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# On counting Z-convex polyominoes

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**Abstract** We show a decomposition that allows to compute the number of convex polyominoes of area $n$ and degree of convexity at most 2 (the so-called Z-convex polyominoes) in polynomial time.

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**Keywords:** integer sequence, convex polyomino, counting problem.

## 1 Introduction

A polyomino is a geometrical figure consisting of a finite set of connected $1 \times 1$ squares (called cells) in the plane $\mathbb{Z} \times \mathbb{Z}$, considered up to translations. Polyominoes gained popularity after the paper of S. Golomb [9]. Nowadays they are widely studied by physicists, mathematicians, computer scientists and also by biologists.

The problem of counting the number $c_n$ of polyominoes with $n$ cells (*i.e.* of area $n$) is probably one of the fundamental open problems in combinatorial geometry (see problem 37 in [6]). The problem has been solved up to $n \leq 56$ [10] and no closed-form expression for $c_n$ is known. Due to the difficulty of the problem, simpler classes of polyominoes have been introduced and widely studied. In particular, the class of convex polyominoes (polyominoes where the intersection with an infinite horizontal or vertical stripe is a finite segment) and some of its subclasses have been deeply investigated [1, 2, 11, 5].

In this paper we consider all convex polyominoes $P$ with the property that any two cells of $P$ can be joined by a path in $P$ with at most two changes of direction. This is the class of Z-convex polyominoes introduced in [8] and studied in [12]. Its generating function with respect to the area is still unknown and the only way to enumerate it (up to now) is to use the CAT algorithm presented in [3], which can not be used to compute the number $z_n$ of Z-convex polyominoes of area $n$ for a large $n$, since $z_n$ grows exponentially. We show how to decompose Z-convex polyominoes in order to obtain a set of equations that can be used to compute $z_n$ in polynomial time with respect to $n$ (under the uniform cost model).

## 2 Notation and preliminaries

Let $P$ be a polyomino with an $r \times c$ minimal bounding rectangle. The $r$ rows (resp., $c$ columns) of $P$ are numbered from bottom to top (resp., from left to right). The *area* of $P$ is the number of its



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cells, denoted by $\mathrm{A}(P)$. If $\mathrm{A}(P) = 0$ then $P$ is null. A cell of $P$ is identified by a pair of integers $(i, j)$, where $i$ (resp., $j$) is the row (resp., column) index. Two cells $a = (i, j)$ and $a' = (i', j')$ are *adjacent* if $|i - i'| + |j - j'| = 1$. Given two cells $a$ and $b$ of $P$, a *path* in $P$ from $a$ to $b$ is a sequence $q_1, q_2, \dots, q_k$ of cells of $P$, with $q_1 = a$ and $q_k = b$, such that $q_i$ and $q_{i+1}$ are adjacent for all $i$ with $1 \leq i < k$. A *step* is a sequence of two adjacent cells $(i, j), (i', j')$. Steps are distinguished according to the directions $\mathsf{N}$ (North), $\mathsf{W}$ (West), $\mathsf{S}$ (South) and $\mathsf{E}$ (East). The number of *changes of direction* in a path $\beta \in \{\mathsf{N}, \mathsf{W}, \mathsf{S}, \mathsf{E}\}^+$ is defined as the number of indices $i$ such that $\beta_i \neq \beta_{i+1}$, with $1 \leq i < |\beta|$. A path is *monotone* if $\beta \in \{\mathsf{N}, \mathsf{W}\}^+$ (NW-path) or $\beta \in \{\mathsf{N}, \mathsf{E}\}^+$ (NE-path) or $\beta \in \{\mathsf{S}, \mathsf{E}\}^+$ (SE-path) or $\beta \in \{\mathsf{S}, \mathsf{W}\}^+$ (SW-path).

A polyomino $P$ is *horizontally convex* (resp., *vertically convex*) if any row (resp., column) of $P$ consists of exactly one segment. The class of *convex* polyominoes contains all polyominoes that are horizontally and vertically convex. It has been proved [4, Proposition 1] that a polyomino $P$ is convex if and only if any two cells of $P$ are joined by a monotone path in $P$. The *degree of convexity* of $P$, denoted by $\deg_c(P)$, is defined as the least integer $k$ such that any two cells of $P$ can be joined by a monotone path with at most $k$ changes of direction. A convex polyomino is called $k$-convex if its degree of convexity is at most $k$. When $k = 2$ we have the class $\mathsf{ZConv}$ of *Z-convex* polyominoes, introduced in [8]. Given a convex polyomino $P$ and its bounding rectangle $B$, we say that $P$ is a *stack* (resp., *Ferrers diagram*, *parallelogram*, *rectangle*) if it shares exactly two adjacent (resp., three, two opposite, four) vertices with $B$. A stack $P$ is a *left* (resp., *right*) stack if the column with the largest area is the last (resp., first) one. Analogously, in a left (resp., right) Ferrers diagram the largest column is the last (resp., first) one. We denote by $\mathsf{L}$ (resp., $\mathsf{R}$) the set of left (resp., right) stacks. The set of left (resp., right) Ferrers diagrams is $\mathsf{F}_L$ (resp., $\mathsf{F}_R$). Furthermore, we indicate by $\mathsf{C}$ (resp., $\mathsf{T}$) the set of parallelograms (resp., rectangles). For a class $\mathsf{A}$ of polyominoes, $\mathsf{A}(n)$ indicates the set of all $P \in A$ of area $n$.

Let $j$ be a column of $P$, by $\textsc{low}(j)$ (resp., $\textsc{high}(j)$) we denote the row index of the lowest cell (resp., highest cell) of $j$. We indicate by $\textsc{first}(P)$ (resp., $\textsc{last}(P)$) the first (resp., last) column of $P$. Two columns $i$ and $j$ are *overlapping*, denoted by $i \updownarrows j$, if and only if $\textsc{low}(j) < \textsc{low}(i) \leq \textsc{high}(j) < \textsc{high}(i)$ or $\textsc{low}(i) < \textsc{low}(j) \leq \textsc{high}(i) < \textsc{high}(j)$. We say that $i$ and $j$ are *disjoint*, denoted by $i \asymp j$, if and only if $\textsc{low}(i) > \textsc{high}(j)$ or $\textsc{low}(j) > \textsc{high}(i)$. Lastly, $i$ *includes* $j$, denoted by $j \subseteq i$, if and only if $\textsc{low}(i) \leq \textsc{low}(j)$ and $\textsc{high}(i) \geq \textsc{high}(j)$. A convex polyomino $P$ is called *descending* (resp., *ascending*) if there exists a column $j$ such that $j > e$, $j \updownarrows e$ and $\textsc{low}(e) > \textsc{low}(j)$ (resp., $\textsc{low}(e) < \textsc{low}(j)$), where $e$ is the rightmost column of $P$ with $c \subseteq e$ for $1 \leq c < e$. The set of descending (resp., descending) convex polyominoes is indicated by $\mathsf{DConv}$ (resp., $\mathsf{AConv}$). If $P$ is neither descending nor ascending then $P \in \mathsf{T} \cup \mathsf{F}_L \cup \mathsf{F}_R \cup \mathsf{L} \cup \mathsf{R}$ or it belongs to the class $\mathsf{LR}$ containing all convex polyominoes $P$ that are the concatenation of two polyominoes, $P = P_1 \cdot P_2$, where $P_1 \in \mathsf{L} \cup \mathsf{F}_L$, $P_2 \in \mathsf{T} \cup \mathsf{R} \cup \mathsf{F}_R$ and $\textsc{first}(P_2) \subsetneq \textsc{last}(P_1)$. Notice that a polyomino $P$ in $\mathsf{LR}$ contains a column $\bar{j}$ such that $j \subseteq \bar{j}$ for all columns $j$, hence $\deg_c(P) \leq 2$.

We denote by $\mathsf{DConv}_2$ (resp., $\mathsf{AConv}_2$) the set of descending (resp., ascending) polyominoes of degree of convexity 2. Clearly, one has $\mathsf{ZConv} = \mathsf{T} \cup \mathsf{F}_L \cup \mathsf{F}_R \cup \mathsf{L} \cup \mathsf{R} \cup \mathsf{LR} \cup \mathsf{AConv}_2 \cup \mathsf{DConv}_2$, and (because of symmetry) $|\mathsf{DConv}_2(n)| = |\mathsf{AConv}_2(n)|$, $|\mathsf{F}_L(n)| = |\mathsf{F}_R(n)|$, $|\mathsf{L}(n)| = |\mathsf{R}(n)|$. This implies that

$$|\mathsf{ZConv}(n)| = |\mathsf{T}(n)| + 2 \cdot |\mathsf{F}_L(n)| + 2 \cdot |\mathsf{L}(n)| + |\mathsf{LR}(n)| + 2 \cdot |\mathsf{DConv}_2(n)|,$$

and the counting problem for $\mathsf{ZConv}$ is reduced to the counting problem for $\mathsf{DConv}_2$ and to some simpler counting problems (easily solved in polynomial time). In particular, we can compute $|\mathsf{LR}(n)|$ in polynomial time as shown in [7]. In order to compute $|\mathsf{DConv}_2(n)|$ we see that any descending polyomino is obtained by concatenating at most four simple polyominoes.

Definition 2.1 (standard decomposition) A polyomino $P \in \mathsf{DConv}$ can be decomposed as $P = L \cdot F \cdot C \cdot R$ (with $F$, $R$ possibly empty) for suitable polyominoes $L \in \mathsf{L} \cup \mathsf{T} \cup \mathsf{F}_L$, $F \in \mathsf{F}_R$, $C \in \mathsf{C} \cup \mathsf{T} \cup \mathsf{F}_R$, and $R \in \mathsf{R} \cup \mathsf{T} \cup \mathsf{F}_R$ such that: $\text{FIRST}(F) \subsetneq \text{LAST}(L)$, $\text{LOW}(\text{LAST}(L)) = \text{LOW}(\text{FIRST}(F))$, $\text{LAST}(F) \upharpoonleft\downharpoonright \text{FIRST}(C)$ (or $\text{LAST}(L) \upharpoonleft\downharpoonright \text{FIRST}(C)$ if $F = \epsilon$), $\text{LOW}(\text{LAST}(L)) > \text{LOW}(\text{FIRST}(C))$ and (if $R \neq \epsilon$) $\text{FIRST}(R) \subsetneq \text{LAST}(C)$, $\text{LOW}(\text{LAST}(C)) < \text{LOW}(\text{FIRST}(R))$.

The standard decomposition of any $P \in \mathsf{DConv}$ is unique (*e.g.* $\text{LAST}(L)$ is the rightmost column $\bar{\jmath}$ of $P$ such that $j \subseteq \bar{\jmath}$ for $j < \bar{\jmath}$). The subset of $\mathsf{DConv}_2$ containing polyominoes decomposed as $L \cdot C \cdot R$ (resp., $L \cdot C$, $L \cdot F \cdot C \cdot R$, $L \cdot F \cdot C$) is $\mathsf{LCR}_2$ (resp., $\mathsf{LC}_2$, $\mathsf{LFCR}_2$, $\mathsf{LFC}_2$). We introduce the following subset of $\mathsf{DConv}_2$.

Definition 2.2 ($\mathsf{Z}_2$) The set $\mathsf{Z}_2$ contains all $P$ in $\mathsf{LCR}_2 \cup \mathsf{LC}_2$ such that $l \subsetneq \text{FIRST}(C)$ for all columns $l$ of $L$ (with $l \neq \text{LAST}(L)$) and $\text{FIRST}(R) \upharpoonleft\downharpoonright \text{LAST}(L)$ (if $P \in \mathsf{LCR}_2$).

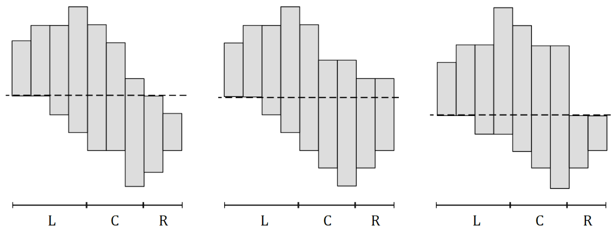


Figure 1: From left to right: $P \in \mathsf{Z}_2^\bullet \setminus \text{s-}\mathsf{Z}_2^\bullet$, $P' \in \mathsf{Z}_2^\circ$, $P'' \in \text{s-}\mathsf{Z}_2^\bullet$.

In the sequel, polyominoes with disjoint columns will be recursively decomposed into simpler polyominoes. Given a class of polyominoes $\mathsf{A}$, we consider the partition $\mathsf{A} = \mathsf{A}^\bullet \cup \mathsf{A}^\circ$, where $\mathsf{A}^\bullet$ (resp., $\mathsf{A}^\circ$) contains those polyominoes in $\mathsf{A}$ that have (resp., do not have) disjoint columns. In a particular, there is a subset of $\mathsf{Z}_2^\bullet$ which plays a special role in the decomposition of a polyomino in $\mathsf{DConv}_2^\bullet$.

Definition 2.3 (s-$\mathsf{Z}_2^\bullet$) The set s-$\mathsf{Z}_2^\bullet$ contains all $P \in \mathsf{Z}_2^\bullet$ that can be written as $P = L \cdot C \cdot R$ with $L \in \mathsf{L} \cup \mathsf{T} \cup \mathsf{F}_L$, $C \in \mathsf{C} \cup \mathsf{T} \cup \mathsf{F}_R$, $R \in \mathsf{R} \cup \mathsf{T} \cup \mathsf{F}_R$ and

- $l \subsetneq \text{LAST}(C)$ for all columns $l$ of $L$, with $l \neq \text{LAST}(L)$;
- $\text{FIRST}(R) \subsetneq \text{LAST}(C)$ and $\text{FIRST}(L) \asymp \text{FIRST}(R)$.

Figure 1 shows examples of polyominoes in the above defined sets.

# 3 Polyominoes Decomposition

We consider a function $\oplus : \mathsf{F}_R \times (\mathsf{T} \cup \mathsf{L} \cup \mathsf{R} \cup \mathsf{F}_L \cup \mathsf{LR}) \times \mathsf{Z}_2 \mapsto \mathsf{DConv}_2 \cup \{\perp\}$ (see Fig. 2) that is used to obtain the main equation for counting $\mathsf{DConv}_2$.

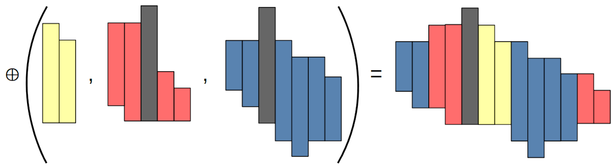


Figure 2: The function $\oplus$.

DEFINITION 3.1 ($\oplus$) Let $F \in \mathsf{F}_R$, $P \in \mathsf{T} \cup \mathsf{L} \cup \mathsf{R} \cup \mathsf{F}_L \cup \mathsf{F}_R \cup \mathsf{LR}$ and $Q \in \mathsf{Z}_2$, with $P = L \cdot d \cdot R$ ($L$, $F$, $R$ possibly null) and $Q = L' \cdot d' \cdot C' \cdot R'$, where $d$ (resp., $d'$) is the rightmost column of $P$ (resp., $Q$) that includes all columns to its left. Then, $\oplus(F, P, Q) = L' \cdot L \cdot d' \cdot F \cdot C' \cdot R' \cdot R$ if and only if $\mathrm{A}(d) = \mathrm{A}(d')$ and

1. $\textsc{high}(d) - \textsc{high}(\textsc{first}(L)) < \textsc{high}(d') - \textsc{high}(\textsc{last}(L'))$ (if $L, L' \neq \epsilon$);
2. $\textsc{high}(d) - \textsc{low}(\textsc{first}(L)) \geq \textsc{high}(d') - \textsc{low}(\textsc{last}(L'))$ (if $L, L' \neq \epsilon$);
3. $\mathrm{A}(\textsc{first}(F)) < \mathrm{A}(d')$ (if $F \neq \epsilon$);
4. $\mathrm{A}(\textsc{last}(F)) \geq \textsc{high}(d') - \textsc{high}(\textsc{first}(C')) + \mathrm{A}(\textsc{first}(C'))$ (if $F \neq \epsilon$);
5. $\textsc{high}(d) - \textsc{high}(\textsc{first}(R)) \geq \textsc{high}(d') - \textsc{high}(\textsc{last}(R'))$ (if $R, R' \neq \epsilon$).

We set $\oplus(F, P, Q) = \perp$ (undefined) if $F$, $P$ and $Q$ do not satisfy conditions 1–5 of Def.3.1 or $\mathrm{A}(d) \neq \mathrm{A}(d')$. Notice that $\oplus(F, P, Q) = Q$ if and only if $F = \epsilon$, $P = d$ and $Q = L \cdot d \cdot C \cdot R$. Moreover, $\oplus$ is extended to sets of polyominoes by setting $\oplus(\mathsf{A}, \mathsf{B}, \mathsf{D}) = \{\oplus(F, P, Q) \mid F \in \mathsf{A}, P \in \mathsf{B}, Q \in \mathsf{D}\}$. We point out that $\oplus$ corresponds to a (unique) decomposition of a polyomino in $\mathsf{DConv}_2$. Thus, from Def. 3.1 one has

$$\mathsf{DConv}_2 = \bigcup_{\substack{F \in \mathsf{F}_R, \mathrm{A}(F) \geq 0 \\ P \in \mathsf{T} \cup \mathsf{L} \cup \mathsf{R} \cup \mathsf{F}_L \cup \mathsf{F}_R \cup \mathsf{LR}, \mathrm{A}(P) > 0 \\ Q \in \mathsf{Z}_2, \mathrm{A}(Q) > 0}} \oplus(F, P, Q). \tag{1}$$

We stress that the union in (1) is disjoint, as $\oplus(F, P, Q) = \oplus(F', P', Q') \neq \perp$ if and only if $F = F'$, $P = P'$ and $Q = Q'$. The set equation (1) allows us to focus on $\mathsf{Z}_2$. In particular, our aim is to find a set equation for $\mathsf{Z}_2^\bullet$ (a set equation for $\mathsf{Z}_2^\circ$ is straightforward). Given $P \in \mathsf{Z}_2^\bullet$, consider its standard decomposition, $P = L \cdot C \cdot R$ ($R$ possibly null), with $L = L' \cdot d$, $d = \textsc{last}(L)$. Let $e$ be the rightmost column of $C$ such that $\textsc{last}(L') \subsetneq e$. Then, let $c$ be the leftmost column in $L'$ such that $c$ is included in $e$ but not in column $e + 1$ (this last condition only if $e \neq \textsc{last}(C)$). Lastly, let $f$ be the leftmost column of $C \cdot R$ such that $f \asymp c$. We stress that if $f$ belongs to $C$, then for any column $f'$ of $C$ to the right of $e$ one has $\textsc{low}(f') = \textsc{low}(e)$. Indeed, if $\textsc{low}(f') < \textsc{low}(e)$ then no column of $P$ includes both $f'$ and $c$, and so $\deg_c(P) > 2$ by [3, thm. 1]. So, there exist two right Ferrers diagrams $F_1$, $F_2$, two parallelograms $C_1$, $C_2$, and two right stacks $R_1$, $R_2$ such that

$$P = L'_2 \cdot L'_1 \cdot d \cdot C_1 \cdot F_2 \cdot F_1 \cdot R \quad (\text{if } f \in C \text{ and } C = C_1 \cdot F_1 \cdot F_2) \tag{2}$$

or

$$P = L'_2 \cdot L'_1 \cdot d \cdot C_1 \cdot C_2 \cdot R_2 \cdot R_1 \quad (\text{if } f \in R \text{ and } R = R_1 \cdot R_2) \tag{3}$$

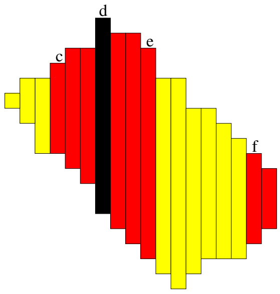


Figure 3: The $\oplus$ decomposition of $P \in \mathsf{Z}_2^\bullet$ gives rise to $P'' \in$ s-$\mathsf{Z}_2^\bullet$ (black+red) and $Q'' \in \mathsf{Z}_2$ (black+yellow). Moreover, $P = P'' \parallel Q''$.

with $c = \text{FIRST}(L_1')$, $e = \text{LAST}(C_1)$, $f = \text{FIRST}(F_1)$ (in case (2)) or $f = \text{FIRST}(R_1)$ (in case (3)), $g \upharpoonleft\downharpoonright c$ for any column $g \in F_2$ (in case (2)) or $g \in C_2 \cdot R_2$ (in case (3)), and $\text{LOW}(h) = \text{LOW}(e)$ for all $h \in F_2 \cdot F_1$. We associate with case (2) the two polyominoes $P' = L_1' \cdot d \cdot C_1 \cdot F_1 \cdot R$ and $Q' = L_2' \cdot d \cdot F_2$. Similarly, in case (3) we consider $P'' = L_1' \cdot d \cdot C_1 \cdot R_1$ and $Q'' = L_2' \cdot d \cdot C_2 \cdot R_2$. By construction one has $P', P'' \in$ s-$\mathsf{Z}_2^\bullet$ and $Q', Q'' \in \mathsf{Z}_2$, with $\text{A}(Q') < \text{A}(P)$ and $\text{A}(Q'') < \text{A}(P)$. Figure 3 illustrates case (3). Furthermore, any $P \in \mathsf{Z}_2^\bullet$ is obtained by combining two polyominoes that are uniquely determined by the decomposition of $P$ stated above. To this aim we define a function $\parallel: \mathsf{Z}_2 \times$ s-$\mathsf{Z}_2^\bullet \mapsto \mathsf{Z}_2^\bullet$, that we call *pseudo-shuffle*, see also Fig. 3.

DEFINITION 3.2 (pseudo-shuffle $\parallel$) Let $P \in \mathsf{Z}_2$ and $P' \in$ s-$\mathsf{Z}_2^\bullet$. Consider the standard decomposition of $P$, $P = L \cdot C \cdot R$ (or $P = L \cdot C$), and write $P'$ as in Def.2.3, $P' = L' \cdot C' \cdot R'$. Lastly, let $L = L'' \cdot d$ and $L' = L''' \cdot d'$, where $d = \text{LAST}(L)$ and $d' = \text{LAST}(L')$. Then, we set $P \parallel P' = L'' \cdot L''' \cdot d' \cdot C' \cdot C \cdot R \cdot R'$ if and only if $\text{A}(d) = \text{A}(d')$ and

1. $\text{HIGH}(d) - \text{HIGH}(\text{LAST}(L'')) > \text{HIGH}(d') - \text{HIGH}(\text{FIRST}(L'''))$ (if $L'' \neq \epsilon$);
2. $\text{HIGH}(d) - \text{LOW}(\text{LAST}(L')) \leq \text{HIGH}(d') - \text{LOW}(\text{FIRST}(L'''))$;
3. $\text{FIRST}(C) \subsetneq \text{LAST}(C')$ or $\text{FIRST}(C) \upharpoonleft\downharpoonright \text{LAST}(C')$, with $\text{HIGH}(d') - \text{HIGH}(\text{LAST}(C')) < \text{HIGH}(d) - \text{HIGH}(\text{FIRST}(C))$ and $\text{HIGH}(d) - \text{LOW}(\text{LAST}(L')) \leq \text{HIGH}(d') - \text{LOW}(\text{FIRST}(L'''))$;
4. $\text{HIGH}(d) - \text{HIGH}(\text{LAST}(R)) \leq \text{HIGH}(d') - \text{HIGH}(\text{FIRST}(R'))$ and $\text{HIGH}(d) - \text{LOW}(\text{LAST}(R)) \geq \text{HIGH}(d') - \text{LOW}(\text{FIRST}(R'))$ (if $R \neq \epsilon$);
5. $\text{LAST}(R) \upharpoonleft\downharpoonright \text{FIRST}(L''')$ and $\text{HIGH}(\text{FIRST}(L''')) > \text{HIGH}(\text{LAST}(R))$ (if $R \neq \epsilon$).

By Def.3.2 it follows that a polyomino $P \in \mathsf{Z}_2^\bullet$ is in s-$\mathsf{Z}_2^\bullet$ or is uniquely written as

$$P = \underbrace{(\cdots(}_{k-1} P_k \parallel P_{k-1}) \parallel P_{k-2}) \parallel \cdots) \parallel P_2) \parallel P_1 \tag{4}$$

with $P_i \in$ s-$\mathsf{Z}_2^\bullet$ for $1 \leq i < k$, and $P_k \in$ s-$\mathsf{Z}_2^\bullet \cup \mathsf{Z}_2^\circ$, see Fig. 4. Let $\mathsf{Z}_2(n, h)$ be the set of all $P \in \mathsf{Z}_2(n)$ where the polyomino $L$ in the standard decomposition of $P$, $P = L \cdot C \cdot R$, has height $h$. Then, from (4) one has

$$\mathsf{Z}_2^\bullet(n, d) = \text{s-}\mathsf{Z}_2^\bullet(n, d) \cup \bigcup_{6 \leq m \leq n-2} \mathsf{Z}_2(n - m + d, d) \parallel \text{s-}\mathsf{Z}_2^\bullet(m, d). \tag{5}$$

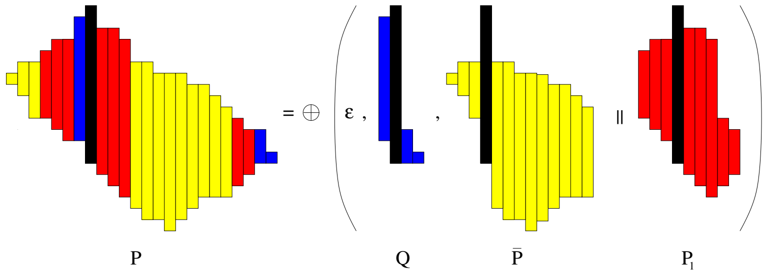


Figure 4: The first two steps in the decomposition of a polyomino $P \in \mathsf{DConv}_2^\bullet$, $P = \oplus(\epsilon, Q, \bar{P} \parallel P_1)$, with $Q \in \mathsf{LR}, \bar{P} \in \mathsf{Z}_2, P_1 \in$ s-$\mathsf{Z}_2^\bullet$.

# 4 Conclusions

We have developed a C++ program that exploits dynamic programming and uses space $O(n^8)$ to solve suitable recurrence equations associated with (1) and (5), and with the set equation for $\mathsf{ZConv}^\circ$. Here is the table of the first 40 entries of the counting sequence of $\mathsf{ZConv}$. We point out that this sequence does not appear in OEIS.

Table 1: $|\mathsf{ZConv}(n)|$ for $0 \leq n \leq 40$.

| |
|---|
| 0, 1, 2, 6, 19, 55, 148, 370, 874, 1966, 4240, 8816, 17773, 34858, 66734, 125014, 229647, 414412 735762, 1286908, 2220035, 3781065, 6363460, 10591124, 17444763, 28453652, 45984090, 73671398, 117061785, 184562194, 288836144, 448846754, 692828996, 1062596751, 1619750728, 2454592300, 3698861168, 5543870866, 8266217558, 12264097608, 18108408216 |

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