permette di ottenere la statistica di Bose per i quanti di gravitazione dalla statistica di Fermi per i neutrini) per sostituire in ogni caso l'azione di un quanto gravitazionale con quella di una coppia di neutrini. In questo modo « il mare dei neutrini con energia negativa » costituirebbe una nuova specie di etere, che determina le geodetiche dell'universo e permette di distinguere localmente i sistemi inerziali da quelli accelerati. Questo punto di vista sta in accordo con l'idea espressa da W. Pauli sull'esistenza di una relazione tra la radice quadrata della costante di gravitazione k e la nuova costante g introdotta da Fermi nella teoria dei raggi β.

Notiamo infine che, se l'interpretazione del movimento oscillatorio dell'elettrone da noi proposta (2) è corretta, le proprietà inerziali dell'elettrone devono essere soggette a fluttuazioni in un intorno dell'ordine di $\left(\frac{\hbar}{mc}\right)^4$ dello spazio-tempo, con la stessa frequenza della « Zitterbevegung ». Di conseguenza, i potenziali di gravità e ogni descrizione spazio temporale degli eventi risulterebbero soggetti ad una nuova specie di indeterminazione in una regione dello spazio-tempo dell'ordine di $\left(\frac{\hbar}{mc}\right)^4$.

Istituto di Fisica dell'Università
San Paolo, Brasile, 26 settembre 1936-XIV.

GLEB WATAGHIN.

(1) Non vogliamo con ciò escludere, che la si possa applicare in ambo i casi.

(2) « La Ricerca Scientifica » (in questo stesso fascicolo, pag. 338)

B. Gleb Wataghin, ‘Sulla teoria quantistica della gravitazione’ (1937)

Reproduction of:

Wataghin, G. Sulla teoria quantica della gravitazione [529]
La Ricerca Scientifica, Serie II, Anno VIII, Vol. 2, n. 5-6, pp. 361-362.

Sulla teoria quantica della gravitazione

Nella relatività generale le equazioni di Dirac assumono la forma seguente (Fock):

$$\gamma^r \nabla_r \psi = \frac{mo}{h} \psi \tag{1}$$

ove γ^r sono matrici a 4 linee e colonne con elementi γ^r_{ik} funzioni delle $x_1\ x_2\ x_3\ x_4 = ict$. Queste determinano anche la metrica di S_4 (Tetrode):

$$\gamma_r \gamma_s + \gamma_s \gamma_r = 2\, g_{rs} \tag{2}$$

∇_r sono le derivate covarianti di Fock e Iwanenko:

$$\nabla_r = \frac{\partial}{\partial x_r} - \Gamma_r \tag{3}$$

Sussiste la seguente importante relazione (Fock):

$$\frac{\partial \gamma_i}{\partial x_k} = \Gamma^{\mu}_{ik}\, \gamma_\mu + \Gamma_k\, \gamma_i - \gamma_i\, \Gamma_k \tag{4}$$

Ci proponiamo di mostrare che dalle (2) e (4) seguono le equazioni gravitazionali nel vuoto:

$$R_{kl} = R^{\cdot\,\cdot\,\cdot\, i}_{kli} = 0 \tag{5}$$

ossia che l'assunzione di Tetrode e Fock costituisce una limitazione nella scelta della metrica. Infatti dalle (4) seguono le relazioni (Fock):

$$R^{\cdot\,\cdot\,\cdot\,\mu}_{kli}\, \gamma_\mu = \Phi_{kl}\, \gamma_i - \gamma_i\, \Phi_{kl} \tag{6}$$

ove $R^{\cdot\,\cdot\,\cdot\,\mu}_{kli}$ è il tensore de Riemann-Christoffel e Φ_{kl} è il tensore emissimetrico formato colle Γ_l e le $\frac{\partial \Gamma_l}{\partial x_k}$.

L'equazione analoga per le γ^ν controvarianti è:

$$R^{\cdot\,\cdot\,\cdot\, s}_{kl\nu}\, \gamma^\nu = \Phi_{kl}\, \gamma^s - \gamma^s\, \Phi_{kl} \tag{6'}$$

Moltiplicando la (6) a destra γ^i e la (6') a sinistra per γ_s e sommando si ha:

$$R^{\cdot\,\cdot\,\cdot\,\mu}_{kli}\, \gamma_\mu\, \gamma^i = \Phi_{kl}\, \gamma_i\, \gamma^i - \gamma_s\, \gamma^s\, \Phi_{kl} \tag{7}$$

Moltiplicando la (6) a sinistra per γ^i e la (6') a destra per γ_s e sommando si trova $R^{\cdot\,\cdot\,\cdot\,\mu}_{kli}\, \gamma^i\, \gamma_\mu$. Sommando quest'ultima espressione colla (7) e valendosi della $\gamma_\mu\, \gamma^i + \gamma^i\, \gamma_\mu = 2\, \delta^i_\mu$ si ottiene la (5).

Questo risultato fa vedere che è possibile fondare la teoria della gravitazione su equazioni del tipo (4) che sono del primo ordine nelle γ^r, e sulla seguente forma dell'elemento lineare:

$$ds = \gamma_r \; dx^r \tag{8}$$

ove ds è una matrice del 4° ordine ($ds^2 = g_{rs} \; dx^r \; dx^s \; . 1$.).

Le equazioni (4) stanno alle equazioni Einsteiniane nella relazione analoga a quella esistente tra le equazioni di Dirac e le equazioni del 2° ordine D'Alembertiane. Nella geometria basata su (8) è necessario associare ad una trasformazione di coordinate puntuale una trasformazione isomorfa delle matrici γ_r determinata dalle Γ_k.

Dall'esame delle (1) si deduce che in esse figurano termini d'interazione del tipo:

$$\gamma^k \; \Gamma_k \; \psi \tag{9}$$

che mostrano l'accoppiamento tra elettroni (ψ), fotoni (essendo la traccia della Γ_k eguale alla componente A_i del quadrivettore potenziale) e campo gravitazionale γ_k. In una recente nota abbiamo messo in evidenza la necessità di assoggettare tutti i campi alla seconda quantizzazione. In particolare assoggettando alla seconda quantizzazione le γ^k, introduciamo la nozione dei corpuscoli gravitazionali con proprietà analoghe a quella dello spin degli elettroni. Vi sono delle ragioni che inducono ad identificare questi corpuscoli coi neutrini di Pauli-Fermi. Infatti, in virtù delle (9), operando con onde piane, in un sistema di riferimento galileiano localmente, si ha conservazione di impulso nei processi elementari in cui ψ e Γ_k inducono transizioni quantiche (mentre γ^k resta costante). Ma in questo stesso processo osservato da un sistema di riferimento accelerato appaiono reazioni di inerzia e non si ha conservazione d'impulso. Questa può essere rispettata solo coll'intervento dei campi gravitazionali (infatti le leggi di conservazione valgono per il tensore somma del tensore della materia del tensore della gravitazione). Ciò mostra che l'emissione e l'assorbimento dei neutrini descritte dalle γ_k quantizzate possono servire per salvaguardare la legge della conservazione dell'impulso. In modo analogo i neutrini di Pauli-Fermi servono per salvaguardare la legge della conservazione dell'energia nella teoria dei raggi β.

Da quanto precede risulta l'esistenza di una connessione fra campi elettromagnetici e gravitazionali: questi sembrano costituire un ente unico, che si manifesta, per esempio, ad un osservatore galileiano come campo elettromagnetico distinto dal campo metrico γ_r pseudoeuclideo, mentre in un sistema accelerato appare come un insieme descritto dagli operatori Γ_k e γ_r.

Le equazioni della teoria unitaria basate su queste idee sono: le (1) quantizzate; le (2) e le (4) completate al secondo membro da un tensore del tipo $K \, . \, T_{ik}$ che si riferisce alla materia, e le relazioni di commutazione delle γ_r e delle Γ_l analoghe delle relazioni relativistiche di Jordan e Pauli per le A_i. Le basi di una tale teoria formano oggetto di una prossima nota.

S. Paulo, Departamento de Physica da Universidade,
30 *de Agosto* 1937-XV

G. WATAGHIN

Unità naturale di corrente e costante di Faraday

Dell'unità pratica di corrente si possono dare diverse definizioni fissando per es. che debba intendersi per Ampère:

1) L'intensità della corrente prodotta dalla forza elettromotrice di 1 Volta in un circuito avente la resistenza di 1 Ohm.; oppure

2) L'intensità della corrente costante che da una soluzione di nitrato di argento separa in 1 secondo 0,001118 gr. di metallo; oppure

3) L'intensità della corrente che trasporta 1 Coulomb in 1 secondo; ecc.

C. On Petrov's classification of Einstein spaces

A. Einstein spaces. The general study of Riemann spaces V_n in n dimensions can be characterized by the algebraic structure of the Riemann curvature tensor:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}\,, \qquad R_{\alpha[\beta\gamma\delta]} = 0. \qquad (\textsc{Bianchi I})$$

$$R_{\alpha\beta\gamma\delta,\eta} = -R_{\beta\alpha\gamma\delta,\eta} = -R_{\alpha\beta\delta\gamma,\eta}\,, \qquad R_{\alpha\beta[\gamma\delta,\eta]} = 0. \qquad (\textsc{Bianchi II})$$

Contracting (Bianchi II) twice, we have

$$(\alpha,\eta): \qquad R^{\alpha}{}_{\beta\gamma\delta,\alpha} + R^{\alpha}{}_{\beta\delta\alpha,\gamma} + R^{\alpha}{}_{\beta\alpha\gamma,\delta} = 0 \quad \longrightarrow \quad R^{\alpha}{}_{\beta\gamma\delta,\alpha} - R_{\beta\delta,\gamma} + R_{\beta\gamma,\delta} = 0 \qquad \text{(C.1)}$$

$$(\beta,\delta): \qquad R^{\alpha\beta}{}_{\gamma\beta,\alpha} - R^{\beta}{}_{\beta,\gamma} + R^{\beta}{}_{\gamma,\beta} = 0 \quad \longrightarrow \quad (R^{\alpha}{}_{\gamma} - \frac{1}{2}R\,\delta^{\alpha}{}_{\gamma})_{,\alpha} = 0. \qquad \text{(C.2)}$$

In particular, a Riemann space V_n restricted to the field equations

$$R_{\mu\nu} = \kappa g_{\mu\nu} \quad \Longrightarrow \quad \kappa = \frac{1}{4}R, \qquad R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}, \qquad \text{(C.3)}$$

is called an *Einstein space*. Note that C.3 is compatible with (Bianchi I, II) only if $\kappa = const$.

B. Bivector spaces. Let V_n be a $n-$ dim spacetime and $p \in V_n$, cf. [414, §15].

- Any tensor with even covariant and contravariant valencies at a point $p \in V_n$, whose indices are subdivided into skew-symmetric pairs, is a *bitensor*.
- The simplest bitensor is a 2nd-order skew-symmetric tensor F at p, with components $F_{\alpha\beta}\,(= -F_{\beta\alpha})$, called a *bivector* at p.
- The set $B(p)$ of all bivectors at p is a $N-$ dim real vector space ($N = n(n-1)/2$).
- The dual of F is defined by $\quad *F_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\gamma\delta}F^{\gamma\delta}$.
- Transformation of $F \in V_n$:

$$F^{\alpha\beta} \quad \longrightarrow \quad F^{\alpha'\beta'} = A^{\alpha'}{}_{\alpha}A^{\beta'}{}_{\beta}F^{\alpha\beta} = 2A^{\alpha'}{}_{[\alpha}A^{\beta'}{}_{\beta]}F^{\alpha\beta} \quad (\alpha,\beta = 0,...,n)$$

$$F^{a} \quad \longrightarrow \quad F^{a'} = A^{a'}{}_{a}F^{a} \quad (a = 1,...,N)$$

$$\therefore \quad A^{a'}{}_{a} = 2A^{\alpha'}{}_{[\alpha}A^{\beta'}{}_{\beta]} = A^{[\alpha'}{}_{[\alpha}A^{\beta']}{}_{\beta]} \quad \text{(centro-affine transformation)} \qquad \text{(C.4)}$$

- (C.4) defines an affine manifold E_N only if a *Klein geometry* (a homogeneous space with a transitive action by a Lie group) satifies the group relations

$$\eta^{a'} = A^{a'}{}_{a}\eta^{a}, \qquad \eta^{a} = A^{a}{}_{a'}\eta^{a'}, \qquad \left|A^{a'}{}_{a}\right| \neq 0, \qquad A^{a}{}_{b'}A^{b'}{}_{c} = \delta^{a}_{c}. \qquad \text{(C.5)}$$

- Thus every local bivector in V_n can be mapped on a centro-affine E_N called *bivector space*.

- It is now possible to metrize the bivector space E_N:

$$g_{ab} \in E_N \quad \longrightarrow \quad g_{\alpha\beta\mu\nu} := g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}, \quad g_{\mu\nu} \in V_n. \tag{C.6}$$

or, in an equivalent way,

$$g_{ab} = B_a^{\ \alpha\beta} B_b^{\ \mu\nu} g_{\alpha\beta\mu\nu}. \tag{C.7}$$

After introducing the tensor g_{ab} into the bivector affine space, E_N becomes a metric space R_N.

- The symmetric tensor $R_{ab} \in R_N$ becomes the image of the Riemann curvature $R_{\alpha\beta\mu\nu} \in V_n$,

$$R_{ab} \in R_N \quad \longrightarrow \quad R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}, \qquad R_{\alpha[\beta\mu\nu]} = 0, \quad R_{\alpha\beta\mu\nu} \in V_n. \tag{C.8}$$

- Introducing a non-holonomic orthonormal system of coordinates at a point $p \in V_4$:

$$g_{\alpha\beta}\Big|_p = \xi_\alpha^{\ \hat{\alpha}} \xi_\beta^{\ \hat{\beta}} e_{\hat{\alpha}} e_{\hat{\beta}} = e_\alpha e_\beta, \qquad g_{\alpha\beta}\Big|_p = \begin{cases} \pm 1 & (\alpha = \alpha) \\ 0 & (\alpha \neq \beta) \end{cases} \tag{C.9}$$

From (C.6), one may write

$$\begin{aligned} V_4 : \qquad g_{\alpha\beta\gamma\delta} &:= g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu} \\ &= \left(\xi_\alpha^{\ \hat{\alpha}} \xi_\mu^{\ \hat{\mu}} e_{\hat{\alpha}} e_{\hat{\mu}}\right)\left(\xi_\beta^{\ \hat{\beta}} \xi_\nu^{\ \hat{\nu}} e_{\hat{\beta}} e_{\hat{\nu}}\right) - \left(\xi_\alpha^{\ \hat{\alpha}} \xi_\nu^{\ \hat{\nu}} e_{\hat{\alpha}} e_{\hat{\nu}}\right)\left(\xi_\beta^{\ \hat{\beta}} \xi_\mu^{\ \hat{\mu}} e_{\hat{\beta}} e_{\hat{\mu}}\right) \\ &= \frac{2^4}{2^2} \xi_{[\alpha}^{\ \ [\hat{\alpha}} \xi_{\beta]}^{\ \ \hat{\beta}]} \xi_{[\gamma}^{\ \ [\hat{\gamma}} \xi_{\delta]}^{\ \ \hat{\delta}]} e_{[\hat{\alpha}} e_{\hat{\beta}]} e_{[\hat{\gamma}} e_{\hat{\delta}]} \equiv \xi_{\alpha\beta}^{\hat{\alpha}\hat{\beta}} \xi_{\gamma\delta}^{\hat{\gamma}\hat{\delta}} e_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \equiv e_{\alpha\beta\gamma\delta}. \end{aligned} \tag{C.10}$$

$$R_6 : \qquad g_{ab} = B_a^{\ \alpha\beta} B_b^{\ \gamma\delta} g_{\alpha\beta\gamma\delta} = \xi_a^{\ \hat{a}} \xi_b^{\ \hat{b}} e_{\hat{a}} e_{\hat{b}} \equiv e_a e_b. \tag{C.11}$$

Thus for Einstein spaces the Riemann tensor in this non-holonomic coordinate system reduces to

$$V_4 : \qquad R_{\alpha\beta\gamma\delta} = \kappa\, g_{\alpha\beta\gamma\delta} \quad \Longrightarrow \quad \begin{cases} (I.)\ R_{\alpha\beta\gamma\delta} e_\alpha e_\gamma = \kappa e_\beta e_\delta & (\beta = \delta), \\ (II.)\ R_{\alpha\beta\gamma\delta} e_\alpha e_\gamma = 0 & (\beta \neq \delta). \end{cases} \tag{C.12}$$

C. Segrè characteristics. According to Petrov's prescription [414, §16],

* The curvature R_{ab} in R_N can be associated with the $\lambda-$matrix as $(R_{ab} - \lambda g_{ab})$.

* We can classify the V_n (valid $\forall n$) by reducing this $\lambda-$matrix to the canonical Jordan form.

* The type of space is determined by the Segrè characteristic of the $\lambda-$matrix.

Summary of Jordan canonical forms, cf.[219, §2.6]:

- A linear map $g : V \longrightarrow V$ is termed *nihilpotent* of index p if $\begin{cases} \underbrace{g \circ \cdots \circ g}_{p \text{ times}} = 0 \in V, \\ \underbrace{g \circ \cdots \circ g}_{p-1 \text{ times}} \neq 0 \in V. \end{cases}$

- Let $\lambda_1, ..., \lambda_r \in \mathbb{C}$ be the distinct eigenvalues of a *diagonalisable* map f with algebraic multiplicity $m_1, ..., m_r$ ($\sum_j m_j = n$). Then one may write

$$V = V_1 \oplus \cdots \oplus V_r,$$

  where, for each j, V_j is the λ_j-eigenspace of f and is an invariant subspace of f of dim m_j. The restriction of f to V_j is a linear map $V_j \longrightarrow V_j$ of the form $\lambda_j \mathbb{I}_j$.

- Jordan prescription: V_j is not necessarily the λ_j-eigenspace of f. The restriction of f to V_j is of the form

$$\lambda_j \mathbb{I}_j + N_j,$$

  where $N_j : V_j \longrightarrow V_j$ is the nihilpotent map.

- One may choose a basis for f in V_j such that the matrix A representing f is of the form

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix}, \qquad A_j = \begin{pmatrix} B_{j1} & & & \\ & B_{j2} & & \\ & & \ddots & \\ & & & B_{jk(j)} \end{pmatrix}_{m_j \times m_j} \tag{C.13}$$

  where

  - A_j is an $m_j \times m_j$ matrix with λ_j in every diagonal position and some arrangement of $\{0, 1\}$ in the superdiagonal;
  - $B_j l$ are $p_{jl} \times p_{jl}$ *basic Jordan blocks* (diagonal entries $\equiv \lambda_j$, superdiagonal entries $\equiv 1$, and $p_{i1} \geq \cdots \geq p_{ik(i)}$. .

- To each L.T. f there is associated the eigenvalues $\lambda_1, \cdots, \lambda_r$, its respective algebraic multiplicities, and for each j ($1 \leq j \leq r$) the numbers p_{jk} ($p_{i1} \geq \cdots \geq p_{ik(i)}$) with $m_j = p_{j1} + \cdots + p_{jk(j)}$. These quantities and their ordering *uniquely* determine the **Jordan canonical form** of the matrix A representing f.

– The general Jordan structure of f can be uniquely characterized by the symbol

$$\left\{(p_{11}, \cdots, p_{1k(1)})(p_{21}, \cdots, p_{2k(2)}) \cdots (p_{r1}, \cdots, p_{rk(r)})\right\} \tag{C.14}$$

called the **Segrè type** (Segrè characteristic or Segrè symbol) of f.

– In particular, if f is diagonalisable over $\mathbb{C}$ with eigenvalues $\lambda_1, \cdots, \lambda_r$ of respective algebraic multiplicity $m_1, \cdots, m_r$, then

  * A_j is an $m_j \times m_j$ diagonal matrix with λ_j in every diagonal position;
  * each $B_{jk(j)}$ is a 1×1 matrix with entry λ_j and $k(j) = m_j \Longrightarrow p_{jk} = 1$;
  * thus the Segrè type is $\{(1 \cdots 1) \cdots (1 \cdots 1)\}$.

– In general, any Jordan form A can be written as

$$A = D + N, \tag{C.15}$$

where

  – D: $n \times n$ diagonal matrix whose entries are the eigenvalues $\lambda_1, \cdots, \lambda_r$;
  – N: nihilpotent matrix, with some arrangement of zeros and ones on the superdiagonal and zeros elsewhere.

Then $f : V \longrightarrow V$ is nihilpotent iff all its eigenvalues are zero.

– **Geometrical interpretation of the Jordan theory.**

  * In $D_j : V_j \longrightarrow V_j$ every non-zero element of V_j can be an eigenvector of f;
  * In $A_j : V_j \longrightarrow V_j$ there is only one independent eigenvector associated with each B_{jl} block within each A_j .

Definition: The *geometric multiplicity* of λ_j is the dimension of the λ_j-eigenspace.

Remark: The algebraic multiplicity is equal to the geometric multiplicity iff $N_j \equiv 0$ and $A_j = D_j$.

– Every Jordan form A satisfies its own *characteristic polynomial*

$$(-1)^n (x - \lambda_1)^{m_1} (x - \lambda_2)^{m_2} \cdots (x - \lambda_r)^{m_r}. \tag{C.16}$$

– There exists a polynomial of least degree m ($1 \leq m \leq n$) which is satisfied by A (and is unique if it is monic), called the *minimal polynomial*:

$$(x - \lambda_1)^{p_{11}} (x - \lambda_2)^{p_{21}} \cdots (x - \lambda_r)^{p_{r1}}. \tag{C.17}$$

– The polynomials $(x - \lambda_j)^{p_{jl}}$ are the *elementary divisors* of f. An elementary divisor associated with λ_j and with $p_{jl} = 1$ is called *simple*. Otherwise it is called non-simple of order p_{jl}.

– An eigenvalue λ is called *non-degenerate* (respectivelly, *degenerate*) is the λ-eigenspace has dimension 1 (resp., > 1).

D. Petrov's theorems. We follow Petrov's prescription [414, §19]:

$$\begin{array}{ccccccc} 10 & \longrightarrow & 1 & & 23 & \longrightarrow & 4 \\ 20 & \longrightarrow & 2 & & 31 & \longrightarrow & 5 \\ 30 & \longrightarrow & 3 & & 12 & \longrightarrow & 6 \end{array}$$

$$\left(g_{\alpha\beta}\right)_p = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \longrightarrow (g_{ab}) = \begin{pmatrix} -\mathbb{I}_3 & \\ & \mathbb{I}_3 \end{pmatrix}.$$

Theorem 1: The matrix (R_{ab}) for the orthonormal tetrad in doubly symmetric.

Proof: From the relations

(I.) $R_{\alpha\beta\gamma\delta} e_\alpha e_\gamma = \kappa e_\beta e_\delta \quad (\beta = \delta) \quad \Longrightarrow \quad R_{o\beta o\delta} - R_{1\beta 1\delta} - R_{2\beta 2\delta} - R_{3\beta 3\delta} = \kappa e_\beta e_\delta$, it follows that

	$R_{\alpha\beta\gamma\delta} \in V_4$	$R_{ab} \in R_6$	
$\beta = \delta = 0:$	$R_{0000} - R_{1010} - R_{2020} - R_{3030} = \kappa\, e_0 e_0 = +\kappa$	$R_{11} + R_{22} + R_{33} = -\kappa$	(I.0)
$\beta = \delta = 1:$	$R_{0101} - R_{1111} - R_{2121} - R_{3131} = \kappa\, e_1 e_1 = -\kappa$	$R_{11} - R_{66} - R_{55} = -\kappa$	(I.1)
$\beta = \delta = 2:$	$R_{0202} - R_{1212} - R_{2222} - R_{3232} = \kappa\, e_2 e_2 = -\kappa$	$R_{22} - R_{66} - R_{44} = -\kappa$	(I.2)
$\beta = \delta = 3:$	$R_{0303} - R_{1313} - R_{2323} - R_{3333} = \kappa\, e_3 e_3 = -\kappa$	$R_{33} - R_{55} - R_{44} = -\kappa$	(I.3)

Solving the system: (I.1)= (I.2) $\Longrightarrow$ $R_{11} - R_{55} = R_{22} - R_{44}$ $\longleftrightarrow$ $R_{11} + R_{44} = R_{22} + R_{55} = 0$ whose compatibility with (I.0) and (I.3) implies $R_{33} + R_{66} = 0$.

(II.) $R_{\alpha\beta\gamma\delta} e_\alpha e_\gamma = 0 \quad (\beta \neq \delta) \quad \Longrightarrow \quad R_{0\beta 0\delta} - R_{1\beta 1\delta} - R_{2\beta 2\delta} - R_{3\beta 3\delta} = 0:$

	$R_{\alpha\beta\gamma\delta} \in V_4$	$R_{ab} \in R_6$
$\beta = 0,\ \delta = 1:$	$R_{0001} - R_{1011} - R_{2021} - R_{3031} = 0$	$-R_{26} + R_{35} = 0$
$\beta = 0,\ \delta = 2:$	$R_{0002} - R_{1012} - R_{2022} - R_{3032} = 0$	$R_{16} - (-R_{34}) = 0$
$\beta = 0,\ \delta = 3:$	$R_{0003} - R_{1013} - R_{2023} - R_{3033} = 0$	$-R_{15} - R_{24} = 0$
$\beta = 1, \delta = 2:$	$R_{0102} - R_{1112} - R_{2122} - R_{3132} = 0$	$R_{12} - (-R_{54}) = 0$
$\beta = 1, \delta = 3:$	$R_{0103} - R_{1013} - R_{2123} - R_{3133} = 0$	$R_{13} - (-R_{64}) = 0$
$\beta = 2, \delta = 3:$	$R_{0203} - R_{1213} - R_{2223} - R_{3233} = 0$	$R_{23} - (-R_{65}) = 0.$

(III.) $R_{\alpha[\beta\gamma\delta]} = 0$ (BIANCHI I) $\iff$ $R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\gamma\beta\delta} = 0$:

$$\begin{array}{lcc} & R_{\alpha\beta\gamma\delta} \in V_4 & R_{ab} \in R_6 \\ \alpha = 0, \beta = 1, \gamma = 2, \delta = 3 : & R_{0123} + R_{0231} + R_{0312} = 0 & -R_{14} - R_{25} - R_{36} = 0. \end{array}$$

□

Theorem 2: There are only three types of spaces defined by gravitational fields in Einstein spaces ($R_{\mu\nu} = kg_{\mu\nu}$) with signature $(+---)$.

Proof: By elementary transformations, we have

$$\begin{aligned}(R_{ab} - \lambda g_{ab}) \; &= \left(\begin{array}{c|c} M + \lambda \mathbb{I}_3 & N \\ \hline N & -M - \lambda \mathbb{I}_3 \end{array}\right) \overset{col1+icol2}{\longrightarrow} \left(\begin{array}{c|c} M + iN + \lambda \mathbb{I}_3 & N \\ \hline -i(M + iN + \lambda \mathbb{I}_3) & -M - \lambda \mathbb{I}_3 \end{array}\right) \\ &\overset{-row2-irow1}{\longrightarrow} \left(\begin{array}{c|c} M + iN + \lambda \mathbb{I}_3 & N \\ \hline 0 & M - iN + \lambda \mathbb{I}_3 \end{array}\right) \\ &\overset{col2+i/2col1}{\longrightarrow} \left(\begin{array}{c|c} M + iN + \lambda \mathbb{I}_3 & \frac{i}{2}(M - iN + \lambda \mathbb{I}_3) \\ \hline 0 & M - iN + \lambda \mathbb{I}_3 \end{array}\right) \\ &\overset{row1-i/2row2}{\longrightarrow} \left(\begin{array}{c|c} M + iN + \lambda \mathbb{I}_3 & 0 \\ \hline 0 & M - iN + \lambda \mathbb{I}_3 \end{array}\right) \\ &\equiv \begin{pmatrix} Q(\lambda) & 0 \\ 0 & \overline{Q}(\lambda) \end{pmatrix}.\end{aligned}$$

Hence the possible Segrè characteristics of the 3×3 matrices $Q(\lambda), \overline{Q}(\lambda)$ are

$$I : \left[1\,1\,1\,, \overline{1\,1\,1}\right], \qquad II : \left[2\,1\,, \overline{2\,1}\right], \qquad III : \; [3\,, 3].$$

□

Theorem 3: There is a real and uniquely defined orthonormal tetrad in all three possible types of $\dot{T}_i$ spaces, relative to which the orthogonal components of the curvature tensor are stated by canonical forms of the matrix

$$(R_{ab}) = \begin{pmatrix} M & N \\ N & -M \end{pmatrix},$$

namely

$$\underline{\dot{T}_1 \text{ space}} : \quad M = \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix}, \quad N = \begin{pmatrix} \beta_1 & & \\ & \beta_2 & \\ & & \beta_3 \end{pmatrix}, \quad \sum_{s=1}^{3} \alpha_s = -\kappa, \quad \sum_{s=1}^{3} \beta_s = 0.$$

$\underline{\dot{T}_2 \text{ space}}$: $M = \begin{pmatrix} \alpha_1 & & \\ & \alpha_2+1 & \\ & & \alpha_2-1 \end{pmatrix}$, $N = \begin{pmatrix} \beta_1 & & \\ & \beta_2 & 1 \\ & 1 & \beta_2 \end{pmatrix}$, $\alpha_1 + 2\alpha_2 = -\kappa$, $\beta_1 + 2\beta_2 = 0$.

$\underline{\dot{T}_3 \text{ space}}$: $M = \begin{pmatrix} -\frac{\kappa}{3} & 1 & \\ 1 & -\frac{\kappa}{3} & \\ & & -\frac{\kappa}{3} \end{pmatrix}$, $N = \begin{pmatrix} 0 & & \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix}$.

Elements of the proof: Let us consider the case of $\dot{T}_1$ space with Segrè type $\left[1\ 1\ 1\ , \overline{1\ 1\ 1}\right]$:

- o Segrè type is simple $\Longrightarrow$ $R_{ab} \in E_6$ has 6 non-isotropic mutually orthogonal eigen-directions, which determine at a given point $p \in \dot{T}_1$ simple bivectors with a specific structure.
- o The real orthonormal tetrad basis at a point $p \in \dot{T}_1$ is denoted by $\xi_\alpha{}^{\hat{\alpha}}$ $(\alpha, \hat{\alpha} = 0, ..., 3)$ and the simple bivectors by $\xi^{\hat{\alpha}\hat{\beta}}_{\alpha\beta} \equiv \xi_{[\alpha}{}^{\hat{\alpha}} \xi_{\beta]}{}^{\hat{\beta}}$. These bivectors define six independent real mutually orthogonal *sechbeins* $\xi_a{}^b$ $(a, b = 1, ...6)$
- o The eigen-directions W^b of R_{ab}, $(R_{ab} - \lambda g_{ab})W^b = 0$, are of the form

$$W^a = \sigma(\xi^a_1 \pm i\xi^a_4) + \mu(\xi^a_2 \pm i\xi^a_5) + \nu(\xi^a_3 \pm i\xi^a_6) \qquad (a, b = 1, ..., 6).$$

- o The double symmetry of (R_{ab}) reduce the set of six equations to the following three:

$$(R_{ab} - \lambda g_{ab})W^b = \left(\begin{array}{c|c} m_{ij} + \lambda\delta_{ij} & n_{i,j+3} \\ \hline n_{i+3,j} & -m_{i+3,j+3} - \lambda\delta_{i+3,j+3} \end{array}\right) \begin{pmatrix} W^j \\ W^{j+3} \end{pmatrix} = 0$$

$$\Longrightarrow \quad (m_{ij} + \lambda\delta_{ij})W^j + n_{i,j+3}W^{j+3} = 0$$

$$(m_{ij} + \lambda\delta_{ij})\left[\sigma\xi^a_1 + \mu\xi^a_2 + \nu\xi^a_3\right] \pm i\, n_{i,j+3}\left[\sigma\xi^{j+3}_4 + \mu\xi^{j+3}_5 + \nu\xi^{j+3}_6\right] = 0$$

$$\sigma(m_{i1} \pm in_{i1} + \lambda\delta_{i1}) + \mu(m_{i2} \pm in_{i2} + \lambda\delta_{i2}) + \nu(m_{i3} \pm in_{i3} + \lambda\delta_{i3}) = 0.$$

σ, μ, ν are non-zero solutions iff λ is a characteristic root of $Q(\lambda) = 0$ or $\overline{Q}(\lambda) = 0$.

- o To $W^a \in R_6$ there corresponds at a given point $p \in \dot{T}_1$ a non-singular bivector $W^{\alpha\beta}$, namely

$$W^{\alpha\beta} = \sigma(\xi^{\alpha\beta}_{10} \pm i\xi^{\alpha\beta}_{23}) + \mu(\xi^{\alpha\beta}_{20} \pm i\xi^{\alpha\beta}_{31}) + \nu(\xi^{\alpha\beta}_{30} \pm i\xi^{\alpha\beta}_{12}) \qquad (\alpha, \beta = 0, ..., 3).$$

- o Any orthogonal transformation converts $W^{\alpha\beta}$ into a bivector of same structure, and replaces $\sigma, \mu, \nu \longrightarrow \dot{\sigma}, \dot{\mu}, \dot{\nu}$, the norm of $W^{\alpha\beta}$ remaining invariant: $\sigma^2 + \mu^2 + \nu^2 = \dot{\sigma}^2 + \dot{\mu}^2 + \dot{\nu}^2$.
- o If λ_j $(j = 1, 2, 3)$ are the roots of $|R_{ab} - \lambda g_{ab}| = 0$, corresponding to the eigenvectors $W^a_{(j)}$, then the roots λ_{j+3} must correspond the complex conjugate eigenvectors $\overline{W}^a_{(j)}$. Namely,

$$W^{\alpha\beta}_{(1)} = V^{\alpha\beta}_{(1)} + i\dot{V}^{\alpha\beta}_{(1)} \quad \Longrightarrow \quad W^{\alpha\beta}_{(4)} = V^{\alpha\beta}_{(4)} - i\dot{V}^{\alpha\beta}_{(4)}.$$

o Setting $\sigma = a_{(1)} + ib_{(1)}, \mu = a_{(2)} + ib_{(2)}, \nu = a_{(3)} + ib_{(3)}, a_{(j)}, b_{(j)} \in \mathbb{R}$, we find that

$$\begin{aligned} W^{\alpha\beta} &= \sigma\xi_{10}^{\alpha\beta} + \mu\xi_{20}^{\alpha\beta} + \nu\xi_{30}^{\alpha\beta} \pm i\left(\sigma\xi_{23}^{\alpha\beta} + \mu\xi_{31}^{\alpha\beta} + \nu\xi_{12}^{\alpha\beta}\right) \\ &= (a_{(1)} + ib_{(1)})\xi_{10}^{\alpha\beta} + (a_{(2)} + ib_{(2)})\xi_{20}^{\alpha\beta} + (a_{(3)} + ib_{(3)})\xi_{30}^{\alpha\beta} \\ &\quad \pm i\left[(a_{(1)} + ib_{(1)})\xi_{23}^{\alpha\beta} + (a_{(2)} + ib_{(2)})\xi_{31}^{\alpha\beta} + (a_{(3)} + ib_{(3)})\xi_{12}^{\alpha\beta}\right] \\ &= a_{(1)}\xi_{10}^{\alpha\beta} + a_{(2)}\xi_{20}^{\alpha\beta} + a_{(3)}\xi_{30}^{\alpha\beta} \mp b_{(1)}\xi_{23}^{\alpha\beta} \mp b_{(2)}\xi_{31}^{\alpha\beta} \mp b_{(3)}\xi_{12}^{\alpha\beta} \\ &\quad + i\left[b_{(1)}\xi_{10}^{\alpha\beta} + b_{(2)}\xi_{20}^{\alpha\beta} + b_{(3)}\xi_{30}^{\alpha\beta} \pm a_{(1)}\xi_{23}^{\alpha\beta} \pm a_{(2)}\xi_{31}^{\alpha\beta} \pm a_{(3)}\xi_{12}^{\alpha\beta}\right] \end{aligned}$$

o Hence $W^{\alpha\beta} = V_{(1)}^{\alpha\beta} + i\dot{V}_{(1)}^{\alpha\beta}$, with

$$\begin{aligned} V_{(1)}^{\alpha\beta} &= a_{(1)}\xi_{10}^{\alpha\beta} + a_{(2)}\xi_{20}^{\alpha\beta} + a_{(3)}\xi_{30}^{\alpha\beta} - b_{(1)}\xi_{23}^{\alpha\beta} - b_{(2)}\xi_{31}^{\alpha\beta} - b_{(3)}\xi_{12}^{\alpha\beta} \\ \dot{V}_{(1)}^{\alpha\beta} &= b_{(1)}\xi_{10}^{\alpha\beta} + b_{(2)}\xi_{20}^{\alpha\beta} + b_{(3)}\xi_{30}^{\alpha\beta} + a_{(1)}\xi_{23}^{\alpha\beta} + a_{(2)}\xi_{31}^{\alpha\beta} + a_{(3)}\xi_{12}^{\alpha\beta} \end{aligned}$$

o $W^a \in R_6$ is non-isotropic : $g_{ab}W_{(1)}^a W_{(1)}^a = 1$, and

$$\begin{aligned} g_{ab}W_{(1)}^a W_{(1)}^a &= -W_{(1)}^1 W_{(1)}^1 - W_{(1)}^2 W_{(1)}^2 - W_{(1)}^3 W_{(1)}^3 + W_{(1)}^4 W_{(1)}^4 + W_{(1)}^5 W_{(1)}^5 + W_{(1)}^6 W_{(1)}^6 \\ &= -(a_{(1)} + ib_{(1)})^2 - (a_{(2)} + ib_{(2)})^2 - (a_{(3)} + ib_{(3)})^2 \\ &\quad + (-b_{(1)} + ia_{(1)})^2 + (-b_{(2)} + ia_{(2)})^2 + (-b_{(3)} + ia_{(3)})^2 \\ &= -\left[a_{(1)}^2 + (ib_{(1)})^2 + a_{(2)}^2 + (ib_{(2)})^2 + a_{(3)}^2 + (ib_{(3)})^2 + 2i(a_{(1)}b_{(1)} + a_{(2)}b_{(2)} + a_{(3)}b_{(3)})\right] \\ &\quad + \left[(b_{(1)}^2 + (ia_{(1)})^2 + b_{(2)}^2 + (ia_{(2)})^2 + b_{(3)}^2 + (ia_{(3)})^2 - 2i(a_{(1)}b_{(1)} + a_{(2)}b_{(2)} + a_{(3)}b_{(3)})\right] \\ &= 2\left[-a_{(1)}^2 - a_{(2)}^2 - a_{(3)}^2 + b_{(1)}^2 + b_{(2)}^2 + b_{(3)}^2\right] - 4i\left[a_{(1)}b_{(1)} + a_{(2)}b_{(2)} + a_{(3)}b_{(3)}\right] = 1. \end{aligned}$$

$$\therefore \quad \sum_{j=1}^{3}(b_j^2 - a_j^2) > 0 \qquad \sum_{j=1}^{3} a_j b_j = 0. \tag{C.18}$$

Definition: Let P, Q be two general non-simple bivectors which lie in flat clusters $\varepsilon_p, \varepsilon_q$, with dimensions $p \leq q \leq 4$, when $n = 4$.

* If $\varepsilon_p, \varepsilon_q$ have k common directions, the*degree of parallelism* of P, Q is k/p.
* If ε_p contains l independent directions orthogonal to ε_q, the respective bivectors are said to have a *degree of orthogonality* l/p.

Lemma 1. $V_{(1)}^{\alpha\beta}$ and $\dot{V}_{(1)}^{\alpha\beta}$ are simple, i.e. lie in 2-dim flat clusters.
(It is sufficient to check that $V_{(1)}^{10}V_{(1)}^{23} + V_{(1)}^{20}V_{(1)}^{31} + V_{(1)}^{30}V_{(1)}^{12} = 0.$, which is satisfied due to the condition C.18.)

Lemma 2. $V^{\alpha\beta}_{(1)}$ and $\dot{V}^{\alpha\beta}_{(1)}$ are 0/2-parallel.
(By contradiction: if $V^{\alpha\beta}_{(1)}$ and $\dot{V}^{\alpha\beta}_{(1)}$ are 1/2-parallel, $W^{\alpha\beta}_{(1)}$ can not be a simple; if they are 2/2-parallel, their components would be proportional and thus reducible to zero.)

Lemma 3. $V^{\alpha\beta}_{(1)}$ and $\dot{V}^{\alpha\beta}_{(1)}$ are 2/2-orthogonal.
(It is necessary and sufficient that $g_{\beta\gamma}V^{\alpha\beta}_{(1)}\dot{V}^{\gamma\delta}_{(1)} = 0$. Compare the following expressions:

$$g_{ab}W^aW^b = \begin{cases} g_{ab}(V^a + i\dot{V}^a)(V^b + i\dot{V}^b) = g_{ab}(V^aV^b - \dot{V}^a\dot{V}^b + 2iV^a\dot{V}^b) \ = \ 1 \\ 2\sum_{j=1}^{3}(b_j^2 - a_j^2) - 4i\sum_{j=1}^{3} a_jb_j \ = \ 1 \end{cases} \qquad \therefore \ g_{ab}V^a\dot{V}^b = 0.) \tag{C.19}$$

Lemma 4. The vector $V^a_{(1)} \in R_6$ associated to the simple bivector $V^{\alpha\beta}_{(1)}$ at a point $p \in \dot{T}_1$ satisfies the following properties:

i. $g_{ab}V^a_{(1)}V^b_{(1)} = \sum_{j=1}^{3}(b_j^2 - a_j^2) > 0;$

ii. $g_{ab}V^a_{(1)}\dot{V}^b_{(1)} = \sum_{j=1}^{3} a_jb_j = 0;$

iii. $g_{ab}\dot{V}^a_{(1)}\dot{V}^b_{(1)} = -\sum_{j=1}^{3}(b_j^2 - a_j^2) < 0.$

o From property $i.$, two real orthogonal and non-isotropic vectors ζ^α and η^α can always be selected in the plane of $V^{\alpha\beta}_{(1)}$ such that its norm is $\left|V^{\alpha\beta}_{(1)}\right| = 2\zeta_\rho\zeta^\rho\,\eta_\sigma\eta\sigma$. Then the two vectors have norms of the same sign.

o Similarly, from property $iii.$ one can determine two orthogonal real and non-isotropic vectors in the plane of $V^{\alpha\beta}_{(1)}$, with norms of opposite sign.

o In this non-holonomic reference system, one may choose the orthonormal tetrad $\{\xi_\alpha{}^{\hat{\alpha}}\}$ to within a rotation in the $\xi_{23}{}^{\hat{\alpha}\hat{\beta}}$-plane and a Lorentz rotation in the $\xi_{10}{}^{\hat{\alpha}\hat{\beta}}$-plane

$$W^{\alpha\beta}_{(1)} = \xi^{\alpha\beta}_{10} + i\xi^{\alpha\beta}_{23}, \qquad W^{\alpha\beta}_{(4)} = \xi^{\alpha\beta}_{10} - i\xi^{\alpha\beta}_{23}$$

Since $W^{\alpha\beta}_{(1)}, \ldots, W^{\alpha\beta}_{(6)}$ are mutually orthogonal at a point $p \in \dot{T}_1$, and taking the complex conjugacy property of the characteristic form, one find

$$W^{\alpha\beta}_{(1)} = \xi^{\alpha\beta}_{10} + i\xi^{\alpha\beta}_{23}, \quad W^{\alpha\beta}_{(2)} = \xi^{\alpha\beta}_{20} + i\xi^{\alpha\beta}_{31}, \quad W^{\alpha\beta}_{(3)} = \xi^{\alpha\beta}_{30} + i\xi^{\alpha\beta}_{12},$$
$$W^{\alpha\beta}_{(4)} = \overline{W}^{\alpha\beta}_{(1)}, \quad W^{\alpha\beta}_{(5)} = \overline{W}^{\alpha\beta}_{(2)}, \quad W^{\alpha\beta}_{(6)} = \overline{W}^{\alpha\beta}_{(3)}.$$

□

Returning to the Petrov's notation for the real and imaginary parts of the bases of the elementary divisors,

$$\alpha_j \equiv a_{(j)}, \quad \beta_j \equiv b_{(j)}, \qquad (j = 1, 2, 3)$$

and collecting the results above, one have the following classification of Einstein spaces $\dot{T}_j$:

$\dot{T}_1$ Space:	I	$\longrightarrow$	D	$\longrightarrow$	O
	$\alpha_1 \neq \alpha_2 \neq \alpha_3$		$\alpha_1, \alpha_2 = \alpha_3$		$\alpha_1 = \alpha_2 = \alpha_3 = -\kappa/3$
	$\beta_1 \neq \beta_2 \neq \beta_3$		$\beta_1, \beta_2 = \beta_3$		$\beta_1 = \beta_2 = \beta_3 = 0$
$\dot{T}_2$ Space:	II	$\longrightarrow$	N		
	$\alpha_1 \neq \alpha_2$		$\alpha_1 = \alpha_2 = -\kappa/3$		
	$\beta_1 \neq \beta_2$		$\beta_1 = \beta_2 = 0$		
$\dot{T}_3$ Space:	III				
	$\alpha = -\kappa/3$				

E. Weyl tensor.

- The Riemann tensor $R_{\alpha\beta\mu\nu} \in V_4$ can be decomposed in its irreducible parts as follows:

$$R_{\alpha\beta\mu\nu} \;=\; C_{\alpha\beta\mu\nu} + M_{\alpha\beta\mu\nu} - \frac{1}{6} R g_{\alpha\beta\mu\nu}, \tag{C.20}$$

  where $C_{\alpha\beta\mu\nu}$ is the Weyl tensor, and $2M_{\alpha\beta\mu\nu} \;:=\; R_{\alpha\mu}\, g_{\beta\nu} + R_{\beta\nu}\, g_{\alpha\mu} - R_{\alpha\nu} g_{\beta\mu} - R_{\beta\mu} g_{\alpha\nu}$.
  In particular, if $R_{\mu\nu} = 0$, then $R_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu}$.

- Debever (1959): the Riemann space V_4 admits the canonical form of $(C_{ab}) \in R_6$ with respect to at least one and not more than four isotropic full vectors $l^{\alpha}_{(N)} \neq 0, N = 1, 2, 3, 4$ (Debever vectors).

Petrov type	Independent eigen-directions	Debever-Sachs symbol	Equations for Debever vectors
I	all distinct	{1 1 1 1}	$l_{[\lambda} C_{\alpha]\beta\gamma[\delta} l_{\eta]} l^{\beta} l^{\gamma} = 0$
D	$l^{\alpha}_{(1)} = l^{\alpha}_{(2)}, l^{\alpha}_{(3)} = l^{\alpha}_{(4)}$	{2 2}	$C_{\alpha\beta\gamma[\delta} l_{\eta]} l^{\beta} l^{\gamma} = 0$
II	$l^{\alpha}_{(1)} = l^{\alpha}_{(2)} \neq l^{\alpha}_{(3)} \neq l^{\alpha}_{(4)}$	{2 1 1}	$C_{\alpha\beta\gamma[\delta} l_{\eta]} l^{\beta} l^{\gamma} = 0$
III	$l^{\alpha}_{(1)} = l^{\alpha}_{(2)} = l^{\alpha}_{(3)} \neq l^{\alpha}_{(4)}$	{3 1}	$C_{\alpha\beta\gamma[\delta} l_{\eta]} l^{\beta} = 0$
N	all identical	{4}	$C_{\alpha\beta\gamma\delta}\, l^{\alpha} = 0$

- To every Petrov type is assigned a gravitational field, according to the prescription:

  a. the Debever vectors $l^{\alpha}_{(N)}$ satisfies the respective equation of the series above;

  b. each Debever vector satisfies only one Petrov type.

C. References

[1] G. 't Hooft, "Past and future of gauge theory", in One Hundred Years of Gauge Theory. Past, Present and Future Perspectives, edited by S. De Bianchi and C. Kiefer, Fundamental Theories of Physics 199 (2020) (cit. on p. 2).

[2] G. 't Hooft, "Projecting local and global symmetries to the Planck scale", Preprint 2202.05367 [hep-th] (cit. on p. 8).

[3] A. Addazi et al., "Quantum gravity phenomenology at the dawn of the multi-messenger era – A review", Prog. Part. Nucl. Phys. **125**, 103948 (2022) (cit. on p. 8).

[4] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity*, 1st Ed., International Series in Pure and Applied Physics (McGraw-Hill Book Company, New York, 1965) (cit. on pp. 10, 11, 37, 45).

[5] R. J. Adler, "Six easy roads to the Planck scale", Am. J. Phys. **78**, 925 (2010) (cit. on p. 8).

[6] S. L. Adler, "Axial-vector vertex in spinor electrodynamics", Phys. Rev. **177**, 2426 (1969) (cit. on p. 64).

[7] J. Agassi, "Einstein's philosophy politely shelved", Phil. Social Scie. **45**, 515–527 (2015) (cit. on p. ii).

[8] S. Alexander and T. Manton, "Pure gauge theory for the gravitational spin connection", Phys. Rev. D **107**, 064065 (2023), Preprint 2212.05069 [gr-qc] (cit. on pp. 29, 52, 64).

[9] S. Alexander, A. Marcianò, and L. Smolin, "Gravitational origin of the weak interaction's chirality", Phys. Rev. D **89**, 065017 (2014) (cit. on p. 32).

[10] W. P. Alford and D. R. Hamilton, "Recoil spectrum in the beta decay of Ne^{19}", Phys. Rev. **95**, 1351 (1954) (cit. on p. 1).

[11] A. Ali, C. Isham, T. Kibble, and Riazuddin, eds., *Selected Papers of Abdus Salam with Commentary*, World Scientific Series in 20th Century Physics 5 (World Scientific, Singapore, 1994) (cit. on p. 1).

[12] E. Amaldi, "Beta decay opens the way to weak interactions", J. Phys. (C8) **43**, 261–300 (1982) (cit. on p. 2).

[13] D. Amati and G. Veneziano, "Metric from matter", Phys. Lett. B **105**, 358–362 (1981) (cit. on p. 37).

[14] V. Ambarzumian and D. Ivanenko, "Zur Frage nach Vermeidung der unendlichen Selbstrückwirkung des Elektrons", Z. Physik **64**, 563–567 (1930), Reproduced in [47] (cit. on p. 13).

[15] J. L. Anderson, *Principles of Relativity Physics* (Academic Press, New York, 1967) (cit. on pp. 34, 37, 52).

[16] J. L. Anderson, "Covariance, invariance, and equivalence: A viewpoint", Gen. Rel. Grav. **2**, 161–172 (1971) (cit. on p. 37).

[17] J. L. Anderson, "Does general relativity require a metric?", Preprint 9912051 [gr-qc] (cit. on p. 67).

[18] T. Appelquist, A. Chodos, and P. G. O. Freund, eds., *Modern Kaluza-Klein Theories*, Frontiers in Physics 65 (Addison-Wesley Publishing, Menlo Park, California, 1987) (cit. on pp. 13, 93, 99, 100, 105).

[19]M. F. Atiyah, R. Bott, and A. Shapiro, "Clifford modules", Topology (Suppl. 1) **3**, 3–38 (1964) (cit. on p. 5).

[20]C. Bambi, D. Malafarina, A. Marciano, and L. Modesto, "Singularity avoidance in classical gravity from four-fermion interaction", Phys. Lett. B **734**, 27–30 (2014) (cit. on p. 32).

[21]M. Bañados, C. Teitelboim, and J. Zanelli, "Dimensionally continued black holes", Phys. Rev. D **49**, 975 (1994) (cit. on p. 10).

[22]C. Barcelo, S. Liberati, and M. Visser, "Analogue gravity", Living Rev. Rel. **14**, 1–159 (2011) (cit. on p. 37).

[23]C. Barcelo, M. Visser, and S. Liberati, "Einstein gravity as an emergent phenomenon?", Int. J. Mod. Phys. D **10**, 799–806 (2001) (cit. on p. 37).

[24]V. Bargmann, "Bemerkungen zur allgemein-relativistischen Fassung der Quantentheorie", Sitzber. kgl.-preuß. Akad. Wiss. Berlin, Sitzung der phys.-math. Klasse **XXIV**, 346–354 (1932) (cit. on pp. 1, 4, 13, 24, 28, 33, 50, 52, 54).

[25]V. Bargmann, "Relativity", Rev. Mod. Phys. **29**, 161–174 (1957) (cit. on p. 37).

[26]V. Bargmann and E. P. Wigner, "Group theoretical discussion of relativistic wave equations", Proc. Nat. Acad. Scie. **34**, 211–223 (1948), Reprinted in [553], pp.82-94 (cit. on pp. 1, 5).

[27]J. Barnett and L. Smolin, "Fermion doubling in loop quantum gravity", Preprint 1507.01232 [gr-qc] (cit. on p. 65).

[28]V. B. Berestetskiĭ, "On the inner parity of the positron", Zh. Eksp. Teoret. Fiz. [JETP] **21**, 93–94 (1951) (cit. on pp. iv, 1).

[29]V. B. Berestetskiĭ, E. Lifshitz, and L. P. Pitaevskiĭ, *Quantum Electrodynamics*, Reprint of the 2nd Ed., 1982, Course of Theoretical Physics 4 (Pergamon Press, Oxford, 1994) (cit. on p. 1).

[30]P. G. Bergmann, "Observables in general relativity", Rev. Mod. Phys. **33**, 510–514 (1961) (cit. on p. 66).

[31]P. G. Bergmann and V. De Sabbata, eds., *Cosmology and Gravitation. Spin, Torsion, Rotation, and Supergravity* (Plenum Press, New York and London, 1980) (cit. on p. 89).

[32]P. G. Bergmann, V. De Sabbata, G. T. Gillies, and P. I. Pronin, eds., *Spin in Gravity. Is it possible to give an experimental basis to torsion?, International School of Cosmology and Gravitation, XV Course, Erice, Italy, May 13-20 1997* (World Scientific, Singapore, 1998) (cit. on p. 5).

[33]Z. Bern, J. J. M. Carrasco, and H. Johansson, "Perturbative quantum gravity as a double copy of gauge theory", Phys. Rev. Lett. **105**, 061602 (2010) (cit. on p. 25).

[34]Z. Bern, T. Dennen, Y. Huang, and M. Kiermaier, "Gravity as the square of gauge theory", Phys. Rev. D **82**, 065003 (2010) (cit. on p. 25).

[35]I. Białynicki-Birula, "Quantum electrodynamics without electromagnetic field", Phys. Rev. **130**, 465 (1963) (cit. on p. 34).

[36]I. Białynicki-Birula, "Nonlinear electrodynamics: Variations on a theme by Born and Infeld", in Quantum Theory of Particles and Fields: Birthday Volume Dedicated to Jan Lopuszanski, edited by B. Jancewicz and J. Lukierski (1984), pp. 31–48 (cit. on p. 14).

[37]I. Białynicki-Birula, "Field theory of photon dust", Acta Phys. Polon. **B23**, 553–559 (1992) (cit. on p. 14).

[38]I. Białynicki-Birula and Z. Białynicki-Birula, *Quantum Electrodynamics*, 1st English Ed., International Series of Monographs in Natural Philosophy 70 (Pergamon Press and Polish Scientific Publishers, Oxford and Warszawa, 1975) (cit. on p. 13).

[39]S. M. Bilenky, *Introduction to the Physics of Massive and Mixed Neutrinos*, 2nd Ed., Lecture Notes in Physics 947 (Springer-Verlag, Berlin Heidelberg, 2018) (cit. on pp. 2, 32).

[40]S. M. Bilenky, "Neutrinos: Majorana or Dirac?", Universe **6**, 134 (2020) (cit. on pp. 1, 32).

[41]S. M. Bilenky, "The prehistory of neutrino oscillations", Preprint 1902.10052 [hist-ph] (cit. on p. 1).

[42]S. M. Bilenky, C. Giunti, J. Grifols, and E. Massó, "Absolute values of neutrino masses: Status and prospects", Phys. Rep. **379**, 69–148 (2003) (cit. on p. 32).

[43]E. Bittencourt, "Dynamical bridges: The electromagnetic case", J. Phys.: Conf. Ser. **566**, 1–5 (2014) (cit. on p. 49).

[44]E. Bittencourt, S. Faci, and M. Novello, "A proposal for the origin of the anomalous magnetic moment", Int. J. Mod. Phys. A **29**, 1450075 (2014) (cit. on p. 49).

[45]E. Bittencourt, S. Faci, and M. Novello, "Chiral symmetry breaking as a geometrical process", Int. J. Mod. Phys. A **29**, 1450145 (2014) (cit. on p. 49).

[46]D. I. Blokhintsev, *Space and Time in the Microworld* (D. Reidel, Dordrecht, 1975) (cit. on pp. 8, 37).

[47]A. S. Blum and D. Rickles, eds., *Quantum Gravity in the First Half of the Twentieth Century: A Sourcebook*, Edition Open Sources 10 (Max Planck Institute for the History of Science, Berlin, 2018) (cit. on pp. 8, 85, 93, 94, 98, 100, 109–112, 115).

[48]T. Blum et al., "Hadronic light-by-light scattering contribution to the muon anomalous magnetic moment from lattice QCD", Phys. Rev. Lett. **124**, 132002 (2020) (cit. on p. 14).

[49]W. Blum, H. Dürr, and H. Rechenberg, eds., *Werner Heisenberg: Gesammelte Werke, Collected Works* (R. Piper GmbH & Co KG Verlag, Berlin, Heidelberg, 1983-1993) (cit. on pp. 15, 16, 19, 24, 26, 37, 92, 97, 98).

[50]N. N. Bogolubov, A. A. Logunov, A. I. Oksak, and I. T. Todorov, *General Principles of Quantum Field Theory*, Mathematical Physics and Applied Mathematics 10 (Kluwer Academic Publishers, Dordrecht, 1990) (cit. on p. 3).

[51]G. Boillat, "Nonlinear electrodynamics: Lagrangians and equations of motion", J. Math. Phys. **11**, 941–951 (1970) (cit. on p. 14).

[52]J. Boos and F. W. Hehl, "Gravity-induced four-fermion contact interaction implies gravitational intermediate *W* and *Z* type gauge bosons", Int. J. Theor. Phys. **56**, 751–756 (2017) (cit. on p. 5).

[53]M. Borba, "Um estudo sobre a teoria espinorial da gravitação", MSci Thesis (CBPF, Rio de Janeiro, 2009) (cit. on p. 49).

[54]M. Born, "On the quantum theory of the electromagnetic field", Proc. R. S. Lond. A **143**, 410 (1934) (cit. on p. 13).

[55]M. Born and L. Infeld, "Foundations of the new field theory", Proc. R. S. Lond. A **144**, 425–451 (1934) (cit. on p. 13).

[56]M. Born and N. S. Nagendra Nath, "The neutrino theory of light", Proc. Indian Acad. Sci. **3**, 318–337 (1936) (cit. on p. 16).

[57]L. Borsten, "Gravity as the square of gauge theory: A review", Riv. Nuovo Cim. **43**, 97–186 (2020) (cit. on p. 25).

[58]S. Böser et al., "Status of light sterile neutrino searches", Prog. Part. Nucl. Phys. **111**, 103736 (2020) (cit. on pp. 32, 64, 67, 68).

[59]R. Brandenberger, "Beyond standard inflationary cosmology", in Beyond Spacetime. The Foundations of Quantum Gravity, edited by N. Huggett, K. Matsubara, and C. Wuethrich (2020), pp. 271–299, Preprint 1809.04926 [hep-ph] (cit. on p. 32).

[60]R. Brauer and H. Weyl, "Spinors in n dimensions", Am. J. Math. **57**, 425–449 (1935) (cit. on p. 1).

[61]P. Brax, "What makes the universe accelerate? A review on what dark energy could be and how to test it", Rep. Prog. Phys. **81**, 016902 (2018) (cit. on p. 32).

[62]D. R. Brill and J. A. Wheeler, "Interaction of neutrinos and gravitational fields", Rev. Mod. Phys. **29**, 465–479 (1957), Errata: Rev. Mod. Phys. **33**, 623-624 (1961) (cit. on pp. 2, 3, 23, 33, 34, 52).

[63]K. A. Bronnikov, Y. P. Rybakov, and B. Saha, "Spinor fields in spherical symmetry: Einstein-Dirac and other space-times", Eur. Phys. J. Plus **135**, 124 (2020) (cit. on p. 39).

[64]L. M. Brown, M. Dresden, and L. Hoddeson, eds., *Pions to Quarks: Particle Physics in the 1950s. Based on a Fermilab Symposium* (Cambridge University Press, Cambridge, 1989) (cit. on p. 2).

[65]M. Bunge, "Phenomenological theories", in Critical Approaches to Science and Philosophy. With a New Introduction, edited by M. Bunge (2017), pp. 234–254 (cit. on p. 7).

[66]S. L. Cacciatori, B. L. Cerchiai, and A. Marrani, "Squaring the magic", Adv. Theor. Math. Phys. **19**, 923–954 (2015), Preprint 1208.6153 [math-ph] (cit. on p. 25).

[67]S. L. Cacciatori, G. Preparata, S. Rovelli, I. Spagnolatti, and S.-S. Xue, "On the ground state of quantum gravity", Phys. Lett. B **427**, 254–260 (1998) (cit. on p. 8).

[68]S. L. Cacciatori and A. Zanzi, *Lectures on Quantum Field Theory* (in preparation, 2023) (cit. on pp. 9, 26).

[69]E. R. Caianiello, "On the universal Fermi-type interaction I", Nuovo Cim. **8**, 534–541 (1951) (cit. on p. 1).

[70]T. Y. Cao and S. S. Schweber, "The conceptual foundations and the philosophical aspects of renormalization theory", Synthese **97**, 33–108 (1993) (cit. on pp. 24, 27).

[71]S. Capozziello, M. De Laurentis, L. Fabbri, and S. Vignolo, "Running coupling in electroweak interactions of leptons from $f(R)$-gravity with torsion", Eur. Phys. J. C **72**, 1–10 (2012) (cit. on pp. 5, 32).

[72]L. Cappiello, O. Catà, G. D'Ambrosio, D. Greynat, and A. Iyer, "Axial-vector and pseudoscalar mesons in the hadronic light-by-light contribution to the muon $(g-2)$", Phys. Rev. D **102**, 016009 (2020) (cit. on p. 14).

[73]E. Cartan, "Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion", Comptes Rendus Acad. Scie. **174**, 593–595 (1922), English translation in [31], pp.489-491 (cit. on p. 50).

[74]E. Cartan, *Leçons sur la Théorie des Spineurs* (Hermann, Paris, 1937), English edition: *The Theory of Spinors*, Hermann, 1966 (cit. on p. 1).

[75]R. Casalbuoni, "The standard model of electroweak interactions. Lectures given at the Otranto School, September 1997", 1–91 (1997) (cit. on p. 2).

[76]D. Cassidy, "Cosmic ray showers, high energy physics, and quantum field theories: Programmatic interactions in the 1930s", Historical Studies in the Physical Sciences **12**, 1–39 (1981) (cit. on p. 15).

[77]L. Castellani, R. D'Auria, and P. Fré, *Supergravity and Superstrings. A Geometric Perspective*, Vol. 1: Mathematical Foundations (World Scientific, Singapore, 1991) (cit. on p. 35).

[78]C. Castro, "On generalized Yang-Mills theories and extensions of the standard model in Clifford (tensorial) spaces", Ann. Phys. **321**, 813–839 (2006) (cit. on p. 5).

[79]C. Castro, "On modified Weyl-Heisenberg algebras, noncommutativity, matrix-valued Planck constant and QM in Clifford spaces", J. Phys. A: Math. Gen. **39**, 14205 (2006) (cit. on p. 5).

[80]E. Chacón, S. Nagy, and C. D. White, "The Weyl double copy from twistor space", J. High Energ. Phys. **2021**, 2239 (2021) (cit. on p. 25).

[81]M. Chaichian, M. Oksanen, and A. Tureanu, "Sakharov's induced gravity and the Poincaré gauge theory", in Jacob Bekenstein: The Conservative Revolutionary (2019), pp. 271–299 (cit. on p. 37).

[82]T.-P. Cheng and L.-F. Li, *Gauge Theory of Elementary Particle Physics* (Oxford University Press, Oxford, 2004) (cit. on p. 2).

[83]D. Chester, A. Marrani, and M. Rios, "Beyond the standard model with six-dimensional spinors", Particles **6**, 144–172 (2023) (cit. on p. 5).

[84]C. Chevalley, *The Algebraic Theory of Clifford Algebras* (Springer, 1997) (cit. on p. 1).

[85]J. S. R. Chisholm and R. S. Farwell, "Electroweak spin gauge theories and the frame field", J. Phys. A: Math. Gen. **20**, 6561–6580 (1987) (cit. on p. 52).

[86]J. S. R. Chisholm and R. S. Farwell, "Gravity and the frame field", Gen. Rel. Grav. **20**, 371–382 (1988) (cit. on p. 52).

[87]Y. Choquet-Bruhat, *General Relativity and the Einstein Equations*, Oxford Mathematical Monographs (Oxford University Press, Oxford, 2009) (cit. on pp. 9, 10, 68).

[88]S. Colin, T. Durt, and R. Willox, "L. de Broglie's double solution program: 90 years later", Ann. de la Found. Louis de Broglie **42**, 19–71 (2017), Preprint 1703.06158 [quant-ph] (cit. on p. 4).

[89]P. Collas and D. Klein, "Dirac particles in a gravitational shock wave", Class. Quantum Grav. **35**, 125006 (2018) (cit. on p. 39).

[90]J. Collins, A. Perez, and D. Sudarsky, "Lorentz invariance violation and its role in quantum gravity phenomenology", in Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, edited by D. Oriti (2009), pp. 528–547, Preprint 0603002 [hep-th] (cit. on pp. 8, 67).

[91]A. W. Conway, "Quaternion treatment of the relativistic wave equation", Proc. R. Soc. Lond. A **162**, 145–154 (1936) (cit. on p. 1).

[92]R. T. Cox, C. G. McIlwraith, and B. Kurrelmeyer, "Apparent evidence of polarization in a beam of β-rays", Proc. Nat. Acad. Scie. **14**, 544–549 (1928) (cit. on p. 1).

[93]J. P. Crawford, "Local automorphism invariance: Gauge boson mass without a Higgs particle", J. Math. Phys. **35**, 2701–2718 (1994) (cit. on pp. 28, 52, 68).

[94]J. P. Crawford, "Spinor metrics, spin connection compatibility and spacetime geometry from spin geometry", Class. Quant. Grav. **20**, 2945–2962 (2003) (cit. on p. 52).

[95]G. Danby et al., "Observation of high-energy neutrino reactions and the existence of two kinds of neutrinos", Phys. Rev. Lett. **9**, 36–44 (1962) (cit. on p. 2).

[96]O. Darrigol, *Relativity Principles and Theories from Galileo to Einstein* (Oxford University Press, Oxford, 2022) (cit. on pp. 36, 37).

[97]A. Das, D. Ghosh, C. Giunti, and A. Thalapillil, "Neutrino charge constraints from scattering to the weak gravity conjecture to neutron stars", Phys. Rev. D **102**, 115009 (2020) (cit. on p. 32).

[98]L. de Broglie, *Théorie Générale des Particules a Spin. Méthode de Fusion*, 1st ed. (Gauthier-Villars, Paris, 1943) (cit. on pp. 1, 4, 16).

[99]L. de Broglie, *Non-linear Wave Mechanics. A Causal Interpretation* (Elsevier Pub. Comp., Amsterdam, 1960) (cit. on pp. 4, 16).

[100]P. De Fabritiis, P. C. Malta, and J. A. Helayël-Neto, "Phenomenology of a Born-Infeld extension of the $U(1)_Y$ sector at lepton colliders", Phys. Rev. D **105**, 016007 (2022) (cit. on p. 14).

[101]V. De Sabbata and M. Gasperini, "Is weak interaction a Cartan force between the elementary particles?", Lett. Nuovo Cim. **21**, 328–332 (1978) (cit. on p. 5).

[102]V. De Sabbata and M. Gasperini, "Strong gravity and weak interactions", Gen. Rel. Grav. **10**, 731–741 (1979) (cit. on p. 5).

[103]B. De Witt, "Gravity: A universal regulator?", Phys. Rev. Lett. **13**, 114 (1964) (cit. on p. 8).

[104]B. De Witt, *The Global Approach to Quantum Field Theory, 2 vols.* International Series of Monographs on Physics 114 (Clarendon Press and Oxford, Oxford, 2003) (cit. on p. 3).

[105]R. Debever, "La super-énergie en relativité générale", Bull. Soc. Math. Belg. **10**, 112–147 (1959) (cit. on p. 4).

[106]R. Debever, ed., *Elie Cartan and Albert Einstein. Letters on Absolute Parallelism, 1929-1932*, Princeton Legacy Library (Princeton University Press, Princeton, 1979) (cit. on p. 11).

[107]H. Dehnen and F. Ghaboussi, "Gauge field theory of gravity", Phys. Rev. D **33**, 2205 (1986) (cit. on p. 5).

[108]H. Dehnen and E. Hitzer, "Spin-gauge theory of gravity with Higgs-field mechanism", Int. J. Theor. Phys. **33**, 575–592 (1994) (cit. on p. 5).

[109]H. Dehnen and E. Hitzer, "$SU(2) \times U(1)$ gauge gravity", Int. J. Theor. Phys. **34**, 1981–2001 (1995) (cit. on p. 5).

[110]S. Deser, "General relativity and the divergence problem in quantum field theory", Rev. Mod. Phys. **29**, 417–423 (1957) (cit. on p. 25).

[111]S. Deser, "Self-interaction and gauge invariance", Gen. Rel. Grav. **1**, 9–18 (1970), Preprint 0411023 [gr-qc] (cit. on p. 25).

[112]S. Deser and B. Zumino, "Broken supersymmetry and supergravity", Phys. Rev. Lett. **38**, 1433 (1977) (cit. on p. 5).

[113]N. Dewar, "General-relativistic covariance", Found. Phys. **50**, 294–318 (2020) (cit. on p. 37).

[114]E. Di Valentino, A. Melchiorri, and J. Silk, "Planck evidence for a closed Universe and a possible crisis for cosmology", Nat. Astron. **4**, 196–203 (2020) (cit. on p. 32).

[115]R. H. Dicke, "Principle of equivalence and the weak interactions", Rev. Mod. Phys. **29**, 355–362 (1957) (cit. on p. 2).

[116]V. E. Didenko and N. K. Dosmanbetov, "Classical double copy and higher-spin fields", Phys. Rev. Lett. **130**, 071603 (2023) (cit. on p. 25).

[117]C. F. Diether III and J. Christian, "On the role of Einstein-Cartan Gravity in the fundamental particle physics", Universe **6**, 112 (2020) (cit. on pp. 5, 9, 32).

[118]J. Dieudonné, *A History of Algebraic and Differential Topology. 1900-1960*, Modern Birkhäuser Classics (Birkhäuser, Boston, 2009) (cit. on p. 3).

[119]P. A. M. Dirac, "Application of quaternions to Lorentz transformations", Proc. R. Irish Acad. A **50**, 261–270 (1945) (cit. on p. 1).

[120]P. A. M. Dirac, "Forms of relativistic dynamics", Rev. Mod. Phys. **21**, 392–399 (1949) (cit. on p. 25).

[121]P. A. M. Dirac, "Electron wave equation in Riemannian space", in Max Planck Festschrift, edited by B. Kockel, W. Macke, and A. Papapetrou (1958), pp. 339–334 (cit. on p. 4).

[122]P. A. M. Dirac, "The quantum theory of the electron", Proc. R. Soc. Lond. A **117**, 610–624 (1928) (cit. on p. 1).

[123]P. A. M. Dirac, "The quantum theory of the electron. Part II", Proc. R. Soc. Lond. A **118**, 351–361 (1928) (cit. on p. 1).

[124]P. A. M. Dirac, "The proton", Nature **126**, 605–606 (1930) (cit. on p. 16).

[125]P. A. M. Dirac, "Quantised singularities in the electromagnetic field", Proc. R. Soc. Lond. A **133**, 60–72 (1931) (cit. on p. 3).

[126]P. A. M. Dirac, "The electron wave equation in De-Sitter space", Ann. Math. **36**, 657–669 (1935) (cit. on pp. 1, 48).

[127]P. A. M. Dirac, “A reformulation of the Born-Infeld electrodynamics”, Proc. R. Soc. Lond. A **257**, 32–43 (1960) (cit. on p. 14).

[128]P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th Ed., Reprint: 1958, International Series of Monographs on Physics 27 (Oxford University Press, Oxford, 2011) (cit. on p. 50).

[129]G. M. Dixon, “Division algebras; spinors; idempotents; the algebraic structure of reality”, Preprint 1012.1304 [hep-th] (cit. on p. 5).

[130]A. D. Dolgov and Y. B. Zeldovich, “Cosmology and elementary particles”, Rev. Mod. Phys. **53**, 1–41 (1981) (cit. on p. 37).

[131]J. F. Donoghue, “Is the spin connection confined or condensed?”, Phys. Rev. D **96**, 044003 (2017) (cit. on pp. 29, 52, 64).

[132]J. F. Donoghue and B. R. Holstein, “Low energy theorems of quantum gravity from effective field theory”, J. Phys. G: Nucl. Part. Phys. **42**, 103102 (2015) (cit. on p. 8).

[133]J. Droste, “The field of a single centre in Einstein’s theory of gravitation, and the motion of a particle in that field”, Koninklijke Nederlandsche Akademie van Wetenshappen Proceedings **19**, 197 (1917), Republication in Gen. Rel. Grav. **34**, 1545-1563 (2002) (cit. on p. 10).

[134]R. Ducrocq, “New solutions in supergravity: A phenomenological study at the LHC”, PhD Thesis (Université de Strasbourg, 2021), 2212.06798 [hep-th] (cit. on p. 5).

[135]G. V. Dunne, “The Heisenberg-Euler effective Lagrangians: Basics and extensions”, in From Fields to Strings: Circumnavigating Theoretical Physics, Vol. 3, edited by M. Shifman, A. Vainshtein, and J. Wheater (2005), pp. 445–522, Preprint 0406216 [hep-th] (cit. on p. 15).

[136]G. V. Dunne, “The Heisenberg-Euler effective action: 75 years on”, Int. J. Mod. Phys.: Conference Series **14**, 42–56 (2012) (cit. on p. 15).

[137]H.-P. Dürr, W. Heisenberg, H. Mitter, S. Schlieder, and K. Yamazaki, “Zur Theorie der Elementarteilchen”, Z. Nat. A **14**, 441–485 (1959), Reprinted in [49], Series A-3, pp.352-396 (cit. on pp. 20, 36).

[138]J. Eccles, “Principles of Scientific Method”, in Notes on Lectures by Dr. K. R. Popper given at the University of Otago from May 22nd to 26th, 1945 (2016), Available at Popper and Prior in New Zealand, accessed March 20, 2023 (cit. on p. 8).

[139]T. Eguchi and K. Nishijima, eds., *Broken Symmetry. Selected Papers of Y. Nambu* (World Scientific, Singapore, 1995) (cit. on pp. 104, 105).

[140]A. Einstein, “Über die Entwicklung unserer Anschauungen über das Wesen und die Konstitution der Strahlung”, Physikalische Zeitschrift **10**, 817–826 (1909), Reprinted in [484], Doc. 60 (cit. on p. 13).

[141]A. Einstein, “Näherungsweise Integration der Feldgleichungen der Gravitation”, Sitzber. kgl.-preuß. Akad. Wiss. Berlin, Sitzung der phys.-math. Klasse **XXXII**, 688–696 (1916), Reprinted in [287], Doc. 32 (cit. on pp. 8, 38, 47).

[142]A. Einstein, “Spielen Gravitationsfelder im Aufbau der materiellen Elementarteilchen eine wesentliche Rolle?”, Sitzber. kgl.-preuß. Akad. Wiss. Berlin, Sitzung der phys.-math. Klasse, 349–356 (1919), English translation in [326], pp.189-198 (cit. on pp. 24, 25, 33, 37).

[143]A. Einstein, "Zu Kaluzas Theorie des Zusammenhanges von Gravitation und Elektrizität (zwei Mitteilungen)", Sitzber. kgl.-preuß. Akad. Wiss. Berlin, Sitzung der phys.-math. Klasse, 23–30 (1927) (cit. on p. 11).

[144]A. Einstein, "Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus", Sitzber. kgl.-preuß. Akad. Wiss. Berlin, Sitzung der phys.-math. Klasse, 217–221 (1928), English translation in 0503046 [physics] (cit. on pp. 11, 24, 25).

[145]A. Einstein, "Autobiographical notes", in Albert Einstein: Philosopher-Scientist, Vol. VII, edited by P. A. Schilpp (1949), pp. 1–95 (cit. on pp. 50, 67).

[146]A. Einstein, "Reply to criticisms", in Albert Einstein: Philosopher-Scientist, Vol. VII, edited by P. A. Schilpp (1949), pp. 665–688 (cit. on pp. 37, 67).

[147]A. Einstein, *Relativity*, 1st ed. 1916, Routledge Classics (Routledge, London and New York, 2003) (cit. on pp. 13, 67).

[148]A. Einstein, *The Meaning of Relativity*, 1st ed. 1922, Routledge Classics (Routledge, London and New York, 2003) (cit. on pp. 11, 38).

[149]A. Einstein and V. Bargmann, "Bivector fields", Ann. Math. **45**, 1–14 (1944) (cit. on pp. 37, 50).

[150]A. Einstein, V. Bargmann, and P. G. Bergmann, "On the five-dimensional representation of gravitation and electricity", in Theodore von Kármán Anniversary Volume: Contributions to Applied Mechanics and Related Subjects by the Friends of Theodore von Kármán on His Sixtieth Birthday (1941), pp. 212–225 (cit. on pp. iv, 12).

[151]A. Einstein and P. Bergmann, "On a generalization of Kaluza's theory of electricity", Ann. Math. **39**, 683–701 (1938), Reprinted in [18], pp.89-107 (cit. on p. 12).

[152]A. Einstein, L. Infeld, and B. Hoffmann, "The gravitational equations and the problem of motion", Ann. Math. **39**, 65–100 (1938), Reproduced in [47] (cit. on pp. 10, 11).

[153]J. Ellis, "100 years of general relativity", in General Relativity and Gravitation: A Centennial Perspective, edited by A. Ashtekar, B. K. Berger, J. Isenberg, and M. Maccallum (2015), pp. 10–48, Preprint 1509.01772 [gr-qc] (cit. on p. 10).

[154]C. P. Enz, *No Time to be Brief. A Scientific Biography of Wolgang Pauli* (Oxford University Press, Oxford, 2002) (cit. on pp. 15, 16, 19, 23, 27, 50).

[155]C. P. Enz and K. von Meyenn, eds., *Wolfgang Pauli Writings on Physics and Philosophy* (Springer-Verlag, Berlin, Heidelberg, 1994) (cit. on pp. 18, 21, 23, 107).

[156]H. Euler, "Über die Streuung von Licht an Licht nach der Diracshen Theorie", Ann. der Phys. (Leipzig) **26**, 398–448 (1936) (cit. on p. 29).

[157]H. Euler and B. Kockel, "Über die Streuung von Licht an Licht nach der Diracshen Theorie", Naturwissenschaften **23**, 246–247 (1935) (cit. on p. 15).

[158]L. Fabbri, "The spin-torsion coupling and causality for the standard model", Mod. Phys. Lett. A **26**, 2091–2100 (2011) (cit. on p. 5).

[159]L. Fabbri, "Fundamental theory of torsion gravity", Universe **7**, 305 (2021) (cit. on p. 5).

[160]L. D. Faddeev, L. A. Khalfin, and I. V. Komarov, eds., *Selected Works of V. A. Fock. Quantum Mechanics and Quantum Field Theory* (CRC Press, Boca Raton, Florida, 2004) (cit. on pp. 32, 94).

[161]E. E. Fairchild Jr., “Yang-Mills formulation of gravitational dynamics”, Phys. Rev. D **16**, 2438 (1977) (cit. on pp. 28, 52).

[162]P. Fayet, “Mixing between gravitational and weak interactions through the massive gravitino”, Phys. Lett. B **70**, 461–464 (1977) (cit. on p. 5).

[163]E. Fermi, “Tentativo di una teoria dei raggi β”, Nuovo Cim. **11**, 1–19 (1934), Reprinted in [165] (cit. on p. 1).

[164]E. Fermi, “Versuch einer Theorie der β-Strahlen. I”, Z. Physik **88**, 161–177 (1934), Reprinted in [165] (cit. on p. 1).

[165]E. Fermi, *Collected Papers (Note e Memorie)*, edited by E. Amaldi et al., Vol. I. Italy 1921-1938 (The University of Chicago Press, Accademia Nazionale dei Lincei, Chicago, Roma, 1962) (cit. on p. 94).

[166]J. G. Fernandes, “Teoria espinorial da gravitação”, MSci Thesis (Universidade de Brasília, 2019) (cit. on p. 49).

[167]R. P. Feynman, *Lectures on Gravitation*, edited by B. Hatfield (Addison-Wesley Pub., Menlo Park, California, 1995) (cit. on pp. 8, 25, 39).

[168]R. P. Feynman and M. Gell-Mann, “Theory of the Fermi interaction”, Phys. Rev. **109**, 193 (1958) (cit. on pp. 1, 35).

[169]M. Fierz, “Zur Fermischen Theorie des β-Zerfalls”, Z. Physik **104**, 553–565 (1937) (cit. on p. 35).

[170]M. Fierz and W. Pauli, “On relativistic wave equations for particles of arbitrary spin in an electromagnetic field”, Proc. R. S. Lond. A **173**, 211–232 (1939) (cit. on pp. 5, 35).

[171]R. Finkelstein, “Spacetime of the elementary particles”, J. Math. Phys. **1**, 440–451 (1960) (cit. on p. 4).

[172]J. G. Fletcher, “Dirac matrices in Riemannian space”, Nuovo Cim. **8**, 451–458 (1958) (cit. on p. 52).

[173]V. A. Fock, “Über die invariante Form der Wellen- und der Bewegungsgleichungen für einen geladenen Massenpunkt”, Z. Physik **39**, 226–232 (1926), English translation in [160], Paper 26-2 (cit. on p. 11).

[174]V. A. Fock, “Dirac wave equation and Riemann geometry”, J. Phys. Radium, Série VI **10**, 392 (1929), Reprinted in [160], Paper 29-4 (cit. on pp. 1, 24, 33, 52).

[175]V. A. Fock, “Geometrisierung der Diracschen theorie des elektrons”, Z. Physik **57**, 261–277 (1929), Reproduced in [47] (cit. on pp. 1, 24, 52).

[176]V. A. Fock, “The principles of relativity and of equivalence in the Einsteinian gravitation theory. I.”, Det Kongelige Norske Videnskabers Selskabs Forhandlinger **63**, 16–21 (1963) (cit. on pp. iv, 11, 25).

[177]V. A. Fock, “The principles of relativity and of equivalence in the Einsteinian gravitation theory. II.”, Det Kongelige Norske Videnskabers Selskabs Forhandlinger **63**, 22–27 (1963) (cit. on pp. iv, 11, 25).

[178]V. A. Fock, *The Theory of Space, Time and Gravitation*, Reprint of 2nd English Ed., 1966 (Pergamon Press, Oxford, 1976) (cit. on pp. 25, 47).

[179]V. A. Fock and D. Ivanenko, "Über eine mögliche geometrische Deutung der relativistischen Quantentheorie", Z. Physik **54**, 798–802 (1929) (cit. on pp. 1, 52).

[180]J. Formiga, "Conservation of the Dirac current in models with a general spin connection", Preprint 1210.0759 [hep-th] (cit. on p. 49).

[181]M. Fouché, R. Battesti, and C. Rizzo, "Limits on nonlinear electrodynamics", Phys. Rev. D **93**, 093020 (2016) (cit. on p. 15).

[182]E. S. Fradkin and A. A. Tseytlin, "Effective field theory from quantized strings", Phys. Lett. B **158**, 316–322 (1985) (cit. on p. 14).

[183]E. S. Fradkin and A. A. Tseytlin, "Nonlinear electrodynamics from quantized strings", Phys. Lett. **163B**, 123–130 (1985) (cit. on p. 14).

[184]D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, "Progress toward a theory of supergravity", Phys. Rev. D **13**, 3214 (1976) (cit. on p. 5).

[185]D. Z. Freedman and A. Van Proeyen, eds., *Supergravity* (Cambridge University Press, Cambridge, 2012) (cit. on p. 5).

[186]F. A. M. Frescura and B. J. Hiley, "Algebras, quantum theory and pre-space", Rev. Bras. Fís. Volume Especial: Os 70 anos de Mario Schönberg, 49–86 (1984) (cit. on p. 3).

[187]A. Friedman, "Über die Krümmung des Raumes", Z. Physik **10**, 377–386 (1922) (cit. on p. 39).

[188]J. I. Friedman and V. L. Telegdi, "Nuclear emulsion evidence for parity nonconservation in the decay chain $\pi^+ \to \mu^+ \to e^+$", Phys. Rev. **106**, 1290 (1957) (cit. on p. 1).

[189]H. Fritzsch and P. Minkowski, "Unified interactions of leptons and hadrons", Ann. Phys. **93**, 193–266 (1975) (cit. on p. 5).

[190]C. Furey, "Unified theory of ideals", Phys. Rev. D **86**, 025024 (2012) (cit. on p. 5).

[191]C. Furey, "Standard model physics from an algebra?", PhD Thesis (University of Waterloo, Waterloo, Ontario, 2015), 1611.09182 [hep-th] (cit. on p. 5).

[192]N. Furey and M. J. Hughes, "Division algebraic symmetry breaking", Phys. Lett. B **831**, 137186 (2022) (cit. on p. 5).

[193]W. I. Fushchich and R. Z. Zhdanov, "Symmetry and exact solutions of nonlinear spinor equations", Phys. Rep. **172**, 123–174 (1989) (cit. on p. 36).

[194]G. Gamow, D. Ivanenko, and L. Landau, "World constants and transition limiting", Zh. Russ. Fiz.-Khim. O-va., Chast Fiz. **60**, 13 (1928), Republication in Phys. Atom. Nuclei **65**, 1373-1375 (2002 (cit. on pp. 13, 20, 24).

[195]R. L. Garwin, L. M. Lederman, and M. Weinrich, "Observations of the failure of conservation of parity and charge conjugation in meson decays: the magnetic moment of the free muon", Phys. Rev. **105**, 1415 (1957) (cit. on p. 1).

[196]L. Geiger, "Quadratically Extended *BF* Theory in the BV Formalism. Extending Theories of Gravity with Torsion", MSci Thesis (ETH Zürich, 2022) (cit. on p. 5).

[197] A. Geitner, M. Hanauske, and E. Hitzer, "New Higgs field ansatz for effective gravity in flat space time", Acta Phys. Pol. B **29**, 971 (1998), Preprint 9801048 [gr-qc] (cit. on p. 5).

[198] A. Geitner, D. Ketterer, and H. Dehnen, "Gravity and parity violation", Nuovo Cim. B **115**, 1357–1378 (2000), Preprint 9712074 [gr-qc] (cit. on pp. 5, 32).

[199] M. Gell-Mann, "Test of the nature of the vector interaction in β decay", Phys. Rev. **111**, 362 (1958) (cit. on p. 1).

[200] M. Gell-Mann and A. Pais, "Behavior of neutral particles under charge conjugation", Phys. Rev. **97**, 1387 (1955) (cit. on p. 1).

[201] H. Georgi and S. L. Glashow, "Unity of all elementary-particle forces", Phys. Rev. Lett. **32**, 438–441 (1974) (cit. on p. 5).

[202] R. Geroch, "Spinor structure of space-times in general relativity. I", J. Math. Phys. **9**, 1739 (1968) (cit. on p. 5).

[203] R. Geroch, "Spinor structure of space-times in general relativity. II", J. Math. Phys. **11**, 343 (1970) (cit. on p. 5).

[204] F. Ghaboussi, "A unified gravito-electroweak gauge field model", Nuovo Cim. A **104**, 1475–1481 (1991) (cit. on p. 5).

[205] C. Giunti and A. Studenikin, "Neutrino electromagnetic interactions: A window to new physics", Rev. Mod. Phys. **87**, 531–591 (2015), Preprint 1403.6344 [hep-ph] (cit. on p. 32).

[206] S. L. Glashow, "Partial-symmetries of weak interactions", Z. Physik **22**, 579–588 (1961), Reprinted in [296], pp.171-180 (cit. on p. 2).

[207] S. L. Glashow, J. Iliopoulos, and L. Maiani, "Weak interactions with lepton-hadron symmetry", Phys. Rev. D **2**, 1285 (1970), Reprinted in [296], pp.203-210 (cit. on p. 2).

[208] H. F. M. Goenner, "On the history of unified field theories", Living Rev. Rel. **7**, 2 (2004) (cit. on pp. 11, 13, 50).

[209] H. F. M. Goenner, "On the history of unified field theories. Part II. (ca. 1936-ca. 1960)", Living Rev. Rel. **17**, 5 (2014) (cit. on pp. 11, 37, 50).

[210] H. S. Green, "Dirac matrices, teleparallelism and parity conservation", Nucl. Phys. **7**, 373–383 (1958) (cit. on pp. 28, 52, 55).

[211] H. S. Green, "Spinor fields in general relativity", Proc. R. Soc. London A **245**, 521–535 (1958) (cit. on pp. 28, 52, 55).

[212] D. J. Gross, "Oscar Klein and gauge theory", in The Oskar Klein Centenary. Proceedings of the Symposium, held 19-21 September 1994 in Stockholm, Sweden, edited by U. Lindström (1995), Preprint 9411233 [hep-th] (cit. on pp. 2, 13).

[213] P. Gulmanelli, *Su una Teoria dello Spin Isotopico*, Pubblicazioni della Sezione di Milano dell'Istituto Nazionale di Fisica Nucleare (Casa Editrice Pleion, Milano, 1954), English translation to appear in [223] (cit. on pp. iv, 1, 52, 54).

[214]N. D. S. Gupta, “The wave equation for zero rest-mass particles”, Nucl. Phys. B **4**, 147–152 (1967) (cit. on p. 5).

[215]K. Habermann, “The Dirac operator on symplectic spinors”, Ann. Global Anal. Geom. **13**, 155–168 (1995) (cit. on p. 4).

[216]K. Habermann and L. Habermann, *Introduction to Symplectic Dirac Operators* (Springer-Verlag, Berlin, Heidelberg, 2006) (cit. on p. 4).

[217]J. Hacki, H. Ruegg, and G. Wanders, eds., *E.C.G. Stueckelberg, An Unconventional Figure of Twentieth Century Physics. Selected Scientific Papers with Commentaries* (Birkhäuser Verlag, Basel, 2009) (cit. on pp. 16, 27).

[218]A. Hagar, “Squaring the circle: Gleb Wataghin and the prehistory of quantum gravity”, Studies in History and Philosophy of Modern Physics **46**, 217–227 (2014) (cit. on p. 22).

[219]G. S. Hall, *Symmetries and Curvature Structure in General Relativity*, World Scientific Lecture Notes in Physics 46 (World Scientific, Singapore, 2004) (cit. on pp. 62, 77).

[220]O. Halpern, “Scattering processes produced by electrons in negative energy states”, Phys. Rev. **44**, 855 (1933) (cit. on pp. 15, 29).

[221]P. Halpern, “Nordström, Ehrenfest, and the role of dimensionality in physics”, Phys. Perspect. **6**, 390–400 (2004) (cit. on p. 11).

[222]A. E. S. Hartmann, “On Pauli’s square root conjecture of the gravitational constant”, Forthcoming Preprint (2023) (cit. on p. 37).

[223]A. E. S. Hartmann, “Translation of P. Gulmanelli’s seminar notes ‘On a theory of isotopic spin’ (1953)”, Forthcoming Preprint (2023) (cit. on p. 96).

[224]A. E. S. Hartmann and S. L. Cacciatori, “On non-Abelian gauge fields in the Clifford bundle”, Forthcoming Preprint (2023) (cit. on pp. 52, 63).

[225]S. Hawking, “Perturbations of an expanding universe”, Astrophys. J. **145**, 544–554 (1966) (cit. on pp. 68, 69).

[226]F. W. Hehl and B. K. Datta, “Nonlinear spinor equation and asymmetric connection in general relativity”, J. Math. Phys. **12**, 1334–1339 (1971) (cit. on p. 5).

[227]W. Heisenberg, “Die Selbstenergie des Elektrons”, Z. Physik **65**, 4–13 (1930), Reprinted in [49], Series A-2, Paper 6.1; English translation in [339] (cit. on pp. 15, 16, 20, 21, 67).

[228]W. Heisenberg, “Bemerkungen zur Diracschen Theorie des Positrons”, Z. Phys. **90**, 209–231 (1934), Reprinted in [49], Series A-2 (cit. on p. 15).

[229]W. Heisenberg, “Über die in der Theorie dr Elementarteilchen auftretende universelle Länge”, Ann. Phys. (Leipzig) **32**, 20–33 (1938), Reprinted in [49], Series A-2, pp.301-314; English translation in [339] (cit. on p. 16).

[230]W. Heisenberg, “Die ‘beobachtbaren Größen’ in der Theorie der Elementarteilchen”, Z. Physik **120**, 513–538 (1943), Reprinted in [49], Series A-2 (cit. on p. 13).

[231] W. Heisenberg, “Quantum theory of fields and elementary particles”, Rev. Mod. Phys. **29**, 269–278 (1957), Reprinted in [49], Series B-1, pp.552-561 (cit. on pp. 25, 36).

[232] W. Heisenberg, *Introduction to the Theory of Unified Field Theory of Elementary Particles* (John Wiley & Sons, London, 1966), Reprinted in [49], Series B-1, pp.677-861 (cit. on p. 25).

[233] W. Heisenberg and E. Euler, “Folgerungen aus der Diracschen Theorie des Positrons”, Z. Physik **98**, 714–732 (1936) (cit. on p. 29).

[234] W. Heisenberg and W. Pauli, “On the isospin group in the theory of the elementary particles”, Unpublished Preprint (1958), Reprinted in [49], Series A-3, pp.337-351 (cit. on pp. 19, 20, 36).

[235] J. A. Helayël-Neto, C. A. Hernaski, B. Pereira-Dias, A. A. Vargas-Paredes, and V. J. Vasquez-Otoya, “Chern-Simons gravity with $(\text{curvature})^2$ and $(\text{torsion})^2$ terms and a basis of degree-of-freedom projection operators”, Phys. Rev. D **82**, 064014 (2010) (cit. on p. 5).

[236] R. Hermann, *Vector Bundles in Mathematical Physics*, 2 vols (W. A. Benjamin, Inc., New York, 1970) (cit. on p. 5).

[237] D. Hestenes, *Space-Time Algebra* (Gordon and Breach, New York, 1966) (cit. on p. 3).

[238] D. Hestenes, “Space-time structure of weak and electromagnetic interactions”, Found. Phys. **12**, 153–168 (1982) (cit. on p. 3).

[239] J. M. Hoff da Silva, D. Beghetto, R. T. Cavalcanti, and R. da Rocha, “Exotic fermionic fields and minimal length”, Eur. Phys. J. C **80**, 727 (2020) (cit. on p. 5).

[240] J. M. Hoff da Silva, R. T. Cavalcanti, D. Beghetto, and G. M. Caires da Rocha, “A geometrical approach to nontrivial topology via exotic spinors”, J. High Energy Phys. **2023**, 59 (2023) (cit. on p. 5).

[241] S. Hossenfelder, “Minimal length scale scenarios for quantum gravity”, Living Rev. Rel. **16**, 2 (2013) (cit. on pp. 8, 15, 20).

[242] L. Infeld and B. L. van der Waerden, “Die Wellengleichung des Elektrons in der allgemeinen Relativitätstheorie”, Sitzber. kgl.-preuß. Akad. Wiss. Berlin, Sitzung der phys.-math. Klasse, 380–401 (1933), Reproduced in [47] (cit. on p. 1).

[243] A. Inomata and W. A. McKinley, “Geometric theory of neutrinos”, Phys. Rep. **140**, 1467 (1965) (cit. on pp. 2, 36).

[244] C. J. Isham, A. Salam, and J. Strathdee, “f-Dominance of gravity”, Phys. Rev. **3**, 867 (1970) (cit. on p. 5).

[245] D. Ivanenko, “On the possible transmutations of ordinary matter in gravitation”, in Les Theories Relativistes de la Gravitation. Colloque International, Royaumont, 21-25 Juin 1959, edited by A. Lichnerowicz and M. A. Tonnelat (1962), pp. 433–439 (cit. on p. 25).

[246] D. Ivanenko, “A compensating treatment of gravitation”, in Relativistic Theories of Gravitation. Proceedings of a conference held in Warsaw and Jabłonna July, 1962, edited by L. Infeld (1964), pp. 212–215 (cit. on pp. 4, 25).

[247] D. Ivanenko and A. Brodskiĭ, “Interaction of gravitation with a particle vacuum”, Deklady Akademii Nauk SSSR **92**, 731–734 (1953) (cit. on p. 25).

[248]D. Ivanenko and A. Sokolow, “Self-interaction of neutrons and protons”, Nature **138**, 684 (1936) (cit. on p. 28).

[249]J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons. The Relativistic Quantum Field Theory of Charged Particles with Spin One-half*, 2nd Expanded Ed., TMP (Springer-Verlag, Berlin, 1980) (cit. on p. 15).

[250]C. Jensen, *Controversy and Consensus: Nuclear Beta Decay 1911-1934*, edited by F. Aaserud, H. Kragh, E. Rüdinger, and R. H. Stuewer, Historical Studies Science Networks (Springer, Basel, 2000) (cit. on p. 21).

[251]J. B. Jiménez, L. Heisenberg, and T. Koivisto, “The coupling of matter and spacetime geometry”, Class. Quantum Grav. **37**, 195013 (2020) (cit. on pp. 38, 67).

[252]J. B. Jiménez, L. Heisenberg, G. J. Olmo, and D. Rubiera-Garcia, “Born-Infeld inspired modifications of gravity”, Phys. Rep. **727**, 1–129 (2018) (cit. on p. 14).

[253]K. Johnson and E. C. G. Sudarshan, “Inconsistency of the local field theory of charged spin $3/2$ particles”, Ann. Phys. **13**, 126–145 (1961) (cit. on p. 5).

[254]H. Joos, “Zur Darstellungstheorie der inhomogenen Lorentzgruppe als Grundlage quantenmechanischer Kinematik”, Fortschr. Phys. **10**, 65–146 (1963) (cit. on p. 5).

[255]P. Jordan, “Die Lichtquantenhypothese”, Ergebnisse der Exakten Naturwissenschaften **7**, 158 (1928) (cit. on p. 16).

[256]P. Jordan, “Zur Neutrinotheorie des Lichtes”, Z. Physik **93**, 464–472 (1935) (cit. on p. 16).

[257]P. Jordan, “Fünfdimensionale Kosmologie”, Astronomische Nachrichten **276**, 193–208 (1948) (cit. on p. 12).

[258]P. Jordan, J. Ehlers, and W. Kundt, “Strenge Lösungen der Feldgleichungen der Allgemeinen Relativitästheorie”, Akad. Wiss. Lit. Abh. Math.-naturwiss. Klasse, 21–105 (1960), Republication in Gen. Rel. Grav. **41**, 2191-2280 (2009) (cit. on p. 69).

[259]G. Juvet, “Opérateurs de Dirac et équations de Maxwell”, Comment. Math. Helv. **2**, 225–235 (1930) (cit. on p. 1).

[260]T. Kaluza, “Zum unitätsproblem der physik”, Sitzber. kgl.-preuß. Akad. Wiss. Berlin, 966–972 (1921), Reprinted and translated to English in [18] (cit. on p. 11).

[261]B. S. Kay, “Editorial note to: Erwin Schrödinger, Dirac electron in the gravitational field I”, Gen. Rel. Grav. **52**, 1–14 (2019), Addendum: Gen. Rel. Grav. **54**, 134 (2022) (cit. on pp. 3, 52, 54).

[262]R. Kerner, “Generalization of the Kaluza-Klein theory for an arbitrary non-abelian gauge group”, Annales de l’I. H. P., section A **9**, 143–152 (1968), Reprinted in [18], pp.115-124 (cit. on p. 13).

[263]I. B. Khriplovich, “Gravitational four-fermion interaction on Planck scale”, Phys. Lett. B **709**, 111–113 (2012) (cit. on p. 32).

[264]T. W. B. Kibble, “Lorentz invariance and the gravitational field”, J. Math. Phys. **2**, 212–221 (1961) (cit. on p. 5).

[265]C. Kiefer, *Quantum Gravity*, 3rd Ed., International Series of Monographs on Physics 155 (Oxford University Press, Oxford, 2012) (cit. on pp. 3, 8, 13, 34, 36).

[266]T. Kimura, "Coupling of Dirac particle and gravitational field", Progr. Theor. Phys. **24**, 386 (1960) (cit. on p. 52).

[267]O. Klein, "Quantentheorie und fünfdimensionale Relativitätstheorie", Z. Physik **37**, 895–906 (1926), English translation in [18], pp.76-87; Reproduced in [47] (cit. on pp. 11, 13).

[268]O. Klein, "The atomicity of electricity as a quantum theory law", Nature **118**, 516 (1926), Reprinted in [18], p.88 (cit. on pp. 11–13, 18).

[269]O. Klein, "Zur fünfdimensionale Darstellung der Relativitätstheorie", Z. Physik **46**, 188–208 (1928), Reproduced in [47] (cit. on pp. 11, 13).

[270]O. Klein, "Meson fields and nuclear interaction", Arkiv för Matematik, Astronomi och Fysik **34**, 1–19 (1948) (cit. on pp. 1, 13, 24, 52).

[271]O. Klein, "Aktuella problem kring fysikens små och stora tal", Kosmos (Svenska Fysikersamfundet) **32**, 33–52 (1954) (cit. on pp. iv, 8, 13, 24).

[272]O. Klein, "Quantum theory and relativity", in Niels Bohr and the Development of Physics. Essays dedicated to Niels Bohr on the occasion of his seventieth birthday, edited by W. Pauli, L. Rosenfeld, and V. Weisskopf (1955), pp. 96–117 (cit. on pp. 8, 13, 24).

[273]O. Klein, "Generalisations of Einstein's theory of gravitation considered from the point of view of quantum field theory", Helv. Phys. Acta, Suppl. 4, Fünzig Jahre Relativitätstheorie **29**, 58–71 (1956) (cit. on pp. 4, 13, 24, 25, 28).

[274]O. Klein, "Some remarks on general relativity and the divergence problem of quantum field theory", Nuovo Cim., Suppl. 1 **6**, 344–348 (1957) (cit. on pp. 8, 13, 25).

[275]O. Klein, "Some remarks on the inversion theorems of quantum field theory", Nucl. Phys. **4**, 677–686 (1957) (cit. on pp. 8, 13).

[276]O. Klein, "The Dirac theory of the electron in general relativity theory", Det Kongelige Norske Videnskabers Selskabs Forhandlinger **31**, 47–51 (1958) (cit. on pp. iv, 4, 13, 24, 28, 52, 68).

[277]O. Klein, "On the treatment of the gravitational field in connection with the generally relativistic Dirac equation", Arkiv för Fysik **17**, 517–520 (1959) (cit. on pp. iv, 4, 13, 25, 52).

[278]O. Klein, "A tentative program for the development of quantum field theory as an extension of the equivalence principle of general relativity theory", Nucl. Phys. B **21**, 253–260 (1970) (cit. on pp. 4, 13, 68).

[279]O. Klein, "Generalization of Einstein's principle of equivalence so as to embrace the field equations of gravitation", Physica Scripta **9**, 69–72 (1974) (cit. on pp. 4, 13, 25, 68).

[280]O. Klein, "Electromagnetic theory treated in analogy to the theory of gravitation", Nucl. Phys. B **92**, 541–546 (1975) (cit. on pp. 4, 13, 68).

[281]O. Klein, "On the theory of charged fields", in The Oskar Klein Memorial Lectures (1991), pp. 85–102, English translation of "Sur la théorie des champs associés à des particules chargées", in: New Theories in Physics (International Institute of Intellectual Cooperation, Paris, 1939) (cit. on pp. 1, 2, 13).

[282]K. Kodama et al., "Observation of tau neutrino interactions", Phys. Lett. B **504**, 218–224 (2001) (cit. on p. 2).

[283]W. Kofink, "Über das magnetische und elektrische Moment des Elektrons nach der Diracschen Theorie", Ann. Phys. (Leipzig) **30**, 91–98 (1937) (cit. on p. 35).

[284]W. Kofink, "Zur Diracshen Theorie des Elektrons, II. Algebraische Identitäten in der Diracschen Theorie des Elektrons, die Differentialquotienten enthalten", Ann. Phys. (Leipzig) **38**, 436–455 (1940) (cit. on p. 35).

[285]W. Kofink, "Zur Mathematik der Diracmatrizen: Die Bargmannsche Hermitisierungsmatrix A und die Paulische Transpositionsmatrix B", Math. Z. **51**, 702–711 (1949) (cit. on p. 35).

[286]B. Kostant, "Symplectic spinors", Symposia mathematica **14**, 139–152 (1974) (cit. on p. 4).

[287]A. J. Kox and R. Schulmann, eds., *The Collected Papers of Albert Einstein*, Vol. 6. The Berlin Years: Writings 1914-1917 (Princeton University Press, Princeton, 1996) (cit. on p. 92).

[288]H. Kragh, *Varying Gravity. Dirac's Legacy in Cosmology and Geophysics* (Birkhäuser, Cham, 2016) (cit. on p. 21).

[289]H. Kragh, "'Let the stars shine in peace!' Niels Bohr and stellar energy, 1929-1934", Annals of Science **74**, 126–148 (2017) (cit. on p. 21).

[290]K. Krasnov, *Formulations of General Relativity: Gravity, Spinors and Differential Forms*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2020) (cit. on p. 51).

[291]K. Krasnov and R. Percacci, "Gravity and unification: A review", Class. Quantum Grav. **35**, 143001 (2018) (cit. on pp. 5, 29, 37, 51, 65).

[292]K. Krasnov, "SO(9) characterization of the standard model gauge group", J. Math. Phys. **62**, 021703 (2021) (cit. on p. 5).

[293]K. Krasnov, "Spin(11,3), particles, and octonions", J. Math. Phys. **63**, 031701 (2022) (cit. on p. 5).

[294]B. Kuchowicz, "Neutrinos in general relativity: Four(?) levels of approach", Gen. Rel. Grav. **5**, 201–234 (1974) (cit. on pp. 2, 5).

[295]W. Kundt, "The plane-fronted gravitational waves", Z. Physik **163**, 77–86 (1961) (cit. on pp. 39, 48).

[296]C. H. Lai, ed., *Selected Papers on Gauge Theory of Weak and Electromagnetic Interactions* (World Scientific, Singapore, 1981) (cit. on pp. 2, 96, 109, 114–116).

[297]C. Lanczos, "Die Erhaltungssätze in der feldmäßigen Darstellungen der Diracschen Theorie", Z. Physik **57**, 484–493 (1929), English translation in 0508013 [physics] (cit. on p. 1).

[298]C. Lanczos, "Die tensoranalytischen Beziehungen der Diracschen Gleichung", Z. Physik **57**, 447–473 (1929), English translation in 0508002 [physics] (cit. on p. 1).

[299]C. Lanczos, "Zur kovarianten Formulierung der Diracschen Gleichung", Z. Physik **57**, 474–483 (1929), English translation in 0508012 [physics] (cit. on p. 1).

[300]C. Lanczos, "Die Wellenmechanik als Hamiltonsche Dynamik des Funktionenraumes. Eine neue Ableitung der Diracschen Gleichung", Z. Physik **81**, 703–732 (1933) (cit. on p. 1).

[301]C. Lanczos, "The splitting of the Riemann tensor", Rev. Mod. Phys. **34**, 379–389 (1962) (cit. on p. 69).

[302]C. Lanczos, "Einstein equations and electromagnetism", J. Math. Phys. **8**, 829–836 (1967) (cit. on p. 38).

[303]L. Landau, "On the conservation laws for weak interactions", Nucl. Phys. **3**, 127–131 (1957) (cit. on p. 1).

[304]L. D. Landau, "On the quantum theory of fields", in Niels Bohr and the Development of Physics. Essays dedicated to Niels Bohr on the occasion of his seventieth birthday, edited by W. Pauli, L. Rosenfeld, and V. Weisskopf (1955), pp. 52–69 (cit. on pp. 2, 8, 13).

[305]B. E. Laurent, "On covariant quantization with application to the scattering of gravitating Dirac particles", Nuovo Cim. **4**, 1445–1460 (1956) (cit. on pp. 4, 28).

[306]B. E. Laurent, "A variational principle and conservation theorems in connexion with the generally relativistic Dirac equation", Arkiv för Fysik **16**, 263–278 (1959) (cit. on pp. iv, 4).

[307]B. E. Laurent, "On a generally covariant quantum theory", Arkiv för Fysik **16**, 237–245 (1959) (cit. on pp. iv, 4).

[308]B. E. Laurent, "Remarks on the functional integration quantization of gravitation", Arkiv för Fysik **16**, 279–283 (1959) (cit. on p. 4).

[309]M. J. Lazo, J. Paiva, J. T. S. Amaral, and G. S. F. Frederico, "Action principle for action-dependent Lagrangians toward nonconservative gravity: Accelerating universe without dark energy", Phys. Rev. D **95**, 101501 (2017) (cit. on p. 32).

[310]K. Lechner, *Classical Electrodynamics. A Modern Perspective*, UNITEXT for Physics (Springer, Switzerland, 2018) (cit. on p. 25).

[311]T. D. Lee, "The weak interaction: Its history and impact on physics", Int. J. Mod. Phys. A **16**, 3633–3658 (2001) (cit. on pp. 2, 28).

[312]T. D. Lee and C. N. Yang, "Question of parity conservation in weak interactions", Phys. Rev. **104**, 254 (1956), Errata: Phys. Rev. **106**, 1371 (1957) (cit. on p. 1).

[313]D. Lehmkuhl, "Why Einstein did not believe that general relativity geometrizes gravity", Studies in History and Philosophy of Modern Physics **46**, 316–326 (2014) (cit. on pp. 18, 25, 33, 36, 67).

[314]J. Leite Lopes, "A model of the universal Fermi interaction", Nucl. Phys. **8**, 234–236 (1958) (cit. on p. 2).

[315]J. Leite Lopes, *Gauge Field Theories: An Introduction* (Pergamon Press, Paris, 1981) (cit. on p. 2).

[316]J. Leite Lopes, "Lectures on weak interactions: From Fermi-Majorana-Perrin to Glasgow-Weinberg-Salam, and some contributions from Latin American physicists", CBPF-MO **02**, 1–65 (1987) (cit. on p. 2).

[317]J. Lewandowski and C. Zhang, "Fermion coupling to loop quantum gravity: canonical formulation", Phys. Rev. D **105**, 124025 (2022) (cit. on p. 65).

[318]A. Lichnerowicz, "Champs spinoriels et propagateurs en relativité générale", Bulletin de la S. M. F., **92**, 11–100 (1964) (cit. on p. 5).

[319]A. Lichnerowicz, "Topics on space-times", in Battelle Rencontres, edited by C. M. DeWitt (1968), pp. 107–116 (cit. on p. 5).

[320]A. Lichnerowicz, "Espaces fibres et espace-temps", Gen. Rel. Grav. **1**, 235–245 (1971) (cit. on p. 5).

[321]A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Électromagnétisme*, Reprint: Masson, 1955 (Éditions Jacques Gabay, 2011) (cit. on p. 12).

[322]I. P. Lobo and C. Romero, "Experimental constraints on the second clock effect", Phys. Lett. B **783**, 306–310 (2018) (cit. on p. 11).

[323]H. G. Loos, "Spin connection in general relativity", Ann. Phys. **25**, 91–108 (1963) (cit. on pp. 28, 52).

[324]H. G. Loos and R. P. Treat, "Conditional dynamic equivalence of free Yang-Mills fields and free gravitational fields", Phys. Lett. A **26**, 91–92 (1967) (cit. on pp. 28, 52).

[325]R. Lopes and R. da Rocha, "New spinor classes on the Graf-Clifford algebra", J. High Energy Phys. **2018**, 1–22 (2018) (cit. on p. 5).

[326]H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The Principle of Relativity. A Collection of Original Memoirs on the Special and General Theory of Relativity* (Dover, New York, 1923) (cit. on p. 92).

[327]E. Lubkin, "Geometric definition of gauge invariance", Ann. Phys. **23**, 233–283 (1963) (cit. on p. 5).

[328]G. Lüders, "Zur Bewegungsumkehr in quantisierten Feldtheorien", Z. Physik **133**, 325–339 (1952) (cit. on p. 1).

[329]G. Lüders, "On the equivalence of invariance under time reversal and under particle-antiparticle conjugation for relativistic field theories", Det Kongelige Danske Videnskabernes Selskab, Mat.-Fys. Medd. **28**, 1–17 (1954) (cit. on p. 1).

[330]G. Lüders and B. Zumino, "Some consequences of TCP-invariance", Phys. Rev. **106**, 385 (1957) (cit. on p. 1).

[331]S. Mandal, "Neutrino oscillations in cosmological spacetime", Nucl. Phys. B **965**, 115338 (2021) (cit. on pp. 5, 32).

[332]H. Mandel, "Zur Herleitung der Feldgleichungen in der allgemeinen Relativitätstheorie", Z. Physik **39**, 136–145 (1926) (cit. on p. 11).

[333]H. Mandel, "Zur Herleitung der Feldgleichungen in der allgemeinen Relativitätstheorie", Z. Physik **45**, 285–306 (1927) (cit. on p. 11).

[334]Y. I. Manin, *Gauge Field Theory and Complex Geometry*, 2nd Ed., Grundlehren der mathematischen Wissenschaften 289 (Springer-Verlag, Berlin Heidelberg, 1997) (cit. on pp. 4, 5).

[335]A. March, "Die Idee einer atomistischen Struktur des Raumes", Naturwissenschaften **26**, 649–656 (1938) (cit. on p. 13).

[336]D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 3rd Ed., Oxford Graduate Texts in Mathematics 27 (Oxford University Press, 2017) (cit. on p. 4).

[337]S. B. Medina, M. Nowakowski, and D. Batic, "Einstein-Cartan cosmologies", Ann. Phys. **400**, 64–108 (2008) (cit. on p. 9).

[338]S. Mereghetti, "The strongest cosmic magnets: soft gamma-ray repeaters and anomalous X-ray pulsars", Astron. Astrophys. Rev. **15**, 225–287 (2008) (cit. on p. 30).

[339]A. I. Miller, ed., *Early Quantum Electrodynamics: A Source Book* (Cambridge University Press, Cambridge, 1994) (cit. on p. 97).

[340]J. Milnor, "Spin structures on manifolds", L'enseignement math. **9**, 198–213 (1963) (cit. on p. 5).

[341]C. W. Misner and J. A. Wheeler, "Classical physics as geometry", Ann. Phys. **2**, 525–603 (1957) (cit. on p. 2).

[342]R. N. Mohapatra and J. C. Pati, "Left-right gauge symmetry and an "isoconjugate" model of CP violation", Phys. Rev. D **11**, 566 (1975) (cit. on p. 5).

[343]C. Møller, "On the localization of the energy of a physical system in the general theory of relativity", Ann. Phys. **4**, 347–371 (1958) (cit. on p. 4).

[344]C. Møller, "Conservation laws and absolute parallelism in general relativity", Det Kongelige Danske Videnskabernes Selskab, Mat.-Fys. Skrifter **10**, 1–50 (1961) (cit. on p. 4).

[345]C. Møller, "Further remarks on the localization of the energy in the general theory of relativity", Ann. Phys. **12**, 118–133 (1961) (cit. on p. 4).

[346]M. J. Mortonson, W. Hu, and D. Huterer, "Falsifying paradigms for cosmic acceleration", Phys. Rev. D **79**, 023004 (2009) (cit. on p. 8).

[347]R. A. Mosna and W. A. Rodrigues Jr, "The bundles of algebraic and Dirac-Hestenes spinor fields", J. Math. Phys. **45**, 2945–2966 (2004) (cit. on p. 3).

[348]F. Muscolino, "Supersymmetry breaking in supergravity via super Yang-Mills squared", MSci Thesis (Università degli Studi di Milano, 2018) (cit. on p. 25).

[349]Y. Mutô and K. Yano, "Sur la théorie des spineurs", Proc. Phys.-Math. Soc. Japan **19**, 413–435 (1937) (cit. on p. 1).

[350]G. L. Naber, *Topology, Geometry, and Gauge Fields*, 2nd Ed., Vol. II. Interactions (Springer, New York, 2011) (cit. on pp. 5, 61).

[351]H. Nakamura and T. Toyoda, "Connection of spinor fields", Nucl. Phys. **3**, 524–528 (1961) (cit. on p. 52).

[352]Y. Nambu, "Axial vector current conservation in weak interactions", Phys. Rev. Lett. **4**, 380 (1960), Reprinted in [139], pp.107-109 (cit. on p. 64).

[353] Y. Nambu, “Thermodynamic analogy in quantum field theory”, in Contribution to the Festschrift of K. Nishijima on the occasion of his sixtieth birthday (1987), pp. 167–173, Reprinted in [139], pp.399-405 (cit. on p. 38).

[354] Y. Nambu, “Energy gap, mass gap, and spontaneous symmetry breaking”, Int. J. Mod. Phys. A **25**, 4141–4148 (2010) (cit. on pp. 2, 20, 27).

[355] J. V. Narlikar, “The quasi-stead-state cosmology”, in Current Issues in Cosmology, edited by J.-C. Pecker and J. V. Narlikar (2006), pp. 139–151 (cit. on p. 32).

[356] T. J. Nelson and R. H. Good, Jr., “Second-quantization process for particles with any spin and with internal symmetry”, Rev. Mod. Phys. **40**, 508 (1968) (cit. on p. 5).

[357] F. Nesti and R. Percacci, “Gravi-weak unification”, J. Phys. A: Math. Theor. **41**, 075405 (2008) (cit. on pp. 32, 65).

[358] E. Newman and R. Penrose, “An approach to gravitational radiation by a method of spin coefficients”, J. Math. Phys. **3**, 566–578 (1961) (cit. on p. 4).

[359] G. Nordström, “Über die Möglichkeit, das elektromagnetische Feld und das Gravitationsfeld zu vereinigen”, Physik. Zeitschr. **XV**, 504–506 (1914), Reprinted and translated to English in [18], and in 0702221 [physics.gen-ph] (cit. on p. 11).

[360] M. Novello, “Gravitation as a consequence of the self-interaction of the Γ fields”, J. Math. Phys. **12**, 1039–1041 (1971) (cit. on p. 52).

[361] M. Novello, “Weak and electromagnetic forces as a consequence of the self-interaction of the γ field”, Phys. Rev. D **8**, 2398 (1973) (cit. on p. 52).

[362] M. Novello, “A spinor theory of gravity and the cosmological framework”, J. Cosm. Astropart. Phys. **06**, 1–18 (2007) (cit. on pp. 27, 28, 49, 52).

[363] M. Novello, “Spinor theory of gravity”, Preprint 0609033 [gr-qc] (cit. on pp. 28, 49, 52).

[364] M. Novello and M. Borba, “Reproducing gravity through spinor fields”, Grav. & Cosm. **17**, 224 (2011) (cit. on pp. 28, 49).

[365] M. Novello and A. E. S. Hartmann, “Is the electromagnetic field responsible for the cosmic acceleration in late times?”, Int. J. Mod. Phys. A **34**, 1950083 (2019) (cit. on p. 32).

[366] M. Novello and A. E. S. Hartmann, “Beyond the equivalence principle: Gravitational magnetic monopoles”, Grav. & Cosm. **27**, 221–225 (2021) (cit. on p. 69).

[367] M. Novello and A. E. S. Hartmann, “From weak interaction to gravity”, Int. J. Mod. Phys. A **36**, 2150051 (2021) (cit. on pp. 8, 27, 28, 32, 38, 52).

[368] M. Novello and A. E. S. Hartmann, “Gravitational waves in the spinor theory of gravity”, Mod. Phys. Lett. A **36**, 2150003 (2021) (cit. on pp. 32, 47).

[369] M. Novello, A. E. S. Hartmann, and E. Bittencourt, “Galaxy rotation curves in the light of the spinor theory of gravity”, Mod. Phys. Lett. A **36**, 2150248 (2021) (cit. on pp. 32, 45).

[370] M. Novello and S. E. Perez Bergliaffa, “Bouncing cosmologies”, Phys. Rep. **463**, 127–213 (2008) (cit. on p. 32).

[371]M. Novello and P. Rotelli, “The cosmological dependence of weak interactions”, J. Phys. A: Gen. Phys. **5**, 1488–1494 (1972) (cit. on pp. 28, 52).

[372]L. O’Raifeartaigh, *The Dawning of Gauge Theory* (Princeton University Press, Princeton, 1997) (cit. on p. 115).

[373]L. O’Raifeartaigh and N. Straumann, “Gauge theory: historical origins and some modern developments”, Rev. Mod. Phys. **72**, 1–23 (2000) (cit. on p. 2).

[374]V. I. Ogievetskiĭ and I. V. Polubarinov, “Spinors in gravitation theory”, Sov. Phys. JETP **48**, 1625–1636 (1965) (cit. on p. 52).

[375]L. B. Okun, “On the article of G. Gamow, D. Ivanenko, and L. Landau ‘World Constants and Limiting Transition’”, Phys. Atom. Nuclei **65**, 1370–1372 (2002) (cit. on p. 24).

[376]R. Onofrio, “Gravitational vacuum polarization phenomena due to the Higgs field”, Eur. Phys. J. C **72**, 1–8 (2012) (cit. on p. 32).

[377]R. Onofrio, “On weak interactions as short-distance manifestations of gravity”, Mod. Phys. Lett. A **28**, 1350022 (2013) (cit. on pp. 28, 32).

[378]R. Onofrio, “Proton radius puzzle and quantum gravity at the Fermi scale”, Europhys. Lett. **104**, 20002 (2013) (cit. on pp. 28, 32).

[379]R. Onofrio, “High-energy density implications of a gravitoweak unification scenario”, Mod. Phys. Lett. A **29**, 1350187 (2014) (cit. on pp. 28, 32).

[380]J. M. Overduin and P. S. Wesson, “Kaluza-Klein gravity”, Phys. Rep. **283**, 303–378 (1997) (cit. on p. 13).

[381]T. Padmanabhan, “Physical significance of the Planck length”, Ann. Phys. **165**, 38–58 (1985) (cit. on p. 8).

[382]T. Padmanabhan, “Planck length as the lower bound to all physical length scales”, Gen. Rel. Grav. **17**, 215–221 (1985) (cit. on p. 8).

[383]T. Padmanabhan, “Planck length: Lost + found”, Phys. Lett. B **809**, 135774 (2020) (cit. on p. 8).

[384]H. Pagels, “Spin and gravitation”, Ann. Phys. **31**, 64–87 (1965) (cit. on p. 52).

[385]A. Pais, “Spherical spinors in a Euclidean 4-space”, Proc. Nat. Acad. Scie. **40**, 835–841 (1954) (cit. on p. 1).

[386]J. A. P. Paiva, M. J. Lazo, and V. T. Zanchin, “Generalized nonconservative gravitational field equations from Herglotz action principle”, Phys. Rev. D **105**, 124023 (2022) (cit. on p. 32).

[387]D. M. Pantoja, “A teoria de Bohm-de Broglie e as singularidades cosmológicas”, PhD Thesis (CBPF, Rio de Janeiro, 2014) (cit. on p. 4).

[388]J. C. Pati and A. Salam, “Unified lepton-hadron symmetry and a gauge theory of basic interactions”, Phys. Rev. D **8**, 1240 (1973) (cit. on p. 5).

[389]J. C. Pati and A. Salam, “Lepton number as the fourth ‘color’”, Phys. Rev. D **10**, 275 (1974) (cit. on p. 5).

[390]W. Pauli, “Über die Formulierung der Naturgesetze mit fünf homogenen Koordinaten. Teil I: Klassische Theorie”, Ann. Phys. **410**, 305–336 (1933) (cit. on pp. 1, 18).

[391]W. Pauli, “Über die Formulierung der Naturgesetze mit fünf homogenen Koordinaten. Teil II: Die Diracschen Gleichung für die Materiewellen”, Ann. Phys. **410**, 337–372 (1933) (cit. on pp. 1, 18, 19).

[392]W. Pauli, “Contributions mathématiques à la théorie des matrices de Dirac”, Annales de l’I. H. P. **6**, 109–136 (1936) (cit. on p. 1).

[393]W. Pauli, “Raum, Zeit und Kausalität in der modernen Physik”, Scientia **59**, 65–79 (1936), Reprinted in [398]; English translation in [155]. (Cit. on pp. 19, 20, 22).

[394]W. Pauli, *The Theory of the Positron and Related Topics. Report of a Seminar*, Notes by Dr. Banesh Hoffmann (CERN-ARCH-PLEC-002) (The Institute for Advanced Study, 1936) (cit. on pp. 13, 16).

[395]W. Pauli, “Einstein’s contributions to quantum theory”, in Albert Einstein: Philosopher-Scientist, edited by P. A. Schilpp (1949), pp. 147–160 (cit. on p. 17).

[396]W. Pauli, “The theory of relativity and science”, Helv. Phys. Acta, Supp. 4 **29**, 282–286 (1956), Reprinted in [155, pp.107-111] (cit. on pp. 10, 27).

[397]W. Pauli, *Theory of Relativity* (Pergamon Press, London and New York, 1958) (cit. on pp. 17–19).

[398]W. Pauli, *Physik und Erkenntnistheorie* (Springer, Braunschweig, 1984) (cit. on p. 107).

[399]W. Pauli and J. Solomon, “La théorie unitaire d’Einstein et Mayer et les équations de Dirac”, J. Phys. Radium **3**, 452–463 (1932) (cit. on p. 1).

[400]W. Pauli and J. Solomon, “La théorie unitaire d’Einstein et Mayer et les équations de Dirac. II”, J. Phys. Radium **3**, 582–589 (1932) (cit. on p. 1).

[401]M. Pavšič, “Kaluza–Klein theory without extra dimensions: Curved Clifford space”, Phys. Lett. B **614**, 85 (2005) (cit. on pp. 28, 64).

[402]M. Pavšič, “Space inversion of spinors revisited: A possible explanation of chiral behavior in weak interactions”, Phys. Lett. B **692**, 212 (2010) (cit. on pp. 28, 64).

[403]C. Pellegrini and J. Plebański, “Tetrad fields and gravitational fields”, Det Kongelige Danske Videnskabernes Selskab, Mat.-Fys. Skrifter **2**, 1–39 (1963) (cit. on p. 4).

[404]R. Penrose, “A spinor approach to general relativity”, Ann. Phys. **10**, 171–201 (1960) (cit. on p. 4).

[405]R. Penrose, “Structure of space-time”, in Battelle Rencontres, edited by C. M. DeWitt (1968), pp. 121–235 (cit. on p. 5).

[406]R. Penrose and W. Rindler, *Spinors and Space-Time, 2vols.* Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 1984-1986) (cit. on pp. 4, 59).

[407]A. Peres, “Spinor fields in generally covariant theories”, Nuovo Cim. Suppl. **24**, 389–452 (1962) (cit. on p. 52).

[408]A. Peres, “Gyro-gravitational ratio of Dirac particles”, Nuovo Cim. **28**, 1091–1092 (1963) (cit. on p. 52).

[409]A. Perez, “The spin foam representation of loop quantum gravity”, in Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, edited by D. Oriti (2009), pp. 272–289, Preprint 0601095 [gr-qc] (cit. on pp. 8, 13).

[410]P. Peter and J.-P. Uzan, *Primordial Cosmology*, Oxford Graduate Texts (Oxford University Press, Oxford, 2009) (cit. on p. 13).

[411]P. Peter and N. Pinto-Neto, “Cosmology without inflation”, Phys. Rev. D **78**, 063506 (2008) (cit. on p. 32).

[412]P. Peter and N. Pinto-Neto, “Bouncing alternatives to inflation”, C. R. Physique **16**, 1038–1047 (2015) (cit. on p. 32).

[413]A. A. Petrov and A. E. Blechman, *Effective Field Theories* (World Scientific, Singapore, 2016) (cit. on pp. 2, 15, 49).

[414]C. Petrov, *Einstein Spaces*, Reprint of the 1st ed., 1969 (Pergamon Press, Oxford, 2013) (cit. on pp. 4, 59, 62, 75, 76, 79).

[415]N. Pinto-Neto and J. C. Fabris, “Quantum cosmology from the de Broglie-Bohm perspective”, Class. Quantum Grav. **30**, 143001 (2013) (cit. on p. 4).

[416]R. Pittau, “On the predictivity of the non-renormalizable quantum field theories”, Fortschr. Physik **63**, 132–141 (2015) (cit. on p. 27).

[417]J. Plebański, *Lectures on Non-linear Electrodynamics. An extended version of lectures given at the Niels Bohr Institute and NORDITA in October 1968* (Nordisk Institut for Teoretisk Atomfysik, Copenhagen, 1970) (cit. on p. 14).

[418]J. Plebański and A. Krasiński, *An Introduction to General Relativity and Cosmology* (Cambridge University Press, Cambridge, 2006) (cit. on pp. 4, 9, 10, 13).

[419]B. Podolsky, “A tensor form of Dirac’s equation”, Phys. Rev. **37**, 1398 (1931) (cit. on p. 1).

[420]B. M. Pontecorvo, “Mesonium and antimesonium”, JETP Lett. **33**, 549–551 (1957) (cit. on p. 1).

[421]K. Popper, *Objective Knowledge. An Evolutionary Approach* (Oxford University Press, London, 1972) (cit. on p. 5).

[422]K. Popper, *The Logic of Scientific Discovery*, Routledge Classics (Routledge, London and New York, 2002) (cit. on pp. 7, 9, 11).

[423]D. Prinz, “Algebraic structures in the coupling of gravity to gauge theories”, Ann. Phys. **426**, 1–50 (2021) (cit. on pp. 5, 39).

[424]C. Quigg, “Electroweak symmetry breaking in historical perspective”, Preprint 1503.01756 [hep-th] (cit. on p. 2).

[425]G. Racah, “Sulla simmetria tra particelle e antiparticelle”, Nuovo Cim. **14**, 322–328 (1937) (cit. on p. 1).

[426]L. Randall and R. Sundrum, “An alternative to compactification”, Phys. Rev. Lett. **83**, 4690 (1999) (cit. on p. 14).

[427] L. Randall and R. Sundrum, “Large mass hierarchy from a small extra dimension”, Phys. Rev. Lett. **83**, 3370 (1999) (cit. on p. 14).

[428] W. Rarita and J. Schwinger, “On a theory of particles with half-integral spin”, Phys. Rev. **60**, 61 (1941) (cit. on p. 5).

[429] F. Reines and C. L. Cowan, “Detection of the free neutrino”, Phys. Rev. **92**, 830 (1953) (cit. on p. 1).

[430] F. Reines and C. L. Cowan, “The neutrino”, Nature **178**, 446–449 (1956) (cit. on p. 1).

[431] F. Reines and C. L. Cowan, “Free antineutrino absorption cross section. I. Measurement of the free antineutrino absorption cross section by protons”, Phys. Rev. **113**, 273 (1959) (cit. on p. 1).

[432] M. Riesz, *Clifford Numbers and Spinors*, edited by E. F. Bolinder and P. Lounesto, Fundamental Theories of Physics 54 (Springer, Dordrecht, 1993) (cit. on p. 1).

[433] A. Rocci, “La storia della Gravità Quantistica: Dalla nascita della Relatività Generale al secondo dopoguerra”, PhD Thesis (Università degli Studi di Padova, Dipartimento di Fisica e Astronomia “Galileo Galilei”, 2015) (cit. on p. 22).

[434] V. I. Rodichev, “Twisted space and nonlinear field equations”, Sov. Phys. JETP **13**, 1029–1031 (1961) (cit. on pp. 4, 52).

[435] N. Rosen, “Statistical geometry and fundamental particles”, Phys. Rev. **72**, 298 (1947), Reproduced in [47] (cit. on p. 13).

[436] G. Rudolph and M. Schmidt, *Differential Geometry and Mathematical Physics*, Vol. II. Fibre Bundles, Topology, and Gauge Fields, TMP (Springer, Dordrecht, 2017) (cit. on pp. 3, 5, 51, 52, 61, 64).

[437] R. Ruffini, G. Vereshchagin, and S.-S. Xue, “Electron–positron pairs in physics and astrophysics: From heavy nuclei to black holes”, Phys. Rep. **487**, 1–140 (2010) (cit. on p. 30).

[438] Y. B. Rumer, *Studies in 5-optics* (State Publishing Office for Technico-Theoretical Literature, Moscow, 1956) (cit. on p. 52).

[439] Y. B. Rumer, “Action as a space coordinate. X”, Sov. Phys. JETP **36**, 1348–1353 (1959) (cit. on pp. 12, 52).

[440] A. D. Sakharov, “Vacuum quantum fluctuations in curved space and the theory of gravitation”, Doklady Akademii Nauk SSSR **177**, 70–71 (1967), Republication in Gen. Rel. Grav. **32**, 365-367 (2000) (cit. on pp. 25, 37, 68).

[441] J. J. Sakurai, “Mass reversal and weak interactions”, Nuovo Cim. **7**, 649–660 (1958) (cit. on p. 2).

[442] A. Salam, “On parity conservation and neutrino mass”, Nuovo Cim. **5**, 299–301 (1957), Reprinted in [444], pp.167-169 (cit. on p. 1).

[443] A. Salam, “Weak and electromagnetic interactions”, in Elementary Particle Theory, edited by N. Svartholm (1968), Reprinted in [296], pp.188-198 (cit. on pp. 2, 27).

[444] A. Salam, *Unification of Fundamental Forces. The first of the 1988 Dirac Memorial Lectures* (Cambridge University Press, Cambridge, 2005) (cit. on pp. 109, 110).

[445]A. Salam and J. C. Ward, “Weak and electromagnetic interactions”, Nuovo Cim. **11**, 568–577 (1959), Reprinted in [444], pp.183-192 (cit. on p. 2).

[446]F. Sauter, “Lösung der Diracschen Gleichungen ohne Spezialisierung der Diracschen Operatoren”, Z. Physik **63**, 803–814 (1930) (cit. on p. 1).

[447]F. Sauter, “Lösung der Diracschen Gleichungen ohne Spezialisierung der Diracschen Operatoren. II”, Z. Physik **64**, 295–303 (1930) (cit. on p. 1).

[448]F. Sauter, “Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs”, Z. Physik **69**, 742–764 (1931) (cit. on p. 29).

[449]A. Schild, “Discrete space-time and integral Lorentz transformations”, Phys. Rev. **73**, 414 (1948), Reproduced in [47] (cit. on p. 13).

[450]A. Schild, “Discrete space-time and integral Lorentz transformations”, Canad. J. Math. **1**, 29–47 (1949) (cit. on p. 13).

[451]E. Scholz, “Local spinor structures in V. Fock’s and H. Weyl’s work on the Dirac equation”, in Géométrie au XXe siècle: histoire et horizons, edited by J. Kouneiher, D. Flament, P. Nabonnand, and J.-J. Szczeciniarz (2005), pp. 184–301, Preprint 0409158 [physics] (cit. on pp. 11, 52).

[452]E. Scholz, “Introducing groups into quantum theory (1926-1930)”, Historia Mathematica **33**, 440–490 (2006) (cit. on p. 3).

[453]E. Scholz, “The unexpected resurgence of Weyl geometry in late 20th-Century physics”, in Beyond Einstein, edited by D. Rowe, T. Sauer, and S. Walter, Einstein Studies 14 (2018), pp. 261–360 (cit. on p. 11).

[454]E. Scholz, “H. Weyl’s and E. Cartan’s proposals for infinitesimal geometry in the early 1920s”, Preprint 2206.07576 [phys.hist-ph] (cit. on p. 11).

[455]M. Schönberg, “On the Grassmann and Clifford algebras I”, Anais Acad. Bras. Ciências **28**, 11–19 (1956) (cit. on p. 3).

[456]M. Schönberg, “Quantum kinematics and geometry”, Nuovo Cim. Supp. **6**, 356–380 (1957) (cit. on p. 3).

[457]M. Schönberg, “Quantum mechanics and geometry. Part I”, Anais Acad. Bras. Ciências **29**, 473–499 (1957) (cit. on pp. iv, 3).

[458]M. Schönberg, “Quantum mechanics and geometry. Part II”, Anais Acad. Bras. Ciências **30**, 1–20 (1958) (cit. on pp. iv, 3).

[459]M. Schönberg, “Quantum mechanics and geometry. Part III”, Anais Acad. Bras. Ciências **30**, 117–131 (1958) (cit. on pp. iv, 3).

[460]M. Schönberg, “Quantum mechanics and geometry. Part IV”, Anais Acad. Bras. Ciências **30**, 259– (1958) (cit. on pp. iv, 3).

[461]M. Schönberg, “Quantum mechanics and geometry. Part V”, Anais Acad. Bras. Ciências **30**, 429–446 (1958) (cit. on pp. iv, 3).

[462]M. Schönberg, "Quantum theory and geometry", in Max Planck Festschrift, edited by B. Kockel, W. Macke, and A. Papapetrou (1958) (cit. on p. 3).

[463]J. A. Schouten, "Dirac equations in general relativity: 1. Four dimensional theory", J. Math. Phys. **10**, 239–271 (1931) (cit. on p. 1).

[464]J. A. Schouten, "Dirac equations in general relativity: 2. Five dimensional theory", J. Math. Phys. **10**, 272–283 (1931) (cit. on p. 1).

[465]J. A. Schouten and D. van Dantzig, "Generelle Feldtheorie", Z. Physik **78**, 639–667 (1932) (cit. on p. 1).

[466]J. A. Schouten and D. van Dantzig, "Zum Unifizierungsproblem der Physik; Skizze einer generellen Feldtheorie", Proc. Koninklijke Akademie van Wetenschappen te Amsterdam **35**, 642–655 (1932) (cit. on p. 1).

[467]J. A. Schouten and D. van Dantzig, "Zur generellen generellen Feldtheorie; Diracshe Gleichung und Hamiltonsche Funktion", Proc. Koninklijke Akademie van Wetenschappen te Amsterdam **35**, 844–852 (1932) (cit. on p. 1).

[468]J. A. Schouten and D. van Dantzig, "On projective connexions and their application to the general field-theory", Ann. Math. **34**, 271–312 (1933) (cit. on p. 1).

[469]E. Schrödinger, "Diracsches Elektron im Schwerefeld I", Sitzber. kgl.-preuß. Akad. Wiss. Berlin, Sitzung der phys.-math. Klasse, 105–128 (1932), Reproduced in [47]; Republication in Gen. Rel. Grav. **52**, 4 (2020) (cit. on pp. 1, 24, 33, 50, 52).

[470]E. Schrödinger, "The relation between metric and affinity", Proc. R. Irish Acad. A **51**, 147–150 (1947) (cit. on p. 36).

[471]E. Schrödinger, *Space-Time Structure* (Cambridge University Press, Cambridge, 1985) (cit. on pp. 37, 50).

[472]K. Schwarzschild, "Über das Gravitationsfeld eines Massenpuktes nach der Einsteinschen Theorie", Sitzber. kgl.-preuß. Akad. Wiss. Berlin, phys.-math. Klasse, 189–196 (1916), Republication in Gen. Rel. Grav. **35**, 951-959 (2003) (cit. on pp. 10, 39).

[473]S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Dover, Mineola, New York, 1989) (cit. on pp. 13, 15, 29).

[474]J. Schwinger, "On gauge invariance and vacuum polarization", Phys. Rev. **82**, 664 (1951), Reprinted in [475], Paper 20 (cit. on pp. 1, 29).

[475]J. Schwinger, ed., *Selected papers on Quantum Electrodynamics*, Reprint: 1st Ed., 1958 (Dover, Mineola, New York, 2012) (cit. on p. 111).

[476]D. Sciama, "On the origin of inertia", Month. Not. R. Astron. Soc. **113**, 34–42 (1953) (cit. on p. 67).

[477]R. Shaw, "The problem of particle types and other contributions to the theory of elementary particles", PhD Thesis (Cambridge University, 1955) (cit. on p. 66).

[478]M. Shifman, "From Heisenberg to supersymmetry", Fortschr. Phys. **50**, 552–561 (2002) (cit. on p. 20).

[479]D. Shirokov, "On inner automorphisms preserving fixed subspaces of Clifford algebras", Adv. Appl. Clifford Algebras **31**, 30 (2021) (cit. on p. 64).

[480]T. P. Singh, "Trace dynamics and division algebras: Towards quantum gravity and unification", Z. Naturforsch. A **76**, 131–162 (2021) (cit. on p. 5).

[481]A. Sirlin and A. Ferroglia, "Radiative correction in precision electroweak physics: A historical perspective", Rev. Mod. Phys. **85**, 263 (2013) (cit. on p. 2).

[482]C. Sivaram and K. P. Sinha, "f-Gravity and spinors in general relativity", Lett. Nuovo Cim. **13**, 357–363 (1975) (cit. on p. 5).

[483]H. S. Snyder, "Quantized space-time", Phys. Rev. **71**, 38 (1947), Reproduced in [47] (cit. on p. 13).

[484]J. Stachel, D. C. Cassidy, J. Renn, and R. Schulmann, eds., *The Collected Papers of Albert Einstein*, Vol. 2. The Swiss Years: Writings 1900-1909 (Princeton University Press, Princeton, 1990) (cit. on p. 92).

[485]H. Stephani, D. Kramer, M. Maccallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 3rd Ed., Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2009) (cit. on pp. 9, 10, 59).

[486]N. Straumann, "On Pauli's invention of non-abelian Kaluza-Klein theory in 1953", in (2002), pp. 1063–1066, Preprint 0012054 [gr-qc] (cit. on pp. 54, 66).

[487]N. Straumann, "Early history of gauge theories and weak interactions, Talk at the PSI Summer School on Physics with Neutrinos, Zuoz, Switzerland, August 4-10", Preprint 9609230 [hep-th] (cit. on p. 2).

[488]E. C. G. Stueckelberg, "Neutrino theory of light", Nature **139**, 189–199 (1937) (cit. on p. 16).

[489]E. C. G. Sudarshan and R. E. Marshak, "Chirality invariance and the universal Fermi interaction", Phys. Rev. **109**, 1860 (1958) (cit. on p. 1).

[490]E. C. G. Sudarshan and R. E. Marshak, "The nature of the four-fermion interaction", in Proceedings of Padua-Venice Conference on Mesons and Newly Discovered Particles, September 1957 (1958) (cit. on p. 1).

[491]A. H. Taub, "Quantum equations in cosmological spaces", Phys. Rev. **51**, 512 (1937) (cit. on p. 1).

[492]A. H. Taub, O. Veblen, and J. von Neumann, "The Dirac equation in projective relativity", Proc. N. A. S. **20**, 383–388 (1934) (cit. on p. 1).

[493]M. Tecchiolli, "On the mathematics of coframe formalism and Einstein-Cartan theory - A brief review", Universe **5**, 206 (2019) (cit. on p. 52).

[494]T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, 2008) (cit. on pp. 8, 13).

[495]W. Thirring, "An alternative approach to the theory of gravitation", Ann. Phys. **16**, 96–117 (1961) (cit. on p. 25).

[496]J. Tiomno, "Mass reversal and the universal interaction", Nuovo Cim. **1**, 226–232 (1955) (cit. on p. 1).

[497]I. Todorov, "Clifford algebras and spinors", Bulg. J. Phys. **38**, 3–28 (2011), Preprint 1106.3197 [math-ph] (cit. on p. 5).

[498]I. Todorov, "Octonion internal space algebra for the standard model", Universe **2023**, 222, Preprint 2206.06912 [hep-th] (cit. on p. 5).

[499]I. Todorov and S. Drenska, "Exceptional Jordan algebra and the role of the group F_4 in particle physics", Adv. Appl. Clifford Algebras **28**, 82 (2018) (cit. on p. 5).

[500]I. Todorov and M. Dubois-Violette, "Deducing the symmetry of the standard model from the automorphism and structure groups of the exceptional Jordan algebra", Int. J. Mod. Phys. A **33**, 1850118 (2018) (cit. on p. 5).

[501]M. A. Tonnelat, *Einstein's Theory of Unified Fields*, Routledge Library Editions: 20th Century Science (Routledge, London, 2014) (cit. on pp. 37, 50).

[502]A. Trautman, "Noether equations and conservation laws", Commun. Math. Phys. **6**, 248–261 (1967) (cit. on p. 5).

[503]A. Trautman, "Fibre bundles associated with space-time", Rep. on Math. Phys. **1**, 29–62 (1970) (cit. on p. 5).

[504]A. Trautman, "Einstein-Cartan theory", in Encyclopedia of Mathematical Physics, Vol. 2, edited by J.-P. Françoise, G. L. Naber, and T. S. Tsun (2006), pp. 189–195, Preprint 0606062 [gr-qc] (cit. on p. 5).

[505]A. Trautman, F. A. E. Pirani, and H. Bondi, eds., *Lectures on General Relativity* (Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1965) (cit. on p. 37).

[506]G. Trayling and W. E. Baylis, "A geometric basis for the standard-model gauge group", J. Phys. A: Math. Gen. **34**, 3309 (2001) (cit. on p. 5).

[507]R. P. Treat, "Anholonomic bases in the study of the spin connection and gauge potentials in a Riemann space", J. Math. Phys. **11**, 2187–2193 (1970) (cit. on pp. 28, 52).

[508]H.-J. Treder, "On the question of a cosmological rest-mass of gravitons", Int. J. Theor. Phys. **1**, 167–169 (1968) (cit. on p. 25).

[509]H.-J. Treder, "General relativity and general Lorentz-covariance", Int. J. Theor. Phys. **3**, 23–31 (1970) (cit. on p. 25).

[510]H.-J. Treder, *Gravitationstheorie und Äquivalenzprinzip. Lorentz-Gruppe, Einstein-Gruppe und Raumstruktur* (Academic-Verlag, Berlin, 1971) (cit. on pp. 25, 67).

[511]H.-J. Treder, H.-H. von Borzeszkowski, A. van der Merwe, and W. Yourgrau, *Fundamental Principles of General Relativity Theories. Local and Global Aspects of Gravitation* (Springer, New York, 1980) (cit. on pp. 25, 67).

[512]A. Unzicker and T. Case, "Translation of Einstein's attempt of a unified field theory with teleparallelism", Preprint 0503046 [phys] (cit. on p. 55).

[513]R. Utiyama, "Invariant theoretical interpretation of interaction", Phys. Rev. **101**, 1597 (1956) (cit. on p. 5).

[514]J.-P. Uzan, "The fundamental constants and their variation: observational and theoretical status", Rev. Mod. Phys. **75**, 403–455 (2003) (cit. on p. 68).

[515] J.-P. Uzan, "Varying constants, gravitation and cosmology", Living Rev. Rel. **14**, 1–155 (2011) (cit. on pp. 32, 68).

[516] A. A. Vargas-Paredes, "Campos de Yang-Mills e a teoria de Einstein-Cartan: da gravitação quântica à supercondutividade", PhD Thesis (CBPF, Rio de Janeiro, 2012) (cit. on p. 5).

[517] V. V. Varlamov, "Spinor structure and matter spectrum", Int. J. Theor. Phys. **55**, 5008–5045 (2016) (cit. on pp. 32, 36).

[518] V. V. Varlamov, "Spinors in $\mathbb{K}$-Hilbert spaces", Preprint 2204.10808 [math-ph] (cit. on p. 5).

[519] O. Veblen, "Spinors in projective relativity", Proc. N. A. S. **19**, 979–989 (1933) (cit. on p. 1).

[520] O. Veblen and A. H. Taub, "Projective differentiation of spinors", Proc. N. A. S. **20**, 85–92 (1934) (cit. on p. 1).

[521] H. Velten and T. R. P. Caramês, "To conserve, or not to conserve: A review of nonconservative theories of gravity", Universe **7**, 38 (2021) (cit. on p. 32).

[522] M. J. G. Veltman, "Nobel lecture: From weak interactions to gravitation", Rev. Mod. Phys. **72**, 341–349 (2000) (cit. on pp. 2, 27, 32).

[523] M. Visser, "Sakharov's induced gravity: A modern perspective", Mod. Phys. Lett. A **17**, 977–991 (2002) (cit. on p. 37).

[524] H.-H. von Borzeszkowski and H.-J. Treder, *The Meaning of Quantum Gravity*, Fundamental Theories of Physics (D. Reidel, Dordrecht, 1987) (cit. on pp. 8, 14, 25).

[525] K. von Meyenn, ed., *Wolfgang Pauli: Scientific Correspondence with Bohr, Einstein, Heisenberg, a.O.* Vol. I-IV, Sources in the History of Mathematics and Physical Sciences (Springer-Verlag, Heidelberg and New York, 1979-2005) (cit. on pp. 21, 23).

[526] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago and London, 1984) (cit. on pp. 10, 34, 51, 59).

[527] G. Wataghin, "Sull'interazione di particelle elementari", La Ricerca Scientifica, Serie II, Anno VII, **2**, 99–100 (1936) (cit. on pp. iv, 16).

[528] G. Wataghin, "Sulle forze d'inerzia secondo la teoria quantistica della gravitazione", La Ricerca Scientifica, Serie II, Anno VII, **2**, 341 (1936), Reproduced in Appendix A (cit. on pp. iv, 16, 22, 68, 70).

[529] G. Wataghin, "Sulla teoria quantica della gravitazione", La Ricerca Scientifica, Serie II, Anno VIII, **2**, 361–362 (1937), Reproduced in Appendix B (cit. on pp. iv, 16, 47, 52, 54, 72).

[530] S. Watanabe, "Chirality of K particle", Phys. Rev. **106**, 1306 (1957) (cit. on p. 1).

[531] S. Weinberg, "Feynman rules for any spin", Phys. Rev. **133B**, 1318 (1963) (cit. on p. 5).

[532] S. Weinberg, "Feynman rules for any spin. II. Massless particles", Phys. Rev. **134B**, 882 (1964) (cit. on p. 5).

[533] S. Weinberg, "A model of leptons", Phys. Rev. Lett. **19**, 1264 (1967), Reprinted in [296], pp.181-187 (cit. on p. 2).

[534] S. Weinberg, *Gravitation and Cosmology. Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972) (cit. on pp. 10, 47).

[535] S. Weinberg, “Conceptual foundations of the unified theory of weak and electromagnetic interactions”, Rev. Mod. Phys. **52**, 515–523 (1980), Reprinted in [296], pp.1-8 (cit. on pp. 2, 20).

[536] S. Weinberg, *The Quantum Theory of Fields*, Vol. I-III (Cambridge University Press, New York, 1996) (cit. on pp. 2, 9, 26, 27).

[537] S. Weinberg, “Effective field theory, past and future”, Int. J. Mod. Phys. A **31**, 1630007 (2016), Preprint 09081964 [hep-th] (cit. on p. 27).

[538] S. Weinberg, “On the development of effective field theory”, Eur. Phys. J. H **46**, 1–6 (2021) (cit. on p. 27).

[539] S. Weinberg, “What is quantum field theory, and what did we think it is?”, Preprint 9702027 [hep-th] (cit. on p. 26).

[540] V. F. Weisskopf, “Fall of parity”, in Physics in the Twenty Century: Selected Essays (1972), pp. 273–293 (cit. on p. 2).

[541] H. A. Weldon, “Fermions without vierbeins in curved space-time”, Phys. Rev. D **63**, 104010 (2001) (cit. on p. 52).

[542] C. Wetterich, “Gravity from spinors”, Phys. Rev. D **70**, 105004 (2004) (cit. on p. 32).

[543] H. Weyl, “Gravitation und Elektrizität”, Sitzber. kgl.-preuß. Akad. Wiss. Berlin, 465–480 (1918), English translation in [372], pp.24-37 (cit. on pp. 11, 24, 36).

[544] H. Weyl, “Reine Infinitesimalgeometrie”, Math. Z. **2**, 384–411 (1918) (cit. on p. 11).

[545] H. Weyl, “Elektron und Gravitation I”, Z. Physik **56**, 330–352 (1929), Reproduced in [47] (cit. on pp. 1, 3, 11, 24, 25, 36).

[546] H. Weyl, “Gravitation and the electron”, Proc. Nat. Acad. Scie. **15**, 323–334 (1929) (cit. on pp. 1, 3, 11, 24, 25, 36).

[547] H. Weyl, “A remark on the coupling of gravitation and electron”, Phys. Rev. **77**, 699 (1950) (cit. on pp. 1, 11, 25, 36).

[548] J. A. Wheeler, “Geons”, Phys. Rev. **97**, 511 (1955) (cit. on pp. 2, 25, 36).

[549] J. A. Wheeler, “Neutrinos, gravitation and geometry”, in Interazioni Deboli. Rendiconti della Scuola Internazionale di Fisica Enrico Fermi, Como 1959, edited by L. A. Radicati (1961), pp. 67–196 (cit. on pp. 2, 23, 25, 36).

[550] J. A. Wheeler, *Geometrodynamics* (Academic Press, New York and London, 1962) (cit. on pp. 2, 25, 36).

[551] J. T. Wheeler, “Weyl geometry”, Gen. Rel. Grav. **80**, 1–36 (2018) (cit. on p. 11).

[552] G. C. Wick, A. S. Wightman, and E. P. Wigner, “The intrinsic parity of elementary particles”, Phys. Rev. **88**, 101 (1952), Reprinted in [553], pp.102-106 (cit. on p. 1).

[553] A. S. Wightman, ed., *The Collected Works of Eugene Paul Wigner, Part I: Particles and Fields, Part II: Foundations of Quantum Mechanics* (Springer-Verlag, Berlin, Heidelberg, 1997) (cit. on pp. 86, 115, 116).

[554] E. P. Wigner, "Eine Bemerkung zu Einsteins neuer Formulierung des allgemeinen Relativitätsprinzips", Z. Physik **53**, 592–596 (1929), Reprinted in [553], pp.25-29 (cit. on p. 1).

[555] E. P. Wigner, "Über die Operation der Zeitumkehr in der Quantenmechanik", Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math.-phys. Klasse, 546–559 (1932) (cit. on p. 3).

[556] C. M. Will, *Theory and Experiment in Gravitational Physics*, 2nd Ed. (Cambridge University Press, Cambridge, 2018) (cit. on p. 25).

[557] E. Witten, "A note on Einstein, Bergmann, and the fifth dimension", Preprint 1401.8048 [physics.hist-ph] (cit. on p. 12).

[558] L. Witten, "Invariants of general relativity and classification of spaces", Phys. Rev. **113**, 357 (1959) (cit. on p. 4).

[559] P. Woit, "Supersymmetric quantum mechanics, spinors and standard model", Nucl. Phys. B **303**, 329–342 (1988) (cit. on p. 5).

[560] D. M. Wolkow, "Über eine Klasse von Lösungen der Diracschen Gleichung", Z. Physik **94**, 250–260 (1935) (cit. on p. 1).

[561] C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson, "Experimental test of parity conservation in beta decay", Phys. Rev. **105**, 1413 (1957) (cit. on p. 1).

[562] C. N. Yang, "The discrete symmetries P, T and C", J. Phys. (C8) **43**, 439–451 (1982) (cit. on p. 1).

[563] C. N. Yang, "Fermi's β-decay theory", Int. J. Mod. Phys. A **27**, 1230005 (2012) (cit. on p. 2).

[564] C. N. Yang and R. L. Mills, "Conservation of isotopic spin and isotopic gauge invariance", Phys. Rev. **96**, 191 (1954), Reprinted in [296], pp.29-33 (cit. on p. 66).

[565] C. N. Yang and J. Tiomno, "Reflection properties of spin $1/2$ fields and a universal Fermi-type interaction", Phys. Rev. **79**, 495 (1950) (cit. on p. 1).

[566] V. D. Zakharov, *Gravitational Waves in Einstein Spaces* (John Wiley & Sons, London, 1973) (cit. on p. 48).

[567] R. Zaycoff, "Über eine allgemeine Form der Diracschen Gleichung", Ann. Phys. **399**, 650–660 (1930) (cit. on p. 1).

[568] A. Zee, *Quantum Field Theory in a Nutshell*, 2nd Ed. (Princeton University Press, Princeton and Oxford, 2010) (cit. on p. 25).

[569] Y. B. Zeldovich, "On the neutrino charge of elementary particles", Doklady Akademii Nauk SSSR **91**, 505–508 (1953), Reprinted in [573], Paper 9 (cit. on p. 1).

[570] Y. B. Zeldovich, "On the decay of charged π-mesons", Doklady Akademii Nauk SSSR **97**, 421–424 (1954), Reprinted in [573], Paper 10 (cit. on p. 1).

[571] Y. B. Zeldovich, “The cosmological constant and elementary particles”, Pis’ma Zh. Eksp. Teoret. Fiz. [JETP Letters] **6**, 883–884 (1967), Reprinted in [573], Paper 27 (cit. on p. 25).

[572] Y. B. Zeldovich, “The cosmological constant and the theory of elementary particles”, Uspekhi Fiz. Nauk **95**, 209–230 (1968), Republication in V. Sahni and A. Krasiński, Gen. Rel. Grav. **40**, 1557-1591 (2008) (cit. on pp. 25, 68).

[573] Y. B. Zeldovich, *Selected Works of Yakov Borisovich Zeldovich*, edited by J. P. Ostriker, Vol. II. Particles, Nuclei, and the Universe, Princeton Legacy Library (Princeton University Press, Princeton and Oxford, 1993) (cit. on pp. 116, 117).

[574] Y. B. Zeldovich and G. M. Gandelman, “Determination of the limits of applicability of quantum electrodynamics by measurement of the electron magnetic moment”, Doklady Akademii Nauk SSSR **105**, 445–447 (1955), Reprinted in [573], Paper 13 (cit. on p. 1).

[575] Y. B. Zeldovich and S. S. Gershtein, “Meson correction in the theory of beta decay”, Zh. Eksp. Teoret. Fiz. [JETP] **29**, 698–699 (1955), Reprinted in [573], Paper 12 (cit. on pp. 1, 2).

[576] Y. B. Zeldovich and I. D. Novikov, *Relativistic Astrophysics*, Revised and Enlarged from the original Russian edition, Vol. 1. Stars and and Relativity (The University of Chicago Press, Chicago and London, 1971) (cit. on pp. 8, 18, 25, 37, 68).

[577] Y. B. Zeldovich and L. P. Pitaevskiĭ, “On the possibility of creation of particles by a classical gravitational field”, Commun. Math. Phys. **23**, 185–188 (1971), Reprinted in [573], Paper 32 (cit. on p. 25).

[578] G. F. Zharkov, “Neutrino and antineutrino”, Sov. Phys. (JETP) **20**, 492 (1950) (cit. on p. 1).

[579] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 5th Ed., International Series of Monographs on Physics 171 (Oxford University Press, Oxford, 2021) (cit. on pp. 2, 3).

[580] B. Zumino, “Yang-Mills theory of weak and electromagnetic interactions. Lectures given at the GIFT Seminar, Madrid, March 1972”, CERN-TH-1479, 1–32 (1972) (cit. on p. 2).