ON THE RELATIONSHIP BETWEEN IDEAL CLUSTER POINTS AND IDEAL LIMIT POINTS

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Abstract. Let X be a first countable space which has a non-isolated point and let $\mathcal{I}$ be an analytic P-ideal. First, it is shown that the sets of $\mathcal{I}$-limit points of all sequences in X are closed if and only if $\mathcal{I}$ is also an F_σ-ideal.

Moreover, let (x_n) be a sequence taking values in a Polish space. It is known that the set A of its statistical limit points is an F_σ-set, the set B of its statistical cluster points is closed, and that the set C of its ordinary limit points is closed, with $A \subseteq B \subseteq C$. It is proved the sets A and B own some additional relationship: indeed, the set S of isolated points of B is contained also in A.

Conversely, if A is an F_σ-set, B is a closed set with a subset S of isolated points such that $B \setminus S \neq \emptyset$, and C is a closed set with $S \subseteq A \subseteq B \subseteq C$, then there exists a sequence (x_n) for which: A is the set of its statistical limit points, B is the set of its statistical cluster points, and C is the set of its ordinary limit points.

Lastly, we discuss topological nature of the set of $\mathcal{I}$-limit points when $\mathcal{I}$ is neither F_σ- nor analytic P-ideal.

1. Introduction

The aim of this article is to establish some relationship between the set of ideal cluster points and the set of ideal limit points of a given sequence.

To this aim, let $\mathcal{I}$ be an ideal on the positive integers $\mathbf{N}$, i.e., a collection of subsets of $\mathbf{N}$ closed under taking finite unions and subsets. It is assumed that $\mathcal{I}$ contains the collection Fin of finite subsets of $\mathbf{N}$ and it is different from the whole power set $\mathcal{P}(\mathbf{N})$. Note that the family $\mathcal{I}_0$ of subsets with zero asymptotic density, that is,

$$\mathcal{I}_0 := \left\{ S \subseteq \mathbf{N} : \lim_{n\to\infty} \frac{|S \cap \{1,\ldots,n\}|}{n} = 0 \right\}$$

is an ideal. Let also $x = (x_n)$ be a sequence taking values in a topological space X, which will be always assumed hereafter to be Hausdorff. We denote by $\Lambda_x(\mathcal{I})$ the set of $\mathcal{I}$-limit points of x, that is, the set of all $\ell \in X$ for which $\lim_{k\to\infty} x_{n_k} = \ell$, for some subsequence (x_{n_k}) such that $\{n_k : k \in \mathbf{N}\} \notin \mathcal{I}$. In addition, let $\Gamma_x(\mathcal{I})$ be the set of $\mathcal{I}$-cluster points of x, that is, the set of all $\ell \in X$ such that $\{n : x_n \in U\} \notin \mathcal{I}$ for every neighborhood U of ℓ. Note that $\mathrm{L}_x := \Lambda_x(\mathrm{Fin})$ is the set of ordinary limit points of x (and coincides with $\Gamma_x(\mathrm{Fin})$ provided that X is first countable); we also shorten $\Lambda_x := \Lambda_x(\mathcal{I}_0)$ and $\Gamma_x := \Gamma_x(\mathcal{I}_0)$.

Statistical limit points and statistical cluster points (i.e., $\mathcal{I}_0$-limit points and $\mathcal{I}_0$-cluster points, resp.) of real sequences were introduced by Fridy [10], cf. also [2, 5, 11, 12, 14, 16].

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We are going to provide in Section 2, under suitable assumptions on X and $\mathcal{I}$, a characterization of the set of $\mathcal{I}$-limit points. Recall that $\Gamma_x(\mathcal{I})$ is closed and contains $\Lambda_x(\mathcal{I})$, see e.g. [4, Section 5]. Then it is shown that:

(i) $\Lambda_x(\mathcal{I})$ is an F_σ-set, provided that $\mathcal{I}$ is an analytic P-ideal (Theorem 2.2);
(ii) $\Lambda_x(\mathcal{I})$ is closed, provided that $\mathcal{I}$ is an F_σ-ideal (Theorem 2.3);
(iii) $\Lambda_x(\mathcal{I})$ is closed for all x if and only if $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I})$ for all x if and only if $\mathcal{I}$ is an F_σ-ideal, provided that $\mathcal{I}$ is an analytic P-ideal (Theorem 2.5);
(iv) For every F_σ-set A, there exists a sequence x such that $\Lambda_x(\mathcal{I}) = A$, provided that $\mathcal{I}$ is an analytic P-ideal which is not F_σ (Theorem 2.7);
(v) Each isolated $\mathcal{I}$-cluster point is also an $\mathcal{I}$-limit point (Theorem 2.8).

In addition, we provide in Section 3 some joint converse results:

(vi) Given $A \subseteq B \subseteq C \subseteq \mathbf{R}$ where A is an F_σ-set and B, C are closed sets such that A contains the set S of isolated points of B and $B \setminus S \neq \emptyset$, then there exists a real sequence x such that $\Lambda_x = A$, $\Gamma_x = B$, and $\mathrm{L}_x = C$ (Theorem 3.1 and Corollary 3.3);
(vii) Given non-empty closed sets $B \subseteq C \subseteq \mathbf{R}$, there exists a real sequence x such that $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I}) = B$ and $\mathrm{L}_x = C$, provided $\mathcal{I}$ is an F_σ-ideal different from Fin (Theorem 3.4).

Lastly, it is shown in Section 4 that:

(viii) $\Lambda_x(\mathcal{I})$ is analytic, provided that $\mathcal{I}$ is a co-analytic ideal (Proposition 4.1).

We conclude by showing that there exists an ideal $\mathcal{I}$ and a real sequence x such that $\Lambda_x(\mathcal{I})$ is not an F_σ-set (Example 4.2).

2. Topological structure of $\mathcal{I}$-limit points

We recall that an ideal $\mathcal{I}$ is said to be a *P-ideal* if it is σ-directed modulo finite, i.e., for every sequence (A_n) of sets in $\mathcal{I}$ there exists $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all n; equivalent definitions were given, e.g., in [1, Proposition 1].

By identifying sets of integers with their characteristic function, we equip $\mathcal{P}(\mathbf{N})$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals on $\mathbf{N}$. In particular, an ideal $\mathcal{I}$ is *analytic* if it is a continuous image of a Borel subset of a Polish space. Moreover, a map $\varphi : \mathcal{P}(\mathbf{N}) \to [0, \infty]$ is a *lower semicontinuous submeasure* provided that: (i) $\varphi(\emptyset) = 0$; (ii) $\varphi(A) \le \varphi(B)$ whenever $A \subseteq B$; (iii) $\varphi(A \cup B) \le \varphi(A) + \varphi(B)$ for all A, B; and (iv) $\varphi(A) = \lim_n \varphi(A \cap \{1, \ldots, n\})$ for all A.

By a classical result of Solecki, an ideal $\mathcal{I}$ is an analytic P-ideal if and only if there exists a lower semicontinuous submeasure φ such that

$$\mathcal{I} = \mathcal{I}_\varphi := \{A \subseteq \mathbf{N} : \|A\|_\varphi = 0\} \tag{1}$$

and $\varphi(\mathbf{N}) < \infty$, where $\|A\|_\varphi := \lim_n \varphi(A \setminus \{1, \ldots, n\})$ for all $A \subseteq \mathbf{N}$, see [18, Theorem 3.1]. Note, in particular, that for every $n \in \mathbf{N}$ it holds

$$\|A\|_\varphi = \|A \setminus \{1, \ldots, n\}\|_\varphi. \tag{2}$$

Hereafter, unless otherwise stated, an analytic P-ideal will be always denoted by $\mathcal{I}_\varphi$, where φ stands for the associated lower semicontinuous submeasure as in (1).

Given a sequence $x = (x_n)$ taking values in a first countable space X and an analytic P-ideal $\mathcal{I}_\varphi$, define

$$\mathfrak{u}(\ell) := \lim_{k\to\infty} \|\{n : x_n \in U_k\}\|_\varphi \tag{3}$$

for each $\ell \in X$, where (U_k) is a decreasing local base of neighborhoods at ℓ. It is easy to see that the limit in (3) exists and its value is independent of the choice of (U_k).

Lemma 2.1. *The map $\mathfrak{u}$ is upper semi-continuous. In particular, the set*

$$\Lambda_x(\mathcal{I}_\varphi, q) := \{\ell \in X : \mathfrak{u}(\ell) \ge q\}.$$

is closed for every $q > 0$.

Proof. We need to prove that $\mathscr{U}_y := \{\ell \in X : \mathfrak{u}(\ell) < y\}$ is open for all $y \in \mathbf{R}$ (hence $\mathscr{U}_\infty$ is open too). Clearly, $\mathscr{U}_y = \emptyset$ if $y \le 0$. Hence, let us suppose hereafter $y > 0$ and $\mathscr{U}_y \neq \emptyset$. Fix $\ell \in \mathscr{U}_y$ and let (U_k) be a decreasing local base of neighborhoods at ℓ. Then there exists $k_0 \in \mathbf{N}$ such that $\|\{n : x_n \in U_k\}\|_\varphi < y$ for every $k \ge k_0$. Fix $\ell' \in U_{k_0}$ and let (V_k) be a decreasing local base of neighborhoods at ℓ'. Fix also $k_1 \in \mathbf{N}$ such that $V_{k_1} \subseteq U_{k_0}$. It follows by the monotonicity of φ that

$$\|\{n : x_n \in V_k\}\|_\varphi \le \|\{n : x_n \in U_{k_0}\}\|_\varphi < y$$

for every $k \ge k_1$. In particular, $\mathfrak{u}(\ell') < y$ and, by the arbitrariness of ℓ', $U_{k_0} \subseteq \mathscr{U}_y$. □

At this point, we provide a useful characterization of the set $\Lambda_x(\mathcal{I}_\varphi)$ (without using limits of subsequences) and we obtain, as a by-product, that it is an F_σ-set.

Theorem 2.2. *Let x be a sequence taking values in a first countable space X and $\mathcal{I}_\varphi$ be an analytic P-ideal. Then*

$$\Lambda_x(\mathcal{I}_\varphi) = \{\ell \in X : \mathfrak{u}(\ell) > 0\}. \tag{4}$$

In particular, $\Lambda_x(\mathcal{I}_\varphi)$ is an F_σ-set.

Proof. Let us suppose that there exists $\ell \in \Lambda_x(\mathcal{I}_\varphi)$ and let (U_k) be a decreasing local base of neighborhoods at ℓ. Then there exists $A \subseteq \mathbf{N}$ such that $\lim_{n\to\infty, n\in A} x_n = \ell$ and $\|A\|_\varphi > 0$. At this point, note that, for each $k \in \mathbf{N}$, the set $\{n \in A : x_n \notin U_k\}$ is finite, hence it follows by (2) that $\mathfrak{u}(\ell) \ge \|A\|_\varphi > 0$.

On the other hand, suppose that there exists $\ell \in X$ such that $\mathfrak{u}(\ell) > 0$. Let (U_k) be a decreasing local base of neighborhoods at ℓ and define $\mathcal{A}_k := \{n : x_n \in U_k\}$ for each $k \in \mathbf{N}$; note that $\mathcal{A}_k$ is infinite since $\|\mathcal{A}_k\|_\varphi \downarrow \mathfrak{u}(\ell) > 0$ implies $\mathcal{A}_k \notin \mathcal{I}_\varphi$ for all k. Set for convenience $\theta_0 := 0$ and define recursively the increasing sequence of integers (θ_k) so that θ_k is the smallest integer greater than both θ_{k-1} and $\min \mathcal{A}_{k+1}$ such that

$$\varphi(\mathcal{A}_k \cap (\theta_{k-1}, \theta_k]) \ge \mathfrak{u}(\ell)\,(1 - 1/k)\,.$$

Finally, define $\mathcal{A} := \bigcup_k (\mathcal{A}_k \cap (\theta_{k-1}, \theta_k])$. Since $\theta_k \ge k$ for all k, we obtain

$$\varphi(\mathcal{A} \setminus \{1, \dots, n\}) \ge \varphi(\mathcal{A}_{n+1} \cap (\theta_n, \theta_{n+1}]) > \mathfrak{u}(\ell)\,(1 - 1/n)$$

for all n, hence $\|\mathcal{A}\|_\varphi \ge \mathfrak{u}(\ell) > 0$. In addition, we have by construction $\lim_{n\to\infty, n\in\mathcal{A}} x_n = \ell$. Therefore ℓ is an $\mathcal{I}_\varphi$-limit point of x. To sum up, this proves (4).

Lastly, rewriting (4) as $\Lambda_x(\mathcal{I}_\varphi) = \bigcup_n \Lambda_x(\mathcal{I}_\varphi, 1/n)$ and considering that each $\Lambda_x(\mathcal{I}_\varphi, 1/n)$ is closed by Lemma 2.1, we conclude that $\Lambda_x(\mathcal{I}_\varphi)$ is an F_σ-set. □

The fact that $\Lambda_x(\mathcal{I}_\varphi)$ is an F_σ-set already appeared in [3, Theorem 2], although with a different argument. The first result of this type was given in [12, Theorem 1.1] for the case $\mathcal{I}_\varphi = \mathcal{I}_0$ and $X = \mathbf{R}$. Later, it was extended in [5, Theorem 2.6] for first countable spaces. However, in the proofs contained in [3, 5] it is unclear why the constructed subsequence $(x_n : n \in \mathcal{A})$ converges to ℓ. Lastly, Theorem 2.2 generalizes, again with a different argument, [13, Theorem 3.1] for the case X metrizable.

A stronger result holds in the case that the ideal is F_σ. We recall that, by a classical result of Mazur, an ideal $\mathcal{I}$ is F_σ if and only if there exists a lower semicontinuous submeasure φ such that

$$\mathcal{I} = \{A \subseteq \mathbf{N} : \varphi(A) < \infty\}, \tag{5}$$

with $\varphi(\mathbf{N}) = \infty$, see [15, Lemma 1.2].

Theorem 2.3. *Let $x = (x_n)$ be a sequence taking values in a first countable space X and let $\mathcal{I}$ be an F_σ-ideal. Then $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I})$. In particular, $\Lambda_x(\mathcal{I})$ is closed.*

Proof. Since it is known that $\Lambda_x(\mathcal{I}) \subseteq \Gamma_x(\mathcal{I})$, the claim is clear if $\Gamma_x(\mathcal{I}) = \emptyset$. Hence, let us suppose hereafter that $\Gamma_x(\mathcal{I})$ is non-empty. Fix $\ell \in \Gamma_x(\mathcal{I})$ and let (U_k) be a decreasing local base of neighborhoods at ℓ. Letting φ be a lower semicontinuous submeasure associated with $\mathcal{I}$ as in (5) and considering that ℓ is an $\mathcal{I}$-cluster point, we have $\varphi(A_k) = \infty$ for all $k \in \mathbf{N}$, where $A_k := \{n : x_n \in U_k\}$.

Then set $a_0 := 0$ and define an increasing sequence of integers (a_k) which satisfies

$$\varphi(A_k \cap (a_{k-1}, a_k]) \geq k$$

for all k (note that this is possible since $\varphi(A_k \setminus S) = \infty$ whenever S is finite). At this point, set $A := \bigcup_k A_k \cap (a_{k-1}, a_k]$. It follows by the monotonocity of φ that $\varphi(A) = \infty$, hence $A \notin \mathcal{I}$. Moreover, for each $k \in \mathbf{N}$, we have that $\{n \in A : x_n \notin U_k\}$ is finite: indeed, if $n \in A_j \cap (a_{j-1}, a_j]$ for some $j \geq k$, then by construction $x_n \in U_j$, which is contained in U_k. Therefore $\lim_{n\to\infty,\, n \in A} x_n = \ell$, that is, $\ell \in \Lambda_x(\mathcal{I})$. □

Since summable ideals are F_σ P-ideals, see e.g. [7, Example 1.2.3], we obtain the following corollary which was proved in [13, Theorem 3.4]:

Corollary 2.4. *Let x be a real sequence and let $\mathcal{I}$ be a summable ideal. Then $\Lambda_x(\mathcal{I})$ is closed.*

It turns out that, within the class of analytic P-ideals, the property that the set of $\mathcal{I}$-limit points is always closed characterizes the subclass of F_σ-ideals:

Theorem 2.5. *Let X be a first countable space which has a non-isolated point. Let also $\mathcal{I}_\varphi$ be an analytic P-ideal. Then the following are equivalent:*

(i) *$\mathcal{I}_\varphi$ is also an F_σ-ideal;*
(ii) *$\Lambda_x(\mathcal{I}_\varphi) = \Gamma_x(\mathcal{I}_\varphi)$ for all sequences x;*
(iii) *$\Lambda_x(\mathcal{I}_\varphi)$ is closed for all sequences x;*
(iv) *there does not exist a partition $\{A_n : n \in \mathbf{N}\}$ of $\mathbf{N}$ such that $\|A_n\|_\varphi > 0$ for all n and $\lim_n \|\bigcup_{k>n} A_k\|_\varphi = 0$.*

Proof. (i) $\Longrightarrow$ (ii) follows by Theorem 2.3 and (ii) $\Longrightarrow$ (iii) is clear.

(iii) $\Longrightarrow$ (iv) By hypothesis, there exists a non-isolated point $\ell \in X$, hence there is a sequence (ℓ_n) converging to ℓ such that $\ell_n \neq \ell$ for all n. Let us suppose that there exists a partition $\{A_n : n \in \mathbf{N}\}$ of $\mathbf{N}$ such that $\|A_n\|_\varphi > 0$ for all n and $\lim_k \|\bigcup_{n\ge k} A_n\|_\varphi = 0$. Define the sequence $x = (x_n)$ by $x_n = \ell_i$ for all $n \in A_i$. Then we have that $\{\ell_n : n \in \mathbf{N}\} \subseteq \Lambda_x(\mathcal{I}_\varphi)$. On the other hand, since X is first countable Hausdorff, it follows that for all $k \in \mathbf{N}$ there exists a neighborhood U_k of ℓ such that

$$\{n : x_n \in U_k\} \subseteq \{n : x_n = \ell_i \text{ for some } i \ge k\} = \textstyle\bigcup_{n\ge k} A_n.$$

Hence, by the monotonicity of φ, we obtain $0 < \|\{n : x_n \in U_k\}\|_\varphi \downarrow 0$, i.e., $\mathfrak{u}(\ell) = 0$, which implies, thanks to Theorem 2.2, that $\ell \notin \Lambda_x(\mathcal{I}_\varphi)$. In particular, $\mathcal{I}_\varphi$ is not closed.

(iv) $\Longrightarrow$ (i) Lastly, assume that the ideal $\mathcal{I}_\varphi$ is not an F_σ-ideal. According to the proof of [18, Theorem 3.4], cf. also [17, pp. 342–343], this is equivalent to the existence, for each given $\varepsilon > 0$, of some set $M \subseteq \mathbf{N}$ such that $0 < \|M\|_\varphi \le \varphi(M) < \varepsilon$. This allows us to define recursively a sequence of sets (M_n) such that

$$\|M_n\|_\varphi > \sum_{k\ge n+1} \varphi(M_k) > 0. \tag{6}$$

for all n and, in addition, $\sum_k \varphi(M_k) < \varphi(\mathbf{N})$. Then, it is claimed that there exists a partition $\{A_n : n \in \mathbf{N}\}$ of $\mathbf{N}$ such that $\|A_n\|_\varphi > 0$ for all n and $\lim_n \|\bigcup_{k>n} A_k\|_\varphi = 0$. To this aim, set $M_0 := \mathbf{N}$ and define $A_n := M_{n-1} \setminus \bigcup_{k\ge n} M_k$ for all $n \in \mathbf{N}$. It follows by the subadditivity and monotonicity of φ that

$$\varphi(M_{n-1} \setminus \{1, \dots, k\}) \le \varphi(A_n \setminus \{1, \dots, k\}) + \varphi\left(\textstyle\bigcup_{k\ge n} M_k\right)$$

for all $n, k \in \mathbf{N}$; hence, by the lower semicontinuity of φ and (6),

$$\|A_n\|_\varphi \ge \|M_{n-1}\|_\varphi - \varphi\left(\textstyle\bigcup_{k\ge n} M_k\right) \ge \|M_{n-1}\|_\varphi - \sum_{k\ge n} \varphi(M_k) > 0$$

for all $n \in \mathbf{N}$. Finally, again by the lower semicontinuity of φ, we get

$$\|\textstyle\bigcup_{k>n} A_k\|_\varphi = \|\bigcup_{k\ge n} M_k\|_\varphi \le \varphi\left(\bigcup_{k\ge n} M_k\right) \le \sum_{k\ge n} \varphi(M_k)$$

which goes to 0 as $n \to \infty$. This concludes the proof. $\square$

It is worth noting that the proof of the implication (iv) $\Longrightarrow$ (i) did not use the properties of the underlying space X. Indeed, conditions (i) and (iv) are equivalent:

Corollary 2.6. *Let $\mathcal{I}_\varphi$ be an analytic P-ideal. Then $\mathcal{I}_\varphi$ is an F_σ-ideal if and only if there does not exist a partition $\{A_n : n \in \mathbf{N}\}$ of $\mathbf{N}$ such that $\|A_n\|_\varphi > 0$ for all n and* $\lim_n \|\bigcup_{k>n} A_k\|_\varphi = 0$.

Proof. Thanks to Theorem 2.5 and the above comment, we only need to show that "only if" part. To this aim, let $\mathcal{I}_\varphi$ be an F_σ P-ideal and let $\{A_n : n \in \mathbf{N}\}$ be a partition of $\mathbf{N}$ such that $\|A_n\|_\varphi > 0$ for all n. According to the proof of [18, Theorem 3.4], there exists $\varepsilon > 0$ such that, for all sets $M \subseteq \mathbf{N}$, it holds either $\|M\|_\varphi = 0$ or $\varphi(M) \ge \varepsilon$. This implies that $\|A_n\|_\varphi \ge \varepsilon$ for all n. In particular, considering (2), we have

$$\varphi(A_n \setminus \{1, \dots, k\}) \ge \|A_n \setminus \{1, \dots, k\}\|_\varphi = \|A_n\|_\varphi > 0$$

for all $n, k \in \mathbf{N}$, so that $\varphi(A_n \setminus \{1, \dots, k\}) \ge \varepsilon$. This implies that $\|A_n\|_\varphi \ge \varepsilon$ for all n. Therefore $\lim_n \|\bigcup_{k>n} A_k\|_\varphi \ge \varepsilon > 0$. $\square$

At this point, thanks to Theorem 2.2 and Theorem 2.5, observe that, if X is a first countable space which has a non-isolated point and $\mathcal{I}_\varphi$ is an analytic P-ideal which is not F_σ, then there exists a sequence x such that $\Lambda_x(\mathcal{I}_\varphi)$ is a non-closed F_σ-set. Indeed, all the F_σ-sets can be obtained:

Theorem 2.7. *Let X be a topological space where all closed sets are separable. Fix also an analytic P-ideal $\mathcal{I}_\varphi$ which is not F_σ and let $B \subseteq X$ be a non-empty F_σ-set. Then there exists a sequence x such that $\Lambda_x(\mathcal{I}_\varphi) = B$.*

Proof. Let (B_k) be a sequence of non-empty closed sets such that $\bigcup_k B_k = B$. Let also $\{b_{k,n} : n \in \mathbf{N}\}$ be a countable dense subset of B_k. Thanks to Corollary 2.6, there exists a partition $\{A_n : n \in \mathbf{N}\}$ of $\mathbf{N}$ such that $\|A_n\|_\varphi > 0$ for all n and $\lim_n \|\bigcup_{k>n} A_k\|_\varphi = 0$. Moreover, for each $k \in \mathbf{N}$, set $\theta_{k,0} := 0$ and it is easily seen that there exists an increasing sequence of positive integers $(\theta_{k,n})$ such that

$$\varphi(A_k \cap (\theta_{k,n-1}, \theta_{k,n}]) \geq \frac{1}{2}\|A_k \setminus \{1, \ldots, \theta_{k,n-1}\}\|_\varphi = \frac{1}{2}\|A_k\|_\varphi$$

for all n. Hence, setting $A_{k,n} := A_k \cap \bigcup_{m \in A_n} (\theta_{k,m-1}, \theta_{k,m}]$, we obtain that $\{A_{k,n} : n \in \mathbf{N}\}$ is a partition of A_k such that $\frac{1}{2}\|A_k\|_\varphi \leq \|A_{k,n}\|_\varphi \leq \|A_k\|_\varphi$ for all n, k.

At this point, let $x = (x_n)$ be defined by $x_n = b_{k,m}$ whenever $n \in A_{k,m}$. Fix $\ell \in B$, then there exists $k \in \mathbf{N}$ such that $\ell \in B_k$. Let (b_{k,r_m}) be a sequence in B_k converging to ℓ. Thus, set $\tau_0 := 0$ and let (τ_m) be an increasing sequence of positive integers such that $\varphi(A_{k,r_m} \cap (\tau_{m-1}, \tau_m]) \geq \frac{1}{2}\|A_{k,r_m}\|_\varphi$ for each m. Setting $A := \bigcup_m A_{k,r_m} \cap (\tau_{m-1}, \tau_m]$, it follows by construction that $\lim_{n\to\infty, n\in A} x_n = \ell$ and $\|A\|_\varphi \geq \frac{1}{4}\|A_k\|_\varphi > 0$. This shows that $B \subseteq \Lambda_x(\mathcal{I}_\varphi)$.

To complete the proof, fix $\ell \notin B$ and let us suppose for the sake of contradiction that there exists $A \subseteq \mathbf{N}$ such that $\lim_{n\to\infty, n\in A} x_n = \ell$ and $\|A\|_\varphi > 0$. For each $m \in \mathbf{N}$, let U_m be an open neighborhood of ℓ which is disjoint from the closed set $B_1 \cup \cdots B_m$. It follows by the subadditivity and the monotonicity of φ that there exists a finite set Y such that

$$\|A\|_\varphi \leq \|Y\|_\varphi + \|\{n \in A : x_n \notin B_1 \cup \cdots \cup B_m\}\|_\varphi \leq \|\textstyle\bigcup_{k>m} A_k\|_\varphi.$$

The claim follows by the arbitrariness of m and the fact that $\lim_m \|\bigcup_{k>m} A_k\|_\varphi = 0$. $\square$

Note that every analytic P-ideal without the Bolzano-Weierstrass property cannot be F_σ, see [8, Theorem 4.2]. Hence Theorem 2.7 applies to this class of ideals.

It was shown in [5, Theorem 2.8 and Theorem 2.10] that if X is a topological space where all closed sets are separable, then for each F_σ-set A and closed set B there exist sequences $a = (a_n)$ and $b = (b_n)$ with values in X such that $\Lambda_a = A$ and $\Gamma_b = B$.

As an application of Theorem 2.2, we prove that, in general, its stronger version with $a = b$ fails (e.g., there are no real sequences x such that $\Lambda_x = \{0\}$ and $\Gamma_x = \{0, 1\}$).

Here, a topological space X is said to be *locally compact* if for every $x \in X$ there exists a neighborhood U of x such that its closure $\overline{U}$ is compact, cf. [6, Section 3.3].

Theorem 2.8. *Let $x = (x_n)$ be a sequence taking values in a locally compact first countable space and fix an analytic P-ideal $\mathcal{I}_\varphi$. Then each isolated $\mathcal{I}_\varphi$-cluster point is also an $\mathcal{I}_\varphi$-limit point.*

Proof. Let us suppose for the sake of contradiction that there exists an isolated $\mathcal{I}_\varphi$-cluster point, let us say ℓ, which is not an $\mathcal{I}_\varphi$-limit point. Let (U_k) be a decreasing local base of open neighborhoods at ℓ such that $\overline{U}_1$ is compact. Let also m be a sufficiently large integer such that $U_m \cap \Gamma_x(\mathcal{I}_\varphi)=\{\ell\}$. Thanks to [6, Theorem 3.3.1] the underlying space is, in particular, regular, hence there exists an integer $r>m$ such that $\overline{U}_r$ is a compact contained in U_m. In addition, since ℓ is an $\mathcal{I}_\varphi$-cluster point and it is not an $\mathcal{I}_\varphi$-limit point, it follows by Theorem 2.2 that

$$0<\|\{n: x_n \in U_k\}\|_\varphi \downarrow \mathfrak{u}(\ell)=0.$$

In particular, there exists $s \in \mathbf{N}$ such that $0<\|\{n: x_n \in U_s\}\|_\varphi<\|\{n: x_n \in U_r\}\|_\varphi$.

Observe that $K:=\overline{U}_r \setminus U_s$ is a closed set contained in $\overline{U}_1$, hence it is compact. By construction we have that $K \cap \Gamma_x(\mathcal{I}_\varphi)=\emptyset$. Hence, for each $z \in K$, there exists an open neighborhood V_z of z such that $V_z \subseteq U_m$ and $\{n: x_n \in V_z\} \in \mathcal{I}_\varphi$, i.e., $\|\{n: x_n \in V_z\}\|_\varphi=0$. It follows that $\bigcup_{z \in K} V_z$ is an open cover of K which is contained in U_m. Since K is compact, there exists a finite set $\{z_1,\ldots,z_t\} \subseteq K$ for which

$$K \subseteq V_{z_1} \cup \cdots \cup V_{z_t} \subseteq U_m. \tag{7}$$

At this point, by the subadditivity of φ, it easily follows that $\|A \cup B\|_\varphi \le \|A\|_\varphi+\|B\|_\varphi$ for all $A, B \subseteq \mathbf{N}$. Hence we have

$$\begin{aligned}\|\{n: x_n \in K\}\|_\varphi &\ge \|\{n: x_n \in \overline{U}_r\}\|_\varphi-\|\{n: x_n \in U_s\}\|_\varphi \\ &\ge \|\{n: x_n \in U_r\}\|_\varphi-\|\{n: x_n \in U_s\}\|_\varphi>0.\end{aligned}$$

On the other hand, it follows by (7) that

$$\|\{n: x_n \in K\}\|_\varphi \le \|\{n: x_n \in \textstyle\bigcup_{i=1}^t V_{z_i}\}\|_\varphi \le \sum_{i=1}^t \|\{n: x_n \in V_{z_i}\}\|_\varphi=0.$$

This contradiction concludes the proof. □

The following corollary is immediate (we omit details):

Corollary 2.9. *Let x be a real sequence for which Γ_x is a discrete set. Then $\Lambda_x=\Gamma_x$.*

3. Joint Converse results

We provide now a kind of converse of Theorem 2.8, specializing to the case of the ideal $\mathcal{I}_0$: informally, if B is a sufficiently smooth closed set and A is an F_σ-set containing the isolated points of B, then there exists a sequence x such that $\Lambda_x=A$ and $\Gamma_x=B$.

To this aim, we need some additional notation: let $\mathrm{d}^\star$, $\mathrm{d}_\star$, and d be the upper asymptotic density, lower asymptotic density, and asymptotic density on $\mathbf{N}$, resp.; in particular, $\mathcal{I}_0=\{S \subseteq \mathbf{N}: \mathrm{d}^\star(S)=0\}$.

Given a topological space X, the interior and the closure of a subset $S \subseteq X$ are denoted by S° and $\overline{S}$, respectively; S is said to be *regular closed* if $S=\overline{S^\circ}$. We let the Borel σ-algebra on X be $\mathcal{B}(X)$. A Borel probability measure $\mu: \mathcal{B}(X) \to [0,1]$ is said to be *strictly positive* whenever $\mu(U)>0$ for all non-empty open sets U. Moreover, μ is *atomless* if, for each measurable set A with $\mu(A)>0$, there exists a measurable subset $B \subseteq A$ such that $0<\mu(B)<\mu(A)$. Then, a sequence (x_n) taking values in X is said to be *μ-uniformly distributed* whenever

$$\mu(F) \ge \mathrm{d}^\star(\{n: x_n \in F\}) \tag{8}$$

for all closed sets F, cf. [9, Section 491B].

Theorem 3.1. *Let X be a separable metric space and fix sets $A \subseteq B \subseteq C \subseteq X$ such that A is an F_σ-set and B, C are closed sets such that the set S of isolated points of B is contained in A and $F := B \setminus S$ is non-empty. Moreover, assume that there exists an atomless strictly positive Borel probability measure $\mu_F : \mathcal{B}(F) \to [0,1]$. Then there exists a sequence x taking values in X such that*

$$\Lambda_x = A, \ \Gamma_x = B, \ \text{ and } \mathrm{L}_x = C. \tag{9}$$

Proof. First, note that by the separability of X, S is at most countable. Let us assume for now that A is non-empty. Since A is an F_σ-set, there exists a sequence (A_k) of non-empty closed sets such that $\bigcup_k A_k = A$. Considering that X is (hereditarily) second countable, then every closed set is separable. Hence, for each $k \in \mathbf{N}$, there exists a countable set $\{a_{k,n} : n \in \mathbf{N}\} \subseteq A_k$ with closure A_k. Considering that F is a separable metric space on its own right, it follows by [9, Exercise 491Xw] that there exists a μ_F-uniformly distributed sequence (b_n) which takes values in F and satisfies (8). Lastly, let $\{c_n : n \in \mathbf{N}\}$ be a countable dense subset of C.

At this point, let $\mathscr{C}$ be the set of non-zero integer squares and note that $\mathrm{d}(\mathscr{C}) = 0$. For each $k \in \mathbf{N}$ define $\mathscr{A}_k := \{2^k n : n \in \mathbf{N} \setminus 2\mathbf{N}\} \setminus \mathscr{C}$ and $\mathscr{B} := \mathbf{N} \setminus (2\mathbf{N} \cup \mathscr{C})$. It follows by construction that $\{\mathscr{A}_k : k \in \mathbf{N}\} \cup \{\mathscr{B}, \mathscr{C}\}$ is a partition of $\mathbf{N}$. Moreover, each $\mathscr{A}_k$ admits asymptotic density and

$$\lim\nolimits_{n \to \infty} \mathrm{d}\left(\bigcup\nolimits_{k \ge n} \mathscr{A}_k\right) = 0. \tag{10}$$

Finally, for each positive integer k, let $\{\mathscr{A}_{k,m} : m \in \mathbf{N}\}$ be the partition of $\mathscr{A}_k$ defined by $\mathscr{A}_{k,1} := \mathscr{A}_k \cap \bigcup_{n \in \mathscr{A}_1 \cup \mathscr{B} \cup \mathscr{C}} [n!, (n+1)!)$ and $\mathscr{A}_{k,m} := \mathscr{A}_k \cap \bigcup_{n \in \mathscr{A}_m} [n!, (n+1)!)$ for all integers $m \ge 2$. Then, it is easy to check that

$$\mathrm{d}^\star(\mathscr{A}_{k,1}) = \mathrm{d}^\star(\mathscr{A}_{k,2}) = \cdots = \mathrm{d}(\mathscr{A}_k) = 2^{-k-1}.$$

Hence define the sequence $x = (x_n)$ by

$$x_n = \begin{cases} a_{k,m} & \text{if } n \in \mathscr{A}_{k,m}, \\ b_m & \text{if } n \text{ is the } m\text{-th term of } \mathscr{B}, \\ c_m & \text{if } n \text{ is the } m\text{-th term of } \mathscr{C}. \end{cases} \tag{11}$$

To complete the proof, let us verify that (9) holds true:

Claim (i): $\mathrm{L}_x = C$. Note that $x_n \in C$ for all $n \in \mathbf{N}$. Since C is closed by hypothesis, then $\mathrm{L}_x \subseteq C$. On the other hand, if $\ell \in C$, then there exists a sequence (c_n) taking values in C converging (in the ordinary sense) to ℓ. It follows by the definition of (x_n) that there exists a subsequence (x_{n_k}) converging to ℓ, i.e., $C \subseteq \mathrm{L}_x$.

Claim (ii): $\Gamma_x = B$. Fix $\ell \notin B$ and let U be an open neighborhood of ℓ disjoint from B (this is possible since, in the opposite, ℓ would belong to $\overline{B} = B$). Then, $\{n : x_n \in U\} \subseteq \mathscr{C}$, which implies that $\Gamma_x \subseteq B$.

At this point, fix $\ell \in F$ and let V be a open neighborhood of ℓ (relative to F). Since (b_n) is μ_F-uniformly distributed and μ_F is strictly positive, it follows by (8) that

$$\begin{aligned} 0 < \mu_F(V) = 1 - \mu_F(V^c) &\le 1 - \mathrm{d}^\star(\{n : b_n \in V^c\}) \\ &= \mathrm{d}_\star(\{n : b_n \in V\}) \le \mathrm{d}^\star(\{n : b_n \in V\}). \end{aligned}$$

Since $\mathrm{d}(\mathscr{B}) = 1/2$, we obtain by standard properties of $\mathrm{d}^\star$ that

$$\mathrm{d}^\star(\{n : x_n \in V\}) \geq \mathrm{d}^\star(\{n \in \mathscr{B} : x_n \in V\}) = \frac{1}{2}\mathrm{d}^\star(\{n : b_n \in V\}) > 0.$$

We conclude by the arbitrariness of V and ℓ that $F \subseteq \Gamma_x$.

Hence we miss only to show that $S \subseteq \Gamma_x$. To this aim, fix $\ell \in S$, thus ℓ is also an isolated point of A. Hence there exist $k, m \in \mathbf{N}$ such that $a_{k,m} = \ell$. We conclude that $\mathrm{d}^\star(\{n : x_n \in U\}) \geq \mathrm{d}^\star(\{n : x_n = \ell\}) \geq \mathrm{d}(\mathscr{A}_k) > 0$ for each neighborhood U of ℓ. Therefore $B = F \cup S \subseteq \Gamma_x$.

CLAIM (III): $\Lambda_x = A$. Fix $\ell \in A$, hence there exists $k \in \mathbf{N}$ for which ℓ belongs to the (non-empty) closed set A_k. Since $\{a_{k,n} : n \in \mathbf{N}\}$ is dense in A_k, there exists a sequence $(a_{k,r_m} : m \in \mathbf{N})$ converging to ℓ. Recall that $x_n = a_{k,r_m}$ whenever $n \in \mathscr{A}_{k,r_m}$ for each $m \in \mathbf{N}$. Set by convenience $\theta_0 := 0$ and define recursively an increasing sequence of positive integers (θ_m) such that θ_m is an integer greater than θ_{m-1} for which

$$\mathrm{d}^\star\left(\mathscr{A}_{k,r_m} \cap (\theta_{m-1}, \theta_m]\right) \geq \frac{\mathrm{d}(\mathscr{A}_k)}{2} = 2^{-k-2}.$$

Then, setting $\mathcal{A} := \bigcup_m \mathscr{A}_{k,r_m} \cap (\theta_{m-1}, \theta_m]$, we obtain that the subsequence $(x_n : n \in \mathcal{A})$ converges to ℓ and $\mathrm{d}^\star(\mathcal{A}) > 0$. In particular, $A \subseteq \Lambda_x$.

On the other hand, it is known that $\Lambda_x \subseteq \Gamma_x$, see e.g. [10]. If $A = B$, it follows by Claim (II) that $\Lambda_x \subseteq A$ and we are done. Otherwise, fix $\ell \in B \setminus A = F \setminus A$ and let us suppose for the sake of contradiction that there exists a subsequence (x_{n_k}) such that $\lim_k x_{n_k} = \ell$ and $\mathrm{d}^\star(\{n_k : k \in \mathbf{N}\}) > 0$. Fix a real $\varepsilon > 0$. Then, thanks to (10), there exists a sufficiently large integer n_0 such that $\mathrm{d}\left(\bigcup_{k>n_0} \mathscr{A}_k\right) \leq \varepsilon$. In addition, since F is a metric space and μ_F is atomless and strictly positive (see Claim (II)), we have

$$\lim_{n\to\infty} \mu_F(V_n) = \mu_F(\{\ell\}) = 0,$$

where V_n is the open ball (relative to F) with center ℓ and radius $1/n$. Hence, there exists a sufficiently large integer m' such that $0 < \mu_F(V_{m'}) \leq \varepsilon$. In addition, there exists an integer m'' such that $V_{m''}$ is disjoint from the closed set $A_1 \cup \cdots \cup A_{n_0}$. Then set $V := V_m$ where m is an integer greater than $\max(m', m'')$ such that $\mu_F(V) < \mu_F(V_{\max(m',m'')})$. In particular, by the monotonicity of μ_F, we have

$$0 < \mu_F(V) \leq \mu_F(\overline{V}) \leq \mu_F(V_{m'}) \leq \varepsilon. \tag{12}$$

At this point, observe there exists a finite set Y such that

$$\begin{aligned}\{n_k : k \in \mathbf{N}\} &= \{n_k : x_{n_k} \in V\} \cup Y \\ &\subseteq \left(\textstyle\bigcup_{k>n_0} \mathscr{A}_k\right) \cup \{n \in \mathscr{B} : x_n \in V\} \cup \mathscr{C} \cup Y.\end{aligned}$$

Therefore, by the subadditivity of $\mathrm{d}^\star$, (8), and (12), we obtain

$$\begin{aligned}\mathrm{d}^\star(\{n_k : k \in \mathbf{N}\}) &\leq \varepsilon + \mathrm{d}^\star(\{n \in \mathscr{B} : x_n \in V\}) \leq \varepsilon + \mathrm{d}^\star(\{n \in \mathscr{B} : b_n \in V\}) \\ &\leq \varepsilon + \mathrm{d}^\star(\{n \in \mathscr{B} : b_n \in \overline{V}\}) \leq \varepsilon + \mu_F(\overline{V}) \leq 2\varepsilon.\end{aligned}$$

It follows by the arbitrariness of ε that $\mathrm{d}(\{n_k : k \in \mathbf{N}\}) = 0$, i.e., $\Lambda_x \subseteq A$.

To complete the proof, assume now that $A=\emptyset$. In this case, note that necessarily $S=\emptyset$, and it is enough to replace (11) with

$$x_n = \begin{cases} b_{n-\lfloor\sqrt{n}\rfloor} & \text{if } n \notin \mathscr{C}, \\ c_{\sqrt{n}} & \text{if } n \in \mathscr{C}. \end{cases}$$

Then, it can be shown with a similar argument that $\Lambda_x=\emptyset$, $\Gamma_x=B$, and $\mathrm{L}_x=C$. □

It is worth noting that Theorem 3.1 cannot be extended to the whole class of analytic P-ideals. Indeed, it follows by Theorem 2.3 that if $\mathcal{I}$ is an F_σ ideal on $\mathbf{N}$ then the set of $\mathcal{I}$-limit points is closed set, cf. also Theorem 3.4 below.

In addition, under suitable hypotheses on F, it is possible to provide sufficient conditions for the existence of μ_F:

Corollary 3.2. *Let X be a separable metric space and assume that there exists an atomless strictly positive Borel probability measure $\mu:\mathcal{B}(X)\to[0,1]$. Fix also sets $A\subseteq B\subseteq C\subseteq X$ such that A is an F_σ-set, and B,C are closed sets such that: (i) $\mu(B)>0$, (ii) the set S of isolated points of B is contained in A, and (iii) $F:=B\setminus S$ is regular closed. Then there exists a sequence x taking values in X which satisfies (9).*

Proof. Thanks to Theorem 3.1, it is sufficient to show that there exists an atomless strictly positive Borel probability measure $\mu_F:\mathcal{B}(F)\to[0,1]$. Since S is at most countable, then $\mu(S)=0$; hence $\mu(F)=\mu(B)>0$. At this point, define the Borel probability measure

$$\mu_F:\mathcal{B}(F)\to[0,1]:Y\mapsto\frac{1}{\mu(F)}\mu(Y).$$

Note that μ_F is clearly atomless. Lastly, given an open set $U\subseteq X$ with non-empty intersection with F, then $U\cap F^\circ\neq\emptyset$: indeed, in the opposite, we would have $F^\circ\subseteq U^c$, which is closed, hence $F=\overline{F^\circ}\subseteq U^c$, contradicting our hypothesis. This proves that every non-empty open set V (relative to F) contains a non-empty open set of X. Therefore μ_F is also strictly positive. □

Finally, the completeness of X is another sufficient condition for the existence of μ_F:

Corollary 3.3. *Let X be a Polish space and fix sets $A\subseteq B\subseteq C\subseteq X$ such that A is an F_σ-set and B,C are closed sets such that the set S of isolated points of B is contained in A and $F:=B\setminus S$ is non-empty. Then there exists a sequence x taking values in X which satisfies (9).*

Proof. First, observe that the restriction $\tilde{\lambda}$ of the Lebesgue measure λ on the set $\mathscr{I}:=(0,1)\setminus\mathbf{Q}$ is an atomless strictly positive Borel probability measure. At this point, F is a perfect Polish space on its own right. Thanks to [6, Exercise 6.2.A(e)], F contains a dense subspace D which is homeomorphic to $\mathbf{R}\setminus\mathbf{Q}$, which is turn is homeomorphic to $\mathscr{I}$, let us say through $\eta:D\to\mathscr{I}$. This embedding can be used to transfer the measure $\tilde{\lambda}$ to the target space by setting

$$\mu_F:\mathcal{B}(F)\to[0,1]:Y\mapsto\tilde{\lambda}(\eta(Y\cap D)).$$

Note that μ_F is a Borel probability measure. Moreover, since $\tilde{\lambda}$ is atomless then μ_F is atomless too. Lastly, considering that η is an open map, then for each non-empty open set U (relative to F) we get $\mu_F(U) = \tilde{\lambda}(\eta(U \cap D)) > 0$. Therefore μ_F is strictly positive. The claim follows by Theorem 3.1. □

Note that, in general, the condition $B \neq \emptyset$ cannot be dropped: indeed, it follows by [5, Theorem 2.14] that, if X is compact, then every sequence (x_n) admits at least one statistical cluster point.

We conclude with another converse result related to ideals $\mathcal{I}$ of the type F_σ (recall that, thanks to Theorem 2.3, every $\mathcal{I}$-limit point is also an $\mathcal{I}$-cluster point):

Theorem 3.4. *Let X be a first countable space where all closed sets are separable and let $\mathcal{I} \neq \mathrm{Fin}$ be an F_σ-ideal. Fix also closed sets $B, C \subseteq X$ such that $\emptyset \neq B \subseteq C$. Then there exists a sequence x such that $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I}) = B$ and $\mathrm{L}_x = C$.*

Proof. By hypothesis, there exists an infinite set $I \in \mathcal{I}$. Let φ be a lower semicontinuous submeasures associated to $\mathcal{I}$ as in (5). Let $\{b_n : n \in \mathbf{N}\}$ and $\{c_n : n \in \mathbf{N}\}$ be countable dense subsets of B and C, respectively. In addition, set $m_0 := 0$ and let (m_k) be an increasing sequence of positive integers such that $\varphi((\mathbf{N} \setminus I) \cap (m_{k-1}, m_k]) \geq k$ for all k (note that this is possible since $\varphi(\mathbf{N} \setminus I) = \infty$ and φ is a lower semicontinuous submeasure). At this point, given a partition $\{H_n : n \in \mathbf{N}\}$ of $\mathbf{N} \setminus I$, where each H_n is infinite, we set

$$M_k := (\mathbf{N} \setminus I) \cap \bigcup\nolimits_{n \in H_k} (m_{n-1}, m_n]$$

for all $k \in \mathbf{N}$. Then it is easily checked that $\{M_k : k \in \mathbf{N}\}$ is a partition of $\mathbf{N} \setminus I$ with $M_k \notin \mathcal{I}$ for all k, and that the sequence (x_n) defined by

$$x_n = \begin{cases} b_k & \text{if } n \in M_k, \\ c_k & \text{if } n \text{ is the } k\text{-th term of } I. \end{cases}$$

satisfies the claimed conditions. □

In particular, Theorem 2.7 and Theorem 3.4 fix a gap in a result of Das [3, Theorem 3] and provide its correct version.

4. Concluding remarks

In this last section, we are interested in the topological nature of the set of $\mathcal{I}$-limit points when $\mathcal{I}$ is neither F_σ- nor analytic P-ideal.

Let $\mathcal{N}$ be the set of strictly increasing sequences of positive integers. Then $\mathcal{N}$ is a Polish space, since it is a closed subspace of the Polish space $\mathbf{N}^{\mathbf{N}}$ (equipped with the product topology of the discrete topology on $\mathbf{N}$). Let also $x = (x_n)$ be a sequence taking values in a first countable regular space X and fix an arbitrary ideal $\mathcal{I}$ on $\mathbf{N}$. For each $\ell \in X$, let $(U_{\ell,m})$ be a decreasing local base of open neighborhoods at ℓ. Then, ℓ is an $\mathcal{I}$-limit point of x if and only if there exists a sequence $(n_k) \in \mathcal{N}$ such that

$$\{n_k : k \in \mathbf{N}\} \notin \mathcal{I} \quad \text{and} \quad \{k : x_{n_k} \notin U_{\ell,m}\} \in \mathrm{Fin} \text{ for all } m. \tag{13}$$

Set $\mathcal{I}^c := \mathcal{P}(\mathbf{N}) \setminus \mathcal{I}$ and define the continuous function

$$\psi : \mathcal{N} \to \{0,1\}^{\mathbf{N}} : (n_k) \mapsto \chi_{\{n_k : k \in \mathbf{N}\}},$$

where χ_S is the characteristic function of a set $S \subseteq \mathbf{N}$. Moreover, for each m, define the function $\zeta_m : \mathcal{N} \times X \to \{0,1\}^{\mathbf{N}}$ as follows: given $j \in \mathbf{N}$, set $\zeta_m((n_k,\ell))(j) = 1$ if $x_{n_j} \notin U_{\ell,m}$, and $\zeta_m((n_k,\ell))(j) = 0$ otherwise. Hence it easily follows by (13) that

$$\Lambda_x(\mathcal{I}) = \pi_X\left(\bigcap_m \left(\psi^{-1}(\mathcal{I}^c) \times X\right) \cap \zeta_m^{-1}(\mathrm{Fin})\right),$$

where $\pi_X : \mathcal{N} \times X \to X$ stands for the projection on X.

Proposition 4.1. *Let $x = (x_n)$ be a sequence taking values in a first countable regular space X and let $\mathcal{I}$ be a co-analytic ideal. Then $\Lambda_x(\mathcal{I})$ is analytic.*

Proof. For each $(n_k) \in \mathcal{N}$ and $\ell \in X$, the sections $\zeta_m((n_k),\cdot)$ and $\zeta_m(\cdot,\ell)$ are continuous. Hence, thanks to [19, Theorem 3.1.30], each function ζ_m is Borel measurable. Since Fin is an F_σ-set, we obtain that each $\zeta_m^{-1}(\mathrm{Fin})$ is Borel. Moreover, since $\mathcal{I}$ is a co-analytic ideal and ψ is continuous, it follows that $\psi^{-1}(\mathcal{I}^c)\times X$ is an analytic subset of $\mathcal{N}\times X$. Therefore $\Lambda_x(\mathcal{I})$ is the projection on X of the analytic set $\bigcap_m \left(\psi^{-1}(I^c) \times X \cap \zeta_m^{-1}(\mathrm{Fin})\right)$, which proves the claim. □

The situation is much different for *maximal ideals*, i.e., ideals which are maximal with respect to inclusion. It is known that if $\mathcal{I}$ is a maximal ideal then every sequence x in a compact space X is $\mathcal{I}$-convergent, i.e., there exists $\ell \in X$ such that $\{n : x_n \notin U\} \in \mathcal{I}$ for every neighborhood U of ℓ and thus $\Gamma_x(\mathcal{I}) = \{\ell\}$. (This can be deduced using the space $\beta\mathbf{N}$, cf. [20, Claim 1, p. 64].) Consequently, $\Lambda_x(\mathcal{I})$ is either empty or a singleton, hence closed.

We conclude by showing that there exist an ideal $\mathcal{I}$ and a real sequence x such that $\Lambda_x(\mathcal{I})$ is not an F_σ-set.

Example 4.2. Fix a partition $\{P_m : m \in \mathbf{N}\}$ of $\mathbf{N}$ such that each P_m is infinite. Then, define the ideal

$$\mathcal{I} := \{A \subseteq \mathbf{N} : \{m : A \cap P_m \notin \mathrm{Fin}\} \in \mathrm{Fin}\},$$

which corresponds to the Fubini product Fin × Fin on $\mathbf{N}^2$ (it is known that $\mathcal{I}$ is an $F_{\sigma\delta\sigma}$-ideal and it is not a P-ideal). Given a real sequence $x = (x_n)$, let us denote by $x \upharpoonright P_m$ the subsequence $(x_n : n \in P_m)$. Hence, a real ℓ is an $\mathcal{I}$-limit point of x if and only if there exists a subsequence (x_{n_k}) converging to ℓ such that $\{n_k : k \in \mathbf{N}\} \cap P_m$ is infinite for infinitely many m. Moreover, for each m of this type, the subsequence $(x_{n_k}) \upharpoonright P_m$ converges to ℓ. It easily follows that

$$\Lambda_x(\mathcal{I}) = \bigcap_k \bigcup_{m \ge k} \mathrm{L}_{x\upharpoonright P_m}. \tag{14}$$

(In particular, since each $\mathrm{L}_{x\upharpoonright P_m}$ is closed, then $\Lambda_x(\mathcal{I})$ is an $F_{\sigma\delta}$-set.)

At this point, let $(q_t : t \in \mathbf{N})$ be the sequence $(0/1, 1/1, 0/2, 1/2, 2/2, 0/3, 1/3, 2/3, 3/3, \ldots)$, where $q_t := a_t/b_t$ for each t, and note that $\{q_t : t \in \mathbf{N}\} = \mathbf{Q} \cap [0,1]$. It follows by construction that $a_t \le b_t$ for all t and $b_t = \sqrt{2t}(1+o(1))$ as $t \to \infty$. In particular, if m is a sufficiently large integer, then

$$\min_{i \le m:\, q_i \ne q_m} |q_i - q_m| \ge \left(\frac{1}{\sqrt{2m}(1+o(1))}\right)^2 > \frac{1}{3m}. \tag{15}$$

Lastly, for each $m \in \mathbf{N}$, define the closed set

$$C_m := [0,1] \cap \bigcap_{t \le m} \left(q_t - \frac{1}{2^m}, q_t + \frac{1}{2^m}\right)^c.$$

We obtain by (15) that, if m is sufficiently large, let us say $\ge k_0$, then

$$C_m \cup C_{m+1} = [0,1] \cap \bigcap_{t \le m} \left(q_t - \frac{1}{2^{m+1}}, q_t + \frac{1}{2^{m+1}}\right)^c.$$

It follows by induction that

$$C_m \cup C_{m+1} \cup \cdots \cup C_{m+n} = [0,1] \cap \bigcap_{t \le m} \left(q_t - \frac{1}{2^{m+n}}, q_t + \frac{1}{2^{m+n}}\right)^c.$$

for all $n \in \mathbf{N}$. In particular, $\bigcup_{m \ge k} C_m = [0,1] \setminus \{q_1, \ldots, q_k\}$ whenever $k \ge k_0$.

Let x be a real sequence such that each $\{x_n : n \in P_m\}$ is a dense subset of C_m. Therefore, it follows by (14) that

$$\Lambda_x(\mathcal{I}) = \bigcap_k \bigcup_{m \ge k} C_m \subseteq \bigcap_{k \ge k_0} \bigcup_{m \ge k} C_m = \bigcap_{k \ge k_0} [0,1] \setminus \{q_1, \ldots, q_k\} = [0,1] \setminus \mathbf{Q}.$$

On the other hand, if a rational q_t belongs to $\Lambda_x(\mathcal{I})$, then $q_t \in \bigcup_{m \ge k} C_m$ for all $k \in \mathbf{N}$, which is impossible whenever $k \ge t$. This proves that $\Lambda_x(\mathcal{I}) = [0,1] \setminus \mathbf{Q}$, which is not an F_σ-set.

We leave as an open question to determine whether there exists a real sequence x and an ideal $\mathcal{I}$ such that $\Lambda_x(\mathcal{I})$ is not Borel measurable.

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