MASSEY PRODUCTS IN GALOIS COHOMOLOGY AND THE ELEMENTARY TYPE CONJECTURE

CLAUDIO QUADRELLI

Abstract. Let p be a prime. We prove that a positive solution to Efrat's Elementary Type Conjecture implies a positive solution to a strengthened version of Minač–Tân's Massey Vanishing Conjecture in the case of finitely generated maximal pro-p Galois groups whose pro-p cyclotomic character has torsion-free image. Consequently, the maximal pro-p Galois group of a field $\mathbb{K}$ containing a root of 1 of order p (and also $\sqrt{-1}$ if $p = 2$) satisfies the strong n-Massey vanishing property for every $n > 2$ (which is equivalent to the cup-defining n-Massey product property for every $n > 2$, as defined by Minač–Tân) in several relevant cases.

1. Introduction

Throughout the paper, p will denote a prime number. Given a field $\mathbb{K}$, let $\bar{\mathbb{K}}_s$ denote a separable closure of $\mathbb{K}$, and let $\mathbb{K}(p)$ denote the maximal pro-p-extension of $\mathbb{K}$ inside $\bar{\mathbb{K}}_s$. The *absolute Galois group* $G_{\mathbb{K}} = \mathrm{Gal}(\bar{\mathbb{K}}_s/\mathbb{K})$ is a profinite group, and the Galois group

$$G_{\mathbb{K}}(p) := \mathrm{Gal}(\mathbb{K}(p)/\mathbb{K}) = \varprojlim_{[\mathbb{L}:\mathbb{K}]=p^k} \mathrm{Gal}(\mathbb{L}/\mathbb{K}),$$

called the *maximal pro-p Galois group* of $\mathbb{K}$, is the maximal pro-p quotient of $G_{\mathbb{K}}$. A major difficult problem in Galois theory is the characterization of profinite groups which occur as absolute Galois groups of fields, and of pro-p groups which occur as maximal pro-p Galois groups (see, e.g., [22, § 3.12] and [40, § 2.2]). Observe that if a pro-p group G does not occur as the maximal pro-p Galois group of a field containing a root of 1 of order p, then it does not occur as the absolute Galois group of any field.

In the '90s, I. Efrat formulated a conjecture — the *Elementary Type Conjecture* on maximal pro-p Galois groups, see [7] — which proposes a description of finitely generated pro-p groups which occur as maximal pro-p Galois groups containing a root of 1 of order p: it predicts that if $\mathbb{K}$ is a field containing a root of 1 of order p with $G_{\mathbb{K}}(p)$ finitely generated, then $G_{\mathbb{K}}(p)$ may be constructed starting from free pro-p groups and Demushkin pro-p groups and iterating free pro-p products and certain semidirect products with $\mathbb{Z}_p$ (see also [33, § 10] and [50, § 7.5]). The pro-p groups which are constructible in this way are called *pro-p groups of elementary type* (see Definition 3.4 below). The Elementary Type Conjecture is verified, for example, if an extension of relative transcendence degree 1 of a pseudo algebraically closed field (see [20, Ch. 11] and [12, § 5]); or if $\mathbb{K}$ is an algebraic extension of a global field of characteristic not

Date: August 30, 2023.
2010 *Mathematics Subject Classification.* Primary 12G05; Secondary 20E18, 20J06, 12F10.
Key words and phrases. Galois cohomology, Massey products, absolute Galois groups, elementary type conjecture.

p (see [8]). Moreover, in [55] I. Snopce and P.A. Zalesskiĭ provided new evidence in support of the Elementary Type Conjecture, as they proved that within the family of *right-angled Artin pro-p groups* — which is an extremely rich family of pro-p groups —, the only members which occur as maximal pro-p Galois groups (and thus as absolute Galois groups) are of elementary type.

The proof of the celebrated *Bloch-Kato conjecture* — now called *Norm Residue Theorem* — by M. Rost and V. Voevodsky, with the so-called "Weibel's patch" (see [25, 52, 56, 60]) provided new insights in the study of maximal pro-p Galois groups and absolute Galois groups of fields (see, e.g., [3, 17, 39] and references therein). Indeed, the Norm Residue Theorem implies that the ring structure of the $\mathbb{F}_p$-cohomology algebra

$$\mathbf{H}^\bullet(G_{\mathbb{K}}(p)) = \coprod_{n\geq 0} \mathrm{H}^n(G_{\mathbb{K}}(p), \mathbb{F}_p)$$

of a field $\mathbb{K}$ containing a root of 1 of order p, endowed with the graded-commutative *cup-product*

$$\_\smile\_ \colon \mathrm{H}^s(G, \mathbb{F}_p) \times \mathrm{H}^t(G, \mathbb{F}_p) \longrightarrow \mathrm{H}^{s+t}(G, \mathbb{F}_p), \qquad s, t \geq 0,$$

is determined by degrees 1 and 2 (see, e.g., [49, § 1]) — observe that this is true also for every closed subgroup of $G_{\mathbb{K}}(p)$, as every closed subgroup is again a maximal pro-p Galois group. It is worth underlining that all pro-p groups of elementary type satisfy hereditarily this cohomological condition (i.e., the ring structure of the $\mathbb{F}_p$-cohomology algebra of every closed subgroup is determined by degrees 1 and 2, see [50, Thm. 1.4]); on the other hand, it is remarkable that we do not know examples of finitely generated pro-p groups satisfying hereditarily this cohomological condition other than pro-p groups of elementary type.

In recent years — especially after the publication of the work of M. Hopkins and K. Wickelgren [27] —, much of the research on absolute Galois groups and maximal pro-p Galois groups focused on the study of *Massey products* in Galois cohomology (see, e.g., [13, 40, 45, 57] and references therein). Given a pro-p group G and an integer $n \geq 2$, the *n-fold Massey product* is a multi-valued map which associates a sequence $\alpha_1, \ldots, \alpha_n$ of elements of $\mathrm{H}^1(G, \mathbb{Z}/p)$ to a (possibly empty) subset

$$\langle \alpha_1, \ldots, \alpha_n \rangle \subseteq \mathrm{H}^2(G, \mathbb{F}_p).$$

If $n = 2$ it coincides with the cup-product, namely, $\langle \alpha_1, \alpha_2 \rangle = \{\alpha_1 \smile \alpha_2\}$. For $n > 2$, a pro-p group G is said to satisfy the *n-Massey vanishing property* if the set $\langle \alpha_1, \ldots, \alpha_n \rangle$ contains 0 whenever it is non-empty. In [41], J. Minač and N.D. Tân conjectured the following.

Conjecture 1.1. *Let $\mathbb{K}$ be a field containing a root of 1 of order p. Then the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ of $\mathbb{K}$ satisfies the n-Massey vanishing property for every $n > 2$.*

One has the following partial — but very remarkable — results:

(a) E. Matzri proved that the maximal pro-p Galois group of every field containing a root of 1 of order p satisfies the 3-Massey vanishing property (see the preprint [34], see also the published works [15, 43]);
(b) J. Minač and N.D. Tân proved Conjecture 1.1 for local fields (see [45]);
(c) Y. Harpaz and O. Wittenberg proved Conjecture 1.1 for number fields (see [26]);

(d) A. Merkurjev and F. Scavia proved that the maximal pro-2 Galois group of every field satisfies the 4-Massey vanishing property (see [37]).

Further interesting results on Massey products in Galois cohomology have been obtained by various authors (see, e.g., [21, 23, 24, 32, 35, 36, 61]).

The purpose of the present work is to prove a *strengthened version* of Conjecture 1.1 for fields whose maximal pro-p Galois group is of elementary type.

Given a pro-p group G and a positive integer $n > 2$, if the set $\langle \alpha_1, \ldots, \alpha_n \rangle$, associated to a sequence $\alpha_1, \ldots, \alpha_n$ of elements of $\mathrm{H}^1(G, \mathbb{F}_p)$, is non-empty, then necessarily

$$\alpha_1 \smile \alpha_2 = \alpha_2 \smile \alpha_3 = \ldots = \alpha_{n-1} \smile \alpha_n = 0 \tag{1.1}$$

— we underline that the triviality condition (1.1) is also sufficient to imply the non-emptiness of $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, in the case $n = 3$. A pro-p group G is said to satisfy the *strong* n-Massey vanishing property, for $n > 2$, if every sequence $\alpha_1, \ldots, \alpha_n$ of elements of $\mathrm{H}^1(G, \mathbb{F}_p)$ satisfying condition (1.1) yields an n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ containing 0 (see [48, Def. 1.2]). The strong n-Massey vanishing property is stronger than the n-Massey vanishing property; observe that, since the two properties coincide for $n = 3$, E. Matzri's result implies that the maximal pro-p Galois group of a field containing a root of 1 of order p satisfies, "for free", also the strong 3-Massey vanishing property.

By construction, a pro-p group of elementary type G comes endowed with an *orientation*, namely, a homomorphism of pro-p groups $G \to 1 + p\mathbb{Z}_p$, where $1 + p\mathbb{Z}_p$ denotes the multiplicative group of principal units of $\mathbb{Z}_p$ (we give a brief review of orientations of pro-p groups in § 3.1). Our main result is the following.

Theorem 1.2. *Let G be a pro-p group of elementary type. If $p = 2$ assume further that the image of the orientation associated to G is a subgroup of $1 + 4\mathbb{Z}_2$. Then G satisfies the strong n-Massey vanishing property for every $n > 2$.*

To prove Theorem 1.2, we exploit a result — whose original formulation, for discrete groups, is due to W. Dwyer, see [6] — which interprets the vanishing of Massey products in the $\mathbb{F}_p$-cohomology of a pro-p group G in terms of the existence of certain upper unitriangular representations of G. Moreover, we use the *Kummerian property* — a formal version of Hilbert 90, introduced in [18] —, which guarantees the vanishing of "cyclic" Massey products (see Theorem 3.10), and which is enjoyed both by pro-p groups of elementary type and maximal pro-p Galois groups of fields containing a root of 1 of order p (we give a brief review of the Kummerian property in § 3.3).

Let $\mathbb{K}$ be a field containing a root of 1 of order p. It has been shown that $G_{\mathbb{K}}(p)$ satisfies the strong n-Massey vanishing property for every $n > 2$, if p is odd and $\mathbb{K}$ is p-rigid, by J. Minač and N.D. Tân (see [41, Thm. 8.5]); and if $\mathbb{K}$ has virtual cohomological dimension at most 1 or it is pseudo p-adically closed, by A. Pál and E. Szabó (see [48]). As a consequence of Theorem 1.2, we obtain the following.

Corollary 1.3. *Let $\mathbb{K}$ be a field containing a root of 1 of order p, such that the quotient $\mathbb{K}^\times/(\mathbb{K}^\times)^p$ is finite. If $p = 2$ suppose further that $\sqrt{-1} \in \mathbb{K}$.Then $G_{\mathbb{K}}(p)$ satisfies the strong n-Massey vanishing property for every $n > 2$ in the following cases:*

(a) $\mathbb{K}$ *is a local field, or an extension of transcendence degree 1 of a local field;*

(b) $\mathbb{K}$ *is a PAC field, or an extension of relative transcendence degree 1 of a PAC field;*
(c) $\mathbb{K}$ *is p-rigid (for the definition of p-rigid fields see* [59, p. 722]*);*
(d) $\mathbb{K}$ *is an algebraic extension of a global field of characteristic not p;*
(e) $\mathbb{K}$ *is a valued p-Henselian field with residue field κ, and $G_\kappa(p)$ satisfies the strong n-Massey vanishing property for every $n > 2$.*

In [44, Question 4.2], J. Minač and N.D. Tân asked the following.

Question 1.4. *Let p be a prime, and let $\mathbb{K}$ be a field containing a root of 1 of order p. Does the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ of $\mathbb{K}$ satisfy the strong n-Massey vanishing property for every $n > 2$?*

The original formulation of [44, Question 4.2] involves the *cup-defining n-fold Massey product property*, which is equivalent to the strong n-Massey vanishing property, if required for all $n \geq 3$ — see [44, Rem. 4.6] and Remark 2.5 below. Question 1.4 has a negative answer in case $p = 2$. Indeed, in [23, Example A.15], O. Wittenberg produced an example (suggested by Y. Harpaz) of a number field $\mathbb{K}$ not containing $\sqrt{-1}$ whose maximal pro-2 Galois group $G_{\mathbb{K}}(2)$ does not satisfy the strong 4-Massey vanishing property. Moreover, recently A. Merkurjev and F. Scavia showed that every field $\mathbb{K}$ has an extension $\mathbb{L}$ whose maximal pro-2 Galois group $G_{\mathbb{L}}(2)$ does not satisfy the strong 4-Massey vanishing property (cf. [37, Thm. 6.3]).

Wittenberg's example and Merkurjev-Scavia's result involve pro-2 groups that are not finitely generated. Thus, we ask whether Question 1.4 may have a positive answer under the further conditions that the maximal pro-p Galois group is *finitely generated*, and $\sqrt{-1} \in \mathbb{K}$ if $p = 2$.

Question 1.5. *Let $\mathbb{K}$ be a field containing a root of 1 of order p, such that the quotient $\mathbb{K}^\times/(\mathbb{K}^\times)^p$ is finite. If $p = 2$ suppose further that $\sqrt{-1} \in \mathbb{K}$. Does the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ of $\mathbb{K}$ satisfy the strong n-Massey vanishing property for every $n > 2$?*

By Theorem 1.2, a positive solution of the Elementary Type Conjecture would yield a positive answer to Question 1.5.

Acknowledgments. The author wishes to thank I. Efrat, E. Matzri, J. Minač, F.W. Pasini, and N.D. Tân, for several inspiring discussions on Massey products in Galois cohomology and pro-p groups of elementary type, which occurred in the past years; A. Pál, F. Scavia, P. Wake, O. Wittenberg, and again J. Minač and N.D. Tân, for their useful comments on this work. Last, but not least, the author is very grateful to the referee for the careful work carried with the manuscript and for the extremely useful comments and suggestions.

2. Massey products and pro-p groups

Let G be a pro-p group, and let $\mathbb{F}_p$ be the finite field with p elements, considered as a trivial G-module. For basic notions on pro-p groups and their $\mathbb{F}_p$-cohomology, we refer to [54, Ch. I, § 4] and to [46, Ch. I, and Ch. III § 3].

Given a pro-p group G, and two subsets S_1, S_2 of $\mathrm{H}^1(G, \mathbb{F}_p)$, we set

$$S_1 \smile S_2 = \{ \alpha_1 \smile \alpha_2 \mid \alpha_1 \in S_1,\ \alpha_2 \in S_2 \}.$$

2.1. **Massey products in Galois cohomology.** Here we give a brief review on Massey products in the Galois cohomology of pro-p groups. Throughout the paper, we will be merely concerned with Massey products of elements in the first cohomology group $\mathrm{H}^1(G,\mathbb{F}_p)$, whose definition will be recalled here below. Our main references are [57] and [45] — for a general definition of Massey products on the level of cochains the reader may consult [6,30].

For $n\in\{1,2,3\}$ let $\mathfrak{C}^n$ denote the $\mathbb{F}_p$-vector spaces of continuous maps

$$c\colon \underbrace{G\times\cdots\times G}_{n\text{ times}}\to\mathbb{F}_p$$

($G\times\cdots\times G$ is to be intended as the cartesian product of topological spaces). These vector spaces come equipped with homomorphisms $\partial^n\colon\mathfrak{C}^n\to\mathfrak{C}^{n+1}$, $n=1,2$, defined by

$$\begin{aligned}\partial^1(c)(g_1,g_2)&=c(g_1)-c(g_1g_2)+c(g_2),\\ \partial^2(c')(g_1,g_2,g_3)&=c'(g_1,g_2)-c'(g_1,g_2g_3)+c'(g_1g_2,g_3)-c'(g_2,g_3),\end{aligned}$$

for every $c\in\mathfrak{C}^1$, $c'\in\mathfrak{C}^2$, and $g_1,g_2,g_3\in G$. We recall that

$$\mathrm{H}^1(G,\mathbb{F}_p)=\mathrm{Ker}(\partial^1)=\mathrm{Hom}(G,\mathbb{F}_p) \tag{2.1}$$

— where the latter is the group of homomorphisms of pro-p groups $G\to\mathbb{F}_p$, with $\mathbb{F}_p$ considered as a cyclic group of order p —, while

$$\mathrm{H}^2(G,\mathbb{F}_p)=\mathrm{Ker}(\partial^2)/\operatorname{Im}(\partial^1). \tag{2.2}$$

For $c,c'\in\mathfrak{C}^1$, one defines $c\cdot c'\in\mathfrak{C}^2$ by $(c\cdot c')(g_1,g_2)=c(g_1)\cdot c'(g_2)$ for every $g_1,g_2\in G$. Then $\partial^2(c\cdot c')=\partial^1(c)\cdot c'-c\cdot\partial^1(c')$ (cf. [46, Prop. 1.4.1]). Consequently, for $\alpha,\alpha'\in\mathrm{H}^1(G,\mathbb{F}_p)$ one has $\alpha\cdot\alpha'\in\mathrm{Ker}(\partial^2)$, so that one defines the cup-product $\alpha\smile\alpha'$ of α and α' to be the class of $\alpha\cdot\alpha'$ in $\mathrm{H}^2(G,\mathbb{F}_p)$.

Remark 2.1. For every $\alpha,\alpha'\in\mathrm{H}^1(G,\mathbb{F}_p)$, one has $\alpha'\smile\alpha=-\alpha\smile\alpha'$ (cf. [46, Prop. 1.4.4]). In particular, if $p\neq 2$ then $\alpha\smile\alpha=0$.

For $n\geq 2$ let $\alpha_1,\dots,\alpha_n$ be a sequence of elements of $\mathrm{H}^1(G,\mathbb{F}_p)$. A collection $\mathfrak{c}=(c_{ij})$, $1\leq i\leq j\leq n$, of elements of $\mathfrak{C}^1$ is called a *defining set* for the n-fold Massey product $\langle\alpha_1,\dots,\alpha_n\rangle$ if the following conditions hold:

(a) $c_{ii}=\alpha_i$ for every $i=1,\dots,n$;
(b) for every couple (i,j) such that $1\leq i<j\leq n$ and $(i,j)\neq(1,n)$, one has

$$\partial^1(c_{ij})=\sum_{h=1}^{j-1}c_{i,h}\cdot c_{h+1,j}. \tag{2.3}$$

Then $\sum_{h=1}^{n-1}c_{1,h}\cdot c_{h+1,n}$ lies in $\mathrm{Ker}(\partial^2)$, and its class in $\mathrm{H}^2(G,\mathbb{F}_p)$ is called the *value* of $\mathfrak{c}$. The n-fold Massey product $\langle\alpha_1,\dots,\alpha_n\rangle$ is the subset of $\mathrm{H}^2(G,\mathbb{F}_p)$ consisting of the values of all its defining sets. Observe that if $n=2$, then

$$\langle\alpha_1,\alpha_2\rangle=\left\{\alpha_1\cdot\alpha_2+\operatorname{Im}(\partial^1)\right\}=\{\alpha_1\smile\alpha_2\}. \tag{2.4}$$

Remark 2.2. Let $\alpha_1,\dots,\alpha_n$ be a sequence of elements of $\mathrm{H}^1(G,\mathbb{F}_p)$, $n>2$. By (2.3), the existence of a defining set for the n-fold Massey product $\langle\alpha_1,\dots,\alpha_n\rangle$ implies that $\alpha_i\cdot\alpha_{i+1}\in\operatorname{Im}(\partial^1)$ for every $i=1,\dots,n-1$, i.e., $\alpha_i\smile\alpha_{i+1}=0$.

Moreover, if $n = 3$ this condition is also sufficient for the existence of a defining set $\mathfrak{c} = (c_{ij})$ for the 3-fold Massey product $\langle \alpha_1, \alpha_2, \alpha_3\rangle$: indeed, if $\alpha_1 \smile \alpha_2 = \alpha_2 \smile \alpha_3 = 0$ then there exist $c_{1,2}, c_{2,3} \in \mathfrak{C}^1$ such that $\partial^1(c_{1,2}) = \alpha_1 \cdot \alpha_2$ and $\partial^1(c_{2,3}) = \alpha_2 \cdot \alpha_3$, and thus $\langle \alpha_1, \alpha_2, \alpha_3\rangle \neq \varnothing$.

Definition 2.3. Let G be a pro-p group, let n be a positive integer, $n \geq 2$, and let $\alpha_1, \ldots, \alpha_n$ be a sequence of elements of $\mathrm{H}^1(G, \mathbb{F}_p)$.

(a) The n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n\rangle$ is said to be *defined* if it is non-empty — i.e., if there exists at least one defining set.

(b) The n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n\rangle$ is said to *vanish* if $0 \in \langle \alpha_1, \ldots, \alpha_n\rangle$.

Definition 2.4. Let G be a pro-p group, and let n be a positive integer, $n \geq 2$.

(a) The group G is said to satisfy the *cup-defining n-fold Massey property* (with respect to $\mathbb{F}_p$) if every n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n\rangle$ in the $\mathbb{F}_p$-cohomology of G is defined whenever

$$\alpha_i \smile \alpha_{i+1} = 0 \qquad \text{for every } i = 1, \ldots, n-1. \tag{2.5}$$

(b) The group G is said to satisfy the *n-Massey vanishing property* (with respect to $\mathbb{F}_p$) if every defined n-fold Massey product in the $\mathbb{F}_p$-cohomology of G vanishes.

(c) The group G is said to satisfy the *strong* n-Massey vanishing property (with respect to $\mathbb{F}_p$) if it satisfies both the cup-defining n-fold Massey property and the n-fold Massey vanishing property; namely, the n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n\rangle$ vanishes whenever condition (2.5) is satisfied.

Remark 2.5. If a pro-p group G has the cup-defining n-fold Massey property, with $n \geq 4$, then G has the vanishing $(n-1)$-fold Massey vanishing property, as observed in [44, Rem. 4.6]. Therefore, G has the strong n-fold Massey vanishing property for every $n \geq 3$ if, and only if, it has the cup-defining n-fold Massey product property for every $n \geq 3$. In particular, [44, Question 4.2] is equivalent to Question 1.4.

Massey products in the $\mathbb{F}_p$-cohomology of pro-p groups enjoy the following properties (cf., e.g., [57, Prop. 1.2.3–1.2.4] and [45, Rem. 2.2]).

Proposition 2.6. *Let G be a pro-p group and let $\alpha_1, \ldots, \alpha_n$ be a sequence of elements of $\mathrm{H}^1(G, \mathbb{F}_p)$, with $n > 2$. Suppose that the n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n\rangle$ is defined.*

(i) *If $\alpha_i = 0$ for some i, then the n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n\rangle$ vanishes.*

(ii) *If the n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n\rangle$ is defined, then for every $a \in \mathbb{F}_p$ and $i \in \{1, \ldots, n\}$ one has*

$$\langle \alpha_1, \ldots, a\alpha_i, \ldots, \alpha_n\rangle \supseteq \{\, a\beta \mid \beta \in \langle \alpha_1, \ldots, \alpha_n\rangle \,\}.$$

(iii) *If the set $\langle \alpha_1, \ldots, \alpha_n\rangle$ is not empty, then it is closed under adding $\alpha_1 \smile \alpha'$ and $\alpha_n \smile \alpha'$ for any $\alpha' \in \mathrm{H}^1(G, \mathbb{F}_p)$.*

2.2. **Massey products and upper unitriangular matrices.** Massey products may be "translated" in terms of unipotent upper-triangular homomorphisms of G as follows.

For $n \geq 2$ let

$$\mathbb{U}_{n+1} = \left\{ \begin{pmatrix} 1 & a_{1,2} & \cdots & & a_{1,n+1} \\ & 1 & a_{2,3} & \cdots & \\ & & \ddots & \ddots & \vdots \\ & & & 1 & a_{n,n+1} \\ & & & & 1 \end{pmatrix} \mid a_{i,j} \in \mathbb{F}_p \right\} \subseteq \mathrm{GL}_{n+1}(\mathbb{F}_p)$$

be the group of unipotent upper-triangular $(n+1) \times (n+1)$-matrices over $\mathbb{F}_p$. Let I_{n+1} denote the $(n+1) \times (n+1)$ identity matrix, and for $1 \leq i < j \leq n+1$, let E_{ij} denote the $(n+1) \times (n+1)$-matrix with 1 at the entry (i,j), and 0 elsewhere. We set $\bar{\mathbb{U}}_{n+1} = \mathbb{U}_{n+1}/Z$, where Z denotes the normal subgroup

$$Z = \{ I_{n+1} + aE_{1,n+1} \mid a \in \mathbb{F}_p \}. \tag{2.6}$$

For a homomorphism of pro-p groups $\rho\colon G \to \mathbb{U}_{n+1}$, and for $1 \leq i \leq n$, let $\rho_{i,i+1}$ denote the projection of ρ on the $(i, i+1)$-entry. Observe that $\rho_{i,i+1}\colon G \to \mathbb{F}_p$ is a homomorphism of pro-p groups, and thus we may consider $\rho_{i,i+1}$ as an element of $\mathrm{H}^1(G, \mathbb{F}_p)$. One has the following "pro-p translation" of Dwyer's result on Massey products (cf., e.g., [18, Lemma 9.3], see also [13, § 8]).

Proposition 2.7. *Let G be a pro-p group and let $\alpha_1, \ldots, \alpha_n$ be a sequence of elements of $\mathrm{H}^1(G, \mathbb{F}_p)$, with $n \geq 2$.*

(i) *The n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined if, and only if, there exists a continuous homomorphism $\bar{\rho}\colon G \to \bar{\mathbb{U}}_{n+1}$ such that $\bar{\rho}_{i,i+1} = \alpha_i$ for every $i = 1, \ldots, n$.*

(ii) *The n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ vanishes if, and only if, there exists a continuous homomorphism $\rho\colon G \to \mathbb{U}_{n+1}$ such that $\rho_{i,i+1} = \alpha_i$ for every $i = 1, \ldots, n$.*

By Proposition 2.6–(i), in order to check that a pro-p group satisfies the n-Massey vanishing property for some $n \geq 2$, it suffices to verify that every defined n-fold Massey product associated to a sequence of *non-trivial* cohomology elements of degree 1 vanishes. Analogously, we use Proposition 2.7 to show that, in order to check that a pro-p group satisfies the strong n-Massey vanishing property, it suffices to verify that every sequence of length at most n of *non-trivial* cohomology elements of degree 1 whose cup-products satisfy the triviality condition (1.1) yields a Massey product containing 0.

Proposition 2.8. *Given $N > 2$, a pro-p group G satisfies the strong n-Massey vanishing property for every $3 \leq n \leq N$ if, and only if, for every $3 \leq n \leq N$, every sequence $\alpha_1, \ldots, \alpha_n$ of non-trivial elements of $\mathrm{H}^1(G, \mathbb{F}_p)$ satisfying the triviality condition* (1.1) *yields an n-fold Massey product containing* 0*.*

Proof. If G satisfies the strong n-Massey vanishing property, then obviously $\langle \alpha_1, \ldots, \alpha_n \rangle$ vanishes for every sequence $\alpha_1, \ldots, \alpha_n$ with $\alpha_1, \ldots, \alpha_n \neq 0$ satisfying the triviality condition (1.1).

So assume that for every sequence $\alpha_1, \ldots, \alpha_n$, with $\alpha_i \neq 0$ for every $i = 1, \ldots, n$ and satisfying the triviality condition (1.1), one has $0 \in \langle \alpha_1, \ldots, \alpha_n \rangle$. Now pick an arbitrary sequence $\alpha_1,\dots,\alpha_m$ with $\alpha_i\in\mathrm{H}^1(G,\mathbb{F}_p)$ and $3\leq m\leq N$ such that $\alpha_i\smile\alpha_{i+1}=0$ for every $i=1,\dots,m-1$. Let

$$0=m_1<m_2<\ldots<m_r<m_{r+1}=m$$

be positive integers such that: either

$$\alpha_1=\alpha_{m_1+1},\dots,\alpha_{m_2}\neq 0,\qquad \alpha_{m_2+1},\dots,\alpha_{m_3}=0,\qquad \alpha_{m_3+1},\dots,\alpha_{m_4}\neq 0,\qquad \ldots$$

and so on, if $\alpha_1\neq 0$; or conversely

$$\alpha_1=\alpha_{m_1+1},\dots,\alpha_{m_2}=0,\qquad \alpha_{m_2+1},\dots,\alpha_{m_3}\neq 0,\qquad \alpha_{m_3+1},\dots,\alpha_{m_4}=0,\qquad \ldots$$

and so on, if $\alpha_1=0$. For every $j=1,\dots,r$ put $n_j=m_{j+1}-m_j$. By hypothesis, for every j such that $\alpha_{m_j+1},\dots,\alpha_{m_{j+1}}\neq 0$, the n_j-fold Massey product $\langle\alpha_{m_j+1},\dots,\alpha_{m_{j+1}}\rangle$ vanishes. Hence, by Proposition 2.7–(ii) there exists a homomorphism $\rho_j\colon G\to\mathbb{U}_{n_j+1}$ such that $(\rho_j)_{i,i+1}=\alpha_{m_j+1}$ for every $i=1,\dots,n_j$. On the other hand, if j is such that $\alpha_{m_j+1},\dots,\alpha_{m_{j+1}}=0$ and $n_j>1$, then we set $\rho_j\colon G\to\mathbb{U}_{n_j-1}$ to be the homomorphism constantly equal to I_{n_j-1}.

Thus, we may define blockwise a homomorphisms $\rho\colon G\to\mathbb{U}_{m+1}$, where

$$\rho=\begin{pmatrix}\rho_1 & & & 0\\ & \rho_2 & & \\ & & \ddots & \\ 0 & & & \rho_r\end{pmatrix},$$

where we omit ρ_j if $\alpha_{m_j+1}=0$, $n_j=1$, and $j\neq 1,r$. For example, if $\alpha_1\neq 0$ and $n_2=1$ then one has

$$\rho=\left(\begin{array}{c|c}\rho_1\left\{\begin{matrix}\ddots & \ddots & \\ & 1 & \alpha_{m_2}\\ & & 1\end{matrix}\right. & \alpha_{m_2+1}\\ \hline & \left.\begin{matrix}1 & \alpha_{m_3+1}\\ & \ddots\end{matrix}\right\}\rho_3\\ & \qquad\qquad\ddots\end{array}\right),$$

where $m_2+1=m_3$ and $\alpha_{m_2+1}=0$. Then

$$\rho_{i,i+1}=(\rho_j)_{i-m_j,i-m_j+1}=\alpha_i,\qquad \text{if } m_j<i\leq m_{j+1},\ \alpha_{m_j+1}\neq 0,$$

and $\rho_{i,i+1}=0=\alpha_i$ otherwise. Therefore, Proposition 2.7–(ii) implies that the m-fold Massey product $\langle\alpha_1,\dots,\alpha_m\rangle$ vanishes. $\square$

2.3. **Commutators of upper unitriangular matrices.** Let $L=\bigoplus_{k\geq 1}L_k$ be a graded Lie algebra over $\mathbb{F}_p$. Suppose that $L_k=0$ for $k>n$, for some positive n, and that for $1\leq k\leq n$ every subspace L_k has dimension $n+1-k$. Suppose further that each non-trivial subspace L_k has a basis

$$\{\,\mathbf{e}_{1,1+k},\ \mathbf{e}_{2,2+k},\ \ldots,\ \mathbf{e}_{n+1-k,n+1}\,\}\subseteq L_k,$$

whose elements satisfy

$$[\mathbf{e}_{i,j},\mathbf{e}_{i',j'}]=\begin{cases}\mathbf{e}_{i,j'}, & \text{if } j=i',\\ -\mathbf{e}_{i',j}, & \text{if } i=j',\\ 0, & \text{otherwise}\end{cases}\tag{2.7}$$

— here $[\text{␣},\text{␣}]$ denotes the Lie brackets of L — for every $1\leq i,i'\leq n$ and $2\leq j,j'\leq n+1$ such that $i<j$ and $i'<j'$. Then one has the following.

Lemma 2.9. *Let $L=\bigoplus_{k\geq 1}L_k$ be a graded Lie $\mathbb{F}_p$-algebra as above, and let a be an element of L_1 such that*

$$a=\sum_{i=1}^{n}a_i\mathbf{e}_{i,i+1},\qquad a_1,\ldots,a_n\in\mathbb{F}_p$$

satisfying $a_1a_2\cdots a_{n+1-k'}\neq 0$ or $a_{k'}a_{k'+1}\cdots a_n\neq 0$ for some k', $2\leq k'\leq n$. Then for every $c\in L_{k'}$ there exists $b\in L_{k'-1}$ such that $[b,a]=c$.

Proof. Write $c=\sum_{l=1}^{n+1-k'}c_l\mathbf{e}_{l,l+k}$, with $c_l\in\mathbb{F}_p$, and

$$b=\sum_{j=1}^{n+2-k'}b_j\mathbf{e}_{j,j+k'-1},\qquad b_j\in\mathbb{F}_p$$

for an arbitrary element $b\in L_{k'-1}$. Then applying (2.7) yields

$$\begin{aligned}[b,a]&=\sum_{j=1}^{n+2-k'}\sum_{i=1}^{n}b_ja_i[\mathbf{e}_{j,j+k'-1},\mathbf{e}_{i,i+1}]\\&=\sum_{j=1}^{n+1-k'}b_ja_{j+k'-1}\mathbf{e}_{j,j+k'}-\sum_{j=2}^{n+2-k'}b_ja_{j-1}\mathbf{e}_{j-1,j+k'-1}\\&=\sum_{j=1}^{n+1-k'}\left(b_ja_{j+k'-1}-b_{j+1}a_j\right)\mathbf{e}_{j,j+k'}.\end{aligned}$$

Therefore, $[b,a]=c$ if, and only if, the system

$$\begin{pmatrix}a_{k'}&-a_1&0&\cdots&0\\&a_{k'+1}&-a_2&\cdots&0\\&&\ddots&\ddots&\vdots\\&&&a_n&-a_{n+1-k'}\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\\vdots\\b_{n+1-k'}\\b_{n+2-k'}\end{pmatrix}=\begin{pmatrix}c_1\\c_2\\\vdots\\c_{n+1-k'}\end{pmatrix}$$

— where $b_1,\ldots,b_{n+2-k'}$ are the indeterminates — is solvable. The condition on the coefficients $a_1,\ldots,a_n$ ensures that rank of the matrix of coefficients is $n+1-k'$; so the system is solvable, yielding a suitable $b\in L_{k'-1}$. $\square$

Given an arbitrary group U, and two elements $g,h\in U$, we write ${}^gh=ghg^{-1}$, and $[g,h]={}^gh\cdot h^{-1}=ghg^{-1}h^{-1}$. Given three elements g_1,g_2,h of U, one has the identities

$$\begin{aligned}[g_1g_2,h]&=[g_1,[g_2,h]]\,[g_2,h][g_1,h],\\ [h,g_1g_2]&=[h,g_1]\,[g_1,[h,g_2]]\,[h,g_2].\end{aligned}\tag{2.8}$$

Let $(U_{(k)})_{k\geq 1}$ denote the descending central series of U, i.e.,

$$U_{(1)}=U\qquad\text{and}\qquad U_{(k+1)}=[U_{(k)},U].$$

Within this subsection, we fix an integer $n\geq 2$, and for simplicity we write $\mathbb{U}$ instead of $\mathbb{U}_{n+1}$, and I instead of I_{n+1}. The following properties of $\mathbb{U}$ are well-known (cf., e.g., [1, Thm. 1.5], [14, § 2], and [48, § 4]). For every $k \geq 1$, the k-th term $\mathbb{U}_{(k)}$ of the descending central sequence of $\mathbb{U}$ is the subgroup

$$\mathbb{U}_{(k)} = \left\{ I + \sum_{j-i\geq k} a_{ij}E_{ij} \mid a_{ij} \in \mathbb{F}_p \right\}.$$

In particular, $\mathbb{U}_{(n)}$ is the subgroup Z of $\mathbb{U}$ defined in (2.6), while $\mathbb{U}_{(k)} = 0$ for $k > n$. Moreover, every quotient $\mathbb{U}_{(k)}/\mathbb{U}_{(k+1)}$ is a p-elementary abelian group. Altogether, the graded $\mathbb{F}_p$-vector space

$$L(\mathbb{U}) = \bigoplus_{k\geq 1} L(\mathbb{U})_k, \qquad L(\mathbb{U})_k = \mathbb{U}_{(k)}/\mathbb{U}_{(k+1)}$$

is a graded Lie algebra over $\mathbb{F}_p$, endowed with the Lie brackets induced by commutators. Moreover, for every $1 \leq k \leq n$ one has $\dim(L(\mathbb{U})_k) = n+1-k$, and $L(\mathbb{U})_k$ comes endowed with a basis $\{\mathbf{e}_{1,1+k}, \ldots, \mathbf{e}_{n+1-k,n+1}\}$ with

$$\mathbf{e}_{i,i+k} = (I + E_{i,i+k})\mathbb{U}_{k+1} \in L(\mathbb{U})_k, \qquad \text{with } 1 \leq i \leq n+1-k.$$

Straightforward computations show that the elements $\mathbf{e}_{i,i+k}$ above satisfy (2.7) — cf., e.g., [48, Lemma 4.3]. From this we deduce the following.

Proposition 2.10. *Let $A \in \mathbb{U}$ be the matrix with coset $a \in L(\mathbb{U})_1$, $a = \sum_{i=1}^{n} a_i\mathbf{e}_{i,i+1}$, with $a_i \neq 0$ for all $i = 1, \ldots, n$, and let k be a positive integer such that $3 \leq k \leq n$. For every $C \in \mathbb{U}_{(k)}$ there exists a matrix $B \in \mathbb{U}_{(k-1)}$ such that*

$$[B, A] = C. \tag{2.9}$$

Proof. For $l \geq 1$ we produce matrices $B_1, \ldots, B_l \in \mathbb{U}_{(k-1)}$ satisfying

$$[B_l \cdots B_2B_1, A] \equiv C \mod \mathbb{U}_{(k+l)}. \tag{2.10}$$

Since $k + l \geq n + 1$ for l sufficiently large, one has $\mathbb{U}_{(k+l)} = \{1\}$, and thus from (2.10) one obtains $[B_l \cdots B_1, A] = C$. So we may put $B = B_l \cdots B_1$, so that B satisfies (2.9).

Observe that the coset $a \in L(\mathbb{U})_1$ of A satisfies the hypothesis of Lemma 2.9. Let $c \in L(\mathbb{U})_k$ be the coset of C. Lemma 2.9 yields an element $b \in L(\mathbb{U})_{k-1}$ such that $[b, a] = c$. Therefore, any matrix $B_1 \in \mathbb{U}_{(k-1)}$ with coset b satisfies (2.10) with $l = 1$. Now suppose that $l \geq 1$, and that we have found l matrices $B_1, \ldots, B_l \in \mathbb{U}_{(k-1)}$ satisfying (2.10). Namely, one has

$$C_l := [B_l \cdots B_2B_1, A]^{-1}C \in \mathbb{U}_{(k+l)}.$$

Then again Lemma 2.9 yields $B_{l+1} \in \mathbb{U}_{(k+l-1)}$ — hence B_{l+1} lies in $\mathbb{U}_{(k-1)}$, too — such that $[B_{l+1}, A] \equiv C_l \bmod \mathbb{U}_{(k+l+1)}$. The commutator identities (2.9) imply

$$\begin{aligned} [B_{l+1} \cdot (B_l \cdots B_2B_1), A] &= [B_{l+1}, [B_l \cdots B_2B_1, A]] \cdot [B_l \cdots B_2B_1, A] \cdot [B_{l+1}, A] \\ &\equiv [B_{l+1}, [B_l \cdots B_2B_1, A]] \cdot C \mod \mathbb{U}_{(k+l+1)} \\ &\equiv C \mod \mathbb{U}_{(k+l+1)}, \end{aligned}$$

as $[B_{l+1}, [B_l \cdots B_2B_1, A]] \in \mathbb{U}_{(k+l-1)+k}$ and $2k + l - 1 \geq k + l + 1$. Altogether, $B_1, \ldots, B_{l+1}$ lie in $\mathbb{U}_{(k-1)}$, and they satisfy (2.10) (with $l + 1$ instead of l). □

3. Oriented pro-p groups

Recall that, given a pro-p group G, the Frattini subgroup $\Phi(G)$ of G is the subgroup $G^p\cdot \mathrm{cl}([G,G])$, where the latter factor is the closure of the commutator subgroup $[G,G]$ with respect to the topology of G. By (2.1), one has an isomorphism of $\mathbb{F}_p$-vector spaces

$$\mathrm{H}^1(G,\mathbb{F}_p)=(G/\Phi(G))^*, \tag{3.1}$$

where $\_^*$ denotes the $\mathbb{F}_p$-dual space (c.f., e.g., [54, Ch. I, § 4.2, p. 29]).

3.1. **Orientations.** Recall that $1+p\mathbb{Z}_p$ denotes the multiplicative group of principal units of the ring of p-adic integers $\mathbb{Z}_p$, i.e.,

$$1+p\mathbb{Z}_p=\{\,1+p\lambda\;\mid\;\lambda\in\mathbb{Z}_p\,\}.$$

If $p=2$ then $1+2\mathbb{Z}_2=\{\pm1\}\times(1+4\mathbb{Z}_2)$, which is isomorphic to $(\mathbb{Z}/2)\times\mathbb{Z}_2$ as an abelian pro-2 group; while $1+p\mathbb{Z}_p$ is a free cyclic pro-p group if $p\neq 2$.

Let G be a pro-p group. A homomorphism of pro-p groups $\theta\colon G\to 1+p\mathbb{Z}_p$ is called an *orientation*, and the pair (G,θ) is called an *oriented pro-p group* (cf. [50]; oriented pro-p groups were introduced by I. Efrat in [9], with the name "cyclotomic pro-p pairs"). An orientation $\theta\colon G\to 1+p\mathbb{Z}_p$ of a pro-p group G is said to be *torsion-free* if $p\neq 2$, or if $p=2$ and $\mathrm{Im}(\theta)\subseteq 1+4\mathbb{Z}_2$.

If (G,θ) and (H,τ) are two oriented pro-p groups, a homomorphism of oriented pro-p groups

$$\phi\colon (G,\theta)\longrightarrow(H,\tau)$$

is a homomorphism of pro-p groups $\phi\colon G\to H$ such that $\theta=\tau\circ\phi$.

Example 3.1. The maximal pro-p Galois group $G_{\mathbb{K}}(p)$ of a field $\mathbb{K}$ containing a root of 1 of order p comes endowed naturally with an orientation: namely, the *p-cyclotomic character* $\theta_{\mathbb{K}}\colon G_{\mathbb{K}}(p)\to 1+p\mathbb{Z}_p$, satisfying

$$g(\zeta)=\zeta^{\theta_{\mathbb{K}}(g)}\qquad\text{for every } g\in G_{\mathbb{K}}(p),$$

for any root $\zeta\in\mathbb{K}(p)$ of 1 of order a power of p (cf. [18, § 4]). The image of $\theta_{\mathbb{K}}$ is $1+p^f\mathbb{Z}_p$, where f is the maximal positive integer such that $\mathbb{K}$ contains the roots of 1 of order p^f — if such a number does not exists, i.e., if $\mathbb{K}$ contains all roots of 1 of p-power order, then $\mathrm{Im}(\theta_{\mathbb{K}})=\{1\}$, and one sets $f=\infty$. Observe that if $p\neq 2$, or if $p=2$ and $\sqrt{-1}\in\mathbb{K}$, then $\theta_{\mathbb{K}}$ is a torsion-free orientation.

From now on, given an orientation $\theta\colon G\to 1+p\mathbb{Z}_p$ of a pro-p group G the notation $\mathrm{Im}(\theta)=1+p^\infty\mathbb{Z}_p$ will mean that the image of θ is trivial.

Example 3.2. A *Demushkin group* is a pro-p group G satisfying the following:

(i) $\dim(\mathrm{H}^1(G,\mathbb{F}_p))<\infty$;
(ii) $\mathrm{H}^2(G,\mathbb{F}_p)\simeq\mathbb{F}_p$;
(iii) the cup-product induces a non-degenerate bilinear form

$$\mathrm{H}^1(G,\mathbb{F}_p)\times\mathrm{H}^1(G,\mathbb{F}_p)\longrightarrow\mathrm{H}^2(G,\mathbb{F}_p)\ ;$$

cf., e.g., [46, Def. 3.9.9]. J-P. Serre proved that every Demushkin group comes endowed with a canonical orientation $\theta_G\colon G\to 1+p\mathbb{Z}_p$ which completes G into an oriented pro-p group (cf. [53]). If the canonical orientation θ_G is torsion-free, then

$$G=\left\langle\, x_1,\ldots,x_d \;\mid\; x_1^{p^f}[x_1,x_2]\cdots[x_{d-1},x_d]=1\,\right\rangle \tag{3.2}$$

for some even positive integer d, and with $f\in\mathbb{N}\cup\{\infty\}$ such that $\mathrm{Im}(\theta_G)=1+p^f\mathbb{Z}_p$ (cf., e.g., [46, Thm. 3.9.11]), and $\theta_G(x_2)=1+p^f$ and $\theta_G(x_h)=1$ for $h\neq 2$ (see also [50, § 5.3]).

Remark 3.3. If $\mathbb{K}$ is an ℓ-adic local field, with ℓ a prime different to p — respectively if $\mathbb{K}$ is a p-adic local field —, containing a root of 1 of order p, then its maximal pro-p Galois group is a Demushkin group G, with

$$\dim(\mathrm{H}^1(G,\mathbb{F}_p))=\begin{cases}2,\\ [\mathbb{K}:\mathbb{Q}_p]+2\end{cases}$$

respectively — in particular, in the former case (i.e., $\mathbb{K}$ is ℓ-adic) one has $G\simeq\mathbb{Z}_p\rtimes\mathbb{Z}_p$ — (cf. [46, Prop. 7.5.9, Thm. 7.5.11]). In this case the canonical orientation θ_G coincides with the pro-p cyclotomic character $\theta_{\mathbb{K}}$ (see Example 3.9–(b) below). Also, $\mathbb{Z}/2$ is the maximal pro-2 Galois group of $\mathbb{R}$. It is still an open problem to determine whether *any other* Demushkin group occurs as the maximal pro-p Galois group of a field containing a root of 1 of order p: for example, the simplest example for which this is not known is the Demushkin pro-2 group

$$G=\langle\, x_1,x_2,x_3 \;\mid\; x_1^2[x_2,x_3]=1\,\rangle$$

(cf. [28, Rem. 5.5]); while the only Demushkin group on 4 generators which is known to be realizable as a maximal pro-p Galois group is the pro-3 group

$$G=\langle\, x_1,x_2,x_3,x_4 \;\mid\; x_1^3[x_1,x_2][x_3,x_4]=1\,\rangle,$$

which occurs as the maximal pro-3 Galois group of $\mathbb{Q}_3(\zeta_3)$, where ζ_3 is a root of 1 of order 3 (cf. [29, p. 254]).

3.2. **Oriented pro-p groups of elementary type.** In the family of oriented pro-p groups one has the following two constructions (cf. [9, § 3]).

(a) Let (G_0,θ) be an oriented pro-p group, and let A be a free abelian pro-p group. The *semidirect product* $(A\rtimes_\theta G_0,\tilde{\theta})$ is the oriented pro-p group where $A\rtimes_\theta G_0$ is the semidirect product of pro-p groups with action $gag^{-1}=a^{\theta(g)}$ for every $g\in G_0$ and $a\in A$, and where
$$\tilde{\theta}\colon A\rtimes_\theta G_0\longrightarrow 1+p\mathbb{Z}_p$$
is the orientation induced by θ, i.e., $\tilde{\theta}=\theta\circ\pi$, where $\pi\colon A\rtimes_\theta G_0\to G_0$ is the canonical projection.

(b) Let $(G_1,\theta_1),(G_2,\theta_2)$ be two oriented pro-p groups. The *free product* (G_1*G_2,θ) is the oriented pro-p group where G_1*G_2 denote the free pro-p product of the two pro-p groups G_1,G_2, while
$$\theta\colon G_1*G_2\longrightarrow 1+p\mathbb{Z}_p$$

is the orientation induced by the orientations θ_1, θ_2 via the universal property of the free pro-p product.

Definition 3.4. The family of *oriented pro-p groups of elementary type* is the smallest family of oriented pro-p groups containing

(a) every oriented pro-p group (F, θ), where F is a finitely generated free pro-p group, and $\theta\colon F \to 1 + p\mathbb{Z}_p$ is arbitrary,
(b) every Demushkin group endowed with its canonical orientation (G, θ_G) (cf. Example 3.2);

and such that

(c) if (G_0, θ) is an oriented pro-p group of elementary type, then also the semidirect product $(\mathbb{Z}_p \rtimes_\theta G_0, \tilde{\theta})$ is an oriented pro-p group of elementary type,
(d) if $(G_1, \theta_1), (G_2, \theta_2)$ are two oriented pro-p groups of elementary type, then also the free product $(G_1 * G_2, \theta)$ is an oriented pro-p group of elementary type.

Remark 3.5. (a) If (G, θ) is an oriented pro-p group of elementary type, and H is a finitely generated subgroup of G, then also the oriented pro-p group $(H, \theta|_H)$ is of elementary type (cf., e.g., [50, Rem. 5.10–(b)]).
(b) Given an oriented pro-p group of elementary type (G, θ), there might be another orientation $\tau\colon G \to 1 + p\mathbb{Z}_p$, $\tau \neq \theta$, such that also (G, τ) is of elementary type — e.g., if $G = F$ is a finitely generated free pro-p group.

I. Efrat's Elementary Type Conjecture asks whether the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ of every field $\mathbb{K}$ containing a root of 1 of order p such that $[\mathbb{K}^\times : (\mathbb{K}^\times)^p] < \infty$ may be obtained in this way. More precisely, the conjecture states the following (cf. [7], see also [8, Question 4.8], [33, § 10], [50, § 7.5] and [39, Conj. 4.8]).

Conjecture 3.6. *In Definition 3.4, replace item* (b) *with*

(b') *every oriented pro-p group $(G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$, where $\mathbb{K}$ is a p-adic field containing a root of 1 (cf. Remark 3.3), and also the oriented pro-2 group $(\mathbb{Z}/2, \theta_{\mathbb{Z}/2})$, with* $\mathrm{Im}(\theta_{\mathbb{Z}/2}) = \{\pm 1\}$*, if $p = 2$.*

Then the family of oriented pro-p groups $(G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$, where $\mathbb{K}$ is a field containing a root of 1 of order p such that $[\mathbb{K}^\times : (\mathbb{K}^\times)^p] < \infty$, coincides with the family of oriented pro-p groups obtained from Definition 3.4, with (b') *instead of item* (b).

On the one hand, one knows that all oriented pro-p groups constructed as in Conjecture 3.6 occur as maximal pro-p Galois groups (endowed with the pro-p cyclotomic character), as the realizability as maximal pro-p Galois group is preserved by semidirect products and free pro-p products (cf. [9, Rem. 3.4]). On the other hand, one has the following.

Proposition 3.7. *Let $\mathbb{K}$ be a field satisfying $[\mathbb{K}^\times : (\mathbb{K}^\times)^p] < \infty$ and containing a root of 1 of order p. Then the oriented pro-p group $(G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$ is of elementary type in the following cases:*

(i) *$\mathbb{K}$ is finite;*
(ii) *$\mathbb{K}$ is a PAC field, or an extension of relative transcendence degree 1 of a PAC field;*

(iii) $\mathbb{K}$ *is a local field, or an extension of transcendence degree 1 of a local field, with characteristic not p;*
(iv) $\mathbb{K}$ *is p-rigid* (*cf.* [59, p. 722])*;*
(v) $\mathbb{K}$ *is an algebraic extension of a global field of characteristic not p;*
(vi) $\mathbb{K}$ *is a valued p-Henselian field with residue field* κ*, where* $(G_\kappa(p), \theta_\kappa)$ *is of elementary type.*
(vii) $\mathbb{K}$ *is a Pythagorean field, if* $p = 2$*.*

For Proposition 3.7 see [39, Thm. D, Prop. 6.2–6.3]. See also: Remark 3.3 and [11] for item (iii); [4, § 3] for item (iv); [8] for item (v); [19, § 1] for item (vi) in case $\mathrm{char}(\kappa) \neq p$, and [10, § 3] for item (vi) in case $\mathrm{char}(\kappa) = p$; and [39, Thm. 6.5] for item (vii).

3.3. **Kummerian oriented pro-p groups.** An oriented pro-p group (G, θ), with θ a torsion-free orientation, is said to be *θ-abelian* if $\mathrm{Ker}(\theta)$ is a free abelian pro-p group, and there exists a complement $G_0 \subseteq G$ to $\mathrm{Ker}(\theta)$ — thus, $G_0 \simeq \mathrm{Im}(\theta)$ —, and

$$(G, \theta) \simeq \mathrm{Ker}(\theta) \rtimes_\theta (G_0, \theta|_{G_0}) \tag{3.3}$$

(cf. [49, § 1]). Equivalently, (G, θ) is θ-abelian if, and only if, G has a presentation

$$G = \left\langle\, x_0, x_h \;\mid\; h \in J,\ {}^{x_0}x_h = x_h^{\theta(x_0)},\ [x_h, x_l] = 1\ \forall h, l \in J \,\right\rangle, \tag{3.4}$$

for some set J (cf. [49, Prop. 3.4]).

The following notion was introduced in [18] (here we use the formulation given in [51, § 2], which is the most useful for our purposes).

Definition 3.8. An oriented pro-p group (G, θ), with torsion-free orientation θ, is said to be *Kummerian* if there exists an epimorphism of oriented pro-p groups

$$\phi\colon (G, \theta) \longrightarrow (\bar{G}, \bar{\theta})$$

with $(\bar{G}, \bar{\theta})$ a $\bar{\theta}$-abelian oriented pro-p group, such that $\mathrm{Ker}(\phi) \subseteq \Phi(G)$.

By (3.1), $\mathrm{Ker}(\phi) \subseteq \Phi(G)$ if, and only if, ϕ yields an isomorphism

$$\phi^*\colon \mathrm{H}^1(\bar{G}, \mathbb{F}_p) \xrightarrow{\sim} \mathrm{H}^1(G, \mathbb{F}_p).$$

Now let $\mathbb{K}$ be a field containing a root of 1 of order p (and also $\sqrt{-1}$ if $p = 2$), and let $\sqrt[p^\infty]{\mathbb{K}}$ be the compositum of all extensions $\mathbb{K}(\sqrt[p^n]{a})$ with $a \in \mathbb{K}^\times$ and $n \geq 1$. Then the restriction

$$\phi\colon G_{\mathbb{K}}(p) = \mathrm{Gal}(\mathbb{K}(p)/\mathbb{K}) \longrightarrow \mathrm{Gal}(\sqrt[p^\infty]{\mathbb{K}}/\mathbb{K})$$

induces an isomorphism $\phi^*\colon \mathrm{H}^1(\mathrm{Gal}(\sqrt[p^\infty]{\mathbb{K}}/\mathbb{K}), \mathbb{F}_p) \xrightarrow{\sim} \mathrm{H}^1(G, \mathbb{F}_p)$, as $\mathrm{Ker}(\phi) \subseteq \Phi(G_{\mathbb{K}}(p))$; moreover, by Kummer theory both cohomology groups are isomorphic to the quotient $\mathbb{K}^\times/(\mathbb{K}^\times)^p$.

Moreover, let $\mathbb{K}(\zeta_{p^\infty})$ be the compositum of all p-power cyclotomic extensions of $\mathbb{K}$. Then either $\mathrm{Gal}(\mathbb{K}(\zeta_{p^\infty})/\mathbb{K})$ is isomorphic to $\mathbb{Z}_p$, or it is trivial (if $\mathbb{K}$ contains all roots of 1 of p-power order). Furthermore, again by Kummer theory one has

$$\mathrm{Gal}(\sqrt[p^\infty]{\mathbb{K}}/\mathbb{K}) \simeq \mathrm{Gal}(\sqrt[p^\infty]{\mathbb{K}}/\mathbb{K}(\zeta_{p^\infty})) \rtimes \mathrm{Gal}(\mathbb{K}(\zeta_{p^\infty})/\mathbb{K}),$$

where $\mathrm{Gal}(\sqrt[p^\infty]{\mathbb{K}}/\mathbb{K}(\zeta_{p^\infty}))$ is a free abelian pro-p group, and the action of the right-hand side factor on the left-hand side factor is induced by the p-cyclotomic character $\theta_{\mathbb{K}}$ — i.e.,

$$ghg^{-1} = h^\lambda \qquad \forall\, g \in \mathrm{Gal}(\mathbb{K}(\zeta_{p^\infty})/\mathbb{K}),\ h \in \mathrm{Gal}(\sqrt[p^\infty]{\mathbb{K}}/\mathbb{K}(\zeta_{p^\infty})),$$

where $\lambda \in 1+p\mathbb{Z}_p$ is defined by $g(\zeta)=\zeta^\lambda$ with $\zeta \in \mathbb{K}(\zeta_{p^\infty})$ a root of 1 of order a power of p. Therefore, the oriented pro-p group $(\mathrm{Gal}(\sqrt[p^\infty]{\mathbb{K}}/\mathbb{K}),\bar\theta)$ is $\bar\theta$-abelian, where $\bar\theta$ is the orientation satisfying $\theta_{\mathbb{K}}=\bar\theta\circ\phi$. Thus, the oriented pro-p group $(G_{\mathbb{K}}(p),\theta_{\mathbb{K}})$ is Kummerian (cf. [18, Thm. 4.2] and [51, Thm. 2.8]). Moreover, for every p-extension $\mathbb{L}/\mathbb{K}$, $\mathbb{K}$ can be replaced by $\mathbb{L}$ and thus also the oriented pro-p group $(G_{\mathbb{L}}(p),\theta_{\mathbb{L}})$ is Kummerian — we underline that $\theta_{\mathbb{L}}$ is the restriction of $\theta_{\mathbb{K}}$ to $G_{\mathbb{L}}(p)$.

One has also the following examples of Kummerian oriented pro-p groups.

Example 3.9. (a) If G is a finitely generated free pro-p group, then the oriented pro-p group (G,θ) is Kummerian for any torsion-free orientation $\theta\colon G\to 1+p\mathbb{Z}_p$ (cf. [18, Prop. 5.5]). Indeed, the subgroup

$$N=\left\{\, [g,h]h^{\theta(g)^{-1}-1} \;\mid\; g\in G,\ h\in \mathrm{Ker}(\theta)\,\right\}\subseteq G$$

is a normal subgroup of G contained in both $\Phi(G)$ and $\mathrm{Ker}(\theta)$, and the quotient G/N has a presentation as in (3.4).

(b) If G is a Demushkin pro-p group whose canonical orientation θ_G is torsion-free, then θ_G is the only orientation which completes G into a Kummerian oriented pro-p group (cf. [31, Thm. 4], see also [18, Thm. 7.6]). In particular, if G occurs as the maximal pro-p Galois group of a field containing a root of 1 of order p, then $\theta_G=\theta_{\mathbb{K}}$.

(c) If (G_0,θ) is a Kummerian oriented pro-p group, with θ a torsion-free orientation, and $A\simeq\mathbb{Z}_p$, then also $(A\rtimes_\theta G_0,\tilde\theta)$ is Kummerian (cf. [18, Prop. 3.6]). Also, if (G_1,θ_1) and (G_2,θ_2) are Kummerian oriented pro-p groups, with θ_1,θ_2 torsion-free orientations, then also (G_1*G_2,θ) is Kummerian (cf. [18, Prop. 7.5]).

(d) By the previous examples, every oriented pro-p group of elementary type with torsion-free orientation θ is Kummerian (cf. [18, § 7] and [51, § 5.3]).

Let $\mathbb{K}$ be a field containing a root of 1 of order p (and also $\sqrt{-1}$ if $p=2$). Then it is well-known that for every $\alpha\in \mathrm{H}^1(G_{\mathbb{K}}(p),\mathbb{F}_p)$ one has $\alpha\smile\alpha=0$ (cf. Remark 2.1, and, e.g., [51, Rem. 4.2] if $p=2$). In [41, Thm. 8.1], J. Minač and N.D. Tân proved the following: for every $\alpha\in \mathrm{H}^1(G_{\mathbb{K}}(p),\mathbb{F}_p)$ and for every $n>2$, the n-fold Massey product $\langle\alpha,\dots,\alpha\rangle$ is defined, and vanishes.

We prove that pro-p groups which may be completed into a Kummerian oriented pro-p group with torsion-free orientation enjoy the same property.

Theorem 3.10. *Let (G,θ) be a Kummerian oriented pro-p group, with torsion-free orientation θ. Then for every $\alpha\in \mathrm{H}^1(G_{\mathbb{K}}(p),\mathbb{F}_p)$ and for every $n>2$, the n-fold Massey product $\langle\alpha,\dots,\alpha\rangle$ vanishes.*

Proof. First of all, observe that, if $p=2$, then $\alpha\smile\alpha=0$ for every $\alpha\in \mathrm{H}^1(G,\mathbb{F}_2)$, as (G,θ) is Kummerian and θ is torsion-free (cf., e.g., [50, Fact. 7.1]), while if $p\neq 2$ this is true anyway (cf. Remark 2.1), so that the sequence $\alpha,\dots,\alpha$ of length n satisfies the triviality condition (1.1).

Suppose first that (G,θ) is θ-abelian. Then G has a presentation

$$G=\langle\, x_h \;\mid\; h\in J,\ [x_h,x_l]=1\ \forall h,l\in J\,\rangle \tag{3.5}$$

for some set J, if $\mathrm{Ker}(\theta)=G$; or

$$G=\langle\, x_0,x_h \;\mid\; h\in J,\ [x_0,x_h]=x_h^q,\ [x_h,x_l]=1\ \forall h,l\in J\,\rangle\,, \tag{3.6}$$

for some set J and $q = p^f$ with $f \geq 1$ (and $f \geq 2$ if $p = 2$), if $\mathrm{Ker}(\theta) \neq G$ (cf. (3.4)).

Put $\mathbb{U} = \mathbb{U}_{n+1}$ and $I = I_{n+1}$, and set

$$A = I + \sum_{i=1}^{n} E_{i,i+1} = \begin{pmatrix} 1 & 1 & & & 0 \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \in \mathbb{U}.$$

For every integers a, b such that $0 \leq a, b \leq p-1$, one has

$$A^a \equiv I + \sum_{i=1}^{n} \bar{a} \cdot E_{i,i+1} \mod \mathbb{U}_{(2)}, \tag{3.7}$$

where $\bar{a} \in \mathbb{F}_p$ denotes the class represented by a, and obviously $[A^a, A^b] = I$. Moreover, one has

$$A^q = \begin{cases} I, & \text{if } q > n, \\ I + \sum_{i=1}^{n+1-q} E_{i,i+q} \in \mathbb{U}_{(q)}, & \text{if } q \leq n, \end{cases}$$

where $q = p^f$ is as in (3.6). In the latter case, by Proposition 2.10 there exists $B \in \mathbb{U}_{(q-1)}$ such that $[B, A] = A^q$.

Now for every $h \in J$ set $a_0, a_h \in \mathbb{Z}$ such that $0 \leq a_j \leq p-1$ and $\bar{a}_j = \alpha(x_j)$ for every $j \in J \cup \{0\}$. Observe that

$$\begin{aligned} [BA^{a_0}, A^{a_h}] &= {}^{B}\,[A^{a_0}, A^{a_h}] \cdot [B, A^{a_h}] \\ &= I \cdot \left({}^{A^{a_h - 1}}[B, A] \cdots {}^{A}[B, A] \cdots [B, A] \right) \\ &= A^{a_h q}. \end{aligned}$$

Then the assignment $\rho(x_h) = A^{a_h}$ for all $h \in J$, and $\rho(x_0) = BA^{a_0}$ if $\mathrm{Ker}(\theta) \neq G$, gives a continuous homomorphism $\rho\colon G \to \mathbb{U}$, which satisfies $\rho_{i,i+1}(x_j) = \bar{a}_j = \alpha(x_j)$ for every $i = 1, \ldots, n$ and every $j \in J \cup \{0\}$ by (3.7) — observe that

$$\rho(x_0) \equiv A^{a_0} \mod \mathbb{U}_{(2)}$$

as $B \in \mathbb{U}_{(2)}$. Hence the n-fold Massey product $\langle \alpha, \ldots, \alpha \rangle$ vanishes by Proposition 2.7–(ii).

If (G, θ) is an arbitrary Kummerian oriented pro-p group, then there exists an epimorphism of oriented pro-p groups $\phi\colon (G, \theta) \to (\bar{G}, \bar{\theta})$ with $(\bar{G}, \bar{\theta})$ a $\bar{\theta}$-abelian pro-p group and $\mathrm{Ker}(\phi) \subseteq \Phi(G)$. Therefore, the inflation map

$$\mathrm{inf}^1_{\bar{G},G}\colon \mathrm{H}^1(\bar{G}, \mathbb{F}_p) \longrightarrow \mathrm{H}^1(G, \mathbb{F}_p)$$

induced by ϕ is an isomorphism, and there exists $\bar{\alpha} \in \mathrm{H}^1(\bar{G}, \mathbb{F}_p)$ such that $\mathrm{inf}^1_{\bar{G},G}(\bar{\alpha}) = \alpha$, i.e. $\alpha = \bar{\alpha} \circ \phi$. The argument above yields a continuous homomorphism $\rho\colon \bar{G} \to \mathbb{U}$ satisfying $\rho_{i,i+1} = \bar{\alpha}$ for every $i = 1, \ldots, n$, and thus the continuous homomorphism $\rho \circ \phi\colon G \to \mathbb{U}$ satisfies

$$(\rho \circ \phi)_{i,i+1} = \bar{\alpha} \circ \phi = \alpha \qquad \text{for every } i = 1, \ldots, n.$$

Hence the n-fold Massey product $\langle \alpha, \ldots, \alpha \rangle$ vanishes by Proposition 2.7–(ii). $\square$

By Example 3.9–(d), Theorem 3.10 implies the following.

Corollary 3.11. *Let (G,θ) be an oriented pro-p group of elementary type with torsion-free orientation θ, and let α be an element of $\mathrm{H}^1(G,\mathbb{F}_p)$. Then for every $n\geq 2$ the n-fold Massey product $\langle\alpha,\ldots,\alpha\rangle$ vanishes.*

4. Oriented pro-p groups and vanishing of Massey products

4.1. **Semidirect products.** Let (G_0,θ_0) be an oriented pro-p group, let $Z\simeq\mathbb{Z}_p$ be a cyclic pro-p group, and set

$$(G,\theta)=Z\rtimes_{\theta_0}(G_0,\theta_0). \tag{4.1}$$

Then the first and second $\mathbb{F}_p$-cohomology groups of G decompose as follow:

$$\mathrm{H}^1(G,\mathbb{F}_p) = \mathrm{H}^1(G_0,\mathbb{F}_p)\oplus\mathrm{H}^1(Z,\mathbb{F}_p), \tag{4.2}$$

$$\mathrm{H}^2(G,\mathbb{F}_p) = \mathrm{H}^2(G_0,\mathbb{F}_p)\oplus\left(\mathrm{H}^1(G_0,\mathbb{F}_p)\smile\mathrm{H}^1(Z,\mathbb{F}_p)\right), \tag{4.3}$$

(cf. [50, Thm. 3.13]) — observe that $\mathrm{H}^n(Z,\mathbb{F}_p)=0$ for every $n\geq 2$. Moreover, if $\{\chi_h\mid h\in J\}$ is a basis of $\mathrm{H}^1(G_0,\mathbb{F}_p)$, and ψ generates $\mathrm{H}^1(Z,\mathbb{F}_p)\simeq\mathbb{F}_p$, then

$$\{\,\chi_h\smile\psi\;\mid\;h\in J\,\}$$

is a basis for $\mathrm{H}^1(G_0,\mathbb{F}_p)\smile\mathrm{H}^1(Z,\mathbb{F}_p)$.

Remark 4.1. The description of the $\mathbb{F}_p$-cohomology of the semidirect product (4.1) has been provided first by A. Wadsworth (cf. [58, Cor. 4.4 and Thm. 3.6]).

Theorem 4.2. *Let (G_0,θ_0) be a Kummerian oriented pro-p group with torsion-free orientation θ_0, and let $Z\simeq\mathbb{Z}_p$ be a cyclic pro-p group. Set $G=Z\rtimes_{\theta_0}G_0$.*

(i) *If G_0 satisfies the n-Massey vanishing property for every $n>2$, then also G satisfies the n-Massey vanishing property for every $n>2$.*
(ii) *If G_0 satisfies the strong n-Massey vanishing property for every $n>2$, then also G satisfies the strong n-Massey vanishing property for every $n>2$.*

Proof. First of all, since (G_0,θ_0) is Kummerian, also the semidirect product $(G,\theta)=Z\rtimes_{\theta_0}(G_0,\theta_0)$ is Kummerian, cf. Example 3.2–(c). Let $\pi\colon G\to G_0$ denote the canonical projection.

Let ψ be a generator of $\mathrm{H}^1(Z,\mathbb{F}_p)$, and let $\alpha_1,\ldots\alpha_n$ be a sequence of non-trivial elements of $\mathrm{H}^1(G,\mathbb{F}_p)$ satisfying (1.1). By (4.2), for every $i=1,\ldots,n$ one has $\alpha_i=\alpha_i|_{G_0}+b_i\psi$ for some $b_i\in\mathbb{F}_p$. Hence

$$0=\alpha_i\smile\alpha_{i+1}=\underbrace{(\alpha_i|_{G_0}\smile\alpha_{i+1}|_{G_0})}_{\in\mathrm{H}^2(G_0,\mathbb{F}_p)}+\underbrace{(b_{i+1}\alpha_i|_{G_0}-b_i\alpha_{i+1}|_{G_0})\smile\psi}_{\in\mathrm{H}^1(G_0,\mathbb{F}_p)\smile\psi} \tag{4.4}$$

for every $i=1,\ldots,n-1$. By (4.3), equality (4.4) holds if, and only if,

$$\alpha_i|_{G_0}\smile\alpha_{i+1}|_{G_0}=0\qquad\text{and}\qquad b_{i+1}\alpha_i|_{G_0}=b_i\alpha_{i+1}|_{G_0}$$

for every $i=1,\ldots,n-1$ — indeed, for any $\alpha\in\mathrm{H}^1(G_0,\mathbb{F}_p)$, $\alpha\smile\psi=0$ implies $\alpha=0$. Altogether, one has two cases:

(a) either $b_i=0$ and
$$\alpha_i=\alpha_i|_{G_0}\circ\pi\neq 0$$
for every $i=1,\ldots,n$;

(b) or
$$b_i \neq 0 \qquad \text{and} \qquad \alpha_i = \frac{b_i}{b_1}\left(\alpha_1|_{G_0} + b_1\psi\right) = \frac{b_i}{b_1}\cdot\alpha_1$$
for every $i = 1,\dots,n$ (recall that we are assuming that $\alpha_i \neq 0$ for every i, so that if $b_i = 0$ for some i then $\alpha_i|_{G_0} \neq 0$, and conversely if $\alpha_i|_{G_0} = 0$ then $b_i \neq 0$).

Case (a). Assume that the n-fold Massey product $\langle\alpha_1,\dots,\alpha_n\rangle$ is defined in $\mathbf{H}^\bullet(G)$, and that $\alpha_i \neq 0$ for every i (cf. Proposition 2.6–(i)), to prove statement (i). By Proposition 2.7–(i), there exists a homomorphism $\bar\rho\colon G \to \bar{\mathbb{U}}_{n+1}$ such that $\bar\rho_{i,i+1} = \alpha_i$ for all $i = 1,\dots,n$. Now consider the restriction
$$\bar\rho|_{G_0}\colon G_0 \longrightarrow \bar{\mathbb{U}}_{n+1}.$$
Then again by Proposition 2.7–(i) the n-fold Massey product $\langle\alpha_1|_{G_0},\dots,\alpha_n|_{G_0}\rangle$ is defined in $\mathbf{H}^\bullet(G_0)$, too, and thus by hypothesis it vanishes. Hence Proposition 2.7–(ii) yields a homomorphism $\rho\colon G_0 \to \mathbb{U}_{n+1}$ satisfying $\rho_{i,i+1} = \alpha_i|_{G_0}$ for all $i = 1,\dots,n$. Then $\rho\circ\pi\colon G \to \mathbb{U}_{n+1}$ is a homomorphism satisfying
$$(\rho\circ\pi)_{i,i+1} = \alpha_i|_{G_0}\circ\pi = \alpha_i \qquad \text{for every } i,$$
and by Proposition 2.7–(ii) the n-fold Massey product $\langle\alpha_1,\dots,\alpha_n\rangle$ vanishes in $\mathbf{H}^\bullet(G)$. This proves (i) in case (a).

Now assume just that $\alpha_i \smile \alpha_{i+1} = 0$ for all $i = 1,\dots,n-1$, and that $\alpha_i \neq 0$ for every i (cf. Proposition 2.8), to prove statement (ii). Since
$$\alpha_i|_{G_0} \smile \alpha_{i+1}|_{G_0} = \mathrm{res}^2_{G,G_0}(\alpha_i \smile \alpha_{i+1}) = \mathrm{res}^2_{G,G_0}(0) = 0$$
(cf. [46, Prop. 1.6.3]) for all $i = 1,\dots,n-1$, the n-fold Massey product $\langle\alpha_1|_{G_0},\dots,\alpha_n|_{G_0}\rangle$ vanishes in $\mathbf{H}^\bullet(G_0)$ by hypothesis. Hence, by Proposition 2.7 there exists a homomorphism $\rho\colon G_0 \to \mathbb{U}_{n+1}$ such that $\rho_{i,i+1} = \alpha_i|_{G_0}$ for every $i = 1,\dots,n$. Then $\rho\circ\pi\colon G \to \mathbb{U}_{n+1}$ is a homomorphism satisfying $(\rho\circ\pi)_{i,i+1} = \alpha_i$ for every $i = 1,\dots,n$, and by Proposition 2.7–(ii) the n-fold Massey product $\langle\alpha_1,\dots,\alpha_n\rangle$ vanishes in $\mathbf{H}^\bullet(G)$. This proves (ii) in case (a).

Case (b). Assume the α_i's are non-trivial multiples of each other. Since (G,θ) is Kummerian, Theorem 3.10 implies that the n-fold Massey product $\langle\alpha_1,\dots,\alpha_1\rangle$ vanishes in $\mathbf{H}^\bullet(G)$. Then by Proposition 2.6–(ii), one has
$$\begin{aligned}\langle\alpha_1,\alpha_2,\dots,\alpha_n\rangle &\supseteq \left\{\, \frac{b_2}{b_1}\cdot\beta \;\middle|\; \beta\in\langle\alpha_1,\alpha_1,\alpha_3,\dots,\alpha_n\rangle \,\right\}\\ &\;\;\vdots\\ &\supseteq \left\{\, \frac{b_2\cdots b_n}{b_1^{n-1}}\cdot\beta \;\middle|\; \beta\in\langle\alpha_1,\dots,\alpha_1\rangle \,\right\} \ni 0,\end{aligned}$$
and thus $\langle\alpha_1,\dots,\alpha_n\rangle$ vanishes in $\mathbf{H}^\bullet(G)$. This proves both (i) and (ii) in case (b). □

From Theorem 4.2 we deduce the following.

Corollary 4.3. *Let $\mathbb{K}$ be a field containing a root of 1 of order p (and $\sqrt{-1}\in\mathbb{K}$, if $p = 2$). Then the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ of $\mathbb{K}$ satisfies the strong n-Massey vanishing property for every $n > 2$ in the following cases:*

(i) $\mathbb{K}$ *is a p-rigid field;*

(ii) $\mathbb{K}$ *is a valued p-Henselian field whose residue field κ has maximal pro-p Galois group satisfying the strong n-Massey vanishing property for every $n > 2$.*

Proof. In the first case, the oriented pro-p group $(G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$ is $\theta_{\mathbb{K}}$-abelian by [4, Cor. 3.17]. In the second case one has $G_{\mathbb{K}}(p) = A \rtimes_{\theta_\kappa} G_\kappa(p)$, with A a free abelian pro-p group, as shown in [19, § 1] if $\mathrm{char}(\kappa) \neq p$ (see also [58, Thm. 3.6]), and [10, § 3] if $\mathrm{char}(\kappa) = p$. □

4.2. **Proof of Theorem 1.2.** We are ready to prove Theorem 1.2.

Theorem 4.4. *Let (G, θ) be an oriented pro-p group of elementary type, and suppose that either θ is a torsion-free orientation. Then G satisfies the strong n-Massey vanishing property for every $n > 2$.*

Proof. We proceed following the inductive construction of the oriented pro-p group of elementary type.

If G is a free pro-p group, then it is straightforward to see that G satisfies the strong n-Massey vanishing property for every $n \geq 0$ by Proposition 2.7–(ii) (cf., e.g., [45, Ex. 4.1]).

If G is a Demushkin group, then G satisfies the strong n-Massey vanishing property as shown by A. Pál and E. Szabó in [48, Thm. 3.5] (see also [44, Prop. 4.1]).

If (G_1, θ_1) and (G_2, θ_2) are two oriented pro-p groups such that both G_1 and G_2 satisfy the strong n-Massey vanishing property, then also the free pro-p product $G_1 * G_2$ satisfies the strong n-Massey vanishing property (cf. [44, Prop. 4.8] and [2, Rem. 5.2]).

Finally, suppose that (G_0, θ) is an oriented pro-p group of elementary type with G_0 satisfying the strong n-Massey vanishing property for every $n > 2$, and consider the semidirect product

$$(G, \tilde{\theta}) = (\mathbb{Z}_p \rtimes_\theta G_0, \tilde{\theta}).$$

Since θ is a torsion-free orientation, $(G, \tilde{\theta})$ is Kummerian by Example 3.9–(d), and thus also $\mathbb{Z}_p \rtimes_\theta G_0$ satisfies the strong n-Massey vanishing property for every $n > 2$ by Theorem 4.2. □

Items (a)–(c) of Corollary 1.3 follow from Proposition 3.7 and Theorem 4.4, and Items (d)–(e) of Corollary 1.3 follow from Corollary 4.3.

Remark 4.5. Let (G, θ) be an oriented pro-p group of elementary type such that θ is a torsion-free orientation. Since $(H, \theta|_H)$ is again an oriented pro-p group of elementary type for every finitely generated subgroup $H \subseteq G$ (cf. Remark 3.5), Theorem 4.4 implies that every finitely generated subgroup of G — in particular, every *open* subgroup (as an open subgroup of a finitely generated pro-p group is again finitely generated; cf., e.g., [5, Prop. 1.7]) — satisfies the strong n-Massey vanishing property for every $n > 2$.

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Department of Science & High-Tech, University of Insubria, Como, Italy EU
Email address: `claudio.quadrelli@uninsubria.it`