A CHARACTERIZATION OF THE VECTOR LATTICE OF MEASURABLE FUNCTIONS

SIMONE CERREIA-VIOGLIO, PAOLO LEONETTI, AND FABIO MACCHERONI

Abstract. Given a probability measure space (X, Σ, μ), it is well known that the Riesz space $L^0(\mu)$ of equivalence classes of measurable functions $f : X \to \mathbf{R}$ is universally complete and the constant function $\mathbf{1}$ is a weak order unit. Moreover, the linear functional $L^\infty(\mu) \to \mathbf{R}$ defined by $f \mapsto \int f \, \mathrm{d}\mu$ is strictly positive and order continuous. Here we show, in particular, that the converse holds true, i.e., any universally complete Riesz space E with a weak order unit $e > 0$ which admits a strictly positive order continuous linear functional on the principal ideal generated by e is lattice isomorphic onto $L^0(\mu)$, for some probability measure space (X, Σ, μ).

1. Introduction

A classical result of Kakutani [17] states that every AL-space, that is, every Banach lattice with the norm additive on pairs of positive disjoint vectors, has to be a space $L^1(\mu) = L^1(X, \Sigma, \mu)$ of equivalence classes of μ-integrable functions $f : X \to \mathbf{R}$, where $\mu : \Sigma \to [0, \infty]$ is a σ-additive measure. In addition, if there exists a weak order unit, then μ can be chosen finite. This is a characterization of the class of integrable functions by properties of the norm and order.

Relying on this result, Masterson [21] proved a classification for the set of (equivalence classes of) real-valued measurable functions (see Section 1.1 for definitions):

Theorem 1.1. *Let E be an Archimedean Riesz space. Then there exists an onto lattice isomorphism $E \to L^0(\mu)$, for some σ-finite measure space (X, Σ, μ), if and only if E is universally complete, has the countable sup property, and the extended order continuous dual of E is separating on E.*

2010 *Mathematics Subject Classification.* Primary: 16G30; Secondary: 18B35, 46A40.

Key words and phrases. Measurable functions, lateral completeness, f-algebra of L^0 type, extended order continuous dual, strictly positive order continuous linear functional.

Note that Theorem 1.1 involves only order properties. Here, the extended order continuous dual of E, usually denoted by $\Gamma(E)$, is the set of equivalence classes of order continuous linear functionals defined on order dense ideals of E, where two functionals are identified whenever they agree on an order dense ideal of E, cf. [20, §1]. It is well known that $\Gamma(E)$ is separating on E if and only if there exists an order dense ideal I of E such that the order continuous dual of I is separating on I, and in that case there exists an order dense ideal which admits a strictly positive order continuous linear functional, see [20, Theorem 2.5]. Other equivalent conditions are provided in [11, Theorem 3.4]; in particular, if $\Gamma(E)$ is separating on E, then there exists a measure space (X, Σ, μ) for which E can be embedded order densely into $L^0(\mu)$.

Related results concerning representations of Archimedean Riesz spaces as spaces of measurable functions can be found, e.g., in Pinsker [22], Fremlin [12], and Labuda [19], and are surveyed by Filter [11, Section 3].

The aim of this work is to obtain a concrete characterization of the space of (equivalence classes of) measurable real-valued functions $L^0(X, \Sigma, \mu)$, where $\mu : \Sigma \to \mathbf{R}$ is a *probability* measure, which is analogous to Theorem 1.1, and relies more on algebraic than on order properties (see also [13, Chapter 36]). Remarkably, this characterization avoids the use of the extended order continuous dual, thus providing an operational criterion to establish when a vector lattice is necessarily a space of random variables, and the proof of our result is self-contained.

In the recent years there has been a lot of research in L^0-modules and their applications. See, for example the works of Cerreia-Vioglio et al. [5, 6, 7], Doldi and Frittelli [8], Filipović et al. [10], Frittelli and Maggis [14, 15], and Hoffmann et al. [16]. An abstract characterization of $L^0(\mu)$ extends the scope of these applications to modules that are not *prima facie* on $L^0(\mu)$, such as the modules on algebras of stochastic processes that are sometimes used in mathematical finance (e.g., modules on the algebras of predictable and progressively measurable processes, see Doob [9]).

Another advantage of this paper is introducing the possibility of working with "the scalars" of L^0-modules from a purely algebraic/functional analytic perspective. Dispensing with the —sometimes cumbersome— techniques needed to consider zero measure sets, a.s. null functions, and the induced quotient spaces.

Dually, our result delivers a concrete representation for f-algebras of L^0 type considered in [5, 6], which was the original motivation for this work (see Section 2 below).

1.1. **Notation.** We refer to [3] for basic aspects of Riesz spaces. Let E be a Riesz space. Then, we denote the positive cone of a Riesz subspace F by $F^+ := \{x \in F : x \geq 0\}$. A net $(x_\alpha)_{\alpha \in A}$ with values in E is said to be order convergent to $x \in E$ if there exists a net $(y_\alpha)_{\alpha \in A}$ with the same index set satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all $\alpha \in A$. A non-empty subset $S \subseteq E$ is said to be solid if $|y| \leq |x|$ implies $y \in S$ whenever $x \in S$. The principal ideal generated by a vector $x \in E$, that is, the smallest solid Riesz subspace containing x, is denoted by E_x. A vector $e > 0$ is called a *strong order unit* if the principal ideal generated by e, namely,

$$E_e = \{y \in E : |y| \leq \lambda e \text{ for some } \lambda \in \mathbf{R}\},$$

coincides with E. Instead, e is said to be a *weak order unit* if, for each $x \in E$, there exists a net $(x_\alpha)_{\alpha \in A}$ with values in E_e which is order convergent to x.

E is said to be *laterally complete* [respectively, *laterally σ-complete*] if the supremum of every disjoint subset [resp., sequence] of E^+ exists in E. If E is also Dedekind complete, then we say that E is *universally complete*. E has the *countable sup property* if for every subset S of E whose supremum exists in E, there exists an at most countable subset of S having the same supremum as S in E. A (not necessarily Hausdorff) topology τ on a Riesz space E is said to be *locally solid* if τ has a base at zero consisiting of solid sets.

As usual, a probability measure space (X, Σ, μ) is a non-empty set X, together with a σ-algebra Σ of subsets of X, and a σ-additive measure $\mu : \Sigma \to \mathbf{R}$ with $\mu(X) = 1$. Moreover, $\mathbf{1}$ stands for the multiplicative unit of $L^0(\mu)$, whenever the underlying measure space is understood. Finally, given an integrable function $f \in L^1(\mu)$, we shorten $\int f \, \mathrm{d}\mu$ with $\mu(f)$.

2. THE CHARACTERIZATION

We start with a preliminary observation, whose proof is given in Section 3.

Lemma 2.1. *Let E be a Riesz space with weak order unit $e > 0$ and let $\varphi : E_e \to \mathbf{R}$ be a strictly positive linear functional. Then*

$$d_\varphi : E \times E \to \mathbf{R} : (x, y) \mapsto \varphi(|x - y| \wedge e) \tag{1}$$

is an invariant metric and the topology τ_φ generated by d_φ is Hausdorff locally solid.

Our main result follows.

Theorem 2.2. *Let E be a Dedekind complete Riesz space with weak order unit $e > 0$. Then the following are equivalent:*

(i) *There exist a probability measure space (X, Σ, μ) and an onto lattice isomorphism $T : E \to L^0(\mu)$ such that $T(e) = \mathbf{1}$.*

(ii) *There exists a strictly positive order continuous linear functional $\varphi : E_e \to \mathbf{R}$ such that the metric d_φ is complete.*

(iii) *There exists a strictly positive order continuous linear functional $\psi : E_e \to \mathbf{R}$ and E is laterally complete.*

Moreover, in such case, E_e is lattice isomorphic onto $L^\infty(\mu)$, the metrics d_φ and d_ψ are topologically equivalent, i.e., $\tau_\varphi = \tau_\psi$, and E has the countable sup property.

The implication (ii) $\implies$ (i) is related to [18, Theorem 6.4], which characterizes norm dense ideals of $L^1(\mu)$. To the best of our knowledge, the equivalence (i) $\iff$ (iii) is completely new.

As an immediate consequence of Theorem 2.2, we obtain a result in the same spirit of Theorem 1.1. Indeed, recalling that $L^0(\mu)$ is universally complete [3, Theorem 7.73] and has a weak order unit $\mathbf{1}$, it follows that (we omit details):

Corollary 2.3. *Let E be an Archimedean Riesz space. Then E is lattice isomorphic onto $L^0(\mu)$, for some probability measure space (X, Σ, μ), if and only if E is universally complete (hence, with weak order unit $e > 0$) and admits a strictly positive order continuous linear functional on E_e.*

Finally, we obtain a charaterization of f-algebras of L^0 type, cf. [6, Definition 6]. In this regard, we recall that an f-algebra is a Riesz algebra E for which $(a \cdot c) \wedge b = (c \cdot a) \wedge b = 0$ for all $a, b, c \geq 0$ such that $a \wedge b = 0$. If, in addition, E is Dedekind complete and admits a non-zero multiplicative unit e, then it is said to be a Stonean algebra, cf. [5, Definition 2]. In such case, the following facts are well known and readily provable: (i) The multiplication is commutative, i.e., $a \cdot b = b \cdot a$ for all $a, b \in E$, (ii) $x^2 := x \cdot x \geq 0$ for all $x \in E$; in particular, $e > 0$, and (iii) e is a weak order unit.

Accordingly, a Stonean algebra E is said to be *f-algebra of L^0 type* whenever the principal ideal E_e is an Arens algebra, i.e., a real commutative Banach algebra such that $\|e\| = 1$ and $\|a\|^2 \le \|a^2 + b^2\|$ for all $a, b \in E_e$, and there exists a strictly positive order continuous linear functional φ on E_e such that the metric d_φ defined in (1) is complete. As an application, Theorem 2.2 implies that f-algebras of L^0 type are (equivalence classes of) spaces of random variables.

Corollary 2.4. *Let E be an Archimedean f-algebra with non-zero multiplicative unit. Then E is an f-algebra of L^0 type if and only if E is lattice and algebra isomorphic onto $L^0(\mu)$, for some probability measure space (X, Σ, μ).*

Finally, it is worth noting that the topological equivalence of d_φ and d_ψ at the end of Theorem 2.2 cannot be strengthened to strongly equivalence, as it is shown in the following example.

Example 2.5. Let μ be the function $\mathcal{P}(\mathbf{N}) \to \mathbf{R} : X \mapsto \sum_{x \in X} 2^{-x}$, where $\mathbf{N}$ is the set of positive integers and $\mathcal{P}(\mathbf{N})$ its powerset. Then $(\mathbf{N}, \mathcal{P}(\mathbf{N}), \mu)$ is a probability measure space, $L^0(\mu)$ is the space of real-valued sequences (indexed by $\mathbf{N}$), and $L^\infty(\mu)$ is the ideal generated by $e = (1, 1, \ldots)$, i.e., the subspace of bounded sequences ℓ^∞. Accordingly, define the strictly positive order continuous linear functionals $\varphi : \ell^\infty \to \mathbf{R}$ and $\psi : \ell^\infty \to \mathbf{R}$ mapping each $x = (x_1, x_2, \ldots)$ into $\sum_{n \ge 1} x_n 2^{-n}$ and $\sum_{n \ge 1} x_n 3^{-n}$, respectively.

With this, let us suppose for the sake of contradiction that there exists a positive constant c such that $d_\varphi(x, y) \le c d_\psi(x, y)$ for all $x, y \in L^0(\mu)$. Moreover, for each $n \in \mathbf{N}$, define $e_n = (0, \ldots, 0, 1, 1, \ldots)$, where 0 is repeated exactly n times. Then, it would follow

$$\textstyle\sum_{k \ge n} 2^{-k} = \varphi(e_n) = d_\varphi(e_n, 0) \le c d_\psi(e_n, 0) = c\psi(e_n) = c \sum_{k \ge n} 3^{-k},$$

which is false whenever n is sufficiently large.

Proofs of Theorem 2.2 and Corollary 2.4 follow in Section 4.

3. Preliminaries

We start with the proof of Lemma 2.1.

Proof of Lemma 2.1. Note that d_φ is well defined since E_e is solid and $0 \le |x - y| \wedge e \le e \in E_e$ for all $x, y \in E$. Since e is a weak order unit, $|x - y| \wedge e = 0$ if and only if $x = y$. Then, the strict positivity of φ implies that $d_\varphi(x, y) = d_\varphi(y, x) \ge 0$ for all $x, y \in E$, with equality if and only if $x = y$. Finally, for each $x, y, z \in E$, we have $|x - z| \le |x - y| + |y - z|$, so that, thanks to [3, Theorem 1.7.(4)],

$$|x - z| \wedge e \le (|x - y| + |y - z|) \wedge e \le |x - y| \wedge e + |y - z| \wedge e.$$

Since φ is a positive operator, we obtain $d_\varphi(x,z) \le d_\varphi(x,y) + d_\varphi(y,z)$. Clearly, d_φ is invariant and (E, τ_φ) is Hausdorff.

Finally, the local solidness follows by the fact each open ball B centered in 0 and with radius $r > 0$ is solid. Indeed, given $x, y \in E$ with $|x| \le |y|$ and $y \in B$, then by the positivity of φ we get $\varphi(|x| \wedge e) \le \varphi(|y| \wedge e)$, that is, $x \in B$. □

The following result is classical, hence we omit its proof.

Lemma 3.1. *Let E, F be Riesz spaces and let $T : E \to F$ be an onto lattice isomorphism. Then T is order continuous.*

Finally, we will use the following characterization of $L^\infty(\mu)$; cf. also Abramovich, Aliprantis, and Zame [1, Corollary 2.2].

Lemma 3.2. *Let E be a Dedekind complete Riesz space with strong order unit $e > 0$ which admits a strictly positive order continuous linear functional φ. Then there exist a probability measure space (X, Σ, μ) and an onto lattice isomorphism $T : E \to L^\infty(\mu)$ such that $T(e) = \mathbf{1}$ and $\varphi(x) = \mu(T(x))$ for all $x \in E^+$.*

Proof. Since φ is strictly positive, then $\varphi(e) > 0$. Hence, dividing by $\varphi(e)$, we can suppose without loss of generality that $\varphi(e) = 1$. It follows that

$$\|\cdot\| : E \to \mathbf{R} : x \mapsto \varphi(|x|)$$

is an order continuous L-norm. Let $\widehat{E}$ be the topological completion of E. Then, $\widehat{E}$ is an AL-space and, according to [1, Footnote 6], E is an (order dense) ideal of $\widehat{E}$. It follows by Kakutani's representation theorem [17, Theorem 7] that there exists an onto lattice and isometric $\widehat{T} : \widehat{E} \to L^1(\mu)$, for some probability measure space (X, Σ, μ), such that $\widehat{T}(e) = 1$. In particular,

$$\varphi(x) = \mu(\widehat{T}(x))$$

for all $x \in E^+$. In addition, since e is unit of E and E is an ideal of $\widehat{E}$, then $E = E_e = \widehat{E}_e$. The claim follows by letting T equal to the restriction of $\widehat{T}$ from E to its direct image. □

4. PROOF OF THE MAIN RESULT

Proof of Theorem 2.2. We are going to show the following chain of equivalences:

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)} \implies \text{(ii)} \implies \text{(i)}.$$

(i) $\Longrightarrow$ (ii). Let us assume that there exist a probability measure space (X, Σ, μ) and an onto lattice isomorphism $T : E \to L^0(\mu)$ such that $T(e) = \mathbf{1}$. In particular, T is a positive operator. It follows that $T([-\lambda e, \lambda e]) = [-\lambda T(e), \lambda T(e)]$, hence

$$T(E_e) = T\left(\bigcup_{\lambda>0}[-\lambda e, \lambda e]\right) = \bigcup_{\lambda>0}[-\lambda \mathbf{1}, \lambda \mathbf{1}] = L^\infty(\mu). \tag{2}$$

Therefore, the restriction of T on E_e, hereafter denoted by T_e, is a lattice isomorphism onto $L^\infty(\mu)$. Note that, thanks to Lemma 3.1, T_e is order continuous.

At this point, define the linear functional

$$\varphi : E_e \to \mathbf{R} : x \mapsto \mu(T(x)).$$

It is routine to check that φ is strictly positive. Moreover, φ is order continuous. To this aim, since φ is a positive operator, it is enough to show that $\varphi(x_\alpha) \downarrow 0$ for every net $(x_\alpha) \downarrow 0$ in E_e. Since $\mathbf{R}$ is an Archimedean Riesz space with the countable sup property and $\varphi : E_e \to \mathbf{R}$ is strictly positive, it follows by [3, Theorem 1.45] that E_e has the countable sup property as well. In particular, it is enough to show that $\varphi(x_n) \downarrow 0$ for every sequence $(x_n) \downarrow 0$ in E_e. Since T_e is order continuous, $T_e(x_n) \downarrow 0$ in $L^\infty(\mu)$. Finally $\varphi(x_n) = \mu(T_e(x_n)) \downarrow 0$ by Lebesgue's dominated convergence theorem.

Finally, we need to prove that the metric space (E, d_φ) is (topologically) complete. Let d be the metric of convergence in measure on $L^0(\mu)$, that is,

$$d : L^0(\mu) \times L^0(\mu) \to \mathbf{R} : (f, g) \mapsto \mu(|f - g| \wedge \mathbf{1}).$$

Hence, for all $x, y \in E$, we obtain

$$\begin{aligned} d_\varphi(x, y) &= \varphi(|x - y| \wedge e) = \mu(T(|x - y| \wedge e)) = \mu(T(|x - y|) \wedge T(e)) \\ &= \mu(|T(x - y)| \wedge \mathbf{1}) = \mu(|T(x) - T(y)| \wedge \mathbf{1}) = d(T(x), T(y)). \end{aligned} \tag{3}$$

Then, fix a Cauchy sequence (x_n) of vectors in E, i.e., for each $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that $d_\varphi(x_n, x_m) \le \varepsilon$ whenever $n, m \ge n_0$. It follows from (3) that $(T(x_n))$ is a Cauchy sequence in $(L^0(\mu), d)$. Since the metric space $(L^0(\mu), d)$ is complete, there exists $f \in L^0(\mu)$ such that $d(T(x_n), f) \to 0$ as $n \to +\infty$. Moreover, T is a bijection, hence there exists $x \in E$ such that $T(x) = f$. Therefore, thanks to (3), we obtain $d_\varphi(x_n, x) \to 0$ as $n \to +\infty$.

(ii) $\Longrightarrow$ (iii). Suppose that there exists a strictly positive order continuous linear functional $\varphi : E_e \to \mathbf{R}$ for which the metric space (E, d_φ) is complete, and set $\varphi = \psi$.

Since e is a weak order unit and E is Dedekind complete, it follows by [3, Theorem 7.39] that it is enough to show that E is laterally σ-complete. To this aim, let (x_n) be a sequence of disjoint vectors in E^+ and define the sequences (y_n) by $y_n := x_n \wedge e$ for each $n \geq 1$. Note that (y_n) is a disjoint sequence of vectors in the order interval $[0, e]$. Moreover, for each positive integer n, define

$$a_n := x_1 + \cdots + x_n \quad \text{and} \quad b_n := y_1 + \cdots + y_n.$$

Since E is Dedekind complete and $b_n = y_1 \vee \cdots \vee y_n \leq e$ for each $n \geq 1$, then the supremum of the sequence (b_n) exists in $[0, e]$, and we denote it by b. Hence $0 \leq b - b_n \downarrow 0$, which implies $0 \leq (b - b_n) \wedge e \downarrow 0$. Since φ is order continuous, then

$$\lim_{n\to\infty} d_\varphi(b_n, b) = 0.$$

In particular, (b_n) is a Cauchy sequence in (E, d_φ). In addition, for all positive integers n, m with $n > m$, it holds

$$\begin{aligned} d_\varphi(a_n, a_m) &= \varphi((a_n - a_m) \wedge e) = \varphi((x_{m+1} \vee \cdots \vee x_n) \wedge e) = \varphi(y_{m+1} \vee \cdots \vee y_n) \\ &= \varphi((y_{m+1} + \cdots + y_n) \wedge e) = \varphi((b_n - b_m) \wedge e) = d_\varphi(b_n, b_m). \end{aligned}$$

It follows that also (a_n) is a Cauchy sequence in (E, d_φ). Since (E, d_φ) is complete by hypothesis, there exists $a \in E$ such that

$$\lim_{n\to\infty} d_\varphi(a_n, a) = 0. \tag{4}$$

Thanks to Lemma 2.1, (E, τ_φ) is a locally solid Hausdorff Riesz space. Therefore, according to [3, Theorem 2.21.(c)] and (4), it follows that $x_1 \vee \cdots \vee x_n = a_n \uparrow a$. By the previous argument, this implies that E is laterally complete.

(iii) $\implies$ (ii). Set $\varphi = \psi$ and note that $\tau := \tau_\psi$ is a Fatou topology on E, i.e., it has a neighborhood base at 0 consisting of solid and order closed sets. Let $(\widehat{E}, \hat{\tau})$ be the topological completion of (E, τ).

According to a classical result of Nakano, see e.g. [3, Theorem 4.28], since (E, τ) is a Dedekind complete locally solid Riesz space with the Fatou property, then the order intervals of E are τ-complete. Fix $x \in E$ and $\hat{x} \in \widehat{E}$ such that $0 \leq \hat{x} \leq x$ in $\widehat{E}$ and let (y_α) be a net of positive vectors in E such that $y_\alpha \xrightarrow{\tau} \hat{x}$. This implies that $x_\alpha \xrightarrow{\tau} \hat{x}$, where $x_\alpha := y_\alpha \wedge x$ for each index α. Since $x_\alpha \in [0, x]$ for each α and the order intervals are τ-complete, then $\hat{x} \in E$. Hence E is an ideal of $\widehat{E}$. (An alternative proof of this fact can be found also in [2, Theorem 2.2].)

Moreover, given $0 \le \hat{x} \in \widehat{E}$ and a net (x_α) of positive vectors in E such that $x_\alpha \xrightarrow{\tau} \hat{x}$, then $x_\alpha \wedge \hat{x}$ belongs to E^+ (since E is an ideal). Hence, considering the finite suprema of the net $(x_\alpha \wedge \hat{x})$, we obtain a net (y_β) of vectors in E^+ such that $y_\beta \xrightarrow{\tau} \hat{x}$ and $y_\beta \uparrow \hat{x}$. This means that E is an order dense ideal of $\widehat{E}$.

Therefore, since E is a universally complete order dense Riesz subspace of the Archimedean Riesz space $\widehat{E}$, then $E = \widehat{E}$ by the uniqueness of the universal completion, see e.g. [3, Theorem 7.15.(ii)].

(ii) $\implies$ (i). Suppose that an increasing net $(x_\alpha)_{\alpha \in A}$ of positive vectors in E_e is upper bounded by some $y \in E_e$. Then $x := \sup\{x_\alpha : \alpha \in A\}$ exists in E and belongs to the order interval $[0, y]$. Since E_e is solid, then $x \in E_e$. Therefore, thanks to [3, Lemma 1.39], E_e is a Dedekind complete Riesz subspace with strong order unit $e > 0$. It follows by Lemma 3.2 that there exist a probability measure space (X, Σ, μ) and an onto lattice isomorphism

$$T_{\mathrm{e}} : E_e \to L^\infty(\mu)$$

such that $T_{\mathrm{e}}(e) = \mathbf{1}$ and $\varphi(x) = \mu(T_{\mathrm{e}}(x))$ for all $0 \le x \in E_e$. Then, for all $x, y \in E_e$ we obtain

$$\begin{aligned} d(T_{\mathrm{e}}(x), T_{\mathrm{e}}(y)) &= \mu(|T_{\mathrm{e}}(x) - T_{\mathrm{e}}(y)| \wedge \mathbf{1}) \\ &= \mu(T_{\mathrm{e}}(|x-y|) \wedge \mathbf{1}) = \mu(T_{\mathrm{e}}(|x-y| \wedge e)) \\ &= \varphi(|x-y| \wedge e) = d_\varphi(x, y). \end{aligned} \tag{5}$$

CLAIM 1. *E is the topological closure of E_e in (E, τ_φ).*

Proof. Given $x \in E^+$, then $(x_n) \uparrow x$, where $x_n := x \wedge ne$, by the fact that e is a weak order unit. This implies that $(|x - x_n| \wedge e) \downarrow 0$. Since φ is order continuous, then

$$d_\varphi(x_n, x) = \varphi(|x - x_n| \wedge e) \downarrow 0,$$

i.e., $x_n \to x$ in (E, τ_φ). The claim follows by the fact that $x = x^+ - x^-$ for each $x \in E$ and the topological limits are linear. $\square$

CLAIM 2. *There exists a positive operator $T : E \to L^0(\mu)$ extending T_{e} for which (5) holds for all $x, y \in E$.*

Proof. Define the operator $T : E \to L^0(\mu)$ as the unique extension of

$$E^+ \to L^0(\mu) : x \mapsto \lim_{n \to \infty} T_{\mathrm{e}}(x_n), \tag{6}$$

where $(x_n)_{n\geq 1}$ is any sequence in E_e^+ such that $x_n \to x$ in (E, τ_φ). The limit in (6) is understood to be in $(L^0(\mu), d)$.

At first, we show that T is well defined. To prove the existence of the limit, fix a sequence (x_n) of vectors in E_e such that $x_n \to x$ (note that such sequence exists by Claim 1). Then (x_n) is a Cauchy sequence. It follows by (5) that $(T_\mathrm{e}(x_n))$ is a Cauchy sequence in $(L^0(\mu), d)$. Then, by the completeness of the latter space, there exists (a unique) $f \in L^0(\mu)$ such that $\lim_{n\to\infty} T_\mathrm{e}(x_n) = f$.

Then, we show that the limit in (6) is independent from the choice of the sequence (x_n). Indeed, let us suppose that (x'_n) is another sequence of vectors such that $x'_n \to x$ in (E, τ_φ). This implies that $x_n - x'_n \to 0$, i.e.,

$$\lim_{n\to\infty} \varphi(|x_n - x'_n| \wedge e) = 0.$$

Since $x_n - x'_n \in E_e$ for each n and E_e is Dedekind complete, there exists $\ell \in E_e$ such that $\ell = \inf\{|x_n - x'_n| : n \geq 1\}$. In particular, there exists a real $\lambda > 0$ such that $\ell \leq \lambda e$. Clearly, $\ell \geq 0$ and, by the strict positivity of φ, it follows that $\varphi(|x_n - x'_n| \wedge e) \geq \varphi(\ell \wedge e)$ for all n, proving that $\ell \wedge e = 0$. Hence $\ell = \ell \wedge \lambda e = 0$. By the same argument, it is easy to see that there does not exist any $y > 0$ in E_e such that $|x_n - x'_n| \geq y$ for infinitely many n. In particular, choosing $y = {}^1\!/_k\, e$, we obtain that $x_n - x'_n$ belongs to the order interval $[-{}^1\!/_k\, e, {}^1\!/_k\, e]$ whenever n is sufficiently large. This implies that $x_n - x'_n$ converges to 0 with respect to the order, i.e.,

$$x_n - x'_n \xrightarrow{o} 0. \tag{7}$$

Since T_e is a lattice isomorphism onto $L^\infty(\mu)$, then it is also order continuous, thanks to Lemma 3.1. Hence $T_\mathrm{e}(x_n - x'_n) \xrightarrow{o} 0$ in $L^\infty(\mu)$, which is equivalent to

$$\lim_{n\to\infty} T_\mathrm{e}(x_n - x'_n)(\omega) = 0$$

for each $\omega \in X$. Since it is well known that puntual convergence implies convergence in measure, then

$$\lim_{n\to\infty} d(T_\mathrm{e}(x_n - x'_n), 0) = 0,$$

which is what we wanted to show.

In addition, it is routine to check that T is a positive operator.

Finally, for each $x, y \in E$, there exist by Claim 1 two sequences of vectors (x_n) and (y_n) in E_e such that $x_n \to x$ and $y_n \to y$ in (E, τ_φ). Thanks to Claim 1 and

(5), we get

$$d_\varphi(x,y) = \lim_{n\to\infty} d_\varphi(x_n,y_n) = \lim_{n\to\infty} d(T_{\mathrm{e}}(x_n), T_{\mathrm{e}}(y_n))$$
$$= d(\lim_{n\to\infty} T_{\mathrm{e}}(x_n), \lim_{n\to\infty} T_{\mathrm{e}}(y_n)) = d(T(x),T(y))$$

for all $x,y \in E^+$, hence also for all $x,y\in E$. □

CLAIM 3. T is an onto lattice isomorphism.

Proof. Fix $0 \le f \in L^0(\mu)$. Since the constant function $\mathbf{1}$ is a weak order unit, then $f_n \uparrow f$, where $f_n := f \wedge n\mathbf{1}$ for each positive integer n. Again, since puntual convergence implies convergence in measure, we get $f_n \to f$ in $(L^0(\mu), d)$. In particular, (f_n) is a Cauchy sequence.

At this point, define $x_n := T_{\mathrm{e}}^{-1}(f_n)$ for each n. Note that (x_n) is a sequence of positive vectors in E_e and, thanks to (5), is a Cauchy sequence in (E, τ_φ). Since the metric d_φ is complete by hypothesis, there exists $x \in E^+$ for which $x_n \to x$. According to (6), we conclude that

$$T(x) = \lim_{n\to\infty} T_{\mathrm{e}}(x_n) = \lim_{n\to\infty} f_n = f,$$

i.e., $f \in T(E)$. Then, by the arbitrariness of f, T is onto, i.e., $T(E) = L^0(\mu)$.

To sum up, $T : E \to L^0(\mu)$ is a one-to-one and onto linear operator such that T and T^{-1} are both positive operators. Therefore, thanks to [3, Exercise 16], T is an onto lattice isomorphism. □

At this point, note that, if one of the equivalent conditions (i)-(iii) hold, then E_e is lattice isomorphic onto $L^\infty(\mu)$, thanks to (2).

Also, the metrics d_φ and d_ψ are topologically equivalent: indeed, a laterally complete Riesz space admits at most one Hausdorff Fatou topology, which must be necessarily a Lebesgue topology (i.e., $x_\alpha \xrightarrow{\tau} 0$ whenever $x_\alpha \downarrow 0$), see e.g. [3, Theorem 7.53].

Finally, suppose that $0 \le x_\alpha \uparrow x$ in E, hence by the Lebesgue property $x - x_\alpha \xrightarrow{\tau} 0$, i.e., $\varphi((x-x_\alpha)\wedge e) \to 0$. Then, there exists a subsequence (x_{α_n}) of the net (x_α) such that $\varphi((x-x_{\alpha_n})\wedge e) \to 0$ as $n\to\infty$, i.e., $x - x_{\alpha_n} \xrightarrow{\tau} 0$. Considering that $x - x_{\alpha_n}$ is a decreasing sequence, we conclude that $x - x_{\alpha_n} \downarrow 0$ by [3, Theorem 2.21.(c)], that is, $x_{\alpha_n} \uparrow x$. This means that E has the countable sup property. □

Let us conclude with the proof of the last corollary.

Proof of Corollary 2.4. If E is lattice and algebra isomorphic onto $L^0(\mu)$, for some probability measure space (X, Σ, μ), then it is easy to check that E is an f-algebra of L^0 type (we omit details).

Conversely, let us suppose that E is an f-algebra of L^0 type. Then, in particular, E is a Dedekind complete Riesz space with weak order unit $e > 0$ and admits a strictly positive order continuous linear functional $\varphi : E_e \to \mathbf{R}$ such that the metric d_φ defined in (1) is complete. It follows by Theorem 2.2 that there exists a lattice isomorphism $T : E \to L^0(\mu)$, for some probability measure space (X, Σ, μ).

Then, we have to prove that T is also an algebra isomorphism. Note that the multiplication $\cdot$ defined by

$$x \cdot y := T^{-1}(T(x)T(y))$$

for all $x, y \in E$ makes E an Archimedean f-algebra with multiplicative unit $e > 0$. The claim follows by the fact that there exists at most one algebra multiplication on an Archimedean Riesz space L that makes L an Archimedean f-algebra with given unit, see e.g. [4, Theorem 2.58]. □

4.1. **Acknowledgements.** The authors are grateful to PRIN 2017 (grant 2017CY2NCA) for financial support, and to an anonymous reviewer for constructive comments which improved the overall presentation.

REFERENCES

1. Y. A. Abramovich, C. D. Aliprantis, and W. R. Zame, *A representation theorem for Riesz spaces and its applications to economics*, Econom. Theory **5** (1995), no. 3, 527–535.
2. C. D. Aliprantis, *On the completion of Hausdorff locally solid Riesz spaces*, Trans. Amer. Math. Soc. **196** (1974), 105–125.
3. C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces with applications to economics*, second ed., Mathematical Surveys and Monographs, vol. 105, American Mathematical Society, Providence, RI, 2003.
4. ———, *Positive operators*, Springer, Dordrecht, 2006, Reprint of the 1985 original.
5. S. Cerreia-Vioglio, M. Kupper, F. Maccheroni, M. Marinacci, and N. Vogelpoth, *Conditional L_p-spaces and the duality of modules over f-algebras*, J. Math. Anal. Appl. **444** (2016), no. 2, 1045–1070.
6. S. Cerreia-Vioglio, F. Maccheroni, and M. Marinacci, *Hilbert A-modules*, J. Math. Anal. Appl. **446** (2017), no. 1, 970–1017.
7. ———, *Orthogonal decompositions in Hilbert A-modules*, J. Math. Anal. Appl. **470** (2019), no. 2, 846–875.
8. A. Doldi and M. Frittelli, *Conditional systemic risk measures*, SIAM J. Financial Math. **12** (2021), no. 4, 1459–1507.

9. J. L. Doob, *Classical potential theory and its probabilistic counterpart*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1984 edition.
10. D. Filipović, M. Kupper, and N. Vogelpoth, *Approaches to conditional risk*, SIAM J. Financial Math. **3** (2012), no. 1, 402–432.
11. W. Filter, *Representations of Archimedean Riesz spaces—a survey*, Rocky Mountain J. Math. **24** (1994), no. 3, 771–851.
12. D. H. Fremlin, *Abstract Köthe spaces. II*, Proc. Cambridge Philos. Soc. **63** (1967), 951–956.
13. ______, *Measure theory. Vol. 3*, Torres Fremlin, Colchester, 2004, Measure algebras, Corrected second printing of the 2002 original.
14. M. Frittelli and M. Maggis, *Conditional certainty equivalent*, Int. J. Theor. Appl. Finance **14** (2011), no. 1, 41–59.
15. ______, *Complete duality for quasiconvex dynamic risk measures on modules of the L^p-type*, Stat. Risk Model. **31** (2014), no. 1, 103–128.
16. H. Hoffmann, T. Meyer-Brandis, and G. Svindland, *Risk-consistent conditional systemic risk measures*, Stochastic Process. Appl. **126** (2016), no. 7, 2014–2037.
17. S. Kakutani, *Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Ann. of Math. (2) **42** (1941), 523–537.
18. W.-C. Kuo, C. C. A. Labuschagne, and B. A. Watson, *Conditional expectations on Riesz spaces*, J. Math. Anal. Appl. **303** (2005), no. 2, 509–521.
19. I. Labuda, *Submeasures and locally solid topologies on Riesz spaces*, Math. Z. **195** (1987), no. 2, 179–196.
20. W. A. J. Luxemburg and J. J. Masterson, *An extension of the concept of the order dual of a Riesz space*, Canad. J. Math. **19** (1967), 488–498.
21. J. J. Masterson, *A characterization of the Riesz space of measurable functions*, Trans. Amer. Math. Soc. **135** (1969), 193–197.
22. A. G. Pinsker, *On concrete representations of linear semi-ordered spaces*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **55** (1947), 379–381.

UNIVERSITÀ "LUIGI BOCCONI", DEPARTMENT OF DECISION SCIENCES, MILAN, ITALY
Email address: simone.cerreia@unibocconi.it

UNIVERSITÀ "LUIGI BOCCONI", DEPARTMENT OF DECISION SCIENCES, MILAN, ITALY
Email address: paolo.leonetti@unibocconi.it

UNIVERSITÀ "LUIGI BOCCONI", DEPARTMENT OF DECISION SCIENCES, MILAN, ITALY
Email address: fabio.maccheroni@unibocconi.it