CHASING MAXIMAL PRO-p GALOIS GROUPS VIA 1-CYCLOTOMICITY

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Abstract. Let p be a prime. We prove that certain amalgamated free pro-p products of Demushkin groups with pro-p-cyclic amalgam cannot give rise to a 1-cyclotomic oriented pro-p group, and thus do not occur as maximal pro-p Galois groups of fields containing a root of 1 of order p. We show that other cohomological obstructions which are used to detect pro-p groups that are not maximal pro-p Galois groups — the quadraticity of $\mathbb{Z}/p\mathbb{Z}$-cohomology and the vanishing of Massey products — fail with the above pro-p groups. Finally, we prove that the Minač-Tân pro-p group cannot give rise to a 1-cyclotomic oriented pro-p group, and we conjecture that every 1-cyclotomic oriented pro-p group satisfy the strong n-Massey vanishing property for $n > 2$.

1. Introduction

Let p be a prime number, and let $1+p\mathbb{Z}_p$ denote the pro-p group of principal units of the ring of p-adic integers $\mathbb{Z}_p$ — namely, $1+p\mathbb{Z}_p = \{1+p\lambda \mid \lambda \in \mathbb{Z}_p\}$. An *oriented pro-$p$ group* is a pair (G,θ) consisting of a pro-p group G and a morphism of pro-p groups $\theta\colon G \to 1+p\mathbb{Z}_p$, called an *orientation* of G (see [30]; oriented pro-p groups were introduced by I. Efrat in [7], with the name "cyclotomic pro-p pairs"). An oriented pro-p group (G,θ) gives rise to the continuous G-module $\mathbb{Z}_p(\theta)$, which is equal to $\mathbb{Z}_p$ as an abelian pro-p group, and which is endowed with the continuous G-action defined by

$$g\cdot\lambda = \theta(g)\cdot\lambda \qquad \text{for all } g\in G \text{ and } \lambda\in\mathbb{Z}_p(\theta).$$

An oriented pro-p group (G,θ) is said to be *Kummerian* if the following cohomological condition is satisfied: for every $n \geq 1$ the natural morphism

$$\mathrm{H}^1(G,\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)) \longrightarrow \mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z}), \tag{1.1}$$

induced by the epimorphism of continuous G-modules $\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta) \twoheadrightarrow \mathbb{Z}/p$ is surjective (see [11]) — here we consider $\mathbb{Z}/p$ as a trivial G-module. Moreover, the oriented pro-p group (G,θ) is said to be *1-cyclotomic* if the above cohomological condition is satisfied also for every closed subgroup of G — namely, the natural morphism (1.1) is surjective also with H instead of G, and the restriction $\theta|_H\colon H\to 1+p\mathbb{Z}_p$ instead of θ for all closed subgroups H of G (in [26, 27] a 1-cyclotomic oriented pro-p group is called a "1-smooth" oriented pro-p group). This cohomological condition was considered first by J. Labute, who showed *ante litteram* that for every Demushkin group G there exists

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precisely one orientation which completes G into a Kummerian oriented pro-p group, namely, the orientation induced by the dualizing module of G (see [14]).

In case of trivial orientations, 1-cyclotomicity translates into a purely group-theoretical statement. Namely, an oriented pro-p group $(G, \mathbf{1})$ — where $\mathbf{1}\colon G \to 1 + p\mathbb{Z}_p$ denotes the orientation which is constantly equal to 1 — is 1-cyclotomic if, and only if, the abelianization of every closed subgroup of G is a free abelian pro-p group. Pro-p groups satisfying this group-theoretic condition are called *absolutely torsion-free* pro-p groups, and they were introduced by T. Würfel in [37].

The main goal of this work is to produce new examples of pro-p groups which no orientations can turn into a 1-cyclotomic oriented pro-p group.

Theorem 1.1. *Let G be a pro-p group with pro-p presentation*

$$G = \langle\, x, y_1, \ldots, y_{d_1}, z_1, \ldots, z_{d_2} \;\mid\; r_1 = r_2 = 1 \,\rangle\,, \tag{1.2}$$

where d_1, d_2 are two positive odd integers, and either:

(1.1.a) $d_1 + d_2 \geq 4$ *and*

$$r_1 = [x, y_1][y_2, y_3] \cdots [y_{d_1-1}, y_{d_1}],$$
$$r_2 = [x, z_1][z_2, z_3] \cdots [z_{d_2-1}, z_{d_2}];$$

(1.1.b) *or p is odd and*

$$r_1 = y_1^p[y_1, x][y_2, y_3] \cdots [y_{d_1-1}, y_{d_1}],$$
$$r_2 = z_1^p[z_1, x][z_2, z_3] \cdots [z_{d_2-1}, z_{d_2}].$$

Then there are no orientations $\theta\colon G \to 1+p\mathbb{Z}_p$ such that the oriented pro-p group (G, θ) is 1-cyclotomic.

It is worth underlining that the pro-p groups described in Theorem 1.1 are amalgamated free pro-p products of two Demushkin groups — the subgroup generated by $x, y_1, \ldots, y_{d_1}$ and the subgroup generated by $x, z_1, \ldots, z_{d_2}$ —, with pro-p-cyclic amalgam, generated by x. Despite Demushkin groups and their free pro-p products are some of the (extremely few) examples of pro-p groups which are known to give rise to 1-cyclotomic oriented pro-p groups, the presence of a pro-p-cyclic amalgam is enough to lose 1-cyclotomicity.

Oriented pro-p groups satisfying 1-cyclotomicity have great prominence in Galois theory. Given a field $\mathbb{K}$, let $\bar{\mathbb{K}}_s$ and $\mathbb{K}(p)$ denote respectively the separable closure of $\mathbb{K}$, and the compositum of all finite Galois p-extensions of $\mathbb{K}$. The *maximal pro-p Galois group of* $\mathbb{K}$, denoted by $G_{\mathbb{K}}(p)$, is the maximal pro-p quotient of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{K}}_s/\mathbb{K})$ of $\mathbb{K}$, and it coincides with the Galois group of the Galois extension $\mathbb{K}(p)/\mathbb{K}$. Detecting maximal pro-p Galois groups among pro-p groups, are crucial problems in Galois theory. Already the pursuit of concrete examples of pro-p groups which do not occur as maximal pro-p Galois groups of fields is already considered a very remarkable challenge (see [12, § 25.16], and, e.g., [1, 3, 4, 25, 34]).

The maximal pro-p Galois group $G_{\mathbb{K}}(p)$ of a field $\mathbb{K}$ containing a root of 1 of order p gives rise to the oriented pro-p group $(G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$, where

$$\theta_{\mathbb{K}}\colon G_{\mathbb{K}}(p) \longrightarrow 1 + p\mathbb{Z}_p$$

denotes the *pro-p cyclotomic character* (see Example 2.4 below). By Kummer theory, the oriented pro-p group $(G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$ is 1-cyclotomic (see [14, p. 131] and [11, § 4]) — in case $p = 2$ we need to assume further that $\sqrt{-1} \in \mathbb{K}$. Therefore, a pro-p group which cannot complete into a 1-cyclotomic oriented pro-p group does not occur as the maximal pro-p group of a field containing a root of 1 of order p — and hence neither as the absolute Galois group of any field (see, e.g., [25, Rem. 3.3]). Hence, the following corollary may be deduced directly from Theorem 1.1.

Corollary 1.2. *A pro-p group G as in Theorem 1.1 does not occur as the maximal pro-p Galois group of any field containing a root of 1 of order p (and also $\sqrt{-1}$ if $p = 2$). Hence, G does not occur as the absolute Galois group of any field.*

In the recent past, other cohomological properties have been used to study maximal pro-p Galois groups — and to find examples of pro-p groups which do not occur as maximal pro-p Galois groups. By the Norm Residue Theorem — proved by M. Rost and V. Voevodsky, with the contribution by Ch. Weibel, see [13,35] — one knows that if $\mathbb{K}$ is a field containing a root of 1 of order p, the *$\mathbb{Z}/p$-cohomology algebra* $\mathbf{H}^\bullet(G_{\mathbb{K}}(p), \mathbb{Z}/p\mathbb{Z})$, endowed with the *cup-product*

$$\_\smile\_ \colon \mathrm{H}^m(G_{\mathbb{K}}(p), \mathbb{Z}/p\mathbb{Z}) \times \mathrm{H}^n(G_{\mathbb{K}}(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{H}^{m+n}(G_{\mathbb{K}}(p), \mathbb{Z}/p\mathbb{Z}),$$

is *quadratic*, i.e., its ring structure is completely determined by the 1st and the 2nd cohomology groups (see, e.g., [23, § 2]). Moreover, it was shown by E. Matzri that if $\mathbb{K}$ is a field containing a root of 1 of order p, then $G_{\mathbb{K}}(p)$ satisfies the *triple Massey vanishing property* (see [9] and references therein) — for an overview on Massey products in Galois cohomology see [20]. These two cohomological properties were used to find examples of pro-p groups which do not occur as maximal pro-p Galois groups of fields containing a root of 1 of order p, for example in [4, § 8] and in [20, § 7].

We prove that the pro-p groups described in Theorems 1.1 cannot be ruled out as maximal pro-p Galois groups employing the above two cohomological obstructions.

Proposition 1.3. *Let G be a pro-p group as in Theorem 1.1.*

(i) *The $\mathbb{Z}/p$-cohomology algebra $\mathbf{H}^\bullet(G, \mathbb{Z}/p\mathbb{Z})$ is quadratic.*

(ii) *The pro-p group G satisfies the cyclic p-Massey vanishing property — namely, the p-fold Massey product*

$$\underbrace{\langle \alpha, \dots, \alpha \rangle}_{p \text{ times}}$$

contains 0 for every $\alpha \in \mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$.

(iii.a) *If G is as in* (1.1.a)*, then G satisfies the 3- and the strong 4-Massey vanishing property.*

(iii.b) *If G is as in* (1.1.b) *and $p > 3$ then G satisfies the 3- and the strong 4-Massey vanishing property.*

(We recall the basic notions on Massey products in Galois cohomology in § 6.1 below.) Hence, Corollary 1.2 provides brand new examples of pro-p groups which do not occur as maximal pro-p Galois groups of fields containing a root of 1 of order p, and as absolute Galois groups. Moreover, we remark that the relations which define the pro-p groups described in Theorem 1.1 are rather "elementary" — just elementary commutators of generator times, possibly, the p-power of a generator —, unlike the examples provided in [1, 4, 20, 25], where the relations involve higher commutators.

Finally, we focus on the *Minač-Tân pro-p group*, i.e., the pro-p group G with pro-p presentation

$$G = \langle\, x_1, \ldots, x_5 \;\mid\; [[x_1, x_2], x_3][x_4, x_5] = 1 \,\rangle.$$

In [20, § 7], J. Minač and N.D. Tân showed that G does not satisfy the 3-Massey vanishing property, and thus it does not occur as the maximal pro-p Galois group of any field containing a root of 1 of order p. We prove that G cannot complete into a 1-cyclotomic oriented pro-p group.

Theorem 1.4. *Let p be an odd prime. Then there are no orientations turning the Minač-Tân pro-p group into a 1-cyclotomic oriented pro-p group.*

Theorem 1.4 has been proved independently by I. Snopce and P. Zalesskiĭ (unpublished). Theorem 1.4 provides a negative answer to the question posed in [30, Rem. 3.7] — namely, the Minač-Tân pro-p group may be ruled out as a maximal pro-p Galois group of a field containing a root of 1 of order p (and thus as an absolute Galois group) in a "Massey-free" way.

Altogether, 1-cyclotomicity of oriented pro-p groups provides a rather powerful tool studying maximal pro-p Galois groups, and it succeeds in detecting pro-p groups which are not maximal pro-p Galois groups when other methods fail, as underlined above. We believe that further investigations in this direction will lead to new obstructions for the realization of pro-p groups as maximal pro-p Galois group.

Actually, Theorem 1.4, and the main result in [34] (see in particular [34, p. 1907]), may lead to the suspect that 1-cyclotomicity is a more restrictive condition in comparison with the vanishing of Massey products. Thus, we formulate the following conjecture.

Conjecture 1.5. *Let (G, θ) be an oriented pro-p group, such that $\mathrm{Im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$ if $p = 2$. If (G, θ) is 1-cyclotomic, then the pro-p group G satisfies the 3-Massey vanishing property; if moreover G is finitely generated, then G satisfies the strong n-Massey vanishing property for every $n \geq 3$.*

After the publication on the arXiv of an earlier version of this paper, A. Merkurjev and F. Scavia proved the first statement of Conjecture 1.5 — see [17, Thm. 1.3] —; while, on the other hand, there are 1-cyclotomic oriented pro-2 groups (G, θ) such that $\mathrm{Im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$, where G is not finitely generated and does not satisfy the strong 4-Massey vanishing property — see [16, Thm. 1.6]. In particular, [17, Thm. 1.3] implies Theorem 1.4 (see also [17, Rem. 6.3]).

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2. Oriented pro-p groups and cohomology

2.1. **Notation and preliminaries.** Throughout the paper, every subgroup of a pro-p group is tacitly assumed to be *closed* with respect to the pro-p topology. Therefore, sets of generators of pro-p groups, and presentations, are to be intended in the topological sense.

Given a pro-p group G, we denote the closed commutator subgroup of G by G' — namely, G' is the closed normal subgroup generated by commutators

$$[h,g] = h^{-1}\cdot h^g = h^{-1}\cdot g^{-1}hg, \qquad g,h\in G.$$

The *Frattini subgroup* of G is denoted by $\Phi(G)$ — namely, $\Phi(G)$ is the closed normal subgroup generated by G' and by p-powers g^p, $g\in G$ (cf., e.g., [5, Prop. 1.13]). A minimal generating set of G gives rise to a basis of the $\mathbb{Z}/p\mathbb{Z}$-vector space $G/\Phi(G)$, and conversely (cf., e.g., [5, Prop. 1.9]).

Finally, we denote the abelianization G/G' of G by G^{ab}. Throughout the paper, we will make use of the following straightforward fact.

Fact 2.1. *Let G be a finitely generated pro-p group. Then a subset $\{x_1,\ldots,x_d\}$ of G is a minimal generating set of G if, and only if, the subset $\{x_1G',\ldots,x_dG'\}$ of G^{ab} is a minimal generating set of the abelian pro-p group G^{ab}.*

2.2. **Oriented pro-p groups.** Let G be a pro-p group. An orientation $\theta\colon G\to 1+p\mathbb{Z}_p$ is said to be *torsion-free* if p is odd, or if $p=2$ and $\mathrm{Im}(\theta)\subseteq 1+4\mathbb{Z}_2$. Observe that one may have an oriented pro-p group (G,θ) where G has non-trivial torsion and θ torsion-free (e.g., if $G\simeq\mathbb{Z}/p$ and $\mathrm{Im}(\theta)=\{1\}$).

A morphism of oriented pro-p groups $(G_1,\theta_1)\to(G_2,\theta_2)$, is a homomorphism of pro-p groups $\phi\colon G_1\to G_2$ such that $\theta_1=\theta_2\circ\phi$ (cf. [30, § 3, p. 1888]).

Within the family of oriented pro-p groups one has the following constructions. Let (G,θ) be an oriented pro-p group.

(a) If N is a normal subgroup of G contained in $\mathrm{Ker}(\theta)$, one has the oriented pro-p group $(G/N,\theta_{/N})$, where $\theta_{/N}\colon G/N\to 1+p\mathbb{Z}_p$ is the orientation such that $\theta_{/N}\circ\pi=\theta$, with $\pi\colon G\to G/N$ the canonical projection.
(b) If A is an abelian pro-p group (written multiplicatively), one has the oriented pro-p group $A\rtimes(G,\theta)=(A\rtimes G,\tilde{\theta})$, with action given by $gag^{-1}=a^{\theta(g)}$ for every $g\in G$, $a\in A$, where the orientation $\tilde{\theta}\colon A\rtimes G\to 1+p\mathbb{Z}_p$ is the composition of the canonical projection $A\rtimes G\to G$ with θ.

2.3. **Kummerianity and 1-cyclotomicity.** Let (G,θ) be an oriented pro-p group. Observe that the G-action on the G-module $\mathbb{Z}_p(\theta)/p\mathbb{Z}_p(\theta)$ is trivial, as $\theta(g)\equiv 1 \bmod p$ for all $g\in G$. Thus, $\mathbb{Z}_p(\theta)/p\mathbb{Z}_p(\theta)$ is isomorphic to $\mathbb{Z}/p$ as a trivial G-module.

An oriented pro-p group (G,θ) comes endowed with the distinguished subgroup

$$K_\theta(G) = \left\langle\, {}^g h\cdot h^{-\theta(g)} \;\mid\; g\in G,\ h\in\mathrm{Ker}(\theta)\,\right\rangle$$

(cf. [11, § 3]). The subgroup $K_\theta(G)$ is normal in G, and it is contained in both $\mathrm{Ker}(\theta)$ and $\Phi(G)$. On the other hand, $K_\theta(G) \supseteq \mathrm{Ker}(\theta)'$, so that $\mathrm{Ker}(\theta)/K_\theta(G)$ is an abelian pro-p group. Moreover, if θ is a torsion-free orientation, $G/\mathrm{Ker}(\theta) \simeq \mathrm{Im}(\theta)$ is torsion-free, and thus either trivial or isomorphic to $\mathbb{Z}_p$. Hence, the epimorphism $G \twoheadrightarrow G/\mathrm{Ker}(\theta)$ splits, and since $ghg^{-1} \equiv h^{\theta(g)} \bmod K_\theta(G)$ for every $g \in G$ and $h \in \mathrm{Ker}(\theta)$, one concludes that

$$\left(G/K_\theta(G), \theta_{/K_\theta(G)}\right) \simeq \frac{\mathrm{Ker}(\theta)}{K_\theta(G)} \rtimes \left(G/\mathrm{Ker}(\theta), \theta_{/\mathrm{Ker}(\theta)}\right)$$

(cf., e.g., [31, § 2.2, eq. (2.1)]).

One has the following result relating the subgroup $K_\theta(G)$ and the surjectivity of the maps (1.1) (cf. [11, Thm. 7.1], see also [31, Prop. 2.6]).

Proposition 2.2. *Let (G, θ) be an oriented pro-p group with θ a torsion-free orientation. The following are equivalent.*

(i) *The natural map*

$$\mathrm{H}^1(G, \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)) \longrightarrow \mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z}),$$

is surjective for every positive integer n.

(ii) *The quotient $\mathrm{Ker}(\theta)/K_\theta(G)$ is a free abelian pro-p group.*

If an oriented pro-p group (G, θ) with torsion-free orientation satisfies the above two equivalent properties, then it is said to be Kummerian. Moreover, (G, θ) is said to be 1-cyclotomic if $(H, \theta|_H)$ is Kummerian for every subgroup $H \subseteq G$.

Remark 2.3. The original definition of 1-cyclotomic oriented pro-p group requires only that for every *open* subgroup U of G, the oriented pro-p group $(U, \theta|_U)$ is Kummerian (cf. [30, § 1]). By a continuity argument, this is enough to imply that the oriented pro-p group $(H, \theta|_H)$ is Kummerian also for every subgroup $H \subseteq G$ (cf. [30, Cor. 3.2]).

If $(G, \mathbf{1})$ is an oriented pro-p group with $\mathbf{1}\colon G \to 1 + p\mathbb{Z}_p$ the orientation constantly equal to 1, then $K_{\mathbf{1}}(G) = G'$, and by Proposition 2.2 (G, θ) is Kummerian if, and only if, $G/G' = \mathrm{Ker}(\mathbf{1})/K_{\mathbf{1}}(G)$ is a free abelian pro-p group (cf. [11, Ex. 3.5–(1)]). Hence, $(G, \mathbf{1})$ is 1-cyclotomic if, and only if, H/H' is a free abelian pro-p group for every subgroup $H \subseteq G$, i.e., G is absolutely torsion-free (cf. [26, Rem. 2.3]).

2.4. **Examples.**

Example 2.4. Let $\mathbb{K}$ be a field containing a root of 1 of order p, and also $\sqrt{-1}$ if $p = 2$. Then the pro-p cyclotomic character $\theta_{\mathbb{K}}$ of $G_K(p)$ — induced by the action of $G_{\mathbb{K}}(p)$ on the roots of 1 of p-power order contained in $\mathbb{K}(p)$ — has image contained in $1 + p\mathbb{Z}_p$. Observe that $\mathrm{Im}(\theta_{\mathbb{K}}) = 1 + p^f\mathbb{Z}_p$, where $f \in \mathbb{N} \cup \{\infty\}$ is maximal such that $\mathbb{K}$ contains a root of 1 of order p^f (if $f = \infty$, we set $p^\infty = 0$). In particular, $\theta_{\mathbb{K}}$ is a torsion-free orientation. The module $\mathbb{Z}_p(\theta_{\mathbb{K}})$ is called the *1st Tate twist of* $\mathbb{Z}_p$ (cf., e.g., [21, Def. 7.3.6]).

For the convenience of the reader, here we recall J. Labute's argument to show that the oriented pro-p group $(G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$ is Kummerian — and thus also 1-cyclotomic, as every subgroup $H \subseteq G_{\mathbb{K}}(p)$ is the maximal pro-p Galois group of an extension of $\mathbb{K}$, with pro-p cyclotomic character $\theta_{\mathbb{K}}|_H$ —, as it is presented in [14, p. 131] (where the module

$\mathbb{Z}_p(\theta_{\mathbb{K}})$ is denoted by $I = I(\chi'))$. For every $n \geq 1$ one has an isomorphism of continuous $G_{\mathbb{K}}(p)$-modules

$$\mathbb{Z}_p(\theta_{\mathbb{K}})/p^n\mathbb{Z}_p(\theta_{\mathbb{K}}) \simeq \mu_{p^n} = \left\{ \zeta \in \mathbb{K}(p) \mid \zeta^{p^n} = 1 \right\}.$$

Let $\mathbb{K}^\times$ and $\mathbb{K}(p)^\times$ denote the multiplicative groups of units of $\mathbb{K}$ and $\mathbb{K}(p)$ respectively. By Hilbert 90, the short exact sequence of continuous $G_{\mathbb{K}}(p)$-modules

$$\{1\} \longrightarrow \mu_{p^n} \longrightarrow \mathbb{K}(p)^\times \xrightarrow{\_^{p^n}} \mathbb{K}(p)^\times \longrightarrow \{1\} \tag{2.1}$$

induces a commutative diagram

$$\begin{array}{ccccc} \mathbb{K}^\times/(\mathbb{K}^\times)^{p^n} & \longrightarrow & H^1(G_{\mathbb{K}}(p),\mu_{p^n}) & \xrightarrow{\sim} & H^1\left(G_{\mathbb{K}}(p),\mathbb{Z}_p(\theta_{\mathbb{K}})/p^n\mathbb{Z}_p(\theta_{\mathbb{K}})\right) \\ \downarrow{\scriptstyle \pi_n} & & \downarrow & & \downarrow \\ \mathbb{K}^\times/(\mathbb{K}^\times)^{p} & \xrightarrow{\sim} & H^1(G_{\mathbb{K}}(p),\mu_{p}) & \xrightarrow{\sim} & H^1\left(G_{\mathbb{K}}(p),\mathbb{Z}/p\mathbb{Z}\right) \end{array}$$

where the left-side vertical arrow π_n and the central vertical arrow are induced by the p^{n-1}-th power map $\_^{p^n}\colon \mathbb{K}(p)^\times \to \mathbb{K}(p)^\times$, and the right-side vertical arrow is induced by the epimorphism of continuous $G_{\mathbb{K}}(p)$-modules $\mathbb{Z}_p(\theta_{\mathbb{K}})/p^n\mathbb{Z}_p(\theta_{\mathbb{K}}) \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$. Since the map π_n is surjective, also the other vertical arrows are surjective.

Example 2.5. Let G be a free pro-p group. Then the oriented pro-p group (G,θ) is 1-cyclotomic for any orientation $\theta\colon G \to 1 + p\mathbb{Z}_p$ (cf. [30, § 2.2]).

Example 2.6. Let G be an infinite Demushkin group (cf., e.g., [21, Def. 3.9.9]). By [14, Thm. 4], G comes endowed with a canonical orientation $\chi\colon G \to 1 + p\mathbb{Z}_p$ which is the only one completing G into a 1-cyclotomic oriented pro-p group. In particular, if $d = \dim(\mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z}))$ is even (which is always the case if $p \neq 2$), then G has a presentation

$$G = \left\langle\, x_1,\ldots,x_d \;\mid\; x_1^{p^f}[x_1,x_2]\cdots[x_{d-1},x_d] = 1 \,\right\rangle,$$

with $f \geq 1$ ($f \geq 2$ if $p = 2$). In this case $\chi(x_2) = (1-p^f)^{-1}$ and $\chi(x_i) = 1$ for $i \neq 2$.

Example 2.7. Let (G,θ) be an oriented pro-p group, with θ a torsion-free orientation. The oriented pro-p group (G,θ) is said to be *θ-abelian* if the subgroup $K_\theta(G)$ is trivial and if $\mathrm{Ker}(\theta)$ is a free abelian pro-p group — in this case G is a free abelian-by-cyclic pro-p group, i.e.,

$$G \simeq \mathrm{Ker}(\theta) \rtimes \frac{G}{\mathrm{Ker}(\theta)}$$

(cf. [31, Rem. 2.2]). In other words, G has a presentation

$$G = \left\langle\, x_0, x_i \;\mid\; i \in I,\ x_i^{x_0} = x_i^{\theta(x_0)^{-1}}, [x_i,x_j] = 1\,\forall\, i,j \in I \,\right\rangle,$$

for some set of indices I, and $\theta(x_i) = 1$ for all $i \in I$ (cf. [23, Prop. 3.4]). A θ-abelian oriented pro-p group (G,θ) is Kummerian by Proposition 2.2, as by definition $K_\theta(G)$ is trivial and $\mathrm{Ker}(\theta)$ is a free abelian pro-p group. Moreover, if H is a subgroup of G, then one has

$$H \simeq (H \cap \mathrm{Ker}(\theta)) \rtimes \frac{H}{\mathrm{Ker}(\theta|_H)}$$

(cf. [31, Rem. 2.4]), so that the oriented pro-p group $(H,\theta|_H)$ is $\theta|_H$-abelian, and thus Kummerian, and consequently (G,θ) is 1-cyclotomic.

One has the following result to check whether an oriented pro-p group is Kummerian (cf. [31, Prop. 2.6, Prop. 3.6]).

Proposition 2.8. *Let (G,θ) be an oriented pro-p group, with θ a torsion-free orientation. Then (G,θ) is Kummerian if, and only if, there exists a normal subgroup N of G such that $N \subseteq \mathrm{Ker}(\theta)\cap\Phi(G)$, and the quotient $(G/N,\theta_{/N})$, is a $\theta_{/N}$-abelian oriented pro-p group. If such a normal subgroup N exists, then $N = K_\theta(G)$.*

2.5. **Kummerianity and 1-cocyles.** Let (G,θ) be an oriented pro-p group. Recall that for $n\in\mathbb{N}\cup\{\infty\}$, a *1-cocycle* $c\colon G\to\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ is a continuous map satisfying

$$c(gh) = c(g) + \overline{\theta(g)}c(h) \qquad \text{for every } g,h\in G, \tag{2.2}$$

where $\overline{\theta(g)}$ denotes the image of $\theta(g)$ modulo p^n. From (2.2) one deduces

$$c([g,h]) = \overline{\theta(gh)^{-1}}\left(c(g)(1-\overline{\theta(h)}) - c(h)(1-\overline{\theta(g)})\right). \tag{2.3}$$

For $n\in\mathbb{N}\cup\{\infty\}$, every element of $\mathrm{H}^1(G,\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta))$ is represented by a 1-cocycle $c\colon G\to\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$. The following result is due to J. Labute (cf. [14, Prop. 6]).

Lemma 2.9. *Let (G,θ) be a finitely generated oriented pro-p group with torsion-free orientation, and let $\mathcal{X}=\{x_1,\dots,x_d\}$ be a minimal generating set of G. The following are equivalent.*

(i) *(G,θ) is Kummerian.*
(ii) *For all $n\in\mathbb{N}\cup\{\infty\}$ and for any sequence $\lambda_1,\dots,\lambda_d$ of elements of $\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ there exists a continuous 1-cocycle $G\to\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ satisfying $c(x_i)=\lambda_i$ for all $i=1,\dots,d$.*

Proposition 2.10. *Let G be a finitely generated pro-p group, and let (G,θ) be a Kummerian oriented pro-p group with torsion-free orientation. If N is a normal subgroup of G such that $N\subseteq\mathrm{Ker}(\theta)$ and the restriction map*

$$\mathrm{res}^1_{G,N}\colon \mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{H}^1(N,\mathbb{Z}/p\mathbb{Z})^G$$

is surjective, then also $(G/N,\theta_{/N})$ is Kummerian.

In order to prove Proposition 2.10 we need the following fact, whose proof — rather straightforward — is left to the reader.

Fact 2.11. *Let G be a finitely generated pro-p group, and let (G,θ) be an oriented pro-p group with torsion-free orientation.*

(i) *If $c\colon G\to\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ is a continuous 1-cocycle, with $n\in\mathbb{N}\cup\{\infty\}$, then $c^{-1}(0)\cap\mathrm{Ker}(\theta)$ is a normal subgroup of G.*
(ii) *Let $N\subseteq G$ be a normal subgroup satisfying $N\subseteq\mathrm{Ker}(\theta)$, with canonical projection $\pi\colon G\to G/N$. For $n\in\mathbb{N}\cup\{\infty\}$ one has the following:*
    (a) *a continuous 1-cocycle $c\colon G\to\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ satisfying $c|_N\equiv 0$ induces a continuous 1-cocycle $\bar{c}\colon G/N\to\mathbb{Z}_p(\theta_{/N})/p^n\mathbb{Z}_p(\theta_{/N})$ such that $c=\bar{c}\circ\pi$;*
    (b) *a continuous 1-cocycle $\bar{c}\colon G/N\to\mathbb{Z}_p(\theta_{/N})/p^n\mathbb{Z}_p(\theta_{/N})$ induces a continuous 1-cocycle $c\colon G\to\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ satisfying $c|_N\equiv 0$ and $c=\bar{c}\circ\pi$.*

Proof of Proposition 2.10. Set $\bar{G} = G/N$ and $\bar{\theta} = \theta_{/N}$. For every $n \geq 1$, the canonical projection $\pi\colon G \to \bar{G}$ induces the inflation maps

$$
\begin{aligned}
f_n\colon \mathrm{H}^1(\bar{G}, \mathbb{Z}_p(\bar{\theta})/p^n\mathbb{Z}_p(\bar{\theta})) &\longrightarrow \mathrm{H}^1(G, \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)), \\
f\colon \mathrm{H}^1(\bar{G}, \mathbb{Z}/p\mathbb{Z}) &\longrightarrow \mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z}),
\end{aligned}
\tag{2.4}
$$

which are injective by [21, Prop. 1.6.7]. Also, the epimorphisms (respectively of continuous $\bar{G}$-modules and continuous G-modules) $\mathbb{Z}_p(\bar{\theta})/p^n\mathbb{Z}_p(\bar{\theta}) \to \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}_p(\theta)/p^n \to \mathbb{Z}/p\mathbb{Z}$ induce, respectively, the morphisms

$$
\begin{aligned}
\tau_n^N\colon \mathrm{H}^1(\bar{G}, \mathbb{Z}_p(\theta)/p^n) &\longrightarrow \mathrm{H}^1(\bar{G}, \mathbb{Z}/p), \\
\tau_n\colon \mathrm{H}^1(G, \mathbb{Z}_p(\theta)/p^n) &\longrightarrow \mathrm{H}^1(G, \mathbb{Z}/p).
\end{aligned}
\tag{2.5}
$$

Altogether, by [21, Prop. 1.5.2] one has the commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}^1\left(\bar{G}, \mathbb{Z}_p(\bar{\theta})/p^n\mathbb{Z}_p(\bar{\theta})\right) & \xrightarrow{\ \tau_n^N\ } & \mathrm{H}^1(\bar{G}, \mathbb{Z}/p\mathbb{Z}) \\
\Big\downarrow f_n & & \Big\downarrow f \\
\mathrm{H}^1\left(G, \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)\right) & \xrightarrow{\ \tau_n\ } & \mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})
\end{array}
$$

Since (G, θ) is Kummerian, τ_n is surjective for every $n \geq 1$. Given $\bar{\beta} \in \mathrm{H}^1(\bar{G}, \mathbb{Z}/p\mathbb{Z})$, $\bar{\beta} \neq 0$, our goal is to find $\alpha \in \mathrm{H}^1(\bar{G}, \mathbb{Z}_p(\bar{\theta})/p^n\mathbb{Z}_p(\bar{\theta}))$ such that $\bar{\beta} = \tau_n^N(\alpha)$.

Set $\beta = \bar{\beta} \circ \pi = f(\bar{\beta})$. Then $\beta\colon G \to \mathbb{Z}/p$ is a non-trivial continuous homomorphism such that $\mathrm{Ker}(\beta) \supseteq N$. By hypothesis, the morphism $N/N^p[G, N] \to G/\Phi(G)$ induced by the inclusion $N \hookrightarrow G$, and dual to $\mathrm{res}^1_{G,N}$, is injective. Thus, one may find a minimal generating set $\mathcal{X}$ of G such that $\mathcal{Y} = \mathcal{X} \cap N$ generates N as a normal subgroup of G. By Lemma 2.9, there exists a continuous 1-cocycle $c\colon G \to \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$ satisfying

$$c(x) \equiv \beta(x) \mod p\mathbb{Z}_p(\theta) \qquad \text{for every } x \in \mathcal{X}$$

— i.e., $\tau_n([c]) = \beta$, where $[c] \in \mathrm{H}^1(G, \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta))$ denotes the cohomology class of c —, and moreover $c(x) = 0$ for every $x \in \mathcal{Y}$. Therefore, by Fact 2.11–(i), the restriction

$$c|_N\colon N \longrightarrow \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)$$

is the map constantly equal to 0. By Fact 2.11–(ii), c induces a continuous 1-cocycle

$$\bar{c}\colon \bar{G} \longrightarrow \mathbb{Z}_p(\bar{\theta})/p^n\mathbb{Z}_p(\bar{\theta})$$

such that $\bar{c} \circ \pi = c$, and $[c] = f_n([\bar{c}])$, where $[\bar{c}] \in \mathrm{H}^1(\bar{G}, \mathbb{Z}_p(\bar{\theta})/p^n\mathbb{Z}_p(\bar{\theta}))$ denotes the cohomology class of $\bar{c}$. Altogether, one has

$$f(\bar{\beta}) = \beta = \tau_n([c]) = \tau_n \circ f_n([\bar{c}]) = f \circ \tau_n^N([\bar{c}]).$$

Since f is injective, one obtains $\bar{\beta} = \tau_n^N([\bar{c}])$. □

Remark 2.12. Proposition 2.10 may be proved also in a purely group-theoretic way, see [3, Rem. 3.9].

3. The $\mathbb{Z}/p\mathbb{Z}$-cohomology of G

The purpose of this section is to prove the first statement of Proposition 1.3, and more in general to describe the $\mathbb{Z}/p\mathbb{Z}$-cohomology algebra $\mathbf{H}^\bullet(G, \mathbb{Z}/p\mathbb{Z})$ with G as in Theorem 1.1.

3.1. **Degree 1 and 2.** Let G be a pro-p group. We set the subgroup $G_{(3)}$ of G as follows:

$$G_{(3)} = \begin{cases} G^p[G,G'] & \text{if } p \neq 2, \\ G^4(G')^2[G,G'] & \text{if } p = 2, \end{cases}$$

i.e., $G_{(3)}$ is the third term of the p-Zassenhaus filtration of G (cf., e.g., [24, § 3.1]). In particular, $G_{(3)}$ is a normal subgroup of the Frattini subgroup $\Phi(G)$, and the quotient $\Phi(G)/G_{(3)}$ is a p-elementary abelian pro-p group — and thus also a $\mathbb{Z}/p$-vector space.

Recall that the cohomology group $\mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z})$ is equal to the group of pro-p group homomorphisms from G to $\mathbb{Z}/p$, namely, one has

$$\mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z}) = \mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z}) \simeq (G/\Phi(G))^*, \tag{3.1}$$

where $\_^*$ denotes the $\mathbb{Z}/p$-dual (cf., e.g., [33, Ch. I, § 4.2]). Thus, the dimension of $\mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z})$ is equal to the cardinality $\mathrm{d}(G)$ of any minimal generating set of G. On the other hand, the dimension of $\mathrm{H}^2(G,\mathbb{Z}/p\mathbb{Z})$ is equal to the number $\mathrm{r}(G)$ of defining relations of G (cf. [33, Ch. I, § 4.3]). Moreover, if both $\mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z})$ and $\mathrm{H}^2(G,\mathbb{Z}/p\mathbb{Z})$ are finite, and if the cup-product yields an epimorphism $\mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z})^{\otimes 2} \twoheadrightarrow \mathrm{H}^2(G,\mathbb{Z}/p\mathbb{Z})$, one has an isomorphism of elementary abelian p-groups

$$\left(\Phi(G)/G_{(3)}\right)^* \xrightarrow{\mathrm{trg}} \mathrm{H}^2(G,\mathbb{Z}/p\mathbb{Z}) \tag{3.2}$$

(cf. [18, Thm. 7.3]). For further properties of the cohomology of pro-p groups we refer to [33, Ch. I, § 4] and to [21, Ch. III, § 9].

3.2. **Amalgams.** Henceforth G will denote a pro-p group as in Theorem 1.1. Set

$$G_1 = \langle\, x, y_1, \ldots, y_{d_1} \mid x^{\epsilon p}[x,y_1]\cdots[y_{d_1-1},y_{d_1}] = 1 \,\rangle,$$
$$G_2 = \langle\, x, z_1, \ldots, z_{d_2} \mid x^{\epsilon p}[x,z_1]\cdots[z_{d_2-1},z_{d_2}] = 1 \,\rangle,$$

with $\epsilon = 0, 1$ depending on whether we are considering case (1.1.a) or (1.1.b). Then G_1, G_2 are Demushkin groups, and G is the amalgamated free pro-p product

$$G = G_1 \amalg_X^{\hat{p}} G_2, \tag{3.3}$$

with amalgam the subgroup $X \subseteq G_1, G_2$ generated by x. Observe that $X \simeq \mathbb{Z}_p$, as X has infinite index in both G_1, G_2, and subgroups of infinite index of Demushkin groups are free pro-p groups (cf. [33, Ch. I, § 4.5, Ex. 5–(b)]). Therefore, the amalgamated free pro-p product is proper, i.e., $G_1, G_2 \subseteq G$ (cf. [32]).

3.3. **Quadratic cohomology.** Let

$$\mathcal{B} = \{\, \chi,\ \varphi_1,\ \ldots,\ \varphi_{d_1},\ \psi_1,\ \ldots,\ \psi_{d_2} \,\}$$

be the basis of $\mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z}) = \mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$ dual to $\mathcal{X} = \{x, y_1, \ldots, z_{d_2}\}$ — i.e.,

$$\chi(w) = \begin{cases} 1 & \text{if } w = x \\ 0 & \text{if } w = y_i, z_j \end{cases} \qquad \text{and}$$

$$\varphi_i(w) = \begin{cases} \delta_{i,i'} & \text{if } w = y_{i'} \\ 0 & \text{if } w = x, z_j, \end{cases} \qquad \psi_j(w) = \begin{cases} \delta_{j,j'} & \text{if } w = z_{j'} \\ 0 & \text{if } w = x, y_i, \end{cases}$$

for every $1 \leq i, i' \leq d_1$ and $1 \leq j, j' \leq d_2$ (cf. (3.1)). With an abuse of notation, we may consider the subsets $\mathcal{B}_1 = \{\chi, \varphi_1, \ldots, \varphi_{d_1}\}$, $\mathcal{B}_2 = \{\chi, \psi_1, \ldots, \psi_{d_2}\}$, and $\mathcal{B}_X = \{\chi\}$, as bases of $\mathrm{H}^1(G_1, \mathbb{Z}/p\mathbb{Z})$, $\mathrm{H}^1(G_2, \mathbb{Z}/p\mathbb{Z})$, and $\mathrm{H}^1(X, \mathbb{Z}/p\mathbb{Z})$ respectively.

Proposition 3.1. *The algebra* $\mathbf{H}^\bullet(G, \mathbb{Z}/p\mathbb{Z})$ *is quadratic.*

Proof. As stated in § 3.2, $G = G_1 \amalg_X^{\hat{p}} G_2$ is a proper amalgamated free pro-p product. Since $\mathcal{B}_X \subseteq \mathcal{B}_1, \mathcal{B}_2$, the restriction maps

$$\mathrm{res}^1_{G_i,X} \colon \mathrm{H}^1(G_i, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{H}^1(X, \mathbb{Z}/p\mathbb{Z}), \qquad \text{with } i = 1, 2,$$

are surjective. Moreover, $\mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z}) = 0$, as $X \simeq \mathbb{Z}_p$, and thus $\mathrm{Ker}(\mathrm{res}^2_{G_i,X}) = \mathrm{H}^2(G_i, \mathbb{Z}/p\mathbb{Z})$ for both $i = 1, 2$. On the other hand, $\mathrm{H}^1(G_1, \mathbb{Z}/p\mathbb{Z})$ and $\mathrm{H}^1(G_2, \mathbb{Z}/p\mathbb{Z})$ are generated by $\chi \smile \varphi_1$ and $\chi \smile \psi_1$ respectively, as G_1, G_2 are Demushkin groups (cf., e.g., [21, Prop. 3.9.16]), and thus

$$\mathrm{Ker}(\mathrm{res}^2_{G_i,X}) = \mathrm{H}^2(G_i, \mathbb{Z}/p\mathbb{Z}) = \mathrm{Ker}(\mathrm{res}^1_{G_i,X}) \smile \mathrm{H}^1(G_i, \mathbb{Z}/p\mathbb{Z}), \qquad \text{with } i = 1, 2,$$

as $\mathrm{res}^1_{G_1,X}(\varphi_1) = 0$ and $\mathrm{res}^1_{G_2,X}(\psi_1) = 0$. Finally, Demushkin groups are well-known to yield a quadratic $\mathbb{Z}/p\mathbb{Z}$-cohomology algebra, while $\mathbf{H}^\bullet(X, \mathbb{Z}/p\mathbb{Z})$ is obviously quadratic, as $X \simeq \mathbb{Z}_p$. Therefore, we may apply [29, Thm. B], so that also $\mathbf{H}^\bullet(G, \mathbb{Z}/p\mathbb{Z})$ is quadratic. □

We describe now more in detail the structure of $\mathbf{H}^\bullet(X, \mathbb{Z}/p\mathbb{Z})$. By duality — cf. [18, Thm. 7.3] and (3.2) —, the set $\{\chi \smile \varphi_1, \chi \smile \psi_1\}$ is a basis of $\mathrm{H}^2(G, \mathbb{Z}/p\mathbb{Z})$, and in $\mathrm{H}^2(G, \mathbb{Z}/p\mathbb{Z})$ one has the relations

$$\chi \smile \varphi_{i'} = \chi \smile \psi_{j'} = \varphi_i \smile \psi_j = 0 \tag{3.4}$$

for all $1 \leq i, i' \leq d_1$ and $1 \leq j, j' \leq d_2$, with $i', j' \neq 1$, and

$$\begin{aligned} \varphi_i \smile \varphi_{i'} &= \begin{cases} (-1)^\epsilon \chi \smile \varphi_1 & \text{if } 2 \mid i = i' - 1, \\ 0 & \text{otherwise,} \end{cases} \\ \psi_j \smile \psi_{j'} &= \begin{cases} (-1)^\epsilon \chi \smile \psi_1 & \text{if } 2 \mid j = j' - 1, \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3.5}$$

(see also [24, § 3.2]).

Finally, one has an exact sequence

$$\cdots \longrightarrow \mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{H}^3(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{H}^3(G_1, \mathbb{Z}/p\mathbb{Z}) \oplus \mathrm{H}^3(G_2, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \cdots$$

(cf. [29, p. 653]). Since $\mathrm{H}^2(X, \mathbb{Z}/p\mathbb{Z}) = \mathrm{H}^3(G_i, \mathbb{Z}/p\mathbb{Z}) = 0$ for both $i = 1, 2$, one has $\mathrm{H}^3(G, \mathbb{Z}/p\mathbb{Z}) = 0$, and thus by quadraticity also $\mathrm{H}^n(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $n \geq 3$.

Remark 3.2. It is well-known that if a pro-p group has non-trivial torsion, then its n-th $\mathbb{Z}/p$-cohomology group is non trivial for every $n > 0$; hence, G is torsion-free.

4. Proof of Theorem 1.1 case (1.1.a)

Let G be a pro-p group as defined in Theorem 1.1, with defining relations as in (1.1.a) — namely,

$$G = \langle\, x, y_1, \ldots, y_{d_1}, z_1, \ldots, z_{d_2} \mid r_1 = r_2 = 1 \,\rangle,$$

with $d_1 + d_2 \geq 4$ and

$$r_1 = [x, y_1] \cdots [y_{d_1-1}, y_{d_1}],$$
$$r_2 = [x, z_1] \cdots [z_{d_2-1}, z_{d_2}].$$

Without loss of generality, we may assume that $d_1 \geq 3$.

4.1. **Kummerianity.** Let G_1, G_2 be the two Demushkin groups as in § 3.2, with $\epsilon = 0$. By Example 2.6, if

$$\theta_1 \colon G_1 \longrightarrow 1 + p\mathbb{Z}_p \qquad \text{and} \qquad \theta_2 \colon G_2 \longrightarrow 1 + p\mathbb{Z}_p$$

are two torsion-free orientations completing respectively G_1 and G_2 into Kummerian oriented pro-p groups, then necessarily $\theta_1(x) = \theta_1(y_1) = \ldots = \theta_1(y_{d_1}) = 1$, and analogously $\theta_2(x) = \theta_2(z_1) = \ldots = \theta_1(z_{d_2}) = 1$.

Proposition 4.1. *Let $\theta \colon G \to 1 + p\mathbb{Z}_p$ be a torsion-free orientation. Then the oriented pro-p group (G, θ) is Kummerian if, and only if, θ is constantly equal to 1.*

Proof. If $\theta \equiv \mathbf{1}$, then $(G, \mathbf{1})$ is Kummerian if, and only if, the abelianization G^{ab} is a free abelian pro-p group. But this is easily verified, as clearly $G^{\mathrm{ab}} \simeq \mathbb{Z}_p^{d_1+d_2-1}$.

Conversely, suppose that (G, θ) is Kummerian. Let N_1 and N_2 denote the normal subgroups of G generated as normal subgroups by $z_1, \ldots, z_{d_2}$ and $y_1, \ldots, y_{d_1}$ respectively. Then $G/N_1 \simeq G_1$ and $G/N_2 \simeq G_2$. Moreover, Proposition 2.10 implies that $(G/N_i, \theta_{/N_i})$ is Kummerian for both $i = 1, 2$. Since $G/N_i \simeq G_i$ for both i, Example 2.6 and the argument before the statement of the proposition imply that the torsion-free orientations $\theta_{/N_1}$ and $\theta_{/N_2}$ are constantly equal to 1. Hence, also θ is constantly equal to 1, as $\theta(w) = \theta_{/N_1}(wN_1)$ for every $w \in G_1$, and analogously $\theta(w) = \theta_{/N_2}(wN_2)$ for every $w \in G_2$. □

Therefore, if G may complete into a 1-cyclotomic oriented pro-p group, then necessarily G is absolutely torsion-free. In order to prove Theorem 1.1 in case (1.1.a), we aim at exhibiting an open subgroup H of G, of index p^2, whose abelianization H^{ab} has non-trivial torsion.

4.2. **The subgroup U.** Set $u = y_3^p$, $t_0 = z_1^{-1} y_3$, and $t_h = t_0 t_0^{y_3} \cdots t_0^{y_3^h}$ for all $h = 0, \ldots, p-1$. A straightforward computation shows that

$$z_1^h = y_3^h \cdot (t_0^{-1})^{y_3^{h-1}} \cdots (t_0^{-1})^{y_3} \cdot t_0^{-1} = y_3^h t_{h-1}^{-1} \tag{4.1}$$

for all $h = 0, \ldots, p-1$.

Let $\phi_G \colon G \to \mathbb{Z}/p$ be the homomorphism of pro-p groups defined by $\phi_G(y_3) = \phi_G(z_1) = 1$ and $\phi_G(x) = \phi_G(y_i) = \phi_G(z_j) = 0$ for all $i = 1, 2, 4, \ldots, d_1$ and $j = 2, \ldots, d_2$, and set $U = \mathrm{Ker}(\phi)$. Then U is an open subgroup of G of index p, generated as a normal subgroup by the subset

$$\mathcal{X} = \{\, u,\ x,\ t_0,\ y_i,\ z_j \mid i = 1, 2, 4, \ldots, d_1,\ j = 2, \ldots, d_2 \,\},$$

and $G/U = \{U, y_3 U, \ldots, y_3^{p-1} U\}$.

Lemma 4.2. *The subset*

$$\mathcal{Y}_U = \left\{ u,\ x,\ y_2,\ t_h,\ y_i^{y_3^h},\ z_j^{y_3^h} \ \mid\ i=1,4,\dots,d_1,\ j=2,\dots,d_2,\ h=0,\dots,p-1 \right\}$$

of U is a minimal generating set of U as a pro-p group.

Proof. Since U is normally generated by $\mathcal{X}$ and $G/U = \{U,\dots,y_3^{p-1}U\}$, U is generated as a pro-p group by the set $\{w^{y_3^h} \mid w\in\mathcal{X}, h=0,\dots,p-1\}$. Also, U is subject to the relations

$$r_1^{y_3^h} = \left[x^{y_3^h}, y_1^{y_3^h}\right]\cdots\left[y_{d_1-1}^{y_3^h}, y_{d_1}^{y_3^h}\right] = 1, \tag{4.2}$$

$$r_2^{y_3^h} = \left[x^{y_3^h}, z_1^{y_3^h}\right]\cdots\left[z_{d_2-1}^{y_3^h}, z_{d_2}^{y_3^h}\right] = 1, \tag{4.3}$$

with $h=0,\dots,p-1$.

Consider the abelianization U^{ab}. Since the only factor in (4.2) which does not lie in U' is $[y_2^{y_3^h}, y_3]$, the relation (4.2) implies that $[y_2^{y_3^h}, y_3]\in U'$ as well, and therefore

$$y_2^{y_3^h} \equiv y_2 \mod U' \qquad \text{for all } h=0,\dots,p-1.$$

Analogously, the only factor in (4.3) which does not lie in U' is $[x^{y_3^h}, z_1^{y_3^h}]$, so that the relation (4.2) implies that $[x^{y_3^h}, z_1^{y_3^h}]\in U'$ as well. Hence, one has

$$\begin{aligned}
[x,z_1]\equiv 1 \bmod U' \ &\Rightarrow\ x^{y_3 t_0^{-1}} \equiv x \bmod U' \\
&\Rightarrow x^{y_3} \equiv x^{t_0} \bmod U', \\
[x^{y_3}, z_1^{y_3}] \equiv 1 \bmod U' \ &\Rightarrow\ (x^{y_3})^{(z_1^{y_3})} = x^{y_3^2 (t_0^{-1})^{y_3}} \equiv x^{y_3} \bmod U' \\
&\Rightarrow\ x^{y_3^2} \equiv x^{t_1} \bmod U',
\end{aligned}$$

and so on. Thus

$$x^{y_3^h} \equiv x^{t_{h-1}} \mod U' \qquad \text{for all } h=1,\dots,p-1.$$

Altogether, U^{ab} is the free abelian pro-p group generated by the cosets $\{wU' \mid w\in\mathcal{Y}_U\}$, so that Fact 2.1 yields the claim. □

Now set $U_1 = G_1\cap U$ and $U_2 = G_2 \cap U$. Then U_1, U_2 are open subgroups of G_1, G_2 respectively of index p, and thus they are again Demushkin groups, on $2+p(d_1-1)$ and $2+p(d_2-1)$ generators respectively (cf. [6]). In particular, the defining relation of U_1 is

$$s_1 = \prod_{h=p-1}^{0}\left(\left[y_4^{y_3^h}, y_5^{y_3^h}\right]\cdots\left[y_{d_1-1}^{y_3^h}, y_{d_1}^{y_3^h}\right]\left[x^{y_3^h}, y_1^{y_3^h}\right]\right)[y_2,u] = 1, \tag{4.4}$$

while the defining relation of U_2 is

$$\begin{aligned}
s_2 &= \prod_{h=p-1}^{0}\left(\left[z_2^{z_1^h}, z_3^{z_1^h}\right]\cdots\left[z_{d_2-1}^{z_1^h}, z_{d_2}^{z_1^h}\right]\right)[x, z_1^p] \\
&= \prod_{h=p-1}^{0}\left(\left[z_2^{y_3^h t_{h-1}^{-1}}, z_3^{y_3^h t_{h-1}^{-1}}\right]\cdots\left[z_{d_2-1}^{y_3^h t_{h-1}^{-1}}, z_{d_2}^{y_3^h t_{h-1}^{-1}}\right]\right)[x, u t_{p-1}^{-1}] = 1.
\end{aligned} \tag{4.5}$$

Also, from the relations (4.4)–(4.5) and from (4.1), one computes

$$\begin{aligned} x^{y_3} &= x^{z_1 t_0} = x^{t_0}([z_{d_2}, z_{d_2-1}]\cdots[z_3,z_2])^{t_0}, \\ x^{y_3^2} &= x^{t_1}([z_{d_2}, z_{d_2-1}]\cdots)^{t_1}\left([z_{d_2}^{y_3}, z_{d_2-1}^{y_3}]\cdots\right)^{t_0^{-1}t_1}, \\ x^{y_3^3} &= x^{t_2}([z_{d_2}, z_{d_2-1}]\cdots)^{t_2}\left([z_{d_2}^{y_3}, z_{d_2-1}^{y_3}]\cdots\right)^{t_0^{-1}t_2}\left(\left[z_{d_2}^{y_3^2}, z_{d_2-1}^{y_3^2}\right]\cdots\right)^{t_1^{-1}t_2}, \end{aligned} \tag{4.6}$$

and so on. In fact, the two relations (4.4)–(4.5) — with the $x^{y_3^h}$'s replaced using (4.6) — are all the defining relations we need to get U, as shown in the following.

Lemma 4.3. *The pro-p group U has $\mathrm{r}(U) = 2$ defining relations.*

Proof. Since $\mathrm{H}^n(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for every $n \geq 3$ (cf. § 3.3) and $[G : U] = p$, one has $\mathrm{H}^n(U, \mathbb{Z}/p\mathbb{Z}) = 0$ for every $n \geq 3$ as well (cf. [21, Prop. 3.3.5]). Moreover, one has

$$\mathrm{r}(U) - \mathrm{d}(U) + 1 = p\,(\mathrm{r}(G) - \mathrm{d}(G) + 1) \tag{4.7}$$

(cf. [21, Prop. 3.3.13]). By definition, $\mathrm{r}(G) = 2$ and $\mathrm{d}(G) = 1 + d_1 + d_2$, while $\mathrm{d}(U) = 3 + p(d_1 + d_2 - 2)$ by Lemma 4.2. Therefore, from (4.7) one computes $\mathrm{r}(U) = 2$. □

4.3. **The subgroup H.** Let $\phi_U : U \to \mathbb{Z}/p$ be the homomorphism of pro-p groups defined by $\phi_U(y_1)$, $\phi_U(y_1^{y_3}) = -1$, and $\phi_U(w) = 0$ for any other element w of $\mathcal{Y}_U$, and put $H = \mathrm{Ker}(\phi_U)$. Then H is an open subgroup of U of index p. Set $v = y_1$. Since $U/H = \{H, vH, \ldots, v^{p-1}H\}$, H is the pro-p group (non-minimally) generated by

$$\mathcal{X}_H = \left\{ v^p,\ (vy_1^{y_3})^{v^h},\ w^{v^h} \ \mid w \in \mathcal{Y}_U,\ w \neq v, y_1^{y_3},\ h = 0, \ldots, p-1 \right\},$$

and subject to the $2p$ relations $s_1^{v^h} = 1$ and $s_2^{v^h} = 1$, with $h = 0, \ldots, p-1$. We claim that the abelianization H^{ab} yields non-trivial torsion.

Proposition 4.4. *The abelian pro-p group H^{ab} is not torsion-free.*

Proof. Since all the elements of $\mathcal{Y}_U$ showing up in the last terms of the equalities (4.6) belong to H, one deduces that $x^{y_3^h} \equiv x \bmod H'$ for all $h = 0, \ldots, p-1$.

Now, each factor of s_2 — cf. (4.5) — is a commutator of elements of H, and thus the relations $s_2^{v^h} = 1$ yield trivial relations in H^{ab}. On the other hand, every factor of s_1 — cf. (4.4) —, but $[x, y_1]$ and $[x^{y_3}, y_1^{y_3}]$, is a commutator of elements of H. From (4.4) one obtains

$$[x^{y_3}, y_1^{y_3}]\,[x, y_1] \equiv \left[x, v^{-1}(vy_1^{y_3})\right][x, v] \equiv [x, v^{-1}][x, v] \equiv 1 \mod H', \tag{4.8}$$

as $vy_1^{y_3} \in H$. Altogether, H^{ab} is the abelian pro-p group (non-minimally) generated by the set $\mathcal{X}_{H^{\mathrm{ab}}} = \{wH' \mid w \in \mathcal{X}_H\}$, and subject to the p relations

$$\left[x^{v^h}H', v^{-1}H'\right]\left[x^{v^h}H', vH'\right] = H', \qquad \text{with } h = 0, \ldots, p-1,$$

as $U/H = \{H, vH, \ldots, v^{p-1}H\}$. From these relations one deduces the equivalences:

$$x^{v^2} \equiv (x^v)^2 \cdot x^{-1} \mod H' \qquad \text{with } h = 1,$$
$$x^{v^3} \equiv \left(x^{v^2}\right)^2 \cdot (x^v)^{-1} \equiv (x^v)^3 \cdot x^{-2} \mod H' \qquad \text{with } h = 2,$$
$$\vdots$$
$$x^{v^{p-1}} \equiv \left(x^{v^{p-2}}\right)^2 \cdot \left(x^{v^{p-3}}\right)^{-1} \equiv (x^v)^{p-1} \cdot x^{2-p} \mod H' \qquad \text{with } h = p-2,$$
$$x^{v^p} \equiv \left(x^{v^{p-1}}\right)^2 \cdot \left(x^{v^{p-2}}\right)^{-1} \equiv (x^v)^p \cdot x^{1-p} \mod H' \qquad \text{with } h = p-1.$$

But $x^{v^p} \equiv x \bmod H'$, as $v^p \in H$, and thus from the last of the above equivalences one obtains

$$x \equiv (x^v)^p x^{1-p} \mod H' \implies (x^v)^p x^{-p} \equiv (x^v x^{-1})^p \equiv 1 \mod H'. \tag{4.9}$$

Altogether, H^{ab} is the abelian pro-p group minimally generated by

$$\mathcal{Y}_{H^{\mathrm{ab}}} = \left\{ v^pH',\, xH',\, x^vH',\, (vy_1^{y_3})^{v^h}H',\, w^{v^h}H' \mid h = 0, \ldots, p-1 \right\},$$

where $w \in \mathcal{Y}_U \smallsetminus \{v, y_1^{y_3}, x\}$, and subject to the relation $((xH')^{-1} \cdot x^vH')^p = H'$ — in particular, H^{ab} is isomorphic to $\mathbb{Z}_p^{2+p+p^2(d_1+d_2-2)} \times \mathbb{Z}/p\mathbb{Z}$. □

5. Proof of Theorem 1.1 case (1.1.b)

Let p be an odd prime, and let G be a pro-p group as defined in Theorem 1.1, with defining relations as in (1.1.b) — namely,

$$G = \langle\, x, y_1, \ldots, y_{d_1}, z_1, \ldots, z_{d_2} \mid r_1 = r_2 = 1 \,\rangle,$$

with

$$r_1 = y_1^p[y_1, x] \cdots [y_{d_1-1}, y_{d_1}],$$
$$r_2 = z_1^p[z_1, x] \cdots [z_{d_2-1}, z_{d_2}].$$

5.1. **Kummerianity.** Let G_1, G_2 be the two Demushkin groups as in § 3.2, with $\epsilon = 1$. By Example 2.6, if

$$\theta_1 \colon G_1 \longrightarrow 1 + p\mathbb{Z}_p \qquad \text{and} \qquad \theta_2 \colon G_2 \longrightarrow 1 + p\mathbb{Z}_p$$

are two torsion-free orientations completing respectively G_1 and G_2 into Kummerian oriented pro-p groups, then necessarily $\theta_1(y_1) = \ldots = \theta_1(y_{d_1}) = 1$, and analogously $\theta_2(z_1) = \ldots = \theta_1(z_{d_2}) = 1$, while $\theta_1(x) = \theta_2(x) = (1-p)^{-1}$.

Proposition 5.1. *An orientation $\theta \colon G \to 1 + p\mathbb{Z}_p$ completes G into a Kummerian oriented pro-p group (G, θ) if, and only if,*

$$\theta(x) = (1-p)^{-1} \qquad \textit{and} \qquad \theta(y_i) = \theta(z_j) = 1$$

for all $i = 1, \ldots, d_1$ and $j = 1, \ldots, d_2$.

Proof. Suppose that $\theta\colon G\to 1+p\mathbb{Z}_p$ is the orientation defined as above, and pick arbitrary p-adic integers $\lambda,\lambda_i,\lambda'_j\in\mathbb{Z}_p$ for $1\leq i\leq d_1$ and $1\leq j\leq d_2$. The assignment $x\mapsto\lambda$, $y_i\mapsto\lambda_i$ and $z_j\mapsto\lambda'_j$ for every i,j yields a well-defined continuous 1-cocycle $c\colon G\to\mathbb{Z}_p(\theta)$, as (2.3) imples that

$$\begin{aligned}c(r_1)&=c(y_1^p)+c([y_1,x])+c([y_2,y_3])+\ldots+c([y_{d_1-1},y_{d_1}])\\&=p\cdot\lambda_1+\theta(x)^{-1}(\lambda_1(1-\theta(x))-0)+0+\ldots+0\\&=0\end{aligned}$$

and

$$\begin{aligned}c(r_2)&=c(z_1^p)+c([z_1,x])+c([z_2,z_3])+\ldots+c([z_{d_2-1},z_{d_2}])\\&=p\cdot\lambda'_1+\theta(x)^{-1}(\lambda'_1(1-\theta(x))-0)+0+\ldots+0\\&=0\end{aligned}$$

Therefore, (G,θ) is Kummerian by Lemma 2.9.

Conversely, suppose that (G,θ) is Kummerian. Let N_1 and N_2 denote the normal subgroups of G generated as normal subgroups by $z_1,\ldots,z_{d_2}$ and $y_1,\ldots,y_{d_1}$ respectively. Then $G/N_1\simeq G_1$ and $G/N_2\simeq G_2$. Moreover, Proposition 2.10 implies that $(G/N_i,\theta_{/N_i})$ is Kummerian for both $i=1,2$.

Since $G/N_i\simeq G_i$ for both i, Example 2.6 and the argument before the statement of the proposition imply that $\theta_{/N_1}(y_1N_1)=\ldots=\theta_{/N_1}(y_{d_1}N_1)=1$, and analogously $\theta_{/N_2}(z_1N_2)=\ldots=\theta_{/N_2}(z_{d_2}N_2)=1$, while $\theta_{/N_1}(xN_1)=\theta_{/N_2}(xN_2)=(1-p)^{-1}$. Hence, θ is as defined above, as $\theta(w)=\theta_{/N_1}(wN_1)$ for every $w\in G_1$, and analogously $\theta(w)=\theta_{/N_2}(wN_2)$ for every $w\in G_2$. □

Henceforth, $\theta\colon G\to 1+p\mathbb{Z}_p$ will denote the orientation as in Proposition 5.1.

5.2. **The subgroup H.** Let $\phi_1\colon G_1\to\mathbb{Z}/p\oplus\mathbb{Z}/p$ and $\phi_2\colon G_2\to\mathbb{Z}/p\oplus\mathbb{Z}/p$ be the homomorphisms of pro-p groups defined by

$$\begin{aligned}&\phi_1(x)=\phi_2(x)=(1,0),\\&\phi_1(y_1)=\phi_2(z_1)=(0,1),\\&\phi_1(y_i)=\phi_2(z_j)=(0,0)\text{ for }i,j\geq 2.\end{aligned}\tag{5.1}$$

Put $U_1=\mathrm{Ker}(\phi_1)$ and $U_2=\mathrm{Ker}(\phi_2)$, and also

$$t=z_1^{-1}y_1,\qquad u=x^p,\qquad v=y_1^p,\qquad w=z_1^p.$$

Then U_1 is an open normal subgroup of G_1 of index p^2, and likewise for U_2 and G_2 — note that by [6] both U_1 and U_2 are Demushkin groups.

Finally, put $N_1=\mathrm{Ker}(\theta|_{U_1})$, $N_2=\mathrm{Ker}(\theta|_{U_2})$, and let T be the subgroup of G generated by t. Observe that N_1 and N_2 are free pro-p groups, as they are subgroups of infinite index of Demushkin groups (cf. [33, Ch. I, § 4.5, Ex. 5–(b)]), while $T\simeq\mathbb{Z}_p$ as G is torsion-free (cf. Remark 3.2).

Let H be the subgroup of G generated by U_1, U_2 and T, and let M be the subgroup of H generated by N_1, N_2 and T. Observe that $M\subseteq\mathrm{Ker}(\theta)$. Our aim is to show that the oriented pro-p group $(H,\theta|_H)$ is not Kummerian. For this purpose, we need the following.

Lemma 5.2. (i) $M=N_1\amalg N_2\amalg T$.

(ii) *M is a normal subgroup of H, and $H \simeq M \rtimes X^p$*

(iii) *One has an isomorphism of p-elementary abelian groups*

$$\frac{G}{\Phi(G)} \simeq \frac{X^p}{X^{p^2}} \times \frac{N_1}{N_1^p[N_1,U_1]} \times \frac{N_2}{N_2^p[N_2,U_2]} \times \frac{T}{T^p}. \tag{5.2}$$

Proof. Consider the pro-p tree $\mathcal{T}$ associated to the amalgamated free pro-p product (3.3). Namely, $\mathcal{T}$ consists of a set vertices $\mathcal{V}$ and a set of edges $\mathcal{E}$, where

$$\begin{aligned}\mathcal{V} &= \{\, hG_1, hG_2 \mid h \in G \,\} = G/G_1 \,\dot\cup\, G/G_2,\\ \mathcal{E} &= \{\, hX \mid h \in G \,\} = G/X,\end{aligned}$$

and it comes endowed with a natural G-action, i.e.,

$$\begin{aligned} g.(hG_1) &= (gh)G_1 && \text{for every } g \in G,\ hG_1 \in G/G_1 \subseteq \mathcal{V}\\ g.(hG_1) &= (gh)G_2 && \text{for every } g \in G,\ hG_2 \in G/G_2 \subseteq \mathcal{V},\\ g.(hX) &= (gh)X && \text{for every } g \in G,\ hX \in G/X = \mathcal{E}.\end{aligned} \tag{5.3}$$

Pick $g \in M$ and $hX \in \mathcal{E}$. Then $g.hX = hX$ if, and only if, $g \in hXh^{-1}$, i.e., $g = hx^\lambda h^{-1}$ for some $\lambda \in \mathbb{Z}_p$. Since $M \subseteq \mathrm{Ker}(\theta)$, it follows that

$$1 = \theta(g) = \theta\left(hx^\lambda h^{-1}\right) = \theta(x)^\lambda = (1-p)^\lambda, \tag{5.4}$$

and therefore $\lambda = 0$, as $1+p\mathbb{Z}_p$ is torsion-free. Hence, the subgroup M intersects trivially the stabilizer $\mathrm{Stab}_G(hX)$ of every edge $hX \in \mathcal{E}$. By [15, Thm. 5.6], M decomposes as free pro-p product as follows:

$$M = \left(\coprod_{\omega\in\mathcal{V}'} \mathrm{Stab}_M(\omega)\right) \amalg F, \tag{5.5}$$

where F is a free pro-p group, and $\mathcal{V}' \subseteq \mathcal{V}$ is a continuous set of representatives of the space of orbits $M\backslash\mathcal{V}$. Clearly, the vertices G_1 and G_2 belong to different orbits, thus in the decomposition (5.5) one finds the two factors

$$\begin{aligned}\mathrm{Stab}_M(G_1) &= \{\, g \in M \mid gG_1 = G_1 \,\} = M \cap G_1,\\ \mathrm{Stab}_M(G_2) &= \{\, g \in M \mid gG_2 = G_2 \,\} = M \cap G_2.\end{aligned}$$

Since $N_1 \subseteq M \cap G_1 \subseteq \mathrm{Ker}(\theta) \cap G_1 = N_1$, one has $\mathrm{Stab}_M(G_1) = N_1$, and analogously $\mathrm{Stab}_M(G_2) = N_2$. Therefore, from (5.5) one obtains

$$M = N_1 \amalg N_2 \amalg \left(\coprod_{\omega\in\mathcal{V}'\smallsetminus\{G_1,G_2\}} \mathrm{Stab}_M(\omega) \amalg F\right). \tag{5.6}$$

It is straightforward to see that $t \notin N_1 \amalg N_2$. Since M is generated as pro-p group by N_1, N_2 and t, the right-side factor in (5.6) is necessarily T, and this proves (i).

In order to prove (ii), we need only to show that $uMu^{-1} = M$, as $H = \langle u, M\rangle$. Since N_1 is normal in U_1, and $u \in U_1$, then $uN_1u^{-1} = N_1$ — analogously, $uN_2u^{-1} = N_2$. Now, observe that the integer

$$(1-p)^p - 1 = \left(1 - \binom{p}{1}p + \binom{p}{2}p^2 - \ldots - p^p\right) - 1$$

is divisible by p^2 but not by p^3, so we put $(1-p)^p = 1 + p^2\lambda$, with $\lambda \in 1 + p\mathbb{Z}_p$. From the relation $r_1 = 1$ one deduces

$$y_1^x = y_1^{1-p} \cdot ([y_2, y_3] \cdots [y_{d_1-1}, y_{d_1}])^{-1}, \tag{5.7}$$

and by iterating (5.7) p times, one obtains $y_1^u = y_1^{(1-p)^p} n_1$ for some $n_1 \in N_1'$ — for this purpose, observe that for every $\nu \geq 0$ and $i \geq 1$, the triple commutator

$$[y_1^\nu, [y_i, y_{i+1}]] = \left[y_i^{y_1^\nu}, y_{i+1}^{y_1^\nu}\right]^{-1} \cdot [y_i, y_{i+1}]$$

belongs to N_1', as $y_i^{y_0^\nu} \in N_1$. Analogously, $z_1^u = z_1^{(1-p)^p} n_2$ for some $n_2 \in N_2'$. Altogether,

$$t^u = (z_1^{-1} y_1)^u = z_1^u y_1^u = n_2^{-1} \cdot w^{-p\lambda} \cdot t \cdot v^{p\lambda} \cdot n_1, \tag{5.8}$$

which belongs to M — here we replaced $z_1^{-(1-p)^p} = w^{-p\lambda} \cdot z_1^{-1}$ and $y_1^{(1-p)^p} = y_1 \cdot v^{p^\lambda}$. Hence, $M \trianglelefteq H$. Finally, by definition $H = M \cdot X^p$, and moreover

$$M \cap X^p \subseteq \mathrm{Ker}(\theta) \cap X^p = \{1\},$$

so that $H = M \rtimes X^p$. This completes the proof of (ii).

Finally, by (i) and (ii) one has the isomorphism of p-elementary abelian groups

$$\begin{aligned} M/\Phi(M) &\simeq N_1/\Phi(N_1) \times N_2/\Phi(N_2) \times T/T^p \\ H/\Phi(H) &\simeq X^p/X^{p^2} \times M/M^p[M,H]. \end{aligned} \tag{5.9}$$

From (5.8) one has that $[T, X^p] \subseteq \Phi(M)$, and since $H = MX^p$, $U_1 = N_1X^p$, and $U_2 = N_2X^p$, form (5.9) one deduces (iii). □

5.3. **The subgroup H and Kummerianity.**

Proposition 5.3. *The oriented pro-p group $(H, \theta|_H)$ is not Kummerian.*

Proof. Let N be the normal subgroup of H generated as a normal subgroup by N_1, N_2, and set $\bar{H} = H/N$. Then $N \subseteq \mathrm{Ker}(\theta|_H)$, and clearly $\bar{H}$ is finitely generated. Moreover, by duality the restriction map $\mathrm{res}^1_{H,N} \colon H^1(H, \mathbb{Z}/p\mathbb{Z}) \to H^1(N, \mathbb{Z}/p\mathbb{Z})^H$ is surjective, as by Lemma 5.2 one has

$$N/N^p[N,H] \simeq N_1/N_1^p[N_1, U_1] \times N_2/N_2^p[N_2, U_2],$$

which embeds in $H/\Phi(H)$. In particular, $\{uN, tN\}$ is a minimal generating set of $\bar{H}$. Thus, by Proposition 2.10 if the oriented pro-p group $(\bar{H}, \bar{\theta})$ is not Kummerian — where $\bar{\theta} = (\theta|_H)_{/N} \colon \bar{H} \to 1 + p\mathbb{Z}_p$ is the orientation induced by $\theta|_H$ —, then also $(H, \theta|_H)$ is not Kummerian.

By (5.8), in H one has that $[t, u^{-1}] \equiv 1 \bmod N$, and thus $\bar{H}$ is abelian. Moreover,

$$\bar{\theta}(uN) = \theta(u) = (1-p)^p \qquad \text{and} \qquad \bar{\theta}(tN) = \theta(t) = 1,$$

so that $\mathrm{Ker}(\bar{\theta}) = \langle tN \rangle$. Therefore, the subgroup $K_{\bar{\theta}}(\bar{H})$ is generated by

$$\left(t^{-\theta(u)} u t u^{-1}\right) N = t^{p^2\lambda} N.$$

Thus, the quotient $\mathrm{Ker}(\bar{\theta})/K_{\bar{\theta}}(\bar{H}) = \langle tN \rangle / \langle tN \rangle^{p^2}$ is not torsion-free, and by Proposition 2.2, $(\bar{H}, \bar{\theta})$ is not Kummerian. □

This completes the proof of Theorem 1.1 case (1.1.b).

Remark 5.4. If $d_1 = d_2 = 1$, case (1.1.b) of Theorem 1.1 is a particular case of [3, Prop. 6.5].

6. Massey products

6.1. **Massey products in Galois cohomology.** Here we recall briefly what we need in order to prove Proposition 1.3. For a detailed account on Massey products for pro-p groups, we direct the reader to [8, 20, 36].

Let G be a pro-p group. For $n \geq 2$, the *n-fold Massey product* on $\mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$ is a multi-valued map

$$\underbrace{\mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z}) \times \ldots \times \mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})}_{n \text{ times}} \longrightarrow \mathrm{H}^2(G, \mathbb{Z}/p\mathbb{Z}).$$

For $n \geq 2$, given a sequence $\alpha_1, \ldots, \alpha_n$ of elements of $\mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$ (with possibly $\alpha_i = \alpha_j$ for some $1 \leq i < j \leq n$), the (possibly empty) subset of $\mathrm{H}^2(G, \mathbb{Z}/p\mathbb{Z})$ which is the value of the n-fold Massey product associated to the sequence $\alpha_1, \ldots, \alpha_n$ is denoted by $\langle \alpha_1, \ldots, \alpha_n \rangle$. If $n = 2$, then the 2-fold Massey product coincides with the cup-product, i.e., for $\alpha_1, \alpha_2 \in \mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$ one has

$$\langle \alpha_1, \alpha_2 \rangle = \{\alpha \smile \alpha_2\} \subseteq \mathrm{H}^2(G, \mathbb{Z}/p\mathbb{Z}). \tag{6.1}$$

A pro-p group G is said to satisfy:

(a) the *n-Massey vanishing property* (with respect to $\mathbb{Z}/p\mathbb{Z}$) if for every sequence $\alpha_1, \ldots, \alpha_n$ of elements of $\mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$, $\langle \alpha_1, \ldots, \alpha_n \rangle \neq \varnothing$ implies $0 \in \langle \alpha_1, \ldots, \alpha_n \rangle$;

(b) the *strong n-Massey vanishing property* (with respect to $\mathbb{Z}/p\mathbb{Z}$) if for every sequence $\alpha_1, \ldots, \alpha_n$ of elements of $\mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$, the condition on the cup-products

$$\alpha_1 \smile \alpha_2 = \alpha_2 \smile \alpha_3 = \ldots = \alpha_{n-1} \smile \alpha_n = 0 \tag{6.2}$$

implies $0 \in \langle \alpha_1, \ldots, \alpha_n \rangle$ (cf. [22, Def. 1.2]) — we remind that the triviality condition (6.2) is satisfied whenever $\langle \alpha_1, \ldots, \alpha_n \rangle \neq \varnothing$, cf., e.g., [20, § 2];

(c) the *cyclic p-Massey vanishing property* if for every element $\alpha \in \mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$, the p-fold Massey product $\langle \alpha, \ldots, \alpha \rangle$ contains 0.

Remark 6.1. Given a sequence $\alpha_1, \ldots, \alpha_n$ of elements of $\mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$, if an element ω of $\mathrm{H}^2(G, \mathbb{Z}/p\mathbb{Z})$ is a value of the n-fiold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$, then

$$\omega + \alpha_1 \smile \beta \in \langle \alpha_1, \ldots, \alpha_n \rangle \qquad \text{and} \qquad \omega + \alpha_n \smile \beta \in \langle \alpha_1, \ldots, \alpha_n \rangle$$

for any $\beta \in \mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$ (cf. [20, Rem. 2.2]).

In [19, Thm. 8.1], J. Minač and N.D. Tân proved that the maximal pro-p Galois group of a field $\mathbb{K}$ containing a root of 1 of order p (and also $\sqrt{-1}$ if $p = 2$) satisfies the cyclic p-Massey vanishing property. The proof of the last property for a pro-p group G as in Theorem 1.1 is rather immediate.

Proof of Proposition 1.3–(ii). By Proposition 4.1 and Proposition 5.1, G may complete into a Kummerian oriented pro-p group with torsion-free orientation. Hence, G satisfies the cyclic p-Massey vanishing property by [28, Thm. 3.10]. □

6.2. **Massey products and unipotent upper-triangular matrices.** Massey products for a pro-p group G may be translated in terms of unipotent upper-triangular representations of G as follows. For $n\geq 2$ let

$$\mathbb{U}_{n+1}=\left\{\begin{pmatrix} 1 & a_{1,2} & \cdots & & a_{1,n+1} \\ & 1 & a_{2,3} & \cdots & \\ & & \ddots & \ddots & \vdots \\ & & & 1 & a_{n,n+1} \\ & & & & 1 \end{pmatrix} \mid a_{i,j}\in\mathbb{Z}/p\right\}\subseteq \mathrm{GL}_{n+1}(\mathbb{Z}/p\mathbb{Z})$$

be the group of unipotent upper-triangular $(n+1)\times(n+1)$-matrices over $\mathbb{Z}/p$. Then $\mathbb{U}_{n+1}$ is a finite p-group. Moreover, for $1\leq h,l\leq n+1$ let $E_{h,l}$ denote the $(n+1)\times(n+1)$ matrix with the (h,l)-entry equal to 1, and all the other entries equal to 0.

Now let $\rho\colon G\to\mathbb{U}_{n+1}$ be a homomorphism of pro-p groups. Observe that for every $h=1,\ldots,n$, the projection $\rho_{h,h+1}\colon G\to\mathbb{Z}/p$ of ρ onto the $(h,h+1)$-entry is a homomorphism, and thus it may be considered as an element of $\mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z})$. One has the following "pro-p translation" of a result of W. Dwyer which interprets Massey product in terms of unipotent upper-triangular representations (cf., e.g., [11, Lemma 9.3]).

Proposition 6.2. *Let G be a pro-p group, and let $\alpha_1,\ldots,\alpha_n$ be a sequence of elements of $\mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z})$, with $n\geq 2$. Then the n-fold Massey product $\langle\alpha_1,\ldots,\alpha_n\rangle$:*

(i) *is not empty if, and only if, there exists a morphism of pro-p groups $\bar\rho\colon G\to\mathbb{U}_{n+1}/\mathrm{Z}(\mathbb{U}_{n+1})$ such that $\bar\rho_{h,h+1}=\alpha_h$ for every $h=1,\ldots,n$;*

(ii) *vanishes if, and only if, there exists a morphism of pro-p groups $\rho\colon G\to\mathbb{U}_{n+1}$ such that $\rho_{h,h+1}=\alpha_h$ for every $h=1,\ldots,n$.*

We recall that

$$\mathrm{Z}(\mathbb{U}_{n+1})=\{\,I_{n+1}+aE_{1,n+1}\;\mid\;a\in\mathbb{Z}/p\mathbb{Z}\,\}\simeq\mathbb{Z}/p\mathbb{Z}.$$

We use this fact to prove statements (iii.a)–(iii.b) of Proposition 1.3. First of all, let G be as in Theorem 1.1, and let $\alpha_1,\ldots,\alpha_n$ be a sequence of elements of $\mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z})$. Keeping the same notation as in § 3.3, for $h=1,\ldots,n$ one has

$$\alpha_h=\alpha_h(x)\cdot\chi+\sum_{i=1}^{d_1}\alpha_h(y_i)\cdot\varphi_i+\sum_{j=1}^{d_2}\alpha_h(z_j)\cdot\psi_j.$$

Therefore, for $h=1,\ldots,n-1$ one obtains

$$\alpha_h\smile\alpha_h=S_h\cdot(\chi\smile\varphi_1)+S_h'\cdot(\chi\smile\psi_1),$$

where

$$\begin{aligned} S_h=&(\alpha_h(x)\alpha_{h+1}(y_1)-\alpha_h(y_1)\alpha_{h+1}(x))+\\ &+(-1)^\epsilon\sum_{2|i}(\alpha_h(y_i)\alpha_{h+1}(y_{i+1})-\alpha_h(y_{i+1})\alpha_{h+1}(y_i)),\\ S_h'=&(\alpha_h(x)\alpha_{h+1}(z_1)-\alpha_h(z_1)\alpha_{h+1}(x))+\\ &+(-1)^\epsilon\sum_{2|j}(\alpha_h(z_j)\alpha_{h+1}(z_{j+1})-\alpha_h(z_{j+1})\alpha_{h+1}(z_j)),\end{aligned}$$

with $\epsilon = 0$ if G is as in (1.1.a), and $\epsilon = 1$ if G is as in (1.1.b). If the sequence $\alpha_1, \ldots, \alpha_n$ satisfies condition (6.2), then one has $S_h = S'_h = 0$ for $h = 1, \ldots, n-1$, as $\{\chi \smile \varphi_1, \chi \smile \psi_1\}$ is a basis of $\mathrm{H}^2(G, \mathbb{Z}/p)$.

From now on, we will assume that $p > 3$ while considering a pro-p group G as in (1.1.b), unless stated otherwise.

6.3. **3-fold Massey products.** We are ready to prove the following.

Proposition 6.3. *A pro-p group G satisfies the 3-Massey vanishing property in the following cases:*

(a) *if G is as in* (1.1.a)*;*
(b) *if G is as in* (1.1.b) *and $p > 3$.*

Proof. Let $\alpha_1, \alpha_2, \alpha_3$ be a sequence of elements of $\mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$ satisfying (6.2). Then $S_1 = S'_1 = S_2 = S'_2 = 0$ (cf. § 6.2). Our goal is to construct a morphism $\rho\colon G \to \mathbb{U}_4$ such that $\rho_{1,2} = \alpha_1$, $\rho_{2,3} = \alpha_2$, $\rho_{3,4} = \alpha_3$.

For every $w \in \mathcal{X}$ set

$$A(w) = I + \alpha_1(w)E_{1,2} + \alpha_2(w)E_{2,3} + \alpha_3(w)E_{3,4} \in \mathbb{U}_4,$$

where I denotes the 4×4 identity matrix. If G is as in (1.1.a), then one computes

$$\begin{aligned} C &= [A(x), A(y_1)] \cdots [A(y_{d_1-1}), A(y_{d_1})] \\ &= I + E_{1,4}\left(\alpha_1(y_1)\alpha_2(x)\alpha_3(y_1) + \sum_{2|i} \alpha_1(y_i)\alpha_2(y_{i+1})\alpha_3(y_i)\right) \\ C' &= [A(x), A(z_1)] \cdots [A(z_{d_2-1}), A(z_{d_2})] \\ &= I + E_{1,4}\left(\alpha_1(z_1)\alpha_2(x)\alpha_3(z_1) + \sum_{2|j} \alpha_1(z_j)\alpha_2(z_{j+1})\alpha_3(z_j)\right); \end{aligned} \tag{6.3}$$

while if G is as in (1.1.b), then one computes

$$\begin{aligned} C &= A(y_1)^p[A(y_1), A(x)] \cdots [A(y_{d_1-1}), A(y_{d_1})] \\ &= I + E_{1,4}\left(\alpha_1(x)\alpha_2(y_1)\alpha_3(x) + \sum_{2|i} \alpha_1(y_i)\alpha_2(y_{i+1})\alpha_3(y_i)\right) \\ C' &= A(z_1)^p[A(z_1), A(x)] \cdots [A(z_{d_2-1}), A(z_{d_2})] \\ &= I + E_{1,4}\left(\alpha_1(x)\alpha_2(z_1)\alpha_3(x) + \sum_{2|j} \alpha_1(z_j)\alpha_2(z_{j+1})\alpha_3(z_j)\right). \end{aligned} \tag{6.4}$$

— observe that the exponent of $\mathbb{U}_4$ is p, as $p > 4$, and thus $A(y_1)^p = A(z_1)^p = I$.

In both cases, $C, C' \in \mathrm{Z}(\mathbb{U}_4)$, and therefore the assignment $w \mapsto A(w)$ for every $w \in \mathcal{X}$ yields a morphism $\bar{\rho}\colon G \to \mathbb{U}_4/\mathrm{Z}(\mathbb{U}_4)$ satisfying $\bar{\rho}_{h,h+1} = \alpha_h$ for $h = 1, 2, 3$. Thus, $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \neq \varnothing$ by Proposition 6.2.

Moreover, if $C = C' = I$ then the same assignment yields a morphism $\rho\colon G \to \mathbb{U}_4$ with the desired properties. In particular, by (6.3)–(6.4) one has $C = I$ if $\alpha_1(w) = \alpha_3(w) = 0$ for every $w = y_1, \ldots, y_{d_1}$, or for every $w = y_2, \ldots, y_{d_1}$ and $w = x$; and analogously

$C' = I$ if $\alpha_1(w) = \alpha_3(w) = 0$ for every $w = z_1, \ldots, z_{d_{d_2}}$, or for every $w = z_2, \ldots, z_{d_2}$ and $w = x$.

On the other hand, if $C \neq I$ then $\chi \smile \varphi_1 = \pm\operatorname{trg}(r_1 G_{(3)})$ belongs to $\langle \alpha_1, \alpha_2, \alpha_3\rangle$, and analogusly if $C' \neq I$ then $\chi \smile \psi_1 = \pm\operatorname{trg}(r_2 G_{(3)})$ belongs to $\langle \alpha_1, \alpha_2, \alpha_3\rangle$ (cf. [20, Lemma 3.7]) — here the sign depends on whether the relations are as in (1.1.a) or in (1.1.b). Now, if $\alpha_h(y_i) \neq 0$ for some $h = 1, 3$ and $i \in \{2, \ldots, d_1\}$, then

$$\chi \smile \varphi_1 = \alpha_h \smile \beta \qquad \text{for some } \beta \in \mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z}).$$

Analogously, if $\alpha_h(z_j) \neq 0$ for some $h = 1, 3$ and $j \in \{2, \ldots, d_2\}$, then

$$\chi \smile \psi_1 = \alpha_h \smile \beta \qquad \text{for some } \beta \in \mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z}).$$

Moreover, if $\alpha_h(x) \neq 0$ for some $h = 1, 3$, then

$$\chi \smile \varphi_1 = \alpha_h \smile \beta \qquad \text{and} \qquad \chi \smile \psi_1 = \alpha_h \smile \beta'$$

for some $\beta, \beta' \in \mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$. Therefore, Remark 6.1 implies that if $C \neq I$ or $C' \neq I$ then $0 \in \langle \alpha_1, \alpha_2, \alpha_3\rangle$ anyway. □

Remark 6.4. If $p = 3$ and G as in (1.1.b), then G does not satisfy the 3-Massey vanishing property. Indeed, set $\alpha_1 = \alpha_3 = \varphi_1 + \psi_1$, and $\alpha_2 = \varphi_1$. Then

$$\alpha_1 \smile \alpha_2 = \alpha_2 \smile \alpha_3 = \pm(\varphi_1 \smile \psi_1) = 0.$$

It is easy to see that one may construct a morphism of pro-p groups $\bar{\rho}\colon G \to \mathbb{U}_4/\mathrm{Z}(\mathbb{U}_4)$ such that $\bar{\rho}_{1,2} = \bar{\rho}_{3,4} = \alpha_1$ and $\bar{\rho}_{2,3} = \alpha_2$ — and thus $\langle \alpha_1, \alpha_2, \alpha_1\rangle \neq \varnothing$ by Proposition 6.2 —; but, on the other hand, one may not construct a morphism of pro-p groups $\rho\colon G \to \mathbb{U}_4$ satisfying $\rho_{1,2} = \rho_{3,4} = \alpha_1$ and $\rho_{2,3} = \alpha_2$ — so that $0 \notin \langle \alpha_1, \alpha_2, \alpha_1\rangle$ by Proposition 6.2.

6.4. **4-fold Massey products.**

Proposition 6.5. *A pro-p group G as in Theorem 1.1 satisfies the strong 4-Massey vanishing property.*

Proof. Let $\alpha_1, \ldots, \alpha_4$ be a sequence of four elements of $\mathrm{H}^1(G, \mathbb{Z}/p\mathbb{Z})$ satisfying (6.2). Our goal is to construct a homomorphism of pro-p groups $\rho\colon G \to \mathbb{U}_5$ such that $\rho_{h,h+1} = \alpha_h$ for $h = 1, \ldots, 5$, so that the claim follows by Proposition 6.2.

Let I denote the identity matrix of the group $\mathbb{U}_5$. For every $w \in \mathcal{X} = \{x, y_1, \ldots, z_{d_2}\}$ set

$$A(w) = \begin{pmatrix} 1 & \alpha_1(w) & 0 & 0 & 0 \\ & 1 & \alpha_2(w) & 0 & 0 \\ & & 1 & \alpha_3(w) & 0 \\ & & & 1 & \alpha_4(w) \\ & & & & 1 \end{pmatrix} \in \mathbb{U}_5.$$

Moreover, put

$$C = (c_{hl}) = A(y_1)^{\epsilon p} \cdot [A(x), A(y_1)]^{(-1)^\epsilon} \cdots [A(y_{d_1-1}), A(y_{d_1})]\,,$$
$$C' = (c'_{hl}) = A(z_1)^{\epsilon p} \cdot [A(x), A(z_1)]^{(-1)^\epsilon} \cdots [A(z_{d_2-1}), A(z_{d_2})]\,.$$

We will consider the matrix C as a function of the matrices $A(x), \ldots, A(y_{d_1})$, and the matrix C' as a function of the matrices $A(x), A(z_1), \ldots, A(z_{d_2})$.

Since $p \geq 5$, the exponent of the p-group $\mathbb{U}_5$ is p, and thus $A(y_1)^p = A(z_1)^p = I$. Moreover, for every $w, w' \in \mathcal{X}$, the $(h, h+1)$-entry of $[A(w), A(w')]$ is 0 for every $h = 1, \ldots, 4$, and thus also $c_{h,h+1} = c'_{h,h+1} = 0$. Moreover, for $h = 1, 2, 3$ one has $c_{h,h+2} = S_h$ and $c'_{h,h+2} = S'_h$ — which are equal to 0 by (6.2).

We split the proof in the analysis of the following three cases. Our aim is to modify suitably the matrices $A(w)$ — without modifying the $(h, h+1)$-entries with $h = 1, \ldots, 4$ — in order to obtain $C = C' = I$.

Case 1. Suppose first that:

(1.a) $\alpha_2(x) = \alpha_2(y_i) = 0$ for all $2 \leq i \leq d_1$; or
(1.b) $\alpha_3(x) = \alpha_3(y_i) = 0$ for all $2 \leq i \leq d_1$.

Since $S_1 = S_2 = S_3 = 0$ by (6.2), one has

$$\alpha_1(x)\alpha_2(y_1) = \alpha_2(y_1)\alpha_3(x) = 0, \tag{6.5}$$

$$\alpha_2(x)\alpha_3(y_1) = \alpha_3(y_1)\alpha_4(x) = 0, \tag{6.6}$$

respectively in case (1.a) and in case (1.b). Applying (6.5)–(6.6), one computes

$$[A(y_1), A(x)] = \begin{cases} I + (\alpha_3(y_1)\alpha_4(x) - \alpha_3(x)\alpha_4(y_1))\, E_{3,5} & \text{in case (1.a)}, \\ I + (\alpha_1(y_1)\alpha_2(x) - \alpha_2(x)\alpha_1(y_1))\, E_{1,3} & \text{in case (1.b)}, \end{cases}$$

and

$$[A(y_i), A(y_{i+1})] = \begin{cases} I + (\alpha_3(y_i)\alpha_4(y_{i+1}) - \alpha_3(y_{i+1})\alpha_4(y_i))\, E_{3,5} & \text{in case (1.a)}, \\ I + (\alpha_1(y_i)\alpha_2(y_{i+1}) - \alpha_2(y_{i+1})\alpha_1(y_i))\, E_{1,3} & \text{in case (1.b)}, \end{cases}$$

for $i = 2, 4, \ldots, d_1 - 1$. Altogether, one has $C = I + S_3E_{3,5}$ in case (1.a) and $C = I + S_1E_{1,3}$ in case (1.b), so that in both cases $C = I$ by (6.2).
Analogously, if $\alpha_2(x) = \alpha_2(z_j) = 0$ for all $2 \leq j \leq d_2$, or if $\alpha_3(x) = \alpha_3(z_j) = 0$ for all $2 \leq j \leq d_2$, then $C' = I$. This completes the analysis of case 1.

Case 2. Now suppose that $\alpha_1(x) = \alpha_4(x) = \alpha_1(y_i) = \alpha_4(y_i) = 0$ for all $2 \leq i \leq d_1$. Since $S_1 = S_2 = S_3 = 0$ by (6.2), one has

$$\alpha_1(y_1)\alpha_2(x) = \alpha_3(x)\alpha_4(y_1) = 0. \tag{6.7}$$

Then one computes

$$[A(y_1), A(x)] = I + (\alpha_2(y_1)\alpha_3(x) - \alpha_2(x)\alpha_3(y_1))\, E_{2,4} + \alpha_2(x)\alpha_3(y_1)\alpha_4(y_1)E_{2,5},$$
$$[A(y_i), A(y_{i+1})] = I + (\alpha_2(y_i)\alpha_3(y_{i+1}) - \alpha_2(y_{i+1})\alpha_3(y_i))\, E_{2,4},$$

where we apply (6.7) to obtain the first equality, and in the second one i runs through the even positive integers between 2 and $d_1 - 1$. If $\alpha_2(x)\alpha_3(y_1)\alpha_4(y_1) = 0$ then it is straightforward to see that $C = I + S_2E_{2,4} = I$. Otherwise, $\alpha_2(x) \neq 0$, so that (6.7) implies that $\alpha_1(y_1) = 0$. In this case, set

$$\tilde{A} = I - \alpha_3(y_1)\alpha_4(y_1)E_{3,5}.$$

Then

$$\left[\tilde{A}, A(x)\right] = I - \alpha_2(x)\alpha_3(y_1)\alpha_4(y_1)E_{2,5},$$

and

$$\left[A(y_1)\tilde{A}, A(x)\right] = \underbrace{\left[A(y_1), [\tilde{A}, A(x)]\right]}_{=I} \left[\tilde{A}, A(x)\right] [A(y_1), A(x)]$$
$$= I + (\alpha_2(y_1)\alpha_3(x) - \alpha_2(x)\alpha_3(y_1))\, E_{2,4}.$$

Therefore, replacing $A(y_1)$ with $A(y_1)\tilde{A}$ yields $c_{2,4} = S_2 = 0$ and $C_{hl} = 0$ for $h < l$, i.e., $C = I$.

An analogous argument yields $C' = I$ — after replacing suitably the matrix $A(z_1)$ if needed — if $\alpha_1(x) = \alpha_3(x) = \alpha_1(z_j) = \alpha_3(z_j) = 0$ for all $1 \leq j \leq d_2$. This completes the analysis of case 2.

Case 3. Finally, if none of the above two assumptions on the triviality of the values $\alpha_h(x)$ and $\alpha_h(y_i)$, with $2 \leq i \leq d_1$, hold true, then

(3.a) there are $w, w' \in \{x, y_2, \ldots, y_{d_1}\}$ — possibly $w = w'$ — such that $\alpha_1(w) \neq 0$ and $\alpha_2(w') \neq 0$, or

(3.b) there are $w, w' \in \{x, y_2, \ldots, y_{d_1}\}$ — possibly $w = w'$ — such that $\alpha_4(w) \neq 0$ and $\alpha_3(w') \neq 0$.

Suppose we are in case (3.a). If $w = x$ or $w = y_i$ with i odd, set

$$\tilde{A} = \begin{cases} I + \frac{c_{1,4}}{\alpha_1(w)} E_{2,4} & \text{if } w \in \{\, x, y_3, \ldots, y_{d_1} \,\} \\ I - \frac{c_{1,4}}{\alpha_1(w)} E_{2,4} & \text{if } w \in \{\, y_i \mid i \text{ is even} \,\}, \end{cases}$$

and replace $A(y_1)$ with $A(y_1)\tilde{A}$, if $w = x$, or $A(y_{i-1})$ with $A(y_{i-1})\tilde{A}$ if $w = y_i$ with i odd, or $A(y_{i+1})$ with $A(y_{i+1})\tilde{A}$, if $w = y$ with i even. After the replacement, one has $c_{hl} = 0$ for $h < l \leq h + 2$, and for $(h, l) = (1, 4)$. Then, set

$$\tilde{A}' = \begin{cases} I + \frac{c_{2,5}}{\alpha_1(w')} E_{3,5} & \text{if } w' \in \{\, x, y_3, \ldots, y_{d_1} \,\} \\ I - \frac{c_{2,5}}{\alpha_1(w')} E_{3,5} & \text{if } w' \in \{\, y_i \mid \text{ is even} \,\}, \end{cases}$$

and replace $A(y_1)$ with $A(y_1)\tilde{A}'$, if $w = x$, or $A(y_{i-1})$ with $A(y_{i-1})\tilde{A}'$ if $w = y_i$ with i odd, or $A(y_{i+1})$ with $A(y_{i+1})\tilde{A}'$, if $w = y$ with i even. After this further replacement, one has $c_{hl} = 0$ for $h < l \leq h + 3$. Finally, set

$$\tilde{A}'' = \begin{cases} I + \frac{c_{1,5}}{\alpha_1(w)} E_{2,5} & \text{if } w \in \{\, x, y_3, \ldots, y_{d_1} \,\} \\ I - \frac{c_{1,5}}{\alpha_1(w)} E_{2,5} & \text{if } w \in \{\, y_i \mid i \text{ is even} \,\}, \end{cases}$$

and replace $A(y_1)$ with $A(y_1)\tilde{A}''$, if $w = x$, or $A(y_{i-1})$ with $A(y_{i-1})\tilde{A}''$ if $w = y_i$ with i odd, or $A(y_{i+1})$ with $A(y_{i+1})\tilde{A}''$, if $w = y$ with i even. After this last replacement, one has $C = I$.

Now suppose we are in case (3.b). If $w = x$ or $w = y_i$ with i odd, set

$$\tilde{A} = \begin{cases} I - \frac{c_{2,5}}{\alpha_4(w)} E_{3,4} & \text{if } w \in \{\, x, y_3, \ldots, y_{d_1} \,\} \\ I + \frac{c_{2,5}}{\alpha_4(w)} E_{3,4} & \text{if } w \in \{\, y_i \mid i \text{ is even} \,\}, \end{cases}$$

and replace $A(y_1)$ with $A(y_1)\tilde{A}$, if $w = x$, or $A(y_{i-1})$ with $A(y_{i-1})\tilde{A}$ if $w = y_i$ with i odd, or $A(y_{i+1})$ with $A(y_{i+1})\tilde{A}$, if $w = y$ with i even. After the replacement, one has

$c_{hl} = 0$ for $h < l \leq h+2$, and for $(h,l) = (2,5)$. Then, set

$$\tilde{A}' = \begin{cases} I - \frac{c_{1,4}}{\alpha_3(w')} E_{1,3} & \text{if } w' \in \{\, x, y_3, \dots, y_{d_1} \,\} \\ I + \frac{c_{1,4}}{\alpha_3(w')} E_{1,3} & \text{if } w' \in \{\, y_i \mid i \text{ is even} \,\}, \end{cases}$$

and replace $A(y_1)$ with $A(y_1)\tilde{A}'$, if $w = x$, or $A(y_{i-1})$ with $A(y_{i-1})\tilde{A}'$ if $w = y_i$ with i odd, or $A(y_{i+1})$ with $A(y_{i+1})\tilde{A}'$, if $w = y$ with i even. After this further replacement, one has $c_{hl} = 0$ for $h < l \leq h+3$. Finally, set

$$\tilde{A}'' = \begin{cases} I - \frac{c_{1,5}}{\alpha_1(w)} E_{1,4} & \text{if } w \in \{\, x, y_3, \dots, y_{d_1} \,\} \\ I + \frac{c_{1,5}}{\alpha_1(w)} E_{1,4} & \text{if } w \in \{\, y_i \mid i \text{ is even} \,\}, \end{cases}$$

and replace $A(y_1)$ with $A(y_1)\tilde{A}''$, if $w = x$, or $A(y_{i-1})$ with $A(y_{i-1})\tilde{A}''$ if $w = y_i$ with i odd, or $A(y_{i+1})$ with $A(y_{i+1})\tilde{A}''$, if $w = y$ with i even. After this last replacement, one has $C = I$.
Moreover, if none of the above two assumptions on the triviality of the values $\alpha_h(x)$ and $\alpha_h(z_j)$, with $2 \leq j \leq d_2$, hold true, the same argument produces suitable matrices $A(z_1), \dots, A(z_{d_2})$ such that the matrix C' is the identity matrix. This concludes the analysis of case 3.

Altogether, the assignment $w \mapsto A(x)$ for every $w \in \mathcal{X}$ — with the matrices $A(w)$'s suitably modified in case of need — yields a homomorphism of pro-p groups $\rho\colon G \to \mathbb{U}_5$ with the desired properties. □

We believe that the answer to the following questions is positive.

Question 6.6. (a) *Let G be as in* (1.1.a). *Does G satisfy the strong n-Massey vanishing property for every $n \geq 3$?*
(b) *Let G be as in* (1.1.b). *Does G satisfy the strong n-Massey vanishing property for every $3 \leq n < p$?*

7. The Minač-Tân pro-p group

We focus now on the Minač-Tân pro-p group

$$G = \langle\, x_1, \dots, x_5 \mid r = 1 \,\rangle \qquad \text{with } r = [[x_1, x_2], x_3]\,[x_4, x_5].$$

Using Proposition 6.2, one may show that G does not satisfy the 3-Massey vanishing property (cf. [20, Ex. 7.2]). Our aim is to show that G cannot complete into a 1-cyclotomic oriented pro-p group with torsion-free orientation.

7.1. **Kummerianity and 1-cyclotomicity.**

Proposition 7.1. *Let G be the Minač-Tân pro-p group, and let $\theta\colon G \to 1 + p\mathbb{Z}_p$ be a torsion-free orientation. Then the oriented pro-p group (G, θ) is Kummerian if, and only if, $x_4, x_5 \in \mathrm{Ker}(\theta)$, and:*
(a) *$x_3 \in \mathrm{Ker}(\theta)$; or*
(b) *$x_1, x_2 \in \mathrm{Ker}(\theta)$.*

Proof. Let $c\colon G \to \mathbb{Z}_p(\theta)$ be an arbitrary continuous 1-cocycle, and set $c(x_i) = \lambda_i$ for $i = 1, \ldots, 5$. Applying (2.2)–(2.3) one computes $c(r) = c([[x_1, x_2], x_3]) + c([x_4, x_5])$, and

$$\begin{aligned} c([[x_1,x_2],x_3]) &= \theta(x_1x_2)^{-1}\left(\theta(x_3)^{-1}-1\right)\left(\lambda_1(1-\theta(x_2))-\lambda_2(1-\theta(x_1))\right), \\ c([x_4,x_5]) &= \theta(x_4x_5)^{-1}\left(\lambda_4(1-\theta(x_5))-\lambda_5(1-\theta(x_4))\right). \end{aligned} \tag{7.1}$$

On the other hand, $c(r) = 0$ as $r = 1$.

Suppose that (G, θ) is Kummerian. Then by Lemma 2.9, we may prescribe arbitrary values to $\lambda_1, \ldots, \lambda_5$. If $\lambda_4 = 1$ and $\lambda_i = 0$ for $i \neq 4$, from (7.1) and from the fact that $c(r) = 0$ one obtains $0 = 1 \cdot (1 - \theta(x_5))$, and thus $\theta(x_5) = 1$. Analogously, if $\lambda_5 = 1$ and $\lambda_i = 0$ for $i \neq 5$, one deduces $\theta(x_4) = 1$. Finally, if $\lambda_4 = \lambda_5 = 0$ from (7.1) one obtains

$$0 = c(r) = \theta(x_1x_2)^{-1}\left(\theta(x_3)^{-1}-1\right)\left(\lambda_1(1-\theta(x_2))-\lambda_2(1-\theta(x_1))\right),$$

and the arbitrariness of λ_1, λ_2 implies that $\theta(x_3) = 1$ or $\theta(x_1) = \theta(x_2) = 1$.

Conversely, suppose that $x_4, x_5 \in \mathrm{Ker}(\theta)$, and at least one of the hypothesis (i)–(ii) holds true. Then for any choice for λ_4, λ_5, by (7.1) one has $c([x_4, x_5]) = 0$. On the other hand, one has

$$c([[x_1,x_2],x_3]) = \begin{cases} 0\cdot\left(\lambda_1(1-\theta(x_2))-\lambda_2(1-\theta(x_1))\right)=0 & \text{if } x_3\in\mathrm{Ker}(\theta), \\ \left(\theta(x_3)^{-1}-1\right)\left(\lambda_1\cdot 0-\lambda_2\cdot 0\right)=0 & \text{if } x_1,x_2\in\mathrm{Ker}(\theta). \end{cases}$$

Altogether, any choice for $\lambda_1, \ldots, \lambda_5$ yields a well-defined continuous 1-cocycle $c\colon G \to \mathbb{Z}_p(\theta)$, and thus (G, θ) is Kummerian by Lemma 2.9. □

Now consider the subgroup H of G generated by x_3, x_4, x_5 and by $y = [x_1, x_2]$. Then H is subject to the relation

$$r = [y, x_3][x_4, x_5] = 1.$$

If (G, θ) is a 1-cyclotomic oriented pro-p group, with θ a torsion-free orientation, then $= (H, \theta|_H)$ is Kummerian. Therefore, if $c'\colon H \to \mathbb{Z}_p(\theta|_H)$ is a continuous 1-cocycle, applying (2.2)–(2.3) one obtains

$$\begin{aligned} 0 = c'(r) &= c'([y,x_3]) + c'([x_4,x_5]) \\ &= \theta(yx_3)^{-1}\left(c'(y)(1-\theta(x_3)) - c'(x_3)(1-\theta(y))\right) + 0 \\ &= \theta(yx_3)^{-1}c'(y)(1-\theta(x_3)), \end{aligned}$$

as $\theta(x_4) = \theta(x_5) = 1$ by Proposition 7.1, and $y \in G' \subseteq \mathrm{Ker}(\theta)$. Since $c'(y)$ may be arbitrarily chosen by Lemma 2.9, one deduces $\theta(x_3) = 1$. This proves the following.

Lemma 7.2. *Let G be the Mináč-Tân pro-p group, and let $\theta\colon G \to 1+p\mathbb{Z}_p$ be a torsion-free orientation. If the oriented pro-p group (G, θ) is 1-cyclotomic then $x_3, x_4, x_5 \in \mathrm{Ker}(\theta)$.*

Moreover, if (G, θ) is 1-cyclotomic we may suppose without loss of generality that $x_2 \in \mathrm{Ker}(\theta)$, too. Indeed, let $v_p\colon \mathbb{Z}_p \to \mathbb{N}$ denote the p-adic valuation, and let $k \geq 1$ be such that $\mathrm{Im}(\theta) = 1 + p^k\mathbb{Z}_p$.

Suppose first that $v_p(\theta(x_2) - 1) = k$ and $v_p(\theta(x_1) - 1) > k$, and set $z = x_2x_1$. Then $\{z, x_2, x_3, x_4, x_5\}$ is a minimal generating set of G, $v_p(\theta(z) - 1) = k$, and G is subject to the relation

$$[[z, x_2], x_3]\,[x_4, x_5] = 1,$$

as $[x_2x_1, x_2] = [x_1, x_2]$. Hence, we may assume $v_p(\theta(x_1) - 1) = k$.

Consequently, there exists $\lambda \in \mathbb{Z}_p$ such that $\theta(x_2) = \theta(x_1)^\lambda$. Now set $z = x_1^{-\lambda}x_2$. Then $\{x_1, z, x_3, x_4, x_5\}$ is a minimal generating set of G, $\theta(z) = \theta(x_2)\theta(x_1)^{-\lambda} = 1$, and G is subject to the relation

$$[[x_1, z], x_3]\,[x_4, x_5] = 1,$$

as $[x_1, x_1^{-\lambda}x_2] = [x_1, x_2]$.

Therefore, from now on $\theta\colon G \to 1+p\mathbb{Z}_p$ will denote a torsion-free orientation satisfying $x_2, \dots, x_5 \in \mathrm{Ker}(\theta)$.

7.2. **The subgroup U.** Put $u = x_1^p$ and $t = x_1^{-1}x_3$. Let $\phi\colon G \to \mathbb{Z}/p$ be the homomorphism defined by $\phi(x_1) = \phi(x_3) = 1$ and $\phi(x_i) = 0$ for $i = 2, 4, 5$, and let U be the kernel of ϕ. Then U is a normal subgroup of G of index p, and it is generated as a normal subgroup of G by $\{u, t, x_2, x_4, x_5\}$. In fact, U is generated as a pro-p group by the set

$$\mathcal{X}_U = \left\{\, u,\ t^{x_1^h},\ x_2^{x_1^h},\ x_4^{x_1^h},\ x_5^{x_1^h} \ \mid\ h = 0, \dots, p-1 \,\right\},$$

as $G/U = \{U, x_1U, \dots, x_1^{p-1}U\}$. We need to find a subset of $\mathcal{X}_U$ which minimally generates U as a pro-p group.

Proposition 7.3. *The set*

$$\mathcal{Y}_U = \left\{\, t,\ x_2,\ x_2^{x_1},\ t^{x_1^h},\ x_4^{x_1^h},\ x_5^{x_1^h} \ \mid\ h = 0, \dots, p-1 \,\right\},$$

is a minimal generating set of U as a pro-p group. Moreover, the abelian pro-p group U^{ab} is not torsion-free.

Proof. The subgroup U is the pro-p group generated by $\mathcal{X}_U$ and subject to the p-relations $r^{x_1^h} = 1$, $h = 0, \dots, p-1$. Since $x_3 = x_1 t$, one computes

$$\begin{aligned} [[x_1, x_2], x_3] &= [x_1, x_2]^{-1} \cdot [x_1, x_2]^{x_3} \\ &= [x_2, x_1] \cdot [x_1, x_2^{x_1}]^t \\ &= x_2^{-1} \cdot x_2^{x_1} \cdot \left(\left(x_2^{x_1^2} \right)^{-1} x_2^{x_1} \right)^t . \end{aligned} \tag{7.2}$$

From (7.2), and from the relation $r = 1$, one deduces the equivalence

$$\left(x_2^{x_1^2} \right)^{-1} \cdot (x_2^{x_1})^2 \cdot x_1^{-1} \equiv 1 \mod U', \tag{7.3}$$

as $[x_4, x_5] \in U'$ and $t \in U$.

Hence, U^{ab} is the abelian pro-p group generated by $\mathcal{X}_{U^{\mathrm{ab}}} = \{wU' \mid w \in \mathcal{X}_U\}$ and subject to the p relations induced by the equivalences $((x_2^{x_1^2})^{-1}(x_2^{x_1})^2 x_1^{-1})^{x_1^h} \equiv 1 \bmod U'$, namely

$$\begin{aligned} x_2^{x_1^2} &\equiv (x_2^{x_1})^2\, x_1^{-1} \mod U', \qquad \text{for } h = 0, \\ x_2^{x_1^3} &\equiv \left(x_2^{x_1^2} \right)^2 (x_1^{x_2})^{-1} \equiv (x_2^{x_1})^3\, x_1^{-2} \mod U', \qquad \text{for } h = 1, \\ &\ \ \vdots \\ x_2^{x_1^p} &\equiv \left(x_2^{x_1^{p-1}} \right)^2 \left(x_1^{p-2} \right)^{-1} \equiv (x_2^{x_1})^p\, x_1^{1-p} \mod U', \qquad \text{for } h = p-2, \\ x_2^{x_1^{p+1}} &\equiv (x_2^{x_1})^2 \cdot x_1^{-1} \equiv (x_2^{x_1})^{p+1}\, x_1^{-p} \mod U', \qquad \text{for } h = p-1. \end{aligned} \tag{7.4}$$

On the one hand, from (7.4) one deduces that the coset $x_2^{x_1^h}U'$ is generated by x_2U' and $x_2^{x_1}U'$ for every $h=2,\ldots,p-1$, so that $\mathcal{Y}_{U^{\mathrm{ab}}}=\{wU'\mid w\in\mathcal{Y}_U\}$ generates U^{ab} as an abelian pro-p group. On the other hand, from the equivalences with $h=p-2$ and $h=p-1$ one deduces that

$$(x_2^{x_1})^p\,x_1^{1-p}\,(x_2^u)^{-1}\equiv(x_2^{x_1})^p\,x_1^{1-p-1}\equiv\left(x_2^{x_1}x_1^{-1}\right)^p\equiv 1\mod U',$$
$$(x_2^{x_1})^{p+1}\,x_1^{-p}\,(x_2^{ux_1})^{-1}\equiv(x_2^{x_1})^{p+1-1}\,x_1^{-p}\equiv\left(x_2^{x_1}x_1^{-1}\right)^p\equiv 1\mod U',$$

as $x_2^u\equiv x_2\bmod U'$; therefore they yield equivalent relations in U^{ab}. Altogether, U^{ab} is the abelian pro-p group minimally generated by $\mathcal{X}_{U^{\mathrm{ab}}}$ and subject to the relation

$$\left((x_2U')^{-1}\cdot x_2^{x_1}U'\right)^p=1.$$

Hence U^{ab} is not torsion-free, and $\mathcal{Y}_U$ is a minimal generating set of U by Fact 2.1. □

From Proposition 7.3, one deduces that G is not absolutely torsion-free, and thus the oriented pro-p group $(G,\mathbf{1})$ is not 1-cyclotomic.

7.3. **1-cyclotomicity and the Minač-Tân pro-p group.** We are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Suppose for contradiction that there exists a torsion free orientation $\theta\colon G\to 1+p\mathbb{Z}_p$ such that the oriented pro-p group (G,θ) is 1-cyclotomic. Then by § 7.1, we may assume without loss of generality that $x_2,\ldots,x_5\in\mathrm{Ker}(\theta)$, while $\theta(x_1)\neq 1$ by § 7.2. Set $\lambda\in p\mathbb{Z}_p\smallsetminus\{0\}$ such that $\theta(x_1)=1+\lambda$.

Consider the oriented pro-p group $(U,\theta|_U)$, and set $K=K_{\theta|_U}(U)$, $\bar U=U/K$. Our goal is to show that the oriented pro-p group $(\bar U,(\theta|_U)_{/K})$ is not $(\theta|_U)_{/K}$-abelian, so that $(U,\theta|_U)$ is not Kummerian by Proposition 2.8, and thus (G,θ) is not 1-cyclotomic.

Since $K\subseteq\Phi(U)$, by Proposition 7.3 the set $\mathcal{Y}_{\bar U}=\{wK\mid w\in\mathcal{Y}_U\}$ is a minimal generating set of $\bar U$. Now, since $\theta(t)=\theta(x_1)=(1+\lambda)^{-1}$, one has $w^t\equiv w^{1+\lambda}\bmod K$ for every $w\in U$. Therefore, from (7.2), and from the fact that $[x_4,x_5]\in\mathrm{Ker}(\theta|_U)'\subseteq K$, one obtains

$$[x_1,x_2]^{-1}\left([x_1,x_2]^{x_1}\right)^t\equiv[x_1,x_2]^{-1}\left([x_1,x_2]^{x_1}\right)^{(1+\lambda)^{-1}}\equiv 1\mod K,$$

and consequently

$$\begin{aligned}[x_1,x_2]^{x_1}&\equiv[x_1,x_2]^{1+\lambda}\mod K,\\ [x_1,x_2]^{x_1^2}&\equiv[x_1,x_2]^{(1+\lambda)^2}\mod K,\\ &\vdots\\ [x_1,x_2]^{x_1^{p-1}}&\equiv[x_1,x_2]^{(1+\lambda)^{p-1}}.\end{aligned}\tag{7.5}$$

Set

$$\mu=(1+\lambda)^0+(1+\lambda)^1+\ldots+(1+\lambda)^{p-1}=\frac{(1+\lambda)^p-1}{\lambda}.$$

Then $\mu\neq 0$ (as $\lambda\neq 0$), and $p\mid\mu$. Since $[x_1,x_2]=(x_2^{x_1})^{-1}x_2$, replacing the coset $x_2^{x_1}K$ with the coset $[x_1,x_2]K$ in $\mathcal{Y}_{\bar U}$ yields another minimal generating set — let us call it $\mathcal{Y}'_{\bar U}$

— of $\bar{U}$. Now, from (7.5) one obtains

$$\begin{aligned}[u, x_2] &= [x_1, x_2]^{x_1^{p-1}} \cdots [x_1, x_2]^{x_1} \cdot [x_1, x_2] \\ &\equiv [x_1, x_2]^{(1+\lambda)^{p-1}} \cdots [x_1, x_2]^{1+\lambda} \cdot [x_1, x_2] \mod K \\ &\equiv [x_1, x_2]^{\mu} \mod K\end{aligned}$$

— observe that $[x_1, x_2]^{x_i^h} \in \mathrm{Ker}(\theta|_U)$ for every h, and thus all such elements commute modulo K. Therefore, one has the relation

$$([x_1, x_2]K)^{\mu} = [uK, x_2K]$$

between elements of the minimal generating set $\mathcal{Y}'_{\bar{U}}$, and by [11, Thm. 8.1] this relation prevents the oriented pro-p group $(\bar{U}, (\theta|_U)_{/K})$ from being Kummerian — and thus also $(\theta|_U)_{/K}$-abelian. □

From Theorem 1.4 we obtain a new family of pro-p groups which cannot complete into 1-cyclotomic oriented pro-p groups.

Corollary 7.4. *Let G be the pro-p group with presentation*

$$G = \langle x_1, \ldots, x_n, x_{n+1}, x_{n+2} \mid [[\ldots[[x_1, x_2], x_3], \ldots x_{n-1}], x_n]\, [x_{n+1}, x_{n+2}] = 1 \rangle,$$

with $n \geq 3$. Then G cannot complete into a 1-cyclotomic oriented pro-p group with torsion-free orientation.

Proof. Set $y = [\ldots[x_1, x_2], \ldots x_{n-2}]$, and let H be the subgroup of G generated by $\{y, x_{n-1}, \ldots, x_{n+2}\}$. Then

$$H = \langle\, y, x_{n-1}, \ldots, x_{n+2} \mid [[y, x_{n-1}], x_n][x_{n+1}, x_{n+2}] \,\rangle$$

is isomorphic to the Minač-Tân pro-p group, and hence it cannot complete into a 1-cyclotomic oriented pro-p group with torsion-free orientation by Theorem 1.4. □

The following question remains open (cf. [2, Ex. 3.2]).

Question 7.5. *Is the Minač-Tân pro-p group G a Bloch-Kato pro-p group? Namely, is the $\mathbb{Z}/p\mathbb{Z}$-cohomology algebra of every closed subgroup of G a quadratic algebra?*

Declarations

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References

[1] D. Benson, N. Lemire, J. Minač, and J. Swallow, *Detecting pro-p-groups that are not absolute Galois groups*, J. Reine Angew. Math. **613** (2007), 175–191.

[2] S. Blumer, A. Cassella, and C. Quadrelli, *Groups of p-absolute Galois type that are not absolute Galois groups*, J. Pure Appl. Algebra **227** (2023), no. 4, Paper No. 107262.

[3] S. Blumer, C. Quadrelli, and Th.S. Weigel, *Oriented right-angled Artin pro-ℓ groups and maximal pro-ℓ Galois groups*, Int. Math. Res. Not. (2024). In press, published on-line.

[4] S.K. Chebolu, I. Efrat, and J. Minač, *Quotients of absolute Galois groups which determine the entire Galois cohomology*, Math. Ann. **352** (2012), no. 1, 205–221.

[5] J.D. Dixon, M.P.F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 61, Cambridge University Press, Cambridge, 1999.

[6] D. Dummit and J. Labute, *On a new characterization of Demuskin groups*, Invent. Math. **73** (1983), no. 3, 413–418.

[7] I. Efrat, *Small maximal pro-p Galois groups*, Manuscripta Math. **95** (1998), no. 2, 237–249.

[8] ———, *The Zassenhaus filtration, Massey products, and representations of profinite groups*, Adv. Math. **263** (2014), 389–411.

[9] I. Efrat and E. Matzri, *Triple Massey products and absolute Galois groups*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 12, 3629–3640.

[10] I. Efrat and J. Minač, *On the descending central sequence of absolute Galois groups*, Amer. J. Math. **133** (2011), no. 6, 1503–1532.

[11] I. Efrat and C. Quadrelli, *The Kummerian property and maximal pro-p Galois groups*, J. Algebra **525** (2019), 284–310.

[12] M. D. Fried and M. Jarden, *Field arithmetic*, 4th ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 11, Springer-Verlag, Berlin, 2023. Revised by Jarden.

[13] C. Haesemeyer and Ch. Weibel, *The norm residue theorem in motivic cohomology*, Annals of Mathematics Studies, vol. 200, Princeton University Press, Princeton, NJ, 2019.

[14] J.P. Labute, *Classification of Demushkin groups*, Canad. J. Math. **19** (1967), 106–132.

[15] O. V. Mel′nikov, *Subgroups and the homology of free products of profinite groups*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 97–120 (Russian); English transl., Math. USSR-Izv. **34** (1990), no. 1, 97–119.

[16] A. Merkurjev and F. Scavia, *Degenerate fourfold Massey products over arbitrary fields*, 2022. Preprint, available at `arXiv:2208.13011`.

[17] ———, *On the Massey Vanishing Conjecture and Formal Hilbert 90*, 2023. Preprint, available at `arXiv:2308.13682`.

[18] J. Minač, F. Pasini, C. Quadrelli, and N. D. Tân, *Koszul algebras and quadratic duals in Galois cohomology*, Adv. Math. **380** (2021), article no. 107569.

[19] J. Minač and N.D. Tân, *The kernel unipotent conjecture and the vanishing of Massey products for odd rigid fields*, Adv. Math. **273** (2015), 242–270.

[20] ———, *Triple Massey products and Galois theory*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 1, 255–284.

[21] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. 323, Springer-Verlag, Berlin, 2008.

[22] A. Pál and E. Szabó, *The strong Massey vanishing conjecture for fields with virtual cohomological dimension at most 1*, 2020. Preprint, available at `arXiv:1811.06192`.

[23] C. Quadrelli, *Bloch-Kato pro-p groups and locally powerful groups*, Forum Math. **26** (2014), no. 3, 793–814.

[24] ———, *Pro-p groups with few relations and universal Koszulity*, Math. Scand. **127** (2021), no. 1, 28–42.

[25] ———, *Two families of pro-p groups that are not absolute Galois groups*, J. Group Theory **25** (2022), no. 1, 25–62.

[26] ———, *Galois-theoretic features for 1-smooth pro-p groups*, Canad. Math. Bull. **65** (2022), no. 2, 525–541.

[27] ———, *1-smooth pro-p groups and Bloch-Kato pro-p groups*, Homology Homotopy Appl. **24** (2022), no. 2, 53–67.

[28] ———, *Massey products in Galois cohomology and the Elementary Type Conjecture*, J. Number Theory **258** (2024), 40–65.

[29] C. Quadrelli, I. Snopce, and M. Vannacci, *On pro-p groups with quadratic cohomology*, J. Algebra **612** (2022), 636–690.

[30] C. Quadrelli and Th.S. Weigel, *Profinite groups with a cyclotomic p-orientation*, Doc. Math. **25** (2020), 1881–1916.

[31] ———, *Oriented pro-ℓ groups with the Bogomolov-Positselski property*, Res. Number Theory **8** (2022), no. 2, Paper No. 21.

[32] L. Ribes, *On amalgamated products of profinite groups*, Math. Z. **123** (1971), 357–364.

[33] J.-P. Serre, *Galois cohomology*, Corrected reprint of the 1997 English edition, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. Translated from the French by Patrick Ion and revised by the author.

[34] I. Snopce and P. Zalesskiĭ, *Right-angled Artin pro-p groups*, Bull. Lond. Math. Soc. **54** (2022), no. 5, 1904-1922.

[35] V. Voevodsky, *On motivic cohomology with* $\mathbf{Z}/l$*-coefficients*, Ann. of Math. (2) **174** (2011), no. 1, 401–438.

[36] D. Vogel, *Massey products in the Galois cohomology of number fields*, 2004, http://www.ub.uni-heidelberg.de/archiv/4418. PhD thesis, University of Heidelberg.

[37] T. Würfel, *On a class of pro-p groups occurring in Galois theory*, J. Pure Appl. Algebra **36** (1985), no. 1, 95–103.

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