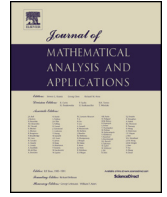




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On generalized limits and ultrafilters

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ABSTRACT

Given an ideal \mathcal{I} on ω , we denote by $SL(\mathcal{I})$ the family of positive normalized linear functionals on ℓ_∞ which assign value 0 to all characteristic sequences of sets in \mathcal{I} . We show that every element of $SL(\mathcal{I})$ is a Choquet average of certain ultrafilter limit functionals. Also, we prove that the diameter of $SL(\mathcal{I})$ is 2 if and only if \mathcal{I} is not maximal, and that the latter claim can be considerably strengthened if \mathcal{I} is meager. Lastly, we provide several applications: for instance, recovering a result of Freedman (1981) [19], we show that the family of bounded sequences for which all functionals in $SL(\mathcal{I})$ assign the same value coincides with the closed vector space of bounded \mathcal{I} -convergent sequences.

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1. Introduction

One of the main purposes of operator theory is to provide representations of certain families of positive normalized linear functionals on the space of bounded sequences. In particular, we show that every functional of this type is a Choquet average of suitable ultrafilter-limit functionals.

For the sake of clarity, we introduce some notation. Let \mathcal{I} be an ideal on the nonnegative integers ω , that is, a family of subsets of ω which is stable under finite unions and subsets. Unless otherwise stated, it is also assumed that $\omega \notin \mathcal{I}$, and that the ideal of finite sets $\text{Fin} := [\omega]^{<\omega}$ is contained in \mathcal{I} . Viewing ideals as subsets of the Cantor space $\{0, 1\}^\omega$ allows us to speak about their descriptive complexity. An ideal \mathcal{I} is a P -ideal if it is σ -directed modulo finite sets, i.e., for every sequence $(A_n : n \in \omega)$ in \mathcal{I} there exists $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all $n \in \omega$. A basic example is the ideal of sets of asymptotic density zero, that is,

$$\mathcal{Z} := \left\{ A \subseteq \omega : \lim_{n \rightarrow \infty} \frac{|A \cap [0, n]|}{n+1} = 0 \right\},$$

which is an $F_{\sigma\delta}$ P -ideal on ω , see e.g. [16].

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We also consider the vector space ℓ_∞ of bounded real sequences and all its subspaces with the supremum norm and the product pointwise order (so that ℓ_∞ is a Banach lattice). We write $e := (1, 1, \dots)$ and $\mathbf{1}_A$ for the characteristic sequence of $A \subseteq \omega$ (hence $e = \mathbf{1}_\omega$).

The following notion has been introduced by Freedman in [19, Section 3]:

Definition 1.1. A linear functional $f : \ell_\infty \rightarrow \mathbf{R}$ is said to be an $S_{\mathcal{I}}$ -limit if:

- (i) f is positive (i.e., $f(x) \geq 0$ for all $x \in \ell_\infty$ with $x \geq 0$);
- (ii) f extends the limit functionals (i.e., $f(x) = \lim x$ for all $x \in c$);
- (iii) $f(\mathbf{1}_A) = 0$ for all $A \in \mathcal{I}$.

The family of $S_{\mathcal{I}}$ -limits is denoted by $SL(\mathcal{I})$.

Special instances of Definition 1.1 can be found also in [32, Definition 1] and [37, Definition 1.1]: in fact, the families \mathcal{I} considered in the latter works are sufficiently “well behaved,” cf. e.g. [5, Proposition 13] and [18,35]. In the case $\mathcal{I} = \mathcal{Z}$, positive linear functionals extending the Cesàro mean (which is stronger than property (ii)) have been studied in [28]. In a different direction, Banach limits are simply elements of $SL(\text{Fin})$ which are, in addition, translation invariant, see e.g. [7,34].

We list below some observations. First, the family $SL(\mathcal{I})$ is nonempty: in fact, if \mathcal{J} is a maximal ideal containing \mathcal{I} , then the linear functional $f_{\mathcal{J}}$ defined by

$$\forall x \in \ell_\infty, \quad f_{\mathcal{J}}(x) := \mathcal{J}\text{-}\lim x \quad (1)$$

is an $S_{\mathcal{I}}$ -limit (it is well known that $f_{\mathcal{J}}$ is well defined).

Second, properties (i)–(iii) in Definition 1.1 are independent of each other. To this aim, if $\mathcal{I} \neq \text{Fin}$, pick an infinite $A \in \mathcal{I}$ (hence, A is not cofinite) and let \mathcal{J} be a maximal ideal containing $\text{Fin} \cup \{\omega \setminus A\}$. Then the linear operator $f_{\mathcal{J}}$ defined in (1) satisfies (i) and (ii), but property (iii) fails. Also, trivially, $f = 0$ satisfies (i) and (iii), but property (ii) fails. Lastly, to prove the independence of property (i), suppose that \mathcal{I} is not maximal, hence it is possible to pick two distinct maximal ideals $\mathcal{J}_1, \mathcal{J}_2$ containing \mathcal{I} . Fix $A \in \mathcal{J}_1 \setminus \mathcal{J}_2$. Then $f := 2f_{\mathcal{J}_1} - f_{\mathcal{J}_2}$ satisfies (ii) and (iii), while property (i) fails since $f(\mathbf{1}_A) = -1$.

Third, it is easy to see that, if f is an $S_{\mathcal{I}}$ -limit, then

$$(i') \quad f \text{ is continuous and } \|f\| = f(e) = 1;$$

cf. e.g. [2, Theorem 4.3]. In fact, also the converse holds, namely, if f is a linear functional on ℓ_∞ satisfying (i'), (ii), and (iii), then also (i) holds. To this aim, suppose that there exists $x \in \ell_\infty$ with $x \geq 0$ such that $\alpha := f(x) < 0$. Pick a sufficiently small $t > 0$ such that $\|e - tx\| \in (0, 1]$. Then

$$1 = \|f\| \geq f(e - tx) / \|e - tx\| \geq f(e - tx) = 1 - t\alpha > 1,$$

which provides the required contradiction; cf. also Remark 4.2 below for additional equivalent definitions of $S_{\mathcal{I}}$ -limits. We remark that a linear functional $f : \ell_\infty \rightarrow \mathbf{R}$ satisfying (i') and (ii) has been termed “extended limit” by Bennett and Kalton in [7, Section 2].

With the above premises, following Freedman [19, Section 1], we consider the vector space $\mathcal{V}(\mathcal{I})$ of sequences $x \in \ell_\infty$ for which all $S_{\mathcal{I}}$ -limits assign the same value:

Definition 1.2. Given an ideal \mathcal{I} on ω , define

$$\mathcal{V}(\mathcal{I}) := \{x \in \ell_\infty : f(x) = g(x) \text{ for all } f, g \in SL(\mathcal{I})\}.$$

Given an ideal \mathcal{I} on ω , a (not necessarily bounded) real sequence $x \in \mathbf{R}^\omega$ is said to be \mathcal{I} -convergent to $\eta \in \mathbf{R}$, shortened as $\mathcal{I}\text{-}\lim x = \eta$, if $\{n \in \omega : |x_n - \eta| \geq \varepsilon\} \in \mathcal{I}$ for all $\varepsilon > 0$. The vector space of \mathcal{I} -convergent sequences is denoted by $c(\mathcal{I})$. Let also $c_{00}(\mathcal{I})$ be the set of sequences which are supported on \mathcal{I} .

Similarly, a real sequence $x \in \mathbf{R}^\omega$ is said to be \mathcal{I}^* -convergent to $\eta \in \mathbf{R}$, shortened as $\mathcal{I}^*\text{-}\lim x = \eta$, if there exists $A \in \mathcal{I}$ such that the subsequence $(x_n : n \in \omega \setminus A)$ is convergent (in the ordinary sense) to η . The vector space of \mathcal{I}^* -convergent sequences is denoted by $c(\mathcal{I}^*)$. It is folklore that \mathcal{I}^* convergence is stronger than \mathcal{I} -convergence, and that they coincide if and only if \mathcal{I} is a P -ideal.

Freedman proved in [19] that $\mathcal{V}(\mathcal{I})$ coincides with the closure of the vector space of bounded \mathcal{I}^* -convergent sequences, namely, $\mathcal{V}(\mathcal{I}) = \overline{c(\mathcal{I}^*)} \cap \ell_\infty$. Here, as an application of a representation of $S_{\mathcal{I}}$ -limits as Choquet averages of the functionals $f_{\mathcal{J}}$ defined in (1) (see Theorem 2.1 below for details), we recover Freedman’s result through a simpler and direct proof.

Theorem 1.3. *Let \mathcal{I} be an ideal on ω . Then $\mathcal{V}(\mathcal{I}) = \overline{c(\mathcal{I}^*)} \cap \ell_\infty = c(\mathcal{I}) \cap \ell_\infty$. If, in addition, \mathcal{I} is a P -ideal, then also $\mathcal{V}(\mathcal{I}) = c + (c_{00}(\mathcal{I}) \cap \ell_\infty)$.*

In our main results, we will provide characterizations of $SL(\mathcal{I})$, some structural properties, and related results. The proof of Theorem 1.3 is given at the end of Section 4.

2. Main results

Consider the Stone–Čech compactification $\beta\omega$ of the nonnegative integers ω , and recall that it is homeomorphic to the space of ultrafilters \mathcal{F} on ω , which we still denote by $\beta\omega$ and is topologized by the base of clopen subsets $\{\{\mathcal{F} \in \beta\omega : A \in \mathcal{F}\} : A \subseteq \omega\}$. For each ultrafilter \mathcal{F} , we write $\mathcal{J}_{\mathcal{F}} := \{A \subseteq \omega : \omega \setminus A \in \mathcal{F}\}$ for its associated maximal ideal. To ease the notation, we will use $\mathcal{J}_{\mathcal{F}}\text{-}\lim x$ or $\mathcal{F}\text{-}\lim x$ interchangeably, and, similarly, $f_{\mathcal{F}}$ in place of $f_{\mathcal{J}_{\mathcal{F}}}$. Also, by $Ult(\mathcal{I})$ we denote the compact subspace of free ultrafilters which contain the dual filter of a given ideal \mathcal{I} , that is, equivalently,

$$Ult(\mathcal{I}) := \{\mathcal{F} \in \beta\omega : \mathcal{I} \subseteq \mathcal{J}_{\mathcal{F}}\}.$$

The space $Ult(\mathcal{I})$ is endowed with its relative topology. Note that the subspace $Ult(\mathcal{I})$, sometimes called “support set,” was introduced by Henriksen [22] and further studied in [3,13,14,33] in the context of ideals generated by nonnegative regular summability matrices, cf. also [5,10,24].

In our first representation of $SL(\mathcal{I})$, we need to recall the notion of Choquet integral. Given a measurable space (S, Σ) and a normalized capacity $\nu : \Sigma \rightarrow \mathbf{R}$ (that is, $\nu(\emptyset) = 0$ and $\nu(A) \leq \nu(B)$ for all $A, B \in \Sigma$ with $A \subseteq B$), we define the *Choquet integral* of a bounded Σ -measurable function $x : S \rightarrow \mathbf{R}$ with respect to ν as the quantity

$$\int x \, d\nu := \int_0^\infty \nu(x \geq t) \, dt + \int_{-\infty}^0 [\nu(x \geq t) - \nu(S)] \, dt,$$

where the integrals on the right hand side are meant to be improper Riemann integrals (which exist since the map $t \mapsto \nu(x \geq t)$ is monotone). This naturally generalizes the standard notion of integral since the two coincide when ν is finitely additive, see [10] and references therein. It is worth remarking that the Choquet functional $x \mapsto \int x \, d\nu$ is the unique translation invariant extension of the map $x \mapsto \int_0^\infty \nu(x \geq t) \, dt$ on the subfamily of positive bounded Σ -measurable functions x , see [30]. Moreover, it is well known that the Choquet functional is monotone, normalized, positive homogeneous, and comonotonic additive (which means that $\int (x+y) \, d\nu = \int x \, d\nu + \int y \, d\nu$ whenever $(x(s) - x(t))(y(s) - y(t)) \geq 0$ for all $s, t \in S$). Nonadditive

set functions and their integrals are widely used in applications: for applications in economics, probability, and statistics, see [11,29,30,36].

In addition, given a topological space X , denote by $\mathcal{B}(X)$ its Borel σ -algebra (recall that if a subspace $Y \subseteq X$ is endowed with its relative topology then $\mathcal{B}(Y) = \{A \cap Y : A \in \mathcal{B}(X)\}$).

Theorem 2.1. *Let \mathcal{I} be an ideal on ω . Then, for each $f \in \text{SL}(\mathcal{I})$, there exists a normalized capacity $\rho : \mathcal{B}(\text{Ult}(\mathcal{I})) \rightarrow \mathbf{R}$ such that*

$$\forall x \in \ell_\infty, \quad f(x) = \int_{\text{Ult}(\mathcal{I})} \mathcal{F}\text{-lim } x \, d\rho(\mathcal{F}). \quad (2)$$

In our second representation of $\text{SL}(\mathcal{I})$, we endow the norm dual ℓ'_∞ with the weak* topology. Hence, for each $Y \subseteq \ell'_\infty$, $\overline{\text{co}}(Y)$ stands for the weak* closed convex hull of Y . Recall also that $f_{\mathcal{F}}(x) := \mathcal{F}\text{-lim } x$ for each $x \in \ell_\infty$ and $\mathcal{F} \in \beta\omega$.

Theorem 2.2. *Let \mathcal{I} be an ideal on ω . Then*

$$\text{SL}(\mathcal{I}) = \overline{\text{co}}(\{f_{\mathcal{F}} : \mathcal{F} \in \text{Ult}(\mathcal{I})\}).$$

Since each $S_{\mathcal{I}}$ -limit has norm one, it follows that the diameter of $\text{SL}(\mathcal{I})$, that is, $\text{diam}(\text{SL}(\mathcal{I})) := \sup\{\|f - g\| : f, g \in \text{SL}(\mathcal{I})\}$ is at most 2. In our next result, we show that this upper bound is optimal in the nonmaximal case.

Theorem 2.3. *Let \mathcal{I} be an ideal on ω . Then $\text{diam}(\text{SL}(\mathcal{I})) = 2$ if and only if \mathcal{I} is not maximal.*

If, in addition, the ideal \mathcal{I} is meager (i.e., it can be regarded as a meager subset of $\{0, 1\}^\omega$), then the above claim can be strengthened.

Theorem 2.4. *Let \mathcal{I} be a meager ideal on ω and fix $f \in \text{SL}(\mathcal{I})$. Then there exists $g \in \text{SL}(\mathcal{I})$ such that $\|f - g\| = 2$.*

In the next result, we characterize the set of differences $\text{SL}(\mathcal{I}) - \text{SL}(\mathcal{I})$:

Theorem 2.5. *Let \mathcal{I} be an ideal on ω and fix $f \in \ell'_\infty$. Then there exist $g, h \in \text{SL}(\mathcal{I})$ such that $f = g - h$ if and only if the following conditions hold:*

- (a) $f(e) = 0$;
- (b) $\|f\| \leq 2$;
- (c) $f(\mathbf{1}_A) = 0$ for all $A \in \mathcal{I}$.

In addition, such decomposition is unique if and only if $\|f\| = 2$ or \mathcal{I} is maximal.

Proofs are given in Section 4. Further results are given in Section 5.

3. Preliminaries

Given an ideal \mathcal{I} on ω and a bounded real sequence $x \in \ell_\infty$, a point $\eta \in \mathbf{R}$ is said to be an \mathcal{I} -cluster point of x if $\{n \in \omega : |x_n - \eta| \leq \varepsilon\} \notin \mathcal{I}$ for all $\varepsilon > 0$ (note that they have been termed *statistical cluster points* if $\mathcal{I} = \mathcal{Z}$, see e.g. [20,21]). For each $x \in \ell_\infty$, we denote by $\Gamma_x(\mathcal{I})$ the set of its \mathcal{I} -cluster points; in addition,

\mathcal{I} - $\lim \inf x := \min \Gamma_x(\mathcal{I})$ and \mathcal{I} - $\lim \sup x := \max \Gamma_x(\mathcal{I})$, which are well defined since $\Gamma_x(\mathcal{I})$ is nonempty compact; see e.g. [27] and references therein for basic properties and characterization of \mathcal{I} -cluster points. Interestingly, the map $x \mapsto \mathcal{I}$ - $\lim \sup x$ provides a complete pseudonorm on ℓ_∞ , see [6,15].

Recall also that each sequence $x \in \ell_\infty$ can be extended uniquely to a continuous function $\hat{x} : \beta\omega \rightarrow \mathbf{R}$ defined by

$$\forall \mathcal{F} \in \beta\omega, \quad \hat{x}(\mathcal{F}) := \mathcal{F}\text{-}\lim x.$$

In particular, if \mathcal{F} is the principal ultrafilter associated with an integer $n \in \omega$, then $\hat{x}(\mathcal{F}) = x_n$. Hereafter, for each $A \subseteq \omega$, we write

$$\tilde{A} := \{ \mathcal{F} \in \beta\omega : A \in \mathcal{F} \text{ and } \mathcal{F} \text{ free ultrafilter} \}.$$

Lemma 3.1. *Let \mathcal{I} be an ideal on ω . Then, for each $A \subseteq \omega$, we have $A \in \mathcal{I}$ if and only if $\tilde{A} \cap \text{Ult}(\mathcal{I}) = \emptyset$.*

Proof. Fix $A \subseteq \omega$ and define the dual filter $\mathcal{G} := \{ B \subseteq \omega : \omega \setminus B \in \mathcal{I} \}$. First, suppose that $A \in \mathcal{I}$ and let \mathcal{F} be a free ultrafilter containing A , i.e., $\mathcal{F} \in \tilde{A}$. Then $\omega \setminus A \notin \mathcal{F}$ while, on the other hand, it belongs to \mathcal{G} . Hence $\tilde{A} \cap \text{Ult}(\mathcal{I}) = \emptyset$.

Conversely, suppose that $A \notin \mathcal{I}$. Considering that every finite subfamily of $\{A\} \cup \mathcal{G}$ has intersection not in \mathcal{I} (hence, infinite intersection), there exists a free ultrafilter \mathcal{F} containing A and \mathcal{G} , hence $\tilde{A} \cap \text{Ult}(\mathcal{I}) \neq \emptyset$. \square

Lemma 3.2. *Let \mathcal{I} be an ideal on ω and fix an infinite set $A \subseteq \omega$ with increasing enumeration $(a_n : n \in \omega)$. Then, for each $x \in \ell_\infty$ and $\eta \in \mathbf{R}$, we have $\lim_n x_{a_n} = \eta$ if and only if $\hat{x}[\tilde{A}] = \{\eta\}$.*

Proof. Suppose that $\lim_n x_{a_n} = \eta$ and fix a free ultrafilter \mathcal{F} containing A , i.e., $\mathcal{F} \in \tilde{A}$. Then for each neighborhood U of η there exists $F \in \text{Fin}$ such that $\{n \in \omega : x_n \in U\} \supseteq A \setminus F$, hence it belongs to \mathcal{F} . Therefore $\hat{x}(\mathcal{F}) = \mathcal{F}\text{-}\lim x = \eta$.

Conversely, suppose that $\hat{x}[\tilde{A}] = \{\eta\}$ and let U be a neighborhood of η . Then $\{n \in \omega : x_n \in U\} \in \mathcal{F}$ for each $F \in \tilde{A}$. Hence $\{n \in \omega : x_n \in U\}$ belongs to $\bigcap \tilde{A}$, which is known to be the filter $\{B \subseteq \omega : A \setminus B \in \text{Fin}\}$. We conclude that the subsequence $(x_{a_n} : n \in \omega)$ converges to η . \square

As a consequence, we provide a characterization of \mathcal{I} -cluster points, which generalizes [14, Theorem 2(3)].

Theorem 3.3. *Let \mathcal{I} be an ideal on ω . Then $\Gamma_x(\mathcal{I}) = \hat{x}[\text{Ult}(\mathcal{I})]$ for each $x \in \ell_\infty$.*

Proof. Fix $x \in \ell_\infty$. First, suppose that there exists a free ultrafilter $\mathcal{F} \in \text{Ult}(\mathcal{I})$ such that $\hat{x}(\mathcal{F}) = \eta$, that is, $\mathcal{F}\text{-}\lim x = \eta$. Hence, for each neighborhood U of η , it holds $\{n \in \omega : x_n \in U\} \in \mathcal{F}$. Considering that $\mathcal{F} \cap \mathcal{I} = \emptyset$, we obtain that η is an \mathcal{I} -cluster point of x . Hence $\hat{x}[\text{Ult}(\mathcal{I})] \subseteq \Gamma_x(\mathcal{I})$.

Conversely, suppose that $\eta \in \Gamma_x(\mathcal{I})$ and let (U_k) be a decreasing local base of neighborhoods at η . Then $A_k := \{n \in \omega : x_n \in U_k\} \notin \mathcal{I}$ for each $k \in \omega$. Thanks to Lemma 3.1, for each $k \in \omega$ there exists a free ultrafilter \mathcal{F}_k containing A_k and the dual filter of \mathcal{I} . In particular, $\hat{x}(\mathcal{F}_k) = \eta$ for each $k \in \omega$. Since $(\mathcal{F}_k : k \in \omega)$ is a sequence in the compact space $\text{Ult}(\mathcal{I})$, there is a subsequence of ultrafilters converging to some $\mathcal{F} \in \text{Ult}(\mathcal{I})$. By the continuity of \hat{x} , we conclude that $\hat{x}(\mathcal{F}) = \eta$. Therefore $\Gamma_x(\mathcal{I}) \subseteq \hat{x}[\text{Ult}(\mathcal{I})]$. \square

As a consequence, we recover an intermediate result of Freedman [19, p. 226]:

Corollary 3.4. *Let \mathcal{I} be an ideal on ω and fix $A \subseteq \omega$ with $A \notin \mathcal{I}$. Then there exists $f \in \text{SL}(\mathcal{I})$ such that $f(\mathbf{1}_A) = 1$.*

Proof. Since $A \notin \mathcal{I}$ then 1 is an \mathcal{I} -cluster of the sequence $\mathbf{1}_A$. We obtain by Theorem 3.3 that there exists $\mathcal{F} \in \text{Ult}(\mathcal{I})$ such that $f_{\mathcal{F}}(\mathbf{1}_A) = \mathcal{F}\text{-lim } \mathbf{1}_A = 1$. The claim follows by the fact that $f_{\mathcal{F}} \in \text{SL}(\mathcal{I})$, as anticipated in Section 1. \square

As another consequence, we generalize a claim contained in [37, Theorem 3.2(i)] and [31, Theorem 2.4(i)]:

Corollary 3.5. *Let \mathcal{I} be an ideal on ω . Then*

$$\forall x \in \ell_{\infty}, \quad \{f(x) : f \in \text{SL}(\mathcal{I})\} = \text{co}(\Gamma_x(\mathcal{I})).$$

In particular, $\mathcal{I}\text{-lim inf } x \leq f(x) \leq \mathcal{I}\text{-lim sup } x$ for all $f \in \text{SL}(\mathcal{I})$ and $x \in \ell_{\infty}$.

Proof. It follows by Theorem 2.2 and Theorem 3.3. \square

We conclude with a computation of the norm of $f \in \ell'_{\infty}$ such that $f(e) = 0$.

Lemma 3.6. *Fix $f \in \ell'_{\infty}$ with $f(e) = 0$. Then*

$$\|f\| = 2 \sup\{f(x) : 0 \leq x \leq e\} = 2 \sup\{f(\mathbf{1}_A) : A \subseteq \omega\}.$$

Proof. Set $s := \sup\{f(x) : 0 \leq x \leq e\}$. On the one hand, we have

$$\begin{aligned} \|f\| &= \sup\{f(x) : -e \leq x \leq e\} \\ &= \sup\{f(a-b) : 0 \leq a, b \leq e, a \wedge b = 0\} \\ &\leq \sup\{f(a) + f(-b) : 0 \leq a, b \leq e\} \\ &= \sup\{f(a) + f(e-b) : 0 \leq a, b \leq e\} \\ &= \sup\{f(a) + f(b) : 0 \leq a, b \leq e\} \leq 2s. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} \|f\| &\geq \sup\{f(\mathbf{1}_A - \mathbf{1}_{\omega \setminus A}) : A \subseteq \omega\} \\ &= \sup\{f(\mathbf{1}_A) + f(e - \mathbf{1}_{\omega \setminus A}) : A \subseteq \omega\} \\ &= 2 \sup\{f(\mathbf{1}_A) : A \subseteq \omega\}. \end{aligned}$$

To conclude, it is enough to observe that the equality $\sup\{f(\mathbf{1}_A) : A \subseteq \omega\} = s$ follows from a result of Abramovich, see [2, Theorem 1.50] and [1, p. 541]. \square

4. Proofs of main results

In this section, we provide the proofs of our main results.

Lemma 4.1. *Let \mathcal{I} be an ideal on ω and fix $f \in \text{SL}(\mathcal{I})$ and $A, B \subseteq \omega$ with $A \triangle B \in \mathcal{I}$. Then $f(\mathbf{1}_A) = f(\mathbf{1}_B)$.*

Proof. This follows by $0 \leq |f(\mathbf{1}_A) - f(\mathbf{1}_B)| \leq f(\mathbf{1}_{A \triangle B}) = 0$. \square

Hereafter, we denote by ba the space of signed finitely additive measures $\mu : \mathcal{P}(\omega) \rightarrow \mathbf{R}$ with finite total variation, that is, such that

$$\sup \left\{ \sum_{i=0}^n |\mu(A_i)| : \{A_0, A_1, \dots, A_n\} \text{ is a partition of } \omega \right\} < \infty,$$

see e.g. [1, Section 10.10]. Recall that the norm dual ℓ'_∞ can be identified with ba via the lattice isomorphism $T : \ell'_\infty \rightarrow ba$ defined by

$$\forall f \in \ell'_\infty, \forall A \subseteq \omega, \quad T(f)(A) := f(\mathbf{1}_A), \tag{3}$$

see e.g. [1, Theorem 14.4]. We endow both ba and the norm dual ℓ'_∞ with the weak* topology. Note that T is continuous and its inverse is given by

$$\forall \mu \in ba, \forall x \in \ell_\infty, \quad T^{-1}(\mu)(x) = \int_\omega x \, d\mu. \tag{4}$$

Remark 4.2. By the above premises, it is easy to check that a linear continuous functional $f \in \ell'_\infty$ belongs to $SL(\mathcal{I})$ if and only if the associated finitely additive measure $\mu := T(f)$ is nonnegative, normalized, and assigns 0 to all sets in \mathcal{I} (or, equivalently, it is \mathcal{I} -invariant).

This implies that item (ii) in Definition 1.1 can be replaced by $f(e) = 1$ (in fact, the continuity of f follows by item (i)). In addition, item (iii) can be replaced by the equivalent (formally stronger) $f(x) = 0$ for all $x \in c_{00}(\mathcal{I}) \cap \ell_\infty$. Lastly, it is easy to see that $SL(\mathcal{I})$ coincides also with the set of positive linear functionals f such that $f(x) = \mathcal{I}\text{-lim } x$ for all $x \in c(\mathcal{I}) \cap \ell_\infty$, cf. also Corollary 3.5 below.

Proof of Theorem 2.1. Fix $f \in SL(\mathcal{I})$ and recall that f is continuous since it satisfies (i'). Using the lattice isomorphism T defined in (3), let $\mu := T(f)$ be the nonnegative finitely additive normalized measure $\mathcal{P}(\omega) \rightarrow \mathbf{R}$ associated with f . Note that μ is \mathcal{I} -invariant, thanks to Lemma 4.1. It follows by (4) that $f(x) = \int_\omega x \, d\mu$ for all $x \in \ell_\infty$. Also, for each ultrafilter \mathcal{F} on ω , let $\mu_{\mathcal{F}}$ be the $\{0, 1\}$ -valued finitely additive probability measure defined by $\mu_{\mathcal{F}}(A) = 1$ if and only if $A \in \mathcal{F}$. Taking again into account (4), we obtain by [10, Theorem 1.1] that there exists a normalized capacity $\rho : \mathcal{B}(\text{Ult}(\mathcal{I})) \rightarrow \mathbf{R}$ such that

$$f(x) = \int_\omega x \, d\mu = \int_{\text{Ult}(\mathcal{I})} \left(\int_\omega x \, d\mu_{\mathcal{F}} \right) d\rho(\mathcal{F}) = \int_{\text{Ult}(\mathcal{I})} \mathcal{F}\text{-lim } x \, d\rho(\mathcal{F})$$

for each $x \in \ell_\infty$. This completes the proof. \square

Proof of Theorem 2.2. It is clear that $SL(\mathcal{I})$ is a weak* closed convex subset of ℓ'_∞ . In addition, since it is included in the closed unit ball of ℓ'_∞ , it is weak* compact by Alaoglu's theorem, see e.g. [1, Theorem 5.105]. By the lattice isomorphism T defined in (3), we have that

$$\text{ext}(SL(\mathcal{I})) = T^{-1} [\text{ext} (T[SL(\mathcal{I})])] = T^{-1} [\{\mu_{\mathcal{F}} : \mathcal{F} \in \text{Ult}(\mathcal{I})\}],$$

see [10, Claim 4] and cf. [1, p. 544] (here, as usual, $\text{ext}(S)$ stands for the set of extreme points of a nonempty $S \subseteq \ell'_\infty$). Hence by the inverse T^{-1} in (4) we obtain $\text{ext}(SL(\mathcal{I})) = \{f_{\mathcal{F}} : \mathcal{F} \in \text{Ult}(\mathcal{I})\}$. The conclusion follows by applying Krein–Milman's theorem, see e.g. [1, Theorem 7.68]. \square

Proof of Theorem 2.3. If \mathcal{I} is maximal then it follows by Theorem 2.2 that $SL(\mathcal{I}) = \{f_{\mathcal{I}}\}$, hence $\text{diam}(SL(\mathcal{I})) = 0$. Conversely, if \mathcal{I} is not maximal, there exists a set $A \subseteq \omega$ such that $A \notin \mathcal{I}$ and $\omega \setminus A \notin \mathcal{I}$. Define $x := \mathbf{1}_A - \mathbf{1}_{\omega \setminus A}$, so that $\Gamma_x(\mathcal{I}) = \{-1, 1\}$. It follows by Theorem 2.2 and Theorem 3.3 that there exist $f, g \in SL(\mathcal{I})$ such that $f(x) = 1$ and $g(x) = -1$. Therefore $\text{diam}(SL(\mathcal{I})) \geq \|f - g\| \geq |f(x) - g(x)| = 2$. \square

Proof of Theorem 2.4. Since $f \in \text{SL}(\mathcal{I})$, it follows by definition that $\mathcal{I} \subseteq \{A \subseteq \omega : f(\mathbf{1}_A) = 0\}$. On the other hand, thanks to [23, Corollary 2.10], the latter inclusion has to be strict, hence there exists $A \subseteq \omega$ such that $A \notin \mathcal{I}$ and $f(\mathbf{1}_A) = 0$. Since $f(\mathbf{1}_\omega) = 1$, we get by Lemma 4.1 that also $\omega \setminus A \notin \mathcal{I}$. At this point, define the sequence $x := \mathbf{1}_A - \mathbf{1}_{\omega \setminus A}$. Then $\Gamma_x(\mathcal{I}) = \{1, -1\}$ and $f(x) = f(\mathbf{1}_A) - f(\mathbf{1}_{\omega \setminus A}) = -1$. It also follows by Theorem 2.2 that there exists $g \in \text{SL}(\mathcal{I})$ such that $g(x) = 1$. Therefore $\|f - g\| \geq |f(x) - g(x)| = 2$. \square

Remark 4.3. As it follows by [23, Proposition 2.11] and the proof above, there exists an ideal \mathcal{I} on ω which is *not* meager and, on the other hand, satisfies the claim of Theorem 2.4.

Before we proceed with the proof of Theorem 2.5, recall that the topological dual ℓ'_∞ is a Dedekind complete Riesz space, and its partial order satisfies

$$(f \vee g)(x) = \sup\{f(u) + g(v) : x = u + v, u, v \geq 0\} \quad (5)$$

for all $f, g \in \ell'_\infty$ and all $x \in \ell'_\infty^+$, see e.g. [2, Theorem 1.18 and Theorem 3.49].

Proof of Theorem 2.5. It is clear that each element of $\text{SL}(\mathcal{I}) - \text{SL}(\mathcal{I})$ satisfies properties (a)–(c). Conversely, suppose that f satisfies properties (a)–(c). Fix an arbitrary $f_0 \in \text{SL}(\mathcal{I})$, and define

$$g := (f \vee 0) + \left(1 - \frac{\|f\|}{2}\right) f_0, \quad (6)$$

where the supremum is computed as in (5). Of course, g is a continuous linear functional on ℓ_∞ . We observe that g is positive, indeed for every $x \geq 0$ we have $(f \vee 0)(x) \geq 0$, $f_0(x) \geq 0$, and $1 - \|f\|/2 \geq 0$, hence $g(x) \geq 0$. In addition, for every $A \in \mathcal{I}$ we have $f_0(\mathbf{1}_A) = 0$ and $(f \vee 0)(\mathbf{1}_A) = \sup\{f(x) : 0 \leq x \leq \mathbf{1}_A\} = \sup\{f(\mathbf{1}_B) : B \subseteq A\} = 0$, cf. again [2, Theorem 1.50] and [1, p. 541]. Lastly, we observe by Lemma 3.6 that

$$\begin{aligned} g(e) &= (f \vee 0)(e) + \left(1 - \frac{\|f\|}{2}\right) f_0(e) \\ &= \sup\{f(x) : 0 \leq x \leq e\} + \left(1 - \frac{\|f\|}{2}\right) = 1. \end{aligned}$$

This implies, thanks to Remark 4.2, that $g \in \text{SL}(\mathcal{I})$.

At this point, it is enough to show that

$$h := g - f = ((-f) \vee 0) + \left(1 - \frac{\|f\|}{2}\right) f_0$$

belongs to $\text{SL}(\mathcal{I})$ as well. The proof goes analogously, by noting that $\sup\{-f(x) : 0 \leq x \leq e\}$ can be rewritten equivalently as $\sup\{f(e - x) : 0 \leq x \leq e\}$ or $\sup\{f(x) : 0 \leq x \leq e\}$ (we omit further details).

For the second part, suppose first that \mathcal{I} is maximal. Note that there exists exactly one $S_{\mathcal{I}}$ -limit, which is $f_0(x) := \mathcal{I}\text{-lim } x$ for all $x \in \ell_\infty$, cf. Theorem 2.2. Then a linear functional $f \in \ell'_\infty$ which satisfies properties (a)–(c) is necessarily the null functional: in fact, for each $A \subseteq \omega$ we have $f(\mathbf{1}_A) = 0$ if $A \in \mathcal{I}$ and $f(\mathbf{1}_A) = f(e - \mathbf{1}_{\omega \setminus A}) = -f(\mathbf{1}_{\omega \setminus A}) = 0$ if $A \notin \mathcal{I}$. It follows that $0 = f = f_0 - f_0$, cf. Remark 4.2, hence the decomposition is unique.

Second, suppose that $\|f\| = 2$ and that $f = g - h$ for some $g, h \in \text{SL}(\mathcal{I})$. Thanks to Lemma 3.6, we have $2 = \|f\| = 2 \sup\{g(\mathbf{1}_A) - h(\mathbf{1}_A) : A \subseteq \omega\}$ and, at the same time, $1 = \|g\| = \sup\{g(\mathbf{1}_A) : A \subseteq \omega\}$. It follows that, for each $n \in \omega$, there exists $A_n \subseteq \omega$ such that

$$1 - 2^{-n} \leq g(\mathbf{1}_{A_n}) \leq 1 \quad \text{and} \quad 0 \leq h(\mathbf{1}_{A_n}) \leq 2^{-n}.$$

Since $g, h \in \text{SL}(\mathcal{I})$, it follows also that

$$0 \leq g(\mathbf{1}_{\omega \setminus A_n}) \leq 2^{-n} \quad \text{and} \quad 1 - 2^{-n} \leq h(\mathbf{1}_{\omega \setminus A_n}) \leq 1.$$

For each $A \subseteq \omega$, we obtain that

$$\begin{aligned} 0 \leq (g \wedge h)(\mathbf{1}_A) &= \inf\{g(u) + h(v) : \mathbf{1}_A = u + v, u, v \geq 0\} \\ &\leq \inf\{g(\mathbf{1}_B) + h(\mathbf{1}_{A \setminus B}) : B \subseteq A\} \\ &\leq \inf\{g(\mathbf{1}_{(\omega \setminus A_n) \cap A}) + h(\mathbf{1}_{A \setminus (\omega \setminus A_n)}) : n \in \omega\} \\ &\leq \inf\{g(\mathbf{1}_{\omega \setminus A_n}) + h(\mathbf{1}_{A_n}) : n \in \omega\} = 0. \end{aligned}$$

It follows, again by Lemma 3.6, that

$$\|g \wedge h\| = 2 \sup\{(g \wedge h)(\mathbf{1}_A) : A \subseteq \omega\} = 0.$$

Therefore $f = g - h$ for some $g, h \in \ell'_\infty$ such that $g \wedge h = 0$. We conclude by [2, Theorem 1.5(b)] that $g = f^+$ and $h = f^-$, hence the decomposition is unique (note that the definition of g coincides with the one provided in (6), hence also the definition of $h := g - f$).

Lastly, suppose $\|f\| < 2$ and that \mathcal{I} is not maximal, so that $1 - \|f\|/2 > 0$ and there exist two distinct maximal ideals $\mathcal{J}_1, \mathcal{J}_2$ which extend \mathcal{I} . Then it is easy to see that the function g defined in (6) with $f_0(x) := \mathcal{J}_1\text{-lim } x$ is different from the one choosing $f_0(x) := \mathcal{J}_2\text{-lim } x$ (note that they are both $S_{\mathcal{I}}$ -limits, as it follows by Theorem 2.2). Hence there exist at least two decompositions of f . \square

The following corollary is immediate:

Corollary 4.4. *Let \mathcal{I} be an ideal on ω and fix $f \in \ell'_\infty$ such that $f(e) = f(\mathbf{1}_A) = 0$ for all $A \in \mathcal{I}$. Then for every real $k \geq \|f\|/2$ there exist $g, h \in \text{SL}(\mathcal{I})$ such that $f = k(g - h)$.*

Proof. If $f = 0$ the claim is clear. Otherwise, fix $k \geq \|f\|/2$ (which is positive) and define $f_0 := f/k$. The claim follows applying Theorem 2.5 to f_0 . \square

We conclude with the proof of Theorem 1.3.

Proof of Theorem 1.3. The equality $\overline{c(\mathcal{I}^*) \cap \ell_\infty} = c(\mathcal{I}) \cap \ell_\infty$ is well known, see e.g. [25, Theorem 2.4]. Hence, it is enough to show that $\mathcal{V}(\mathcal{I}) = c(\mathcal{I}) \cap \ell_\infty$. To this aim, suppose that $x \in \mathcal{V}(\mathcal{I})$. Since $x \in \ell_\infty$, then $\Gamma_x(\mathcal{I}) \neq \emptyset$, see e.g. [27, Lemma 3.1(vi)]. Suppose that $\eta_1, \eta_2 \in \Gamma_x(\mathcal{I})$. Thanks to Theorem 3.3, there exist maximal ideals $\mathcal{J}_1, \mathcal{J}_2$ containing \mathcal{I} such that $\mathcal{J}_1\text{-lim } x = \eta_1$ and $\mathcal{J}_2\text{-lim } x = \eta_2$. It follows by Theorem 2.2 that $\eta_1 = \eta_2$, hence $|\Gamma_x(\mathcal{I})| = 1$. This implies that x is \mathcal{I} -convergent, see e.g. [27, Corollary 3.4], therefore $\mathcal{V}(\mathcal{I}) \subseteq c(\mathcal{I}) \cap \ell_\infty$.

Viceversa, pick a sequence $x \in c(\mathcal{I}) \cap \ell_\infty$ and define $\eta := \mathcal{I}\text{-lim } x$. Of course, $\mathcal{F}\text{-lim } x = \eta$ for every ultrafilter $\mathcal{F} \in \text{Ult}(\mathcal{I})$. Pick also a $S_{\mathcal{I}}$ -limit f . Thanks to Theorem 2.1, there exists a normalized capacity $\rho : \mathcal{B}(\text{Ult}(\mathcal{I})) \rightarrow \mathbf{R}$ which satisfies (2). This implies that

$$f(x) = \int_{\text{Ult}(\mathcal{I})} \mathcal{F}\text{-lim } x \, d\rho(\mathcal{F}) = \int_{\text{Ult}(\mathcal{I})} \eta \, d\rho(\mathcal{F}) = \eta,$$

so that $x \in \mathcal{V}(\mathcal{I})$. Hence $c(\mathcal{I}) \cap \ell_\infty \subseteq \mathcal{V}(\mathcal{I})$.

For the second part, suppose that \mathcal{I} is a P -ideal. Then $c(\mathcal{I}) = c + c_{00}(\mathcal{I})$ (the case $\mathcal{I} = \mathcal{Z}$ can be found in [12, Theorem 2.3], the general case goes analogously). We conclude that $\mathcal{V}(\mathcal{I}) = c(\mathcal{I}) \cap \ell_\infty = (c + c_{00}(\mathcal{I})) \cap \ell_\infty = c + (c_{00}(\mathcal{I}) \cap \ell_\infty)$. \square

5. Further results

As an application of Theorem 2.5, we can compute the distance of sequences in ℓ_∞ from the space of bounded \mathcal{I} -convergent sequences. To this aim, for each nonempty $Y \subseteq \ell_\infty$ and $x \in \ell_\infty$, define

$$\text{dist}(x, Y) := \inf\{\|x - y\| : y \in Y\}.$$

(The case $\mathcal{I} = \text{Fin}$ of the next result can be found, e.g., in [8, Proposition 1.18].)

Proposition 5.1. *Let \mathcal{I} be an ideal on ω . Then*

$$\forall x \in \ell_\infty, \quad \text{dist}(x, c(\mathcal{I}) \cap \ell_\infty) = \frac{1}{2} (\mathcal{I}\text{-lim sup } x - \mathcal{I}\text{-lim inf } x). \quad (7)$$

Proof. Fix $x \in \ell_\infty$ and define $\eta_0 := \frac{1}{2}(\eta_+ + \eta_-)$ and $\delta_0 := \frac{1}{2}(\eta_+ - \eta_-)$, where $\eta_+ := \mathcal{I}\text{-lim sup } x$ and $\eta_- := \mathcal{I}\text{-lim inf } x$. Set also $Y := c(\mathcal{I}) \cap \ell_\infty$ and recall that Y is closed, see e.g. Theorem 1.3. If x is a bounded \mathcal{I} -convergent sequence then both sides of (7) are zero. Hence, we suppose hereafter that $x \notin Y$.

On the one hand, it is known (and easy to check) that, if $y \in Y$ is \mathcal{I} -convergent to κ , then $\Gamma_{x-y}(\mathcal{I}) = \Gamma_{x-\kappa e}(\mathcal{I}) = \Gamma_x(\mathcal{I}) - \kappa$, cf. also [26, Proposition 3.2] for a more general result in this direction. Hence

$$\begin{aligned} \text{dist}(x, Y) &\geq \inf\{\max\{|\eta| : \eta \in \Gamma_{x-y}(\mathcal{I})\} : y \in Y\} \\ &= \inf\{\max\{|\eta| : \eta \in \Gamma_x(\mathcal{I}) - \kappa\} : \kappa \in \mathbf{R}\} \\ &\geq \inf\{\max\{|\eta_+ - \kappa|, |\eta_- - \kappa|\} : \kappa \in \mathbf{R}\} \geq \delta_0. \end{aligned}$$

On the other hand, it follows by a consequence of Hahn–Banach theorem that there exists $f \in \ell'_\infty$ such that

$$\|f\| = 1, \quad f(x) = \text{dist}(x, Y), \quad \text{and } f[Y] = \{0\},$$

see e.g. [9, Theorem 2.3.22]. In particular, since $e \in Y$, we have $f(e) = 0$. It follows by Theorem 2.5 that there exists (unique) $g, h \in \text{SL}(\mathcal{I})$ such that $2f = g - h$. Considering that $\eta_- \leq w(x) \leq \eta_+$ for every $w \in \text{SL}(\mathcal{I})$, see Corollary 3.5, we obtain that $\text{dist}(x, Y) = f(x) = (g(x) - h(x))/2 \leq (\eta_+ - \eta_-)/2 = \delta_0$. \square

Remark 5.2. It is worth noting that the upper bound $\text{dist}(x, Y) \leq \delta_0$ can be obtained also by elementary means without the aid of $\text{S}_{\mathcal{I}}$ -limits. To this aim, for each $k \in \omega$, define $A_k := \{n \in \omega : |x_n - \eta_0| \geq \delta_0(1 + 2^{-k})\}$, and note that $A_k \in \mathcal{I}$. For each $k \in \omega$, define also the sequence $y^k \in Y$ by $y_n^k := x_n$ if $n \in A_k$ and $y_n^k := \eta_0$ otherwise. It follows that

$$\begin{aligned} \text{dist}(x, Y) &\leq \inf\{\|x - y^k\| : k \in \omega\} \\ &= \inf\{\sup\{|x_n - \eta_0| : n \in \omega \setminus A_k\} : k \in \omega\} \\ &\leq \inf\{\delta_0(1 + 2^{-k}) : k \in \omega\} = \delta_0. \end{aligned}$$

For our last application, a point $\eta \in \mathbf{R}$ is an \mathcal{I} -limit point of a sequence $x \in \ell_\infty$ if there exists a strictly increasing sequence $(a_n : n \in \omega)$ in ω such that $\{a_n : n \in \omega\} \notin \mathcal{I}$ and $\lim_n x_{a_n} = \eta$; see e.g. [20, 21] in the case $\mathcal{I} = \mathcal{Z}$. We denote by $\Lambda_x(\mathcal{I})$ the set of \mathcal{I} -limit points of $x \in \ell_\infty$. Analogously to Theorem 3.3, we provide a characterization of \mathcal{I} -limit points, which generalizes [14, Theorem 2(2)].

Theorem 5.3. *Let \mathcal{I} be an ideal on ω , and fix $x \in \ell_\infty$ and $\eta \in \mathbf{R}$. Then $\eta \in \Lambda_x(\mathcal{I})$ if and only if there exists $A \subseteq \omega$ such that $\hat{x}[\tilde{A}] = \{\eta\}$ and $\tilde{A} \cap \text{Ult}(\mathcal{I}) \neq \emptyset$.*

Proof. It follows by Lemma 3.1 and Lemma 3.2. \square

Differently from the case of \mathcal{I} -cluster points, the behavior of \mathcal{I} -limit points is really wild: for instance, there exists $x \in \ell_\infty$ such that $\Lambda_x(\mathcal{Z}) = \emptyset$, see [20, Example 4]; cf. also [17] and references therein. However, if the complexity of \mathcal{I} is sufficiently low, then we can show that the notions of \mathcal{I} -cluster points and \mathcal{I} -limit points coincide, hence recovering [4, Theorem 2.3] in the real case:

Proposition 5.4. *Let \mathcal{I} be an F_σ -ideal on ω . Then $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I})$ for all $x \in \ell_\infty$.*

Proof. Fix $x \in \ell_\infty$. It is known (and it follows also by Theorem 3.3 and Theorem 5.3) that every \mathcal{I} -limit point of x is also an \mathcal{I} -cluster point of x . Hence it is sufficient to show that $\Gamma_x(\mathcal{I}) \subseteq \Lambda_x(\mathcal{I})$. To this aim, fix $\eta \in \Gamma_x(\mathcal{I})$. Thanks to Theorem 3.3, there exists a free ultrafilter $\mathcal{F}_0 \in \text{Ult}(\mathcal{I})$ such that $\hat{x}(\mathcal{F}_0) = \eta$. It follows that

$$A_k := \{n \in \omega : |x_n - \eta| < 2^{-k}\} \in \mathcal{F}_0$$

for each $k \in \omega$. In particular, $A_k \notin \mathcal{I}$.

At this point, since \mathcal{I} is an F_σ -ideal, there exists a sequence $(F_n : n \in \omega)$ of closed subsets of $\mathcal{P}(\omega)$ such that $\mathcal{I} = \bigcup_n F_n$. Replacing F_n with $\bigcup_{k \leq n} \bigcup_{A \in F_k} \mathcal{P}(A)$, we can assume, in addition, that F_n is hereditary and $F_n \subseteq F_{n+1}$ for all $n \in \omega$. Note also that, for each $k, n \in \omega$ and $S \in \text{Fin}$, we have $A_k \setminus S \notin \mathcal{I}$, so that $A_k \setminus S \notin F_n$.

Lastly, define recursively the increasing sequence $(m_k : k \in \omega)$ in ω such that $m_0 := 0$ and, for each $k \geq 1$, m_k is the smallest integer $m > m_{k-1}$ such that $A_k \cap [m_{k-1}, m) \notin F_k$ (this is well defined because $B_k := A_k \setminus [0, m_k) \notin F_k$ and F_k is hereditary closed). It follows by construction that the subsequence $(x_n : n \in B)$ converges to η , where $B := \bigcup_k B_k$. In addition, $B \notin \mathcal{I}$: indeed, in the opposite, there would exist $k \in \omega$ such that $B \in F_k$ hence, in particular, $B_k \in F_k$ which is impossible. This proves that η is an \mathcal{I} -limit point of x , completing the proof. \square

6. Concluding remarks

We leave as an open question for the interested reader to characterize the class of ideals \mathcal{I} on ω which satisfy the claim of Theorem 2.4.

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