



# Asymptotic Behavior of Ground States and Local Uniqueness for Fractional Schrödinger Equations with Nearly Critical Growth

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## Abstract

We study quantitative aspects and concentration phenomena for ground states of the following nonlocal Schrödinger equation

$$(-\Delta)^s u + V(x)u = u^{2_s^* - 1 - \epsilon} \text{ in } \mathbb{R}^N,$$

where  $\epsilon > 0$ ,  $s \in (0, 1)$ ,  $2_s^* := \frac{2N}{N-2s}$  and  $N > 4s$ , as we deal with finite energy solutions. We show that the ground state  $u_\epsilon$  blows up and precisely with the following rate  $\|u_\epsilon\|_{L^\infty(\mathbb{R}^N)} \sim \epsilon^{-\frac{N-2s}{4s}}$ , as  $\epsilon \rightarrow 0^+$ . We also localize the concentration points and, in the case of radial potentials  $V$ , we prove local uniqueness of sequences of ground states which exhibit a concentrating behavior.

**Keywords** Nonlocal equations · Fractional Laplacian · Blow-up phenomena · Ground states · Critical growth

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## 1 Introduction

In this paper we consider the following class of nonlocal equations

$$(-\Delta)^s u + V(x)u = u^{2_s^* - 1 - \epsilon} \text{ in } \mathbb{R}^N, \quad (1.1)$$

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where  $\epsilon \rightarrow 0^+, s \in (0, 1), 2_s^* := \frac{2N}{N-2s}, N > 4s, V : \mathbb{R}^N \rightarrow \mathbb{R}$  is a potential function and

$$(-\Delta)^s u(x) = c_{N,s} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy$$

is the fractional Laplacian. Here,  $c_{N,s}$  is a normalizing constant, PV stands for the Cauchy principal value. As we are going to see, the restriction on the dimension is due to the fact that we look for finite  $L^2$ -energy solutions.

For fixed  $\epsilon \in (0, 2_s^* - 2)$ , under suitable conditions on  $V(x)$ , it is known that equation (1.1) admits a positive ground state  $u_\epsilon$ , see for instance [2, 21–23]. Moreover, if  $V(x) = 1$ , then  $u_\epsilon$  is spherically symmetric, see [12, 18]. However, when  $\epsilon = 0$  it follows from a Pohozaev type identity that (1.1) has no solutions in  $H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  if  $V(x) + \frac{1}{2s} x \cdot \nabla V(x) \geq 0$  (and  $\not\equiv 0$ ), see Theorem 9 or [10] in the special case  $V(x) \equiv 1$ .

Therefore, it is natural to wonder what happens to the ground state  $u_\epsilon$  as  $\epsilon \rightarrow 0^+$ . The main motivation of this paper is to achieve a better understanding of this phenomenon. This type of problems for semilinear equations, with the so-called nearly critical growth, were first studied in the unit ball of  $\mathbb{R}^3$  by Atkinson and Peletier in [3] and then extended to spherical domains by Brezis and Peletier in [6] and non-spherical domains by Han in [25]. Indeed, they proved the solution  $u_\epsilon$  blows up in the sense that  $\|u_\epsilon\|_{L^\infty(\mathbb{R}^N)} \sim \epsilon^{-\frac{1}{2}}$  as  $\epsilon \rightarrow 0^+$ . More recently, their results were extended to nonlocal problems in bounded domains in [13]. Precisely, the authors in [13] study the following nonlocal problem

$$\begin{cases} \mathcal{A}_s u = u^{2_s^*-1-\epsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where  $\mathcal{A}_s$  denotes the fractional Laplace operator  $(-\Delta)^s$  in  $\Omega$  defined in terms of the spectrum of the Laplacian subject to Dirichlet boundary conditions. They proved that if  $u_\epsilon$  is a solution to (1.2) satisfying

$$\lim_{\epsilon \rightarrow 0} \frac{\int_{\Omega} |\mathcal{A}_s^{\frac{1}{2}} u_\epsilon|^2 dx}{\|u_\epsilon\|_{2_s^*}^2} = S$$

where  $S$  is the best Sobolev constant in the embedding  $H^s \hookrightarrow L^{s^*}$ , then

$$\lim_{\epsilon \rightarrow 0} \epsilon \|u_\epsilon\|_{L^\infty(\Omega)}^2 = b_{n,s} |\tau(x_0)|,$$

where  $b_{n,s}$  is a normalizing constant and  $x_0 \in \Omega$  is a critical point of the Robin function  $\tau(x)$ . Besides, in [27] the authors study the asymptotic behavior of solutions to the nonlocal nonlinear problem

$$\begin{cases} (-\Delta_p)^s u = |u|^{p_s^*-2-\epsilon} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

where  $p_s^* = \frac{Np}{N-ps}, N > ps, p > 1$ . They prove that ground state solutions concentrate at a single point in  $\Omega$  and analyze the asymptotic behavior for sequences of solutions at higher energy levels as  $\epsilon \rightarrow 0$ . In particular, in the semi-linear case  $p = 2$ , they prove that for smooth domains the concentration point cannot lie on the boundary, and identify its location in the case of annular domains. Regarding the nonlocal problem (1.3) for  $p = 2$ , we also refer to [29] for a profile decomposition approach and to [30] for  $\Gamma$ -convergence methods.

The purpose of this paper is twofold: on one side, under suitable conditions on  $V(x)$ , we give a complete description of the blow up behavior of the ground states of (1.1); on the other side, we identify the location of the concentration points and then we establish local uniqueness of ground states.

Before stating our main results, let us make a few assumptions on  $V(x)$ . Throughout this paper, we assume that  $V(x)$  satisfies the following conditions:

- (V<sub>1</sub>)  $V \in C^2, 0 < V_0 \leq V(x) \leq V_\infty := \sup_{x \in \mathbb{R}^N} V(x) = \liminf_{|x| \rightarrow +\infty} V(x) < +\infty;$
- (V<sub>2</sub>) The function  $x \cdot \nabla V(x)$  stays bounded in  $\mathbb{R}^N$ .

We consider here the fractional Sobolev space

$$H^s_V(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N, V(x) dx) : [u]_s^2 := \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\}$$

endowed with the norm

$$\|u\|_{s,V} := \left( [u]_s^2 + \int_{\mathbb{R}^N} V(x)u^2 dx \right)^{\frac{1}{2}}.$$

Notice that under (V<sub>1</sub>),  $H^s(\mathbb{R}^N)$  which corresponds to the choice  $V \equiv 1$  and  $H^s_{V(x)}(\mathbb{R}^N)$  turn out to be equivalent in terms of norms as well as of elements. Denote by  $D^s(\mathbb{R}^N)$  the closure of  $C^\infty_0(\mathbb{R}^N)$  with respect to the norm  $[u]_s$ . As usual  $\|\cdot\|_p$  denotes  $\|\cdot\|_{L^p(\mathbb{R}^N)}$  for  $1 \leq p \leq \infty$ .

Let

$$S_{2^*_s-\epsilon} := \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_s^2}{\|u\|_{2^*_s-\epsilon}^2} = \inf_{u \in H^s(\mathbb{R}^N)} \{ \|u\|_s^2 : \|u\|_{2^*_s-\epsilon}^{2^*_s-\epsilon} = 1 \}. \tag{1.4}$$

By Lions' concentration compactness, minimizers for  $S_{2^*_s-\epsilon}$  always exist and one may assume they do not change sign, [18, 23]. Moreover, they are radially symmetric, see [24]. Here, we will consider only positive minimizers.

Recall also the Sobolev constant

$$S := \inf_{u \in D^s(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_s^2}{\|u\|_{2^*_s}^2}. \tag{1.5}$$

For each fixed  $\epsilon \in (0, 2^*_s - 2)$ , let  $w_\epsilon$  be a positive minimizer for

$$S_{2^*_s-\epsilon}^V = \inf_{u \in H^s_V(\mathbb{R}^N)} \{ \|u\|_{s,V}^2 : \|u\|_{2^*_s-\epsilon}^{2^*_s-\epsilon} = 1 \}, \tag{1.6}$$

or equivalently,

$$S_{2^*_s-\epsilon}^V = \inf_{u \in H^s_V(\mathbb{R}^N) \setminus \{0\}} I_\epsilon(u), \tag{1.7}$$

where

$$I_\epsilon(u) = \frac{\|u\|_{s,V}^2}{\|u\|_{2^*_s-\epsilon}^2}.$$

Our main results are the following:

**Theorem 1** Assume (V<sub>1</sub>), (V<sub>2</sub>),  $N > 4s$  and that  $u_\epsilon$  is a ground state of (1.1), namely satisfying (1.7), which has a maximum point  $x_\epsilon$  such that  $x_\epsilon \rightarrow x_0$  as  $\epsilon \rightarrow 0^+$ . Then,

- (1)  $\lim_{\epsilon \rightarrow 0^+} S_{2^*_s-\epsilon}^V = S;$
- (2)

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \|u_\epsilon\|_\infty^{\frac{4s}{N-2s}} = A_{N,s} \left[ V(x_0) + \frac{1}{2s} x_0 \cdot \nabla V(x_0) \right]$$

where

$$A_{N,s} := \frac{2^{2(N+1)} N^2 \pi^{\frac{N}{2}} \Gamma\left(\frac{N-4s}{2}\right)}{(N-2s)^2 \Gamma(N-2s)} S^{-\frac{N}{2s}}.$$

In particular we have

**Corollary 1** *Assume  $N > 4s$  and that  $u_\epsilon$  is a minimizer for (1.4). Then, we have:*

- (1)  $\lim_{\epsilon \rightarrow 0^+} S_{2s^* - \epsilon} = S;$
- (2)

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \|u_\epsilon\|_\infty^{\frac{4s}{N-2s}} = A_{N,s}.$$

Notice that in Theorem 1, we assume that the maximum point  $x_\epsilon$  does converge. However, under conditions  $(V_1)$  and  $(V_2)$ , one of the main difficulties is that  $x_\epsilon$  may actually escape to infinity as  $\epsilon \rightarrow 0^+$ . In what follows, we prove that if  $N > 6s$ , then the maximum point  $x_\epsilon$  must be bounded, and therefore converging, up to a subsequence, to a global minimum point of  $V(x)$  provided  $\inf_{x \in \mathbb{R}^N} V(x) < V_\infty$ . More precisely, we have the following

**Theorem 2** *Assume  $(V_1)$ ,  $(V_2)$  with  $\inf_{x \in \mathbb{R}^N} V(x) < V_\infty$ ,  $N > 6s$  and that  $u_\epsilon$  is a ground state of (1.1) (in the sense of (1.7)) which has a maximum point  $x_\epsilon$ . Then, there exists a subsequence  $\{x_{\epsilon_j}\}$  of  $\{x_\epsilon\}$  such that:*

- (1)  $\lim_{j \rightarrow \infty} x_{\epsilon_j} = x_0$ , where  $x_0$  is a global minimum point of  $V(x)$ ;
- (2)  $\lim_{j \rightarrow \infty} S_{2s^* - \epsilon_j}^V = S;$
- (3)

$$\lim_{j \rightarrow \infty} \epsilon_j \|u_{\epsilon_j}\|_\infty^{\frac{4s}{N-2s}} = A_{N,s} V(x_0).$$

What stated in Theorem 3 opens a natural question: *is there more than one blow-up ground state sequence such that the maxima concentrate at the same point?*

We do not have a full answer, however let us consider a special case. Assume  $V(x)$  is radial and that there exist two radial ground state sequences  $u_{\epsilon_j}^1$  and  $u_{\epsilon_j}^2$  of (1.1) such that

$\|u_{\epsilon_j}^i\|_\infty = u_{\epsilon_j}^i(0)$ ,  $i = 1, 2$ . Set  $\mu_j^i := \|u_{\epsilon_j}^i\|_\infty^{-\frac{2s^* - 2 - \epsilon_j}{2s}}$ ,  $i = 1, 2$ . We have the following local uniqueness result.

**Theorem 3** *Assume  $(V_1)$ ,  $(V_2)$ , that  $V(x) = V(|x|)$  is radial,  $N > 4s$  and there exist two radial ground state sequences  $u_{\epsilon_j}^1$  and  $u_{\epsilon_j}^2$  of (1.1) satisfying  $\|u_{\epsilon_j}^i\|_\infty = u_{\epsilon_j}^i(0)$ ,  $i = 1, 2$ , (1), (2) and (3) of Theorem 3. Then, there exists  $\epsilon_0 > 0$  such that for any  $\epsilon_j \in (0, \epsilon_0)$ , we have  $u_{\epsilon_j}^1 = u_{\epsilon_j}^2$ , provided  $\mu_j^1 = \mu_j^2$ . More precisely, up to rescaling, we have the following local uniqueness result*

$$u_{\epsilon_j}^2(x) = \left( \frac{\mu_j^1}{\mu_j^2} \right)^{\frac{2s}{2s^* - 2 - \epsilon_j}} u_{\epsilon_j}^1 \left( \frac{\mu_j^1}{\mu_j^2} x \right)$$

or equivalently

$$u_{\epsilon_j}^2(x) = \frac{\|u_{\epsilon_j}^2\|_\infty}{\|u_{\epsilon_j}^1\|_\infty} u_{\epsilon_j}^1 \left[ \left( \frac{\|u_{\epsilon_j}^2\|_\infty}{\|u_{\epsilon_j}^1\|_\infty} \right)^{\frac{2_s^*-2-\epsilon_j}{2_s}} x \right].$$

*Remark 1* In Theorems 4 we assume that  $\|u_j^i\|_\infty = u_j^i(0)$ ,  $i = 1, 2$ . Indeed, the results are still true if  $\|u_j^i\|_\infty = u_j^i(x_j)$  for some  $x_j \in \mathbb{R}^N$  and  $\lim_{j \rightarrow \infty} x_j = x_0$ . Let  $w_j^i(x) = u_j^i(x + x_j)$ , then  $w_j^i$  satisfies

$$(-\Delta)^s w_j^i + V(x + x_j)w_j^i = (w_j^i)^{2_s^*-1-\epsilon_j} \text{ in } \mathbb{R}^N.$$

Define

$$v_j^i(x) = \mu_j^{\frac{2_s}{2_s^*-2-\epsilon_j}} w_j^i(\mu_j^i x).$$

Then  $0 < v_j^i(x) \leq 1$ ,  $v_j^i(0) = 1$ , and satisfies

$$(-\Delta)^s v_j^i + (\mu_j^i)^{2_s} V(x_j + \mu_j^i x)v_j^i = (v_j^i)^{2_s^*-1-\epsilon_j} \text{ in } \mathbb{R}^N.$$

### 1.1 Overview

The asymptotic behavior of ground states to nonlocal problems has attracted remarkable attention in recent years. In [22], the authors studied the singularly perturbed fractional Schrödinger equation

$$\epsilon^{2s}(-\Delta)^s u + V(x)u = u^p \text{ in } \mathbb{R}^N, \tag{1.8}$$

where  $1 < p < 2_s^* - 1$ . They proved that concentration points turn out to be critical points for  $V$ . Moreover, they proved that if the potential  $V$  is coercive and has a unique global minimum, then ground states concentrate at that minimum point as  $\epsilon \rightarrow 0$ . In [16], by means of a Lyapunov-Schmidt reduction method, the authors proved the existence of various type of concentrating solutions, such as multiple spikes and clusters, such that each of the local maxima converge to a critical point of  $V$  as  $\epsilon \rightarrow 0$ , see also [1, 20]. In [5], the authors considered the nonlocal scalar field equation

$$(-\Delta)^s u + \epsilon u = |u|^{p-2}u - |u|^{q-2}u \text{ in } \mathbb{R}^N, \tag{1.9}$$

where  $2 < p < q$ . For  $\epsilon$  small, they proved the existence and qualitative properties of positive solutions when  $p$  is subcritical, supercritical or critical Sobolev exponent. For the existence of positive solutions of nonlocal equations with a small parameter see also [7, 19].

Loosely speaking, all the results mentioned above were concerned with the characterization of concentration of ground states. The purpose of this paper is quite different as we focus on quantitative aspects of concentrating solutions. Let us emphasize that Theorem 3 can be seen as a nonlocal analog of the results in [31, 40]. In [31], the authors studied the behavior of the ground states of equation

$$-\Delta u + K(x)u = u^{2_s^*-1-\epsilon} \text{ in } \mathbb{R}^N. \tag{1.10}$$

Under some geometric assumptions on  $K(x)$ , they proved the existence of ground states  $u_\epsilon$ . Moreover, the maximum point  $x_\epsilon$  of  $u_\epsilon$  is bounded and  $\|u\|_{L^\infty(\mathbb{R}^N)} \sim \epsilon^{-\frac{N-2}{4}}$  as  $\epsilon \rightarrow 0^+$ . In [40], the author further identified the location of the blow-up point. In the present paper, though conditions  $(V_1)$  and  $(V_2)$  guarantee the existence of the ground state solution  $u_\epsilon$ , it is not true in general that the maximum point  $x_\epsilon$  of  $u_\epsilon$  stays bounded as  $\epsilon \rightarrow 0^+$  and this yields a major difficulty.

The paper is organized as follows: existence of minimizers, local boundedness estimates of solutions and a Pohozaev type identity are established in the preliminary Section 2. In Section 3, we study the asymptotic behavior of ground states, including a uniform bound up to rescaling. Section 4 is devoted to identify the location of blow-up points, whence in Section 5 we prove the local uniqueness of ground states.

Throughout this paper,  $C$  will denote a positive constant which may vary from line to line.

## 2 Preliminaries

Here for the convenience of the reader we prove some auxiliary results. Consider first the following constrained minimization:

$$S_{2_s^*-\epsilon}^V := \inf_{u \in H_V^s(\mathbb{R}^N)} \{ \|u\|_{s,V}^2 : \|u\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} = 1 \}. \quad (2.1)$$

In the special case  $V(x) = 1$ , minimizers for  $S_{2_s^*-\epsilon}^V$  always exist and do not change sign, see e.g. [18, 23]. Moreover, they are radially symmetric, see [24].

**Theorem 4** *Assume  $(V_1)$  holds. Then,  $S_{2_s^*-\epsilon}^V$  is achieved at some  $w_\epsilon \in H_V^s(\mathbb{R}^N)$ .*

*Proof* We assume that  $V(x) \not\equiv V_\infty$ , otherwise the result is obvious. Let  $\{w_n\}$  be a minimizing sequence for  $S_{2_s^*-\epsilon}^V$ . Since  $|w_n| \in H_V^s(\mathbb{R}^N)$  and  $[|w_n|]_s \leq [w_n]_s$ , we may assume that  $w_n$  is nonnegative. Clearly,  $\{w_n\}$  is bounded in  $H_V^s(\mathbb{R}^N)$  and  $\|w_n\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} = 1$ . Therefore, up to subsequences if necessary, there exists  $w \in H_V^s(\mathbb{R}^N)$  such that  $w_n \rightharpoonup w$  in  $H_V^s(\mathbb{R}^N)$  as  $n \rightarrow +\infty$ . Let  $\ell = \|w\|_{2_s^*-\epsilon}^{2_s^*-\epsilon}$ , then  $0 \leq \ell \leq 1$ . We next claim that actually  $\ell = 1$ .

Indeed, let

$$S_{2_s^*-\epsilon}^\infty := \inf_{u \in H_{V_\infty}^s(\mathbb{R}^N)} \{ \|u\|_{s,V_\infty}^2 : \|u\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} = 1 \}.$$

Then minimizer  $u$  for  $S_{2_s^*-\epsilon}^\infty$  exists and does not change sign, see e.g. [18, 23]. Without loss of generality, we assume  $u$  is positive. Using this  $u$  as a test function we can show that if  $V(x)$  is not identically equal to  $V_\infty$ , then  $S_{2_s^*-\epsilon}^V < S_{2_s^*-\epsilon}^\infty$ .

Set  $v_n = w_n - w$ , then  $v_n \rightarrow 0$  in  $H_V^s(\mathbb{R}^N)$  and  $v_n \rightarrow 0$  in  $L_{loc}^2(\mathbb{R}^N)$  as  $n \rightarrow +\infty$  and by Brezis-Lieb lemma, we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|v_n\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} &= \lim_{n \rightarrow +\infty} \|w_n\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} - \|w\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} \\ &= 1 - \|w\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} \\ &= 1 - \ell. \end{aligned} \quad (2.2)$$

On the one hand we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|w_n\|_{s,V}^2 &= \lim_{n \rightarrow +\infty} \|v_n\|_{s,V}^2 + \|w\|_{s,V}^2 \\ &+ 2 \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(v_n(x) - v_n(y))(w(x) - w(y))}{|x - y|^{N+2s}} dx dy \\ &+ \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} V(x)v_n w dx \\ &= \lim_{n \rightarrow +\infty} \|v_n\|_{s,V}^2 + \|w\|_{s,V}^2. \end{aligned} \tag{2.3}$$

On the other hand, by  $(V_1)$  and  $v_n \rightarrow 0$  in  $L^2_{loc}(\mathbb{R}^N)$  as  $n \rightarrow +\infty$ , we have

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} [V(x) - V_\infty]w_n^2 dx = 0.$$

Thus, we have

$$\lim_{n \rightarrow +\infty} \|v_n\|_{s,V(x)}^2 = \lim_{n \rightarrow +\infty} \|v_n\|_{s,V_\infty}^2. \tag{2.4}$$

For  $0 \leq \ell \leq 1$ , by the definitions of  $S_{2^*_s-\epsilon}^V$  and  $S_{2^*_s-\epsilon}^\infty$ , we have

$$\|w\|_{s,V(x)}^2 \geq \ell^{\frac{2}{2^*_s-\epsilon}} S_{2^*_s-\epsilon}^V \tag{2.5}$$

and by (2.2), we get

$$\lim_{n \rightarrow +\infty} \|v_n\|_{s,V_\infty}^2 \geq \lim_{n \rightarrow +\infty} \|v_n\|_{2^*_s-\epsilon}^2 S_{2^*_s-\epsilon}^\infty = (1 - \ell)^{\frac{2}{2^*_s-\epsilon}} S_{2^*_s-\epsilon}^\infty. \tag{2.6}$$

Therefore, by (2.3), (2.4), (2.5) and (2.6), we have

$$S_{2^*_s-\epsilon}^V \geq \ell^{\frac{2}{2^*_s-\epsilon}} S_{2^*_s-\epsilon}^V + (1 - \ell)^{\frac{2}{2^*_s-\epsilon}} S_{2^*_s-\epsilon}^\infty \tag{2.7}$$

which gives

$$1 - \ell^{\frac{2}{2^*_s-\epsilon}} \geq (1 - \ell)^{\frac{2}{2^*_s-\epsilon}}. \tag{2.8}$$

Thus, from (2.8), we deduce that  $\ell = 0$  or  $\ell = 1$ . If  $\ell = 0$ , then from (2.7), we get  $S_{2^*_s-\epsilon}^V \geq S_{2^*_s-\epsilon}^\infty$ , which is a contradiction. Thus,  $\ell = 1$ , that is,  $\|w\|_{2^*_s-\epsilon} = 1$  and thus  $w$  is a minimizer of  $S_{2^*_s-\epsilon}^V$ .  $\square$

*Remark 2* Notice that in the proof of Theorem 6, condition  $S_{2^*_s-\epsilon}^V < S_{2^*_s-\epsilon}^\infty$  plays an important role. This is guaranteed by condition  $(V_1)$  with  $V(x) \not\equiv V_\infty$ .

By the Lagrange multiplier rule, there exists some  $\lambda_\epsilon > 0$  such that  $w_\epsilon$  is a solution of the following equation

$$(-\Delta)^s u + V(x)u = \lambda_\epsilon u^{2^*_s-1-\epsilon} \text{ in } \mathbb{R}^N. \tag{2.9}$$

By the maximum principle  $w_\epsilon > 0$ . In fact,  $w_\epsilon \geq 0$ , and if there exists some  $x_0$  such that  $w_\epsilon(x_0) = 0$ , then

$$0 \leq (-\Delta)^s w_\epsilon(x_0) + V(x_0)w_\epsilon(x_0) = c_{N,s}PV \int_{\mathbb{R}^n} \frac{-u(y)}{|x_0 - y|^{N+2s}} dy < 0, \tag{2.10}$$

thus a contradiction.

*Remark 3* If  $V(x)$  is radial, by means of symmetric rearrangement techniques, we may assume that  $w_n$  is radially symmetric (cf. [32]). Thus, the minimizer  $w$  is radial.

Next we proof a Pohozaev type identity for the nonlocal equation

$$(-\Delta)^s u = f(x, u) \text{ in } \mathbb{R}^N. \quad (2.11)$$

The argument is similar to [4, 33, 34], where the Pohozaev identity for autonomous nonlocal equations was established, hence we just stress the differences.

**Theorem 5** (Pohozaev identity) *Let  $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  be a positive solution to (2.11) and  $F(x, t) \in L^1(\mathbb{R}^N)$ , where  $F(x, t) = \int_0^t f(x, s) ds$ . Then we have*

$$\frac{N-2s}{2} \int_{\mathbb{R}^N} f(x, u) u dx = \int_{\mathbb{R}^N} [NF(x, u) + (x \cdot \nabla_x F(x, u))] dx. \quad (2.12)$$

*Proof* Let  $u$  be a bounded weak nontrivial solution. Suppose that  $w$  is the harmonic extension of  $u$ , see e.g. [9]. Then,  $w$  satisfies

$$\begin{cases} -\operatorname{div}(y^{1-2s} \nabla w) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \frac{\partial w}{\partial \nu^s} = f((\cdot, 0), w(\cdot, 0)) & \text{in } \mathbb{R}^N \times \{y = 0\}. \end{cases} \quad (2.13)$$

For  $r > 0$ , let

$$B_r := \{(x, y) \in \mathbb{R}^{N+1} : |(x, y)| < r\}$$

and

$$B_r^+ = B_r \cap \mathbb{R}_+^{N+1}, \quad Q_r = B_r^+ \cup (B_r \cap (\mathbb{R}^N \times \{0\})).$$

Let  $\phi \in C_0^\infty(\mathbb{R}^{N+1})$  with  $0 \leq \phi \leq 1$ ,  $\phi = 1$  in  $B_1$  and  $\phi = 0$  in  $B_2^c$ ,  $|\nabla \phi| \leq 2$ . For  $R > 0$ , let

$$\varphi_R(x, y) = \varphi\left(\frac{(x, y)}{R}\right),$$

where  $\varphi := \phi|_{\mathbb{R}_+^{N+1}}$ .

Then, multiplying (2.13) by  $((x, y) \cdot \nabla w) \varphi_R$  and integrating in  $\mathbb{R}_+^{N+1}$ , we have,

$$\int_{Q_{2r}} \operatorname{div}(y^{1-2s} \nabla w)[((x, y) \cdot \nabla w) \varphi_R] dx dy = 0. \quad (2.14)$$



From (2.14), by integrating by parts, we get

$$\begin{aligned}
 & \int_{Q_{2r}} y^{1-2s} \nabla w \nabla [((x, y) \cdot \nabla w) \varphi_R] dx dy \\
 &= \int_{\partial Q_{2r}} y^{1-2s} (\nabla w \cdot \mathbf{n}) [((x, y) \cdot \nabla w) \varphi_R] dS \\
 &= - \lim_{y \rightarrow 0^+} \int_{B_{2R} \cap (\mathbb{R}^N \times \{y\})} y^{1-2s} \frac{\partial w}{\partial y} [((x, y) \cdot \nabla w) \varphi_R] dx \\
 &= k_s^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} (x \cdot \nabla_x w) \varphi_R \frac{\partial w}{\partial \nu^s} dx \\
 &= k_s^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} (x \cdot \nabla_x w) \varphi_R f(x, w) dx \\
 &= k_s^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} (x \cdot \nabla F(x, u)) \varphi_R dx - k_s^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} (x \cdot \nabla_x F(x, u)) \varphi_R dx \\
 &= -N k_s^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} F(x, u) \varphi_R dx - k_s^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} F(x, u) (x \cdot \nabla_x \varphi_R) dx \\
 &\quad - k_s^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} (x \cdot \nabla_x F(x, u)) \varphi_R dx.
 \end{aligned} \tag{2.15}$$

For the second integral in the last equality in (2.15), we have

$$\begin{aligned}
 \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} F(x, u) (x \cdot \nabla_x \varphi_R) dx &\leq C \int_{(B_{2R} \setminus B_R) \cap (\mathbb{R}^N \times \{0\})} F(x, u) \frac{|x|}{R} dx \\
 &\leq C \int_{(B_{2R} \setminus B_R) \cap (\mathbb{R}^N \times \{0\})} F(x, u) dx \rightarrow 0, \text{ as } R \rightarrow +\infty
 \end{aligned} \tag{2.16}$$

since  $F(x, t) \in L^1(\mathbb{R}^N)$ . As a consequence, from (2.15) we have

$$\begin{aligned}
 & \lim_{R \rightarrow +\infty} \int_{Q_{2r}} y^{1-2s} \nabla w \nabla [((x, y) \cdot \nabla w) \varphi_R] dx dy \\
 &= -k_s^{-1} \int_{\mathbb{R}^N} [NF(x, u) + (x \cdot \nabla_x F(x, u))] dx.
 \end{aligned} \tag{2.17}$$

On the other hand, similar to the proof of Theorem A.1 in [4], we have

$$\lim_{R \rightarrow +\infty} \int_{Q_{2r}} y^{1-2s} \nabla w \nabla [((x, y) \cdot \nabla w) \varphi_R] dx dy = \frac{2s - N}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy. \tag{2.18}$$

Thanks to (2.17) and (2.18), we have

$$\frac{N - 2s}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy = k_s^{-1} \int_{\mathbb{R}^N} [NF(x, u) + (x \cdot \nabla_x F(x, u))] dx. \tag{2.19}$$

Multiply (3.26) by  $w\varphi_R$  and integrate by parts to get

$$\begin{aligned} \int_{Q_{2r}} y^{1-2s} \nabla w \nabla (w\varphi_R) dx dy &= \int_{\partial Q_{2r}} y^{1-2s} (\nabla w \cdot \mathbf{n}) w\varphi_R dS \\ &= k_s^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} \frac{\partial w}{\partial \nu^s} w\varphi_R dx \\ &= k_s^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} f(x, u) u\varphi_R dx \end{aligned}$$

Proceed now as above to get

$$\int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla w|^2 dx dy = k_s^{-1} \int_{\mathbb{R}^N} f(x, u) u dx. \tag{2.20}$$

Combining (2.18) and (2.20), we deduce that

$$\frac{N - 2s}{2} \int_{\mathbb{R}^N} f(x, u) u dx = \int_{\mathbb{R}^N} [NF(x, u) + (x \cdot \nabla_x F(x, u))] dx.$$

□

Finally, we prove a crucial local estimate. This type of estimate has been studied in Proposition 3.1 and Proposition 2.4 in [38, 39]. Their methods relies on a localization method introduced by Caffarelli and Silvestre in [9, 36] and the standard Moser iteration. However, these estimates contain the extension local domain  $Q_R$ , which has no clear interpretation in terms of the original problem in  $\mathbb{R}^N$  that is our context. We now give another version of this estimate based on a more direct test function method and Moser’s iteration.

**Theorem 6** Assume  $a(x) \in L^t_{loc}(\mathbb{R}^N)$  for some  $t > \frac{N}{2s}$  and that  $u \geq 0$  satisfies

$$(-\Delta)^s u \leq a(x)u, \quad x \in \mathbb{R}^N.$$

Then

$$\max_{B_r} u(x) \leq C \left( \int_{B_R} |u|^{2^*_s} dx \right)^{\frac{1}{2^*_s}}, \quad 0 < r < R, \tag{2.21}$$

where the constant  $C > 0$  depends only on  $N, s, R, t$  and  $\|a(x)\|_{L^t_{loc}(\mathbb{R}^N)}$ .

*Proof* For  $\beta > 1$  and  $T > 0$ , define the function

$$\varphi(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ t^\beta & \text{if } 0 < t \leq T, \\ \beta T^{\beta-1}(t - T) + T^\beta & \text{if } t \geq T. \end{cases} \tag{2.22}$$

Notice that  $\varphi(t)$  is a convex and differentiable function and thus

$$(-\Delta)^s \varphi(u) \leq \varphi'(u) (-\Delta)^s u. \tag{2.23}$$

Let  $\eta(x) = \eta(|x|)$  be a smooth cut-off function satisfying  $\eta(x) = 1$  in  $B_r, 0 \leq \eta(x) \leq 1, \eta(x) = 0$  in  $B^c_R$  and  $|\eta'| \leq \frac{C}{R-r}$  for some constant  $C > 0$ , where  $0 < r < R$  has to be determined. For simplicity, in the following, we denote by  $\varphi := \varphi(u(x))$  and  $\varphi' := \varphi_u(x)$ .

Choose as test function  $\phi(x) = \eta^2 \varphi'$  to obtain

$$\int_{\mathbb{R}^N} \eta^2 \varphi \varphi'(x) (-\Delta)^s u dx \leq \int_{\mathbb{R}^N} \eta^2 \varphi \varphi' a u dx. \tag{2.24}$$

However, by (2.23), we have

$$\int_{\mathbb{R}^N} \eta^2 \varphi(-\Delta)^s \varphi dx \leq \int_{\mathbb{R}^N} \eta^2 \varphi'(-\Delta)^s u dx. \tag{2.25}$$

Using (2.24) and (2.25), the fact  $u\varphi' \leq \beta\varphi$ , by Sobolev embedding theorem and Cauchy inequality, we get

$$\begin{aligned} S(n, s) \|\eta\varphi\|_{2_s^*}^2 &\leq \int_{\mathbb{R}^N} \eta\varphi(-\Delta)^s [\eta\varphi] dx \\ &= \int_{\mathbb{R}^N} \eta^2 \varphi(-\Delta)^s \varphi dx + \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} \varphi(x)\varphi(y) dx dy \\ &\leq \beta \int_{\mathbb{R}^N} a\eta^2 \varphi^2 dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} \varphi(x)^2 dx dy. \end{aligned} \tag{2.26}$$

In what follows, we assume  $u \in L^{\frac{2\beta t}{t-1}}(\mathbb{R}^N)$  where  $\beta$  has to be chosen later on. From  $a(x) \in L^t_{loc}(\mathbb{R}^N)$  and  $\varphi(t) \leq t^\beta$ , we have

$$\int_{\mathbb{R}^N} a\eta^2 u^{2\beta} dx \leq \left[ \int_{\mathbb{R}^N} (\eta a)^t dx \right]^{\frac{1}{t}} \left[ \int_{\mathbb{R}^N} (\eta u^{2\beta})^{\frac{t}{t-1}} dx \right]^{\frac{t-1}{t}}. \tag{2.27}$$

Set

$$\begin{aligned} &\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} dy \right)^t dx \\ &= \int_{|x| \leq R} \left( \int_{\mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} dy \right)^t dx + \int_{|x| \geq R} \left( \int_{\mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} dy \right)^t dx \\ &:= I_1 + I_2. \end{aligned} \tag{2.28}$$

We obtain

$$\begin{aligned} I_1 &\leq \frac{C^t}{(R - r)^{2t}} \int_{|x| \leq R} \left( \int_{|y-x| \leq R} \frac{1}{|x - y|^{N+2s-2}} dy \right)^t dx \\ &\quad + C^t \int_{|x| \leq R} \left( \int_{|y-x| \geq R} \frac{1}{|x - y|^{N+2s}} dy \right)^t dx \\ &= \frac{(2 - 2s)^{-t} C^t}{(R - r)^{2t}} (N\omega_{N-1})^{t+1} R^{N+(2-2s)t} + (2s)^{-t} C^t (N\omega_{N-1})^{t+1} R^{N-2st} \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} I_2 &= \int_{|x| \geq R} \left( \int_{|y| \leq R} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} dy \right)^t dx \\ &= \int_{R \leq |x| \leq 2R} \left( \int_{|y| \leq R} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} dy \right)^t dx + \int_{|x| \geq 2R} \left( \int_{|y| \leq R} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} dy \right)^t dx \\ &:= I_3 + I_4 \end{aligned} \tag{2.30}$$

The estimate of  $I_3$  is similar to the one for  $I_1$ . Finally,

$$\begin{aligned}
 I_4 &\leq \int_{|x|\geq 2R} \left( \int_{|y|\leq R} \frac{1}{(|x| - R)^{N+2s}} dy \right)^t dx \\
 &= (N\omega_{N-1})^t R^{(N-1)t} \int_{|x|\geq 2R} (|x| - R)^{-(N+2s)t} dx \\
 &\leq CR^{N-(1+2s)t}
 \end{aligned}
 \tag{2.31}$$

By combining (2.25)-(2.31), we obtain

$$\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} dy \right)^t dx \leq \frac{C}{(R - r)^{2t}}.$$

Hence

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} \varphi(x)^2 dx dy \leq \left[ \left( \int_{\mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} dy \right)^t dx \right]^{\frac{1}{t}} \left( \int_{\mathbb{R}^N} \eta^2 u^{2\beta \frac{t-1}{t}} dx \right)^{\frac{t-1}{t}}.
 \tag{2.32}$$

Set

$$C := \left( \int_{\mathbb{R}^N} (\eta a(x))^t dx \right)^{\frac{1}{t}} + \left[ \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} dy \right)^t dx \right]^{\frac{1}{t}}.$$

Combining (2.26), (2.27), (2.30) and (2.32), we get

$$S(n, s) \|\eta\varphi\|_{2_s^*}^2 \leq \frac{C}{(R - r)^2} \beta \left( \int_{\mathbb{R}^N} (\eta u^{2\beta})^{\frac{t-1}{t}} dx \right)^{\frac{t-1}{t}}.
 \tag{2.33}$$

Now let  $T \rightarrow +\infty$ , to obtain

$$\left( \int_{B_r} u \beta 2_s^* dx \right)^{\frac{1}{\beta 2_s^*}} \leq \left[ \frac{C\beta}{(R - r)^2} \right]^{\frac{1}{2\beta}} \left( \int_{B_r} u^{2\beta \frac{t-1}{t}} dx \right)^{\frac{t-1}{2t\beta}}.
 \tag{2.34}$$

Since  $t > \frac{N}{2s}$ , we set  $\beta_i = (\frac{2_s^*(t-1)}{2t})^i, i = 1, 2, \dots, r_i = r_0 + \frac{1}{2t}$ . By iterating, we get

$$\left( \int_{B_{r_m}} u \beta_m 2_s^* dx \right)^{\frac{1}{\beta_m 2_s^*}} \leq \left[ \frac{2_s^*(t-1)C}{2t} \right]^{\frac{1}{2} \sum_{i=1}^m i/\beta_i} \left( \int_{B_R} u^{2_s^*} dx \right)^{\frac{1}{2_s^*}}
 \tag{2.35}$$

Let  $m \rightarrow \infty$  to have

$$\max_{B_{r_0}} u(x) \leq C \left( \int_{B_R} u^{2_s^*} dx \right)^{\frac{1}{2_s^*}}.
 \tag{2.36}$$

□

### 3 Asymptotic Behavior of Ground States

Let  $w_\epsilon$  be a positive minimizer for  $S_{2_s^*-\epsilon}^V$  obtained in Theorem 6. Then, by the Lagrange multiplier rule, there exists  $\lambda_\epsilon > 0$  such that  $w_\epsilon$  is a solution to the equation

$$(-\Delta)^s u + V(x)u = \lambda_\epsilon u^{2_s^*-1-\epsilon} \text{ in } \mathbb{R}^N.
 \tag{3.1}$$

By multiplying both sides of equation (3.1) by  $w_\epsilon$  and then integrating, we get  $\lambda_\epsilon = S_{2_s^*-\epsilon}^V$ .

The energy associated with equation (3.1) is given by

$$J_\epsilon(u) = \frac{1}{2} \|u\|_{s,V}^2 - \frac{1}{2_s^*-\epsilon} S_{2_s^*-\epsilon}^V \|u\|_{2_s^*-\epsilon}^{2_s^*-\epsilon}. \tag{3.2}$$

Thus, on the one hand we have

$$J_\epsilon(w_\epsilon) = \frac{1}{2} \|w_\epsilon\|_{s,V}^2 - \frac{1}{2_s^*-\epsilon} S_{2_s^*-\epsilon}^V \|w_\epsilon\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} = \frac{2_s^*-\epsilon-2}{2(2_s^*-\epsilon)} S_{2_s^*-\epsilon}^V. \tag{3.3}$$

On the other hand, if  $v$  is a nontrivial solution of (3.1), then it satisfies  $\|v\|_{s,V}^2 = S_{2_s^*-\epsilon}^V \|v\|_{2_s^*-\epsilon}^{2_s^*-\epsilon}$  and thus

$$\begin{aligned} J_\epsilon(v) &= \frac{1}{2} \|v\|_{s,V}^2 - \frac{1}{2_s^*-\epsilon} S_{2_s^*-\epsilon}^V \|v\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} \\ &= \frac{2_s^*-\epsilon-2}{2(2_s^*-\epsilon)} S_{2_s^*-\epsilon}^V \|v\|_{2_s^*-\epsilon}^{2_s^*-\epsilon}. \end{aligned} \tag{3.4}$$

Besides, we have

$$S_{2_s^*-\epsilon}^V \leq I_\epsilon(v) = \frac{\|v\|_{s,V}^2}{\|v\|_{2_s^*-\epsilon}^2} = \frac{S_{2_s^*-\epsilon}^V \|v\|_{2_s^*-\epsilon}^{2_s^*-\epsilon}}{\|v\|_{2_s^*-\epsilon}^2} = S_{2_s^*-\epsilon}^V \|v\|_{2_s^*-\epsilon}^{2_s^*-\epsilon-2}, \tag{3.5}$$

which yields that  $\|v\|_{2_s^*-\epsilon} \geq 1$ . Thus, we have  $J_\epsilon(v) \geq \frac{2_s^*-\epsilon-2}{2(2_s^*-\epsilon)} S_{2_s^*-\epsilon}^V$  by (3.4). This fact together with (3.3) implies that  $w_\epsilon$  is a ground state of equation (3.1). Furthermore, if we set

$$u_\epsilon = \left( S_{2_s^*-\epsilon}^V \right)^{-\frac{1}{1-2_s^*+\epsilon}} w_\epsilon$$

then,  $u_\epsilon$  is a ground state of equation (1.1). Observe that  $I_\epsilon(u_\epsilon) = S_{2_s^*-\epsilon}^V$ .

For each fixed  $\epsilon \in (0, 2_s^* - 2)$ , by means of the mountain-pass theorem, (1.1) admits a positive ground state (see e.g. Theorem 1.4 in [23]). However, we don't know whether the mountain-pass solution and the minimal solution  $u_\epsilon$  obtained above do agree since uniqueness is not known. Anyway, in what follows, we will focus on the minimal solution  $u_\epsilon$ . We remark that in the special case  $V(x) = 1$ , the ground state is unique and radially symmetric, see [24].

**Lemma 1** *For any fixed  $\epsilon \in (0, 2_s^* - 2)$ , any nontrivial  $u_\epsilon$  of (1.1) satisfies*

$$\|u_\epsilon\|_\infty \geq V_0^{\frac{1}{2_s^*-2}}. \tag{3.6}$$

*Proof* Since  $u_\epsilon$  enjoys (1.1), we have

$$\|u_\epsilon\|_{s,V_0}^2 \leq \|u_\epsilon\|_{s,V}^2 = \|u_\epsilon\|_{2_s^*-\epsilon}^{2_s^*-\epsilon}$$

which yields  $V_0 \|u_\epsilon\|_2^2 \leq \|u_\epsilon\|_{2_s^*-\epsilon}^{2_s^*-\epsilon}$ , that is,

$$\int_{\mathbb{R}^N} u_\epsilon^2 (V_0 - u_\epsilon^{2_s^*-2-\epsilon}) dx \leq 0.$$

Thus, we get  $\|u_\epsilon\|_\infty \geq V_0^{\frac{1}{2_s^*-2-\epsilon}} \geq V_0^{\frac{1}{2_s^*-2}}$ , and the result follows. □

We next need the following result proved in [14].

**Lemma 2** [14] *The infimum in (1.5) is attained, that is*

$$S = \frac{[\tilde{u}]_s^2}{\|\tilde{u}\|_{2_s^*}^2},$$

where

$$\tilde{u}(x) := \kappa(\mu^2 + |x - x_0|^2)^{\frac{2_s^*-N}{2}}, \quad x \in \mathbb{R}^N$$

with  $\kappa \in \mathbb{R} \setminus \{0\}$ ,  $\mu > 0$  and  $x_0 \in \mathbb{R}^N$  fixed constant. Equivalently, the function  $\bar{u}$  defined by  $\bar{u}(x) := \frac{\tilde{u}}{\|\tilde{u}\|_{2_s^*}}$  is such that

$$S = \inf_{u \in D^s(\mathbb{R}^N)} \{[u]_s^2 : \|u\|_{2_s^*} = 1\}. \tag{3.7}$$

Furthermore, the function

$$u^*(x) := \bar{u}\left(S^{-\frac{1}{2_s}}x\right), \quad x \in \mathbb{R}^N$$

is a solution of

$$(-\Delta u)^s u = |u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^N$$

satisfying the property

$$\|u\|_{2_s^*}^{2_s^*} = S^{\frac{N}{2_s}}.$$

**Proposition 1**  $\lim_{\epsilon \rightarrow 0^+} S_{2_s^*-\epsilon}^V = S.$

*Proof* Choose  $\phi \in C_0^\infty(\mathbb{R}^N)$ ,  $\phi \geq 0$  such that  $\inf_{0 < \epsilon < 2_s^*-2} \|\phi\|_{2_s^*-\epsilon} > 0$ , to get

$$0 < S_{2_s^*-\epsilon}^V \leq \frac{\|\phi\|_{s,V}^2}{\inf_{0 < \epsilon < 2_s^*-2} \|\phi\|_{2_s^*-\epsilon}^2} < +\infty. \tag{3.8}$$

This means that  $\{S_{2_s^*-\epsilon}^V\}$  is uniformly bounded with respect to  $\epsilon$ . Next, we further prove that

$$\lim_{\epsilon \rightarrow 0^+} S_{2_s^*-\epsilon}^V = S.$$

Let  $w \in H_V^s(\mathbb{R}^n)$  be such that  $\|w\|_{2_s^*-\epsilon} = 1$  and  $\|w\|_{s,V}^2 = S_{2_s^*-\epsilon}^V$ . Then,

$$\|w\|_2^2 \leq \|w\|_s^2 \leq \max\left\{1, \frac{1}{V_0}\right\} \|w\|_{s,V}^2 =: CS_{2_s^*-\epsilon}^V. \tag{3.9}$$

By Hölder’s inequality we have

$$1 = \|w\|_{2_s^*-\epsilon}^{2_s^*-\epsilon} \leq \|w\|_2^{\frac{2\epsilon}{2_s^*-2}} \|w\|_{2_s^*}^{\frac{2_s^*(2_s^*-2-\epsilon)}{2_s^*-2}} \leq (CS_{2_s^*-\epsilon}^V)^{\frac{\epsilon}{2_s^*-2}} \|w\|_{2_s^*}^{\frac{2_s^*(2_s^*-2-\epsilon)}{2_s^*-2}}. \tag{3.10}$$

Thanks to (3.8) and (3.10), we get

$$1 \leq \liminf_{\epsilon \rightarrow 0^+} \|w\|_{2_s^*}. \tag{3.11}$$

On the other hand, by (1.5), we have

$$S \leq \frac{[w]_s^2}{\|w\|_{2_s^*}^2} \leq \frac{\|w\|_{s,V}^2}{\|w\|_{2_s^*}^2} = \frac{S_{2_s^*-\epsilon}^V}{\|w\|_{2_s^*}^2}, \tag{3.12}$$

Thanks to (3.11) and (3.12), we have

$$S \leq \liminf_{\epsilon \rightarrow 0^+} S_{2_s^* - \epsilon}^V. \tag{3.13}$$

Next we prove that

$$\limsup_{\epsilon \rightarrow 0^+} S_{2_s^* - \epsilon}^V \leq S. \tag{3.14}$$

Once (3.14) is proved, the result follows from (3.13).

Let

$$U_\epsilon(x) = \epsilon^{\frac{2s-N}{2}} u^*(x/\epsilon),$$

where  $u^*$  is defined in Lemma 12. Furthermore, let  $\eta(x) \in C_0^\infty(\mathbb{R}^N)$  be such that  $0 \leq \eta(x) \leq 1$  in  $\mathbb{R}^N$ ,  $\eta(x) \equiv 1$  in  $B_{1/2}$  and  $\eta(x) \equiv 0$  in  $B_1^c$ . Set  $u_\epsilon(x) := \eta(x)U_\epsilon(x)$ ,  $x \in \mathbb{R}^N$ . Then, as  $\epsilon \rightarrow 0^+$  we have

$$\|u_\epsilon\|_s^2 \leq S^{\frac{N}{2s}} + O(\epsilon^{N-2s}), \tag{3.15}$$

$$\int_{\mathbb{R}^N} V(x)|u_\epsilon(x)|^2 dx = C_s \epsilon^{2s} + O(\epsilon^{N-2s}), \text{ if } N > 4s \tag{3.16}$$

and

$$\|u_\epsilon(x)\|_{2_s^*}^2 = S^{\frac{N}{2s}} + O(\epsilon^N), \tag{3.17}$$

for some positive constant  $C_s$  depending only on  $s$ , see Propositions 21 and 22 in [35] or Lemma 2.4 in [20]. By Taylor’s expansion we get

$$\|u_\epsilon(x)\|_{2_s^* - \epsilon}^2 = \|u_\epsilon(x)\|_{2_s^*}^2 + O(\epsilon). \tag{3.18}$$

Hence, we deduce from (3.15)–(3.18) that

$$\limsup_{\epsilon \rightarrow 0^+} S_{2_s^* - \epsilon}^V \leq \limsup_{\epsilon \rightarrow 0^+} \frac{\|u_\epsilon\|_{s,V}^2}{\|u_\epsilon\|_{2_s^* - \epsilon}^2} \leq S, \tag{3.19}$$

which implies (3.14). □

Recalling that  $u_\epsilon$  is a solution to (1.1) and that  $u_\epsilon$  attains  $S_{2_s^* - \epsilon}^V$ , we get

$$\|u_\epsilon\|_{s,V}^2 = \|u_\epsilon\|_{2_s^* - \epsilon}^{\frac{2_s^* - \epsilon}{s}} \text{ and } \|u_\epsilon\|_{s,V}^2 = S_{2_s^* - \epsilon}^V \|u_\epsilon\|_{2_s^* - \epsilon}^2. \tag{3.20}$$

So, we have

$$\|u_\epsilon\|_{s,V} = \left(S_{2_s^* - \epsilon}^V\right)^{\frac{2_s^* - \epsilon}{2(2_s^* - 2 - \epsilon)}} \text{ and } \|u_\epsilon\|_{2_s^* - \epsilon} = \left(S_{2_s^* - \epsilon}^V\right)^{\frac{1}{2_s^* - 2 - \epsilon}}. \tag{3.21}$$

These facts together with Lemma 13 imply the following

**Lemma 3**

$$\lim_{\epsilon \rightarrow 0^+} \|u_\epsilon\|_{s,V} = S^{\frac{N}{4s}}, \quad \lim_{\epsilon \rightarrow 0^+} \|u_\epsilon\|_{2_s^* - \epsilon} = S^{\frac{N-2s}{4s}}. \tag{3.22}$$

Now let us prove that  $\|u_\epsilon\|_\infty$  blows up as  $\epsilon \rightarrow 0^+$ , namely

**Lemma 4**  $\lim_{\epsilon \rightarrow 0^+} \|u_\epsilon\|_\infty = +\infty$ .

*Proof* Suppose by contradiction the claim does not hold true. Then, there exists a sequence  $\epsilon_j \rightarrow 0^+$  such that  $\|u_{\epsilon_j}\|_\infty$  stays bounded. Let  $x_{\epsilon_j}$  be a maximum point of  $u_{\epsilon_j}$ . Define  $w_{\epsilon_j}(x) = u_{\epsilon_j}(x + x_{\epsilon_j})$ , then  $\|w_{\epsilon_j}\|_\infty$  is bounded as well and

$$(-\Delta)^s w_{\epsilon_j} = -V(x + x_{\epsilon_j})w_{\epsilon_j} + w_{\epsilon_j}^{2_s^* - 1 - \epsilon_j} \text{ in } \mathbb{R}^N.$$

Now, since  $V \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , we have that  $\|(-\Delta)^s w_{\epsilon_j}\|_\infty$  is uniformly bounded with respect to  $\epsilon_j$ . As a consequence of this fact and of standard regularity results (see e.g. Lemma 4.4 in [8]), we deduce that  $\|w_{\epsilon_j}\|_{C^{2,\alpha}}$  is uniformly bounded with respect to  $\epsilon_j$ , for some  $\alpha \in (0, 1)$ .

By (3.22),  $[w_{\epsilon_j}]_s = [u_{\epsilon_j}]_s$  and  $\|w_{\epsilon_j}\|_2 = \|u_{\epsilon_j}\|_2$  are bounded. Thus,  $\{w_{\epsilon_j}\}$  is bounded in  $H^s_V(\mathbb{R}^N)$ . Up to extracting a subsequence, which we still denote by  $\{w_{\epsilon_j}\}$ , one has  $w_{\epsilon_j} \rightharpoonup w_0$  in  $H^s_V(\mathbb{R}^N)$ ,  $w_{\epsilon_j} \rightarrow w_0$  a.e. in  $\mathbb{R}^N$  and  $w_{\epsilon_j} \rightarrow w_0$  in  $C^{2,\alpha}_{loc}(\mathbb{R}^N)$ . Moreover, by (3.6) one has  $w_0(0) \geq V_0^{\frac{1}{2^*_s-2}} > 0$ .

Let us now distinguish two cases:

*Case 1.*  $\{x_{\epsilon_j}\}_j$  is bounded. Up to a subsequence, we may assume that  $x_{\epsilon_j} \rightarrow x_0$ . Then,  $w_0$  is a nonnegative classical solution of

$$(-\Delta)^s w_0 = -V(x + x_0)w_0 + w_0^{2^*_s-1} \quad \text{in } \mathbb{R}^N. \tag{3.23}$$

It follows from the maximum principle that  $w_0 > 0$ . Thus, by Lemma 14 we have

$$S \leq \frac{[w_0]_s^2}{\|w_0\|_{2^*_s}^2} < \|w_0\|_{2^*_s}^{2^*_s-2} \leq \liminf_{j \rightarrow \infty} \|w_{\epsilon_j}\|_{2^*_s}^{2^*_s-2} = S, \tag{3.24}$$

which is a contradiction.

*Case 2.*  $\{x_{\epsilon_j}\}_j$  is unbounded. Up to a subsequence, we may assume that  $x_{\epsilon_j} \rightarrow \infty$ . Then, by (3.20) and the dominated convergence theorem, we have

$$\begin{aligned} [w_0]_s^2 &\leq -V_0\|w_0\|_2^2 + \|w_0\|_{2^*_s}^{2^*_s} + \lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} (V(x) - V(x + x_{\epsilon_j}))w_{\epsilon_j}w_0 dx \\ &\leq -V_0\|w_0\|_2^2 + \|w_0\|_{2^*_s}^{2^*_s}, \end{aligned} \tag{3.25}$$

which yields  $[w_0]_s^2 < \|w_0\|_{2^*_s}^{2^*_s}$  and similarly to the proof of (3.24), we get a contradiction.  $\square$

As  $\epsilon \rightarrow \epsilon_0 \in (0, \frac{2^*_s-2}{2})$ , we have that  $\{u_\epsilon\}$  is uniformly bounded with respect to  $\epsilon$ , as established in the following

**Lemma 5** *There exists  $K > 0$ , which does not depend on  $\epsilon$ , such that any solutions  $u_\epsilon$  of (1.1) satisfies  $\|u_\epsilon\|_\infty \leq K$  as  $\epsilon \rightarrow \epsilon_0$ .*

*Proof* The claim can be achieved via Moser’s iteration. Indeed, let  $w_\epsilon$  be the harmonic extension of  $u_\epsilon$ , see e.g. [9]. Then,  $w_\epsilon$  satisfies

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w_\epsilon) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \frac{\partial w_\epsilon}{\partial \nu^s} = -V(\cdot)w_\epsilon(\cdot, 0) + w_\epsilon^{2^*_s-1-\epsilon}(\cdot, 0) & \text{in } \mathbb{R}^N \times \{y = 0\}, \end{cases} \tag{3.26}$$

where

$$\frac{\partial w_\epsilon}{\partial \nu^s} := -\frac{1}{k_s} \lim_{y \rightarrow 0^+} y^{1-2s} \frac{\partial w_\epsilon}{\partial y}(x, y).$$

and  $k_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$ .

Following Corollary 2.1 in [1], for each  $L > 0$ , we set

$$w_{\epsilon,L}(x, y) = \begin{cases} w_\epsilon(x, y) & \text{if } w_\epsilon(x, y) \leq L, \\ L & \text{if } w_\epsilon(x, y) \geq L, \end{cases} \quad u_{\epsilon,L}(x) = w_{\epsilon,L}(x, 0), \tag{3.27}$$



and  $\psi_{\epsilon,L} = w_{\epsilon,L}^{2(\beta-1)} w_\epsilon$ , where  $\beta > 1$  to be determined later on. By testing with  $\psi_{\epsilon,L}$ , we get

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s} \nabla w_\epsilon \nabla \psi_{\epsilon,L} dx dy = k_s^{-1} \int_{\mathbb{R}^N} [-V(x)u_\epsilon(x) + u_\epsilon^{2_s^*-1-\epsilon}(x)] \psi_{\epsilon,L}(x, 0) dx. \tag{3.28}$$

Thus,

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s} \nabla w_\epsilon \nabla (w_{\epsilon,L}^{2(\beta-1)} w_\epsilon) dx dy \leq k_s^{-1} \int_{\mathbb{R}^N} u_\epsilon^{2_s^*-\epsilon} w_{\epsilon,L}^{2(\beta-1)} dx. \tag{3.29}$$

Note that

$$\nabla w_\epsilon \nabla (w_{\epsilon,L}^{2(\beta-1)} w_\epsilon) = \begin{cases} (2\beta - 1) w_{\epsilon,L}^{2(\beta-1)}(x, y) |\nabla w_\epsilon|^2 & \text{if } w_\epsilon(x, y) \leq L, \\ L^{2\beta-1} |\nabla w_\epsilon|^2 & \text{if } w_\epsilon(x, y) \geq L. \end{cases} \tag{3.30}$$

Thus, from (3.29), Sobolev imbedding (see e.g. (2.9) in [13]) and Hölder’s inequality,

$$\begin{aligned} \left( \int_{\mathbb{R}^N} |u_{\epsilon,L}^{\beta-1} u_\epsilon|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} &\leq C(N, s) \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla (w_{\epsilon,L}^{\beta-1} w_\epsilon)|^2 dx dy \\ &\leq \beta C(N, s) \int_{\mathbb{R}^N} u_\epsilon^{2_s^*-2-\epsilon} u_\epsilon^{2_s^*} w_{\epsilon,L}^{2(\beta-1)} dx \\ &\leq \beta C(N, s) \left( \int_{\mathbb{R}^N} u_\epsilon^{2_s^*} dx \right)^{\frac{2_s^*-2-\epsilon}{2_s^*}} \left( \int_{\mathbb{R}^N} (u_{\epsilon,L}^{\beta-1} u_\epsilon)^{\frac{22_s^*}{2+\epsilon}} dx \right)^{\frac{2+\epsilon}{2_s^*}}. \end{aligned} \tag{3.31}$$

Since  $\|u_\epsilon\|_{2_s^*}$  is bounded, from (3.31) we get

$$\|u_{\epsilon,L}^{\beta-1} u_\epsilon\|_{2_s^*}^2 \leq \beta C(N, s) \left( \int_{\mathbb{R}^N} (u_{\epsilon,L}^{\beta-1} u_\epsilon)^{\frac{22_s^*}{2+\epsilon}} dx \right)^{\frac{2+\epsilon}{2_s^*}}. \tag{3.32}$$

As  $u_\epsilon \in L^{\frac{22_s^* \beta}{\epsilon}}(\mathbb{R}^N)$ , by using the fact that  $w_{\epsilon,L} \leq w_\epsilon$ , we get

$$\|u_{\epsilon,L}^{\beta-1} u_\epsilon\|_{2_s^*}^2 \leq \beta C(N, s) \left( \int_{\mathbb{R}^N} u_\epsilon^{\frac{22_s^* \beta}{2+\epsilon}} dx \right)^{\frac{2+\epsilon}{2_s^*}}. \tag{3.33}$$

Let  $L \rightarrow +\infty$  and apply Fatou’s lemma to get

$$\|u_\epsilon\|_{2_s^* \beta}^2 \leq \beta^{\frac{1}{\beta}} C^{\frac{1}{\beta}}(N, s) \|u_\epsilon\|_{\frac{22_s^*}{2+\epsilon} \beta}^2. \tag{3.34}$$

The claim now follows by iteration: let  $\beta_i = (\frac{2+\epsilon}{2})^i$ ,  $i = 1, 2, \dots$ , then

$$\|u_\epsilon\|_{2_s^* \beta_{m+1}} \leq \left( \frac{2+\epsilon}{2} \right)^{\frac{1}{2} \sum_{i=1}^m i (\frac{2+\epsilon}{2})^{-i}} C^{\frac{1}{2} \sum_{i=1}^m (\frac{2+\epsilon}{2})^{-i}}(N, s) \|u_\epsilon\|_{2_s^*}. \tag{3.35}$$

Passing to the limit as  $m \rightarrow +\infty$  in (3.35), we have

$$\|u_\epsilon\|_\infty \leq C \|u_\epsilon\|_{2_s^*}.$$

which concludes the proof. □

**Lemma 6** *Let  $\epsilon_0 > 0$ , then  $\limsup_{\epsilon \rightarrow \epsilon_0} S_{2_s^*-\epsilon}^V \leq S_{2_s^*-\epsilon_0}^V$ .*

*Proof* Let  $\phi > 0$  be such that  $S_{2_s^*-\epsilon_0}^V = \frac{\|\phi\|_{s,V}^2}{\|\phi\|_{2_s^*-\epsilon_0}^2}$ . Then,

$$\int_{\mathbb{R}^N} |\phi|^{2^*-\epsilon} dx = \int_{\mathbb{R}^N} |\phi|^{2^*-\epsilon_0} dx + (\epsilon - \epsilon_0) \int_{\mathbb{R}^N} |\phi|^{2^*-\epsilon_0+t(\epsilon_0-\epsilon)} \ln \phi dx, \tag{3.36}$$

where  $t \in (0, 1)$ . Since  $|\phi^\alpha \ln \phi| \leq C$  for any  $\alpha > 0$  as  $\phi \rightarrow 0^+$  and  $\ln \phi \leq 1 + \phi$  as  $\phi \geq 1$ , recalling that  $\|\phi\|_\infty$  is bounded by Lemma 16, we get

$$\int_{\mathbb{R}^N} |\phi|^{2^*-\epsilon} dx = \int_{\mathbb{R}^N} |\phi|^{2^*-\epsilon_0} dx + O(\epsilon - \epsilon_0). \tag{3.37}$$

Now, by (3.37) and the definition of  $S_{2_s^*-\epsilon}^V$ , we have

$$\begin{aligned} \limsup_{\epsilon \rightarrow \epsilon_0} \left( S_{2_s^*-\epsilon}^V \right)^{\frac{2_s^*-\epsilon_0}{2}} &\leq \limsup_{\epsilon \rightarrow \epsilon_0} \frac{\|\phi\|_{s,V}^{2_s^*-\epsilon_0}}{\|\phi\|_{2_s^*-\epsilon}^{2_s^*-\epsilon_0}} \\ &= \limsup_{\epsilon \rightarrow \epsilon_0} \frac{\|\phi\|_{s,V}^{2_s^*-\epsilon_0}}{[\|\phi\|_{2_s^*-\epsilon_0} + O(\epsilon - \epsilon_0)]^{2_s^*-\epsilon_0}} = \left( S_{2_s^*-\epsilon_0}^V \right)^{\frac{2_s^*-\epsilon_0}{2}}. \end{aligned} \tag{3.38}$$

The proof of Lemma 17 is complete. □

Let  $x_\epsilon$  be the global maximum point of  $u_\epsilon$  and let  $\mu_\epsilon > 0$  be such that

$$u_\epsilon(x_\epsilon) = \|u_\epsilon\|_\infty = \mu_\epsilon^{-\frac{2s}{2_s^*-2-\epsilon}}.$$

Clearly, from Lemma 15  $\mu_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0^+$ . Set

$$v_\epsilon(x) = \mu_\epsilon^{\frac{2s}{2_s^*-2-\epsilon}} u_\epsilon(x_\epsilon + \mu_\epsilon x).$$

Then  $0 < v_\epsilon(x) \leq 1$ ,  $v_\epsilon(0) = 1$  and  $v_\epsilon$  satisfies the following

$$(-\Delta)^s v_\epsilon + \mu_\epsilon^{2s} V(x_\epsilon + \mu_\epsilon x) v_\epsilon = v_\epsilon^{2_s^*-1-\epsilon} \text{ in } \mathbb{R}^N. \tag{3.39}$$

We have that  $\|(-\Delta)^s v_\epsilon\|_\infty$  is uniformly bounded with respect to  $\epsilon$ . As a consequence of this fact and regularity results, we deduce that also  $\|v_\epsilon\|_{C^{2,\alpha}}$  is uniformly bounded with respect to  $\epsilon$ , for some  $\alpha \in (0, 1)$ . Similarly to the proof of Lemma 15, there exists a sequence  $\epsilon$ , still denoted by  $v_\epsilon$ , such that  $v_\epsilon \rightarrow U$  in  $C_{loc}^{2,\alpha}(\mathbb{R}^N)$ , where  $U$  is the positive solution of equation

$$(-\Delta)^s u = u^{2_s^*-1} \text{ in } \mathbb{R}^N \tag{3.40}$$

and  $U(0) = \|U\|_\infty = 1$ . From Theorem 1.2 in [11],

$$U(x) = \left( 1 + \frac{|x|^2}{\lambda^2} \right)^{\frac{2s-N}{2}}, \quad \text{where } \lambda = 2 \left( \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma\left(\frac{N-2s}{2}\right)} \right)^{\frac{1}{2}}. \tag{3.41}$$

Since

$$S = \frac{[U]_s^2}{\|U\|_{2_s^*}^2} = \|U\|_{2_s^*}^{2_s^*-2} = [U]_s^{2-\frac{4}{2_s^*}},$$

we conclude that

$$\|U\|_{2_s^*}^{2_s^*} = [U]_s^2 = S^{\frac{N}{2s}}.$$

By Lemma 14, we have

$$\begin{aligned}
 S^{\frac{N}{2s}} &= [U]_s^2 \leq \liminf_{\epsilon \rightarrow 0^+} [v_\epsilon]_s^2 \\
 &\leq \limsup_{\epsilon \rightarrow 0^+} [v_\epsilon]_s^2 \\
 &\leq \limsup_{\epsilon \rightarrow 0^+} \left[ \int_{\mathbb{R}^N} \frac{|v_\epsilon(x) - v_\epsilon(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x_\epsilon + \mu_\epsilon x) v_\epsilon^2 dx \right] \\
 &= \limsup_{\epsilon \rightarrow 0^+} \mu_\epsilon^{\frac{(N-2s)\epsilon}{2s^* - 2 - \epsilon}} \left[ \int_{\mathbb{R}^N} \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x) u_\epsilon^2 dx \right] \\
 &\leq \limsup_{\epsilon \rightarrow 0^+} \left[ \int_{\mathbb{R}^N} \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x) u_\epsilon^2 dx \right] \\
 &= S^{\frac{N}{2s}}.
 \end{aligned}
 \tag{3.42}$$

Finally, from Lemma 14 and (3.42), we obtain the following convergences

**Lemma 7**  $[v_\epsilon - U]_s \rightarrow 0, \|v_\epsilon - U\|_{2s^*} \rightarrow 0, [v_\epsilon]_s^2 \rightarrow S^{\frac{N}{2s}},$  and  $\mu_\epsilon^\epsilon \rightarrow 1$  as  $\epsilon \rightarrow 0^+.$

**Proposition 2** *If  $x_\epsilon \rightarrow x_0, as \epsilon \rightarrow 0^+.$  Then*

$$|u_\epsilon|^{2s^*}(x) \rightarrow S^{\frac{N}{2s}} \delta(x - x_0), \quad \epsilon \rightarrow 0$$

*in the sense of distributions.*

*Proof* For any  $\phi \in C_0^\infty(\mathbb{R}^N),$  we get

$$\begin{aligned}
 \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N} |u_\epsilon|^{2s^*} \phi dx &= \lim_{\epsilon \rightarrow 0^+} \left[ \mu_\epsilon^{-\frac{\epsilon N(N-2s)}{4s - \epsilon(N-2s)}} \int_{\mathbb{R}^N} |v_\epsilon|^{2s^*} \phi(x_\epsilon + \mu_\epsilon x) dx \right] \\
 &= \phi(x_0) \int_{\mathbb{R}^N} |U|^{2s^*} dx \\
 &= \phi(x_0) S^{\frac{N}{2s}}.
 \end{aligned}
 \tag{3.43}$$

□

Notice that up to now we do not know whether the global maximum point  $x_\epsilon$  turns out to be bounded or unbounded.

**Lemma 8** *Suppose that  $\{x_\epsilon\}$  is bounded. Then,*

$$\sup_{\epsilon \in (0, \frac{2s^* - 2}{2})} \int_{|x| \geq R} u_\epsilon^{2s^*} dx \rightarrow 0 \text{ as } R \rightarrow +\infty. \tag{3.44}$$

*Proof* Assume by contradiction that Eq. 3.44 does not hold. Then, there exist two sequences  $\epsilon_j \rightarrow \epsilon_0$  and  $R_j \rightarrow +\infty$  such that

$$\int_{|x| \geq R_j} u_{\epsilon_j}^{2s^*} dx \geq \delta, \tag{3.45}$$

for some  $\delta > 0$  and  $j = 1, 2, \dots$ . We distinguish two cases:

*Case 1.*  $\varepsilon_0 > 0$ . By (3.22),  $\{u_\epsilon\}$  is bounded in  $H_V^s(\mathbb{R}^N)$ , passing to a subsequence  $\{u_{\epsilon_j}\}$  if necessary, we may assume  $u_{\epsilon_j} \rightharpoonup u_{\epsilon_0}$  in  $H_V^s(\mathbb{R}^N)$ . On the other hand, from Lemma 16, we know that  $\|u_{\epsilon_j}\|_\infty$  is bounded and by regularity we deduce that  $\|u_{\epsilon_j}\|_{C^{2,\alpha}}$  is uniformly bounded with respect to  $\epsilon_j$ , for some  $\alpha \in (0, 1)$ . Up to extracting again a subsequence, still denoted by  $\{u_{\epsilon_j}\}$ , we have  $u_{\epsilon_j} \rightarrow u_{\epsilon_0}$  in  $C_{loc}^{2,\alpha}(\mathbb{R}^N)$ . Thus,  $u_{\epsilon_0}$  is a classical nonnegative solution of the equation

$$(-\Delta)^s u + V(x)u = u^{2_s^*-1-\epsilon_0} \text{ in } \mathbb{R}^N. \tag{3.46}$$

By (3.6), we get  $u_{\epsilon_0}(x) \not\equiv 0$ . Moreover, if there is  $x_0 \in \mathbb{R}^N$  such that  $u_{\epsilon_0}(x_0) = 0$ , then from (3.46),  $(-\Delta)^s u(x_0) = 0$ . However, by the very definition

$$(-\Delta)^s u(x_0) = c_{N,s} \text{PV} \int_{\mathbb{R}^N} \frac{-u(y)}{|x_0 - y|^{N+2s}} dy < 0,$$

which is a contradiction. Thus,  $u_{\epsilon_0}(x) > 0$  for all  $x \in \mathbb{R}^N$ .

Now, by (3.21) and Lemma 17, observe that

$$\begin{aligned} S_{2_s^*-\epsilon_0}^V &\leq \frac{\|u_{\epsilon_0}\|_{s,V}^2}{\|u_{\epsilon_0}\|_{2_s^*-\epsilon_0}^2} = \|u_{\epsilon_0}\|_{s,V}^{2-\frac{4}{2_s^*-\epsilon_0}} \\ &\leq \liminf_{j \rightarrow \infty} \|u_{\epsilon_j}\|_{s,V}^{2-\frac{4}{2_s^*-\epsilon_0}} \\ &= \liminf_{j \rightarrow \infty} S_{2_s^*-\epsilon_j}^V \\ &\leq \limsup_{\epsilon \rightarrow \epsilon_0} S_{2_s^*-\epsilon}^V \\ &\leq S_{2_s^*-\epsilon_0}^V. \end{aligned} \tag{3.47}$$

Therefore, we get

$$\lim_{j \rightarrow \infty} S_{2_s^*-\epsilon_j}^V = S_{2_s^*-\epsilon_0}^V. \tag{3.48}$$

Similarly to the proof of Lemma 17, from (3.48) we get  $\|u_{\epsilon_j}\|_{s,V} \rightarrow \|u_{\epsilon_0}\|_{s,V}$  as  $j \rightarrow +\infty$  and hence  $u_{\epsilon_j} \rightarrow u_{\epsilon_0}$  in  $L^{2_s^*}(\mathbb{R}^N)$  as  $j \rightarrow +\infty$ . This contradicts (3.45).

*Case 2.*  $\varepsilon_0 = 0$ . Thanks to Lemma 18, we obtain a contradiction from (3.45). Indeed, we have

$$\begin{aligned} \delta &\leq \int_{|x| \geq R_j} u_{\epsilon_j}^{2_s^*} dx = \mu_{\epsilon_j}^{N-\frac{2s2_s^*}{2_s^*-2-\epsilon_j}} \int_{|x_{\epsilon_j} + \mu_{\epsilon_j} x| \geq R_j} v_{\epsilon_j}^{2_s^*}(x) dx \\ &\leq (\mu_{\epsilon_j}^{\epsilon_j})^{\frac{N(N-2)}{4s-N\epsilon_j+2s\epsilon_j}} \int_{|x| \geq \frac{R_j-|x_{\epsilon_j}|}{\mu_{\epsilon_j}}} v_{\epsilon_j}^{2_s^*}(x) dx \\ &\rightarrow 0 \text{ as } j \rightarrow \infty, \end{aligned} \tag{3.49}$$

since  $\mu_{\epsilon_j}^{\epsilon_j} \rightarrow 1$ ,  $\frac{R_j-|x_{\epsilon_j}|}{\mu_{\epsilon_j}} \rightarrow +\infty$  and  $v_{\epsilon_j} \rightarrow U$  in  $L^{2_s^*}(\mathbb{R}^N)$  as  $j \rightarrow \infty$ .

The proof is now complete. □

**Lemma 9** *Suppose that  $\{x_\epsilon\}$  is unbounded. Then, for any fixed  $R > 0$ ,*

$$\limsup_{\epsilon \rightarrow 0^+} \int_{|x-x_\epsilon| \geq R} u_\epsilon^{2_s^*} dx = 0. \tag{3.50}$$

*Proof* The proof is similar to Lemma 20. Suppose that the claim is not true. Then there exist a sequence  $\epsilon_j \rightarrow \epsilon_0$  such that

$$\int_{|x-x_{\epsilon_j}| \geq R} u_{\epsilon_j}^{2_s^*} dx \geq \delta, \tag{3.51}$$

for some  $\delta > 0$  and  $j = 1, 2, \dots$ .

The proof of the case  $\epsilon_0 > 0$  is similar to Lemma 20. For  $\epsilon_0 = 0$ , we have

$$\begin{aligned} \delta &\leq \int_{|x-x_{\epsilon_j}| \geq R} u_{\epsilon_j}^{2_s^*} dx = \mu_{\epsilon_j}^{N - \frac{2_s 2_s^*}{2_s^* - 2 - \epsilon_j}} \int_{|x| \geq \frac{R}{\mu_{\epsilon_j}}} v_{\epsilon_j}^{2_s^*}(x) dx \\ &= (\mu_{\epsilon_j}^{\epsilon_j})^{\frac{N(N-2)}{4s - N\epsilon_j + 2s\epsilon_j}} \int_{|x| \geq \frac{R}{\mu_{\epsilon_j}}} v_{\epsilon_j}^{2_s^*}(x) dx \\ &\rightarrow 0, \text{ as } j \rightarrow \infty, \end{aligned} \tag{3.52}$$

since  $\mu_{\epsilon_j}^{\epsilon_j} \rightarrow 1, \frac{R}{\mu_{\epsilon_j}} \rightarrow +\infty$  and  $v_{\epsilon_j} \rightarrow U$  in  $L^{2_s^*}(\mathbb{R}^N)$  as  $j \rightarrow \infty$ . □

The following lemmas will play an important role in our analysis.

**Lemma 10** *Assume that  $\{x_\epsilon\}$  stays bounded. Then, there exist constants  $C, R > 0$  independent of  $\epsilon$ , such that*

$$|u_\epsilon(x)| \leq \frac{C}{|x|^{N+2s}}, \text{ for } |x| \geq R. \tag{3.53}$$

*Proof* We observe that

$$(-\Delta)^s u_\epsilon \leq u_\epsilon^{2_s^* - 2 - \epsilon} u_\epsilon.$$

Since  $u_\epsilon^{2_s^* - 2 - \epsilon} \in L^t_{loc}(\mathbb{R}^N)$  for some  $t > \frac{N}{2s}$ , from Theorem 10 we have

$$\max_{B_r(y)} u_\epsilon(x) \leq C \left( \int_{B_R(y)} |u_\epsilon|^{2_s^*} dx \right)^{\frac{1}{2_s^*}}, \quad \forall y \in \mathbb{R}^N, 0 < r < R, \tag{3.54}$$

where  $C$  is independent of  $\epsilon$ . Thus, we conclude from (3.44) and (3.54) that

$$\sup_{\epsilon \in (0, \frac{2_s^* - 2}{2})} u_\epsilon(y) \rightarrow 0, \text{ as } |y| \rightarrow +\infty. \tag{3.55}$$

This fact together with Lemma C.2 in [24] imply

$$|u_\epsilon(x)| \leq \frac{C}{|x|^{N+2s}}. \tag{3.56}$$

Actually, we first fix  $\epsilon > 0$  to applying Lemma C.2 in [24], and then we take the supremum with respect to  $\epsilon$ . Finally, from Lemmas 17 and 14, we get (3.56). □

**Lemma 11** *Suppose that  $\{x_\epsilon\}$  is unbounded. Then there exists a constant  $C > 0$  independent of  $\epsilon$  such that for small  $\epsilon > 0$ ,*

$$|u_\epsilon(x)| \leq \frac{C}{|x - x_\epsilon|^{N+2s}}, \text{ for } |x - x_\epsilon| \geq R, \tag{3.57}$$

for any  $R > 0$ .

*Proof* From (3.50), for any  $\delta, R > 0$ , there exists a small  $\epsilon_0 > 0$  such that if  $0 < \epsilon < \epsilon_0$  and  $|y - x_\epsilon| \geq R$ , then

$$u_\epsilon(y) \leq \delta. \tag{3.58}$$

Let  $w_\epsilon(x) = u_\epsilon(y)$ ,  $y = V_0^{-\frac{1}{2s}}x$ . Then  $w_\epsilon(x)$  enjoys the following

$$(-\Delta)^s w_\epsilon + V(y)V_0^{-1}w_\epsilon = V_0^{-1}w_\epsilon^{2^*_s-1-\epsilon}. \tag{3.59}$$

Furthermore, by condition  $(V_1)$ , if we choose  $\delta > 0$  sufficiently small and  $R_1 > 0$  large enough, we have

$$(-\Delta)^s w_\epsilon + w_\epsilon = f_\epsilon(x) := \left[1 - V(y)V_0^{-1}\right]w_\epsilon(x) + V_0^{-1}w_\epsilon^{2^*_s-1-\epsilon}(x) \leq 0 \tag{3.60}$$

for small  $\epsilon > 0$ ,  $|x| \geq R_1$  and  $|V_0^{-\frac{1}{2s}}x - x_\epsilon| \geq R$ .

Borrowing some results from [23], we also have

$$w_\epsilon(x) = \mathcal{K} * f_\epsilon(x) = \int_{\mathbb{R}^N} \mathcal{K}(x - y)f_\epsilon(y)dy, \tag{3.61}$$

where  $\mathcal{K}$  is the Bessel kernel and which enjoys the following properties:

- (K<sub>1</sub>)  $\mathcal{K}$  is positive, radially symmetric and smooth in  $\mathbb{R}^N \setminus \{0\}$ ;
- (K<sub>2</sub>) There is  $C_1, C_2 > 0$  such that

$$\mathcal{K}(x) \leq \frac{C_1}{|x|^{N+2s}}, \text{ if } |x| \geq 1 \tag{3.62}$$

and

$$\mathcal{K}(x) \leq \frac{C_2}{|x|^{N-2s}}, \text{ if } |x| \leq 1. \tag{3.63}$$

From (3.60) and (3.61) we have

$$w_\epsilon(x) \leq \int_{\{|V_0^{-\frac{1}{2s}}y - x_\epsilon| \leq R, |y| \geq R_1\}} \mathcal{K}(x - y)f_\epsilon(y)dy + \int_{\{|y| \leq R_1\}} \mathcal{K}(x - y)f_\epsilon(y)dy \tag{3.64}$$

Note that  $|V_0^{-\frac{1}{2s}}y - x_\epsilon| > R \Leftrightarrow |y - V_0^{\frac{1}{2s}}x_\epsilon| > V_0^{\frac{1}{2s}}R$ . Since  $\{x_\epsilon\}$  is unbounded, then there exists  $0 < \epsilon_1 < \epsilon_0$  such that  $|x_\epsilon| \geq R + R_1$  for  $0 < \epsilon < \epsilon_1$ . Thus, for  $|y| \leq R_1$ , we get  $|y - x_\epsilon| \geq |x_\epsilon| - |y| \geq R$ . So, from (3.59) and (3.60), we obtain

$$\int_{\{|y| \leq R_1\}} \mathcal{K}(x - y)f_\epsilon(y)dy \leq C \int_{\{|y| \leq R_1\}} \mathcal{K}(x - y)dy. \tag{3.65}$$

By (3.62) and (3.63), we have

$$\begin{aligned} \int_{\{|y| \leq R_1\}} \mathcal{K}(x - y)dy &= \int_{\{|y| \leq R_1, |x-y| < 1\}} \mathcal{K}(x - y)dy + \int_{\{|y| \leq R_1, |x-y| \geq 1\}} \mathcal{K}(x - y)dy \\ &\leq \frac{N\omega_{N-1}C_1}{2s} + C_2N\omega_{N-1}R_1^N. \end{aligned} \tag{3.66}$$

Besides, for  $R_2 > R$ ,  $|x - V_0^{-\frac{1}{2s}} x_\epsilon| > V_0^{-\frac{1}{2s}} R_2$  and  $|y - V_0^{-\frac{1}{2s}} x_\epsilon| \leq V_0^{-\frac{1}{2s}} R$ , we get  $|x - y| \geq \frac{R_2 - R}{R} |x - V_0^{-\frac{1}{2s}} x_\epsilon|$  and thus

$$\begin{aligned} & \int_{\{|V_0^{-\frac{1}{2s}} y - x_\epsilon| \leq R, |y| \geq R_1\}} \mathcal{K}(x - y) f_\epsilon(y) dy \\ & \leq C \int_{\{|V_0^{-\frac{1}{2s}} y - x_\epsilon| \leq R, |y| \geq R_1\}} \mathcal{K}(x - y) u_\epsilon^{2^* - 1 - \epsilon}(y) dy \\ & \leq C \|u_\epsilon\|_{2^* - \epsilon}^{2^* - 1 - \epsilon} \left( \int_{\{|V_0^{-\frac{1}{2s}} y - x_\epsilon| \leq R, |y| \geq R_1\}} \mathcal{K}(x - y)^{\frac{2N - (N - 2s)\epsilon}{N - 2s}} dy \right)^{\frac{N - 2s}{2N - (N - 2s)\epsilon}} \tag{3.67} \\ & \leq \frac{C}{|V_0^{-\frac{1}{2s}} x - x_\epsilon|^{N + 2s}}. \end{aligned}$$

Combining (3.64)–(3.67), for  $|V_0^{-\frac{1}{2s}} x - x_\epsilon| > R_2$  and small  $\epsilon > 0$ , we have

$$|w_\epsilon(x)| \leq \frac{C}{|V_0^{-\frac{1}{2s}} x - x_\epsilon|^{N + 2s}}.$$

That is,

$$|u_\epsilon(x)| \leq \frac{C}{|x - x_\epsilon|^{N + 2s}}, \quad |x - x_\epsilon| > R_2.$$

Since  $R$  is arbitrary, as well as  $R_2$  is arbitrary, the proof is complete. □

*Remark 4* By using standard comparison arguments as in [23], we can prove results similar to (3.44) and (3.50). However, the constant  $C$  obtained there may depend on  $\epsilon$ .

**Lemma 12** *There exists a positive constant  $C$  independent of  $\epsilon$ , such that*

$$v_\epsilon(x) \leq CU(x), \quad x \in \mathbb{R}^N. \tag{3.68}$$

*Proof* Note that we do not assume that  $\{x_\epsilon\}$  is bounded or unbounded. From the definition of  $v_\epsilon$  and  $U$ ,  $v_\epsilon(0) = U(0) = 1$ , and since  $v_\epsilon(x) \in C^{2,\alpha}$ , by choosing some large  $C$ , (3.68) holds in a neighborhood of zero. Therefore, it is enough to establish (3.68) ifor  $|x|$  bounded away from zero. For this purpose, let  $\Phi_\epsilon(x)$  be the Kelvin transform of  $v_\epsilon$ , namely

$$\Phi_\epsilon(x) = |x|^{2s - N} v_\epsilon\left(\frac{x}{|x|^2}\right). \tag{3.69}$$

Then,  $\Phi_\epsilon$  satisfies

$$(-\Delta)^s \Phi_\epsilon + \mu_\epsilon^{2s} |x|^{-4s} V\left(x_\epsilon + \mu_\epsilon \frac{x}{|x|^2}\right) \Phi_\epsilon = |x|^{(2s - N)\epsilon} \Phi_\epsilon^{2^* - 1 - \epsilon} \text{ in } \mathbb{R}^N. \tag{3.70}$$

Now, we aim at proving that  $\{\Phi_\epsilon\}$  is uniformly bounded with respect to  $\epsilon$  in a neighborhood of zero, and this will imply (3.68) by (3.69).

From (3.70), we obtain

$$(-\Delta)^s \Phi_\epsilon \leq |x|^{(2s - N)\epsilon} \Phi_\epsilon^{2^* - 1 - \epsilon} := a(x) \Phi_\epsilon, \quad a(x) = |x|^{(2s - N)\epsilon} \Phi_\epsilon^{2^* - 2 - \epsilon}. \tag{3.71}$$

*Claim:*  $a(x) \in L^t_{loc}(\mathbb{R}^N)$  with some  $t > \frac{N}{2s}$ .

Assume for the moment the claim holds true and let us complete the proof. By Theorem 10, for any compact set  $K$ , we have

$$\begin{aligned} \max_K \Phi_\epsilon(x) &\leq C \left( \int_K |\Phi_\epsilon|^{2_s^*} dx \right)^{\frac{1}{2_s^*}} \leq C \left( \int_{\mathbb{R}^N} |v_\epsilon|^{2_s^*} dx \right)^{\frac{1}{2_s^*}} \\ &\leq C(\mu_\epsilon^\epsilon)^{\frac{N}{2_s^*(2_s^*-2-\epsilon)}} \left( \int_{\mathbb{R}^N} |u_\epsilon|^{2_s^*} dx \right)^{\frac{1}{2_s^*}} \leq C. \end{aligned} \tag{3.72}$$

The last inequality follows from the facts  $\mu_\epsilon^\epsilon \rightarrow 1$  and  $\|u_\epsilon\|_{2_s^*} \leq C\|u_\epsilon\|_{s, V(x)} \rightarrow CS^{\frac{N}{4s}}$  as  $\epsilon \rightarrow 0^+$ .

Thus, it remains to prove the claim. On the one hand, for  $r > 0$  we get

$$\begin{aligned} \int_{\mu_\epsilon^2 \leq |x| \leq r} a(x)^t dx &\leq (\mu_\epsilon^\epsilon)^{2(2s-N)t} \int_{\mu_\epsilon^2 \leq |x| \leq r} \Phi_\epsilon^{(2_s^*-2-\epsilon)t} dx \\ &\leq (\mu_\epsilon^\epsilon)^{2(2s-N)t} |B_r|^{1-\frac{(2_s^*-2-\epsilon)t}{2_s^*}} \left( \int_{B_r} \Phi_\epsilon^{2_s^*} dx \right)^{\frac{(2_s^*-2-\epsilon)t}{2_s^*}} \leq C, \end{aligned} \tag{3.73}$$

since  $\mu_\epsilon^\epsilon \rightarrow 1$  as  $\epsilon \rightarrow 0^+$  and  $\Phi_\epsilon \rightarrow \bar{U}$  in  $L^{2_s^*}(\mathbb{R}^N)$ , where  $\bar{U}(x) = |x|^{2s-N} U\left(\frac{x}{|x|^2}\right)$ .

On the other hand, if  $\{x_\epsilon\}$  is bounded and  $|x| \leq \frac{\mu_\epsilon}{R-|x_\epsilon|}$ , or if  $\{x_\epsilon\}$  is unbounded and  $|x| \leq \frac{\mu_\epsilon}{R}$ , by Lemmas 22 and 23, we have

$$\begin{aligned} \Phi_\epsilon(x) &= |x|^{2s-N} v_\epsilon \left( \frac{x}{|x|^2} \right) \\ &= \mu_\epsilon^{\frac{2s}{2_s^*-2-\epsilon}} |x|^{2s-N} u_\epsilon \left( x_\epsilon + \mu_\epsilon \frac{x}{|x|^2} \right) \\ &\leq C\mu_\epsilon^{\left[ \frac{2s}{2_s^*-2-\epsilon} - (N+2s) \right]} |x|^{4s}. \end{aligned} \tag{3.74}$$

Thus, we have

$$\int_{|x| \leq \mu_\epsilon^2} a(x)^t dx \leq C\mu_\epsilon^{\left[ \frac{2s}{2_s^*-2-\epsilon} - (N+2s) \right] (2_s^*-2-\epsilon)t} \int_{|x| \leq \mu_\epsilon^2} |x|^{[4s(2_s^*-2-\epsilon) + (2s-N)\epsilon]t} dx \leq C \tag{3.75}$$

and the proof is complete. □

**Proposition 3** Assume  $N > 4s$  and suppose that  $x_\epsilon \rightarrow x_0$  as  $\epsilon \rightarrow 0^+$ . Then,

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \|u_\epsilon\|_\infty^{\frac{4s}{N-2}} = A_{N,s} \left[ V(x_0) + \frac{1}{2s} x_0 \cdot \nabla V(x_0) \right],$$

where

$$A_{N,s} = \frac{2^{2(N+1)} N^2 \pi^{\frac{N}{2}} \Gamma\left(\frac{N-4s}{2}\right)}{(N-2s)^2 \Gamma(N-2s)} S^{-\frac{N}{2s}}.$$



*Proof* By Pohozaev identity (2.12), we have

$$\begin{aligned}
 & \left( \frac{1}{2_s^* - \epsilon} - \frac{1}{2_s^*} \right) \int_{\mathbb{R}^N} u_\epsilon^{2_s^* - \epsilon} dx \\
 &= \int_{\mathbb{R}^N} \left[ V(x) + \frac{1}{2_s} x \cdot \nabla V(x) \right] u_\epsilon^2 dx \\
 &= \mu_\epsilon^N \int_{\mathbb{R}^N} \left[ V(x_\epsilon + \mu_\epsilon x) + \frac{1}{2_s} (x_\epsilon + \mu_\epsilon x) \cdot \nabla V(x_\epsilon + \mu_\epsilon x) \right] u_\epsilon^2(x_\epsilon + \mu_\epsilon x) dx \\
 &= \mu_\epsilon^{N - \frac{4s}{2_s^* - 2 - \epsilon}} \int_{\mathbb{R}^N} \left[ V(x_\epsilon + \mu_\epsilon x) + \frac{1}{2_s} (x_\epsilon + \mu_\epsilon x) \cdot \nabla V(x_\epsilon + \mu_\epsilon x) \right] v_\epsilon^2 dx.
 \end{aligned} \tag{3.76}$$

Since  $N > 4s$ , by Lemma 25 and the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
 & \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N} \left[ V(x_\epsilon + \mu_\epsilon x) + \frac{1}{2_s} (x_\epsilon + \mu_\epsilon x) \cdot \nabla V(x_\epsilon + \mu_\epsilon x) \right] v_\epsilon^2 dx \\
 &= \left[ V(x_0) + \frac{1}{2_s} x_0 \cdot \nabla V(x_0) \right] \int_{\mathbb{R}^N} U^2 dx.
 \end{aligned} \tag{3.77}$$

By direct calculations, we deduce that

$$\begin{aligned}
 \int_{\mathbb{R}^N} U^2 dx &= \lambda^{2N} \int_{\mathbb{R}^N} (1 + |x|^2)^{2s - N} dx \\
 &= \lambda^{2N} \omega_N \int_0^\infty (1 + r^2)^{2s - N} r^{N-1} dr \\
 &= \frac{1}{2} \lambda^{2N} \omega_N \int_0^\infty (1 + s)^{2s - N} s^{-1 + \frac{N}{2}} ds \\
 &= \lambda^{2N} \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} B\left(\frac{N}{2}, \frac{N}{2} - 2s\right) \\
 &= 2^{2N} \pi^{\frac{N}{2}} \left[ \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma\left(\frac{N-2s}{2}\right)} \right]^N \frac{\Gamma\left(\frac{N-4s}{2}\right)}{\Gamma(N - 2s)}.
 \end{aligned} \tag{3.78}$$

Finally, combine (3.76)–(3.78) to have as  $\epsilon \rightarrow 0$

$$\begin{aligned}
 \epsilon \mu_\epsilon^{2s} &= \left( \frac{2N}{N - 2s} \right)^2 \left[ V(x_0) + \frac{1}{2_s} x_0 \cdot \nabla V(x_0) \right] S^{-\frac{N}{2s}} \int_{\mathbb{R}^N} U^2 dx + o(1) \\
 &= \frac{2^{2(N+1)} N^2 \pi^{\frac{N}{2}} \Gamma\left(\frac{N-4s}{2}\right)}{(N - 2s)^2 \Gamma(N - 2s)} \left[ \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma\left(\frac{N-2s}{2}\right)} \right]^N \left[ V(x_0) + \frac{1}{2_s} x_0 \cdot \nabla V(x_0) \right] S^{-\frac{N}{2s}} \\
 &\quad + o(1).
 \end{aligned} \tag{3.79}$$

This concludes the proof of Lemma 26. □

*Remark 5* From the proof of Proposition 26, assuming  $N > 4s$ , no matter  $x_\epsilon$  stays bounded or not, we still have

$$\epsilon = O(\mu_\epsilon^{2s}).$$

*Proof of Theorem 1.* The conclusions (1) and (2) in Theorem 1 follow from Propositions 13 and 26. Clearly, Corollary 2 is a particular case of Theorem 1.

### 4 Localizing Blow up Points

We next recall for convenience of the reader a few basic facts on fractional Sobolev spaces. Let  $\beta > 0$  and  $p \in [1, \infty)$ ,

$$\mathcal{W}^{\beta,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : \mathcal{F}^{-1}[(1 + |\xi|^\beta)\hat{u}] \in L^p(\mathbb{R}^N)\}$$

endowed with the norm

$$\|u\|_{\mathcal{W}^{\beta,p}(\mathbb{R}^N)} = \|\mathcal{F}^{-1}[(1 + |\xi|^\beta)\hat{u}]\|_p.$$

We refer to [23] for the following results.

**Proposition 4** *The following properties hold true:*

- (1) *If  $0 < \beta < 1$ ,  $1 < p \leq q \leq \frac{Np}{N-\beta p} < \infty$  or  $p = 1$  and  $1 \leq q < \frac{N}{N-\beta}$ , then  $\mathcal{W}^{\beta,p}(\mathbb{R}^N)$  is continuously embedded in  $L^q(\mathbb{R}^N)$ .*
- (2) *Assume that  $0 \leq \beta \leq 2$  and  $\beta > \frac{N}{p}$ . If  $\beta - \frac{N}{p} > 1$  and  $0 < \mu \leq \beta - 1 - \frac{N}{p}$ , then  $\mathcal{W}^{\beta,p}(\mathbb{R}^N)$  is continuously embedded in  $C^{1,\mu}(\mathbb{R}^N)$ . If  $\beta - \frac{N}{p} < 1$  and  $0 < \mu \leq \beta - \frac{N}{p}$ , then  $\mathcal{W}^{\beta,p}(\mathbb{R}^N)$  is continuously embedded in  $C^{0,\mu}(\mathbb{R}^N)$ .*

For  $p \in [1, +\infty)$  and  $\beta > 0$ , consider the Bessel potential space

$$\mathcal{L}^{\beta,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : \mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{\beta}{2}}\hat{u}] \in L^p(\mathbb{R}^N)\}.$$

Then,  $\mathcal{L}^{\beta,p}(\mathbb{R}^N) = \mathcal{W}^{\beta,p}(\mathbb{R}^N)$ , see Theorem 3.1 in [23]. On the other hand, from Theorem 5 in Chapter V of [37], for  $p \in [2, \infty)$  and  $0 < \beta < 1$ , one has  $\mathcal{W}^{\beta,p}(\mathbb{R}^N) \subset W^{\beta,p}(\mathbb{R}^N)$ , where  $W^{\beta,p}(\mathbb{R}^N)$  is the usual fractional Sobolev space defined by

$$W^{\beta,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\beta p}} dx dy \right\}.$$

Our next target is to identify the location of the blow up points. For this purpose we adapt the method developed in [28], where the basic idea is to get an asymptotic expansion of the ground state energy and then to compare it with an upper bound of  $S_{2_s^*-\epsilon}^V$ . This method has been used to deal with the localization of blow-up points of ground states to semilinear problems in [40].

Let us begin with establishing an upper bound for  $S_{2_s^*-\epsilon}^V$ .

**Theorem 7** *Assume  $N > 4s$  and that  $u_{\epsilon_j}$  is a ground state of (1.1) satisfying (1.6) which has a maximum point  $x_{\epsilon_j}$  which enjoys  $x_{\epsilon_j} \rightarrow x_0$  as  $j \rightarrow \infty$ . Then*

$$\begin{aligned} S_{2_s^*-\epsilon_j}^V &\leq S \\ &+ \mu_j^{2s} \left\{ S^{\frac{2s-N}{2s}} V(\hat{x}_0) \int_{\mathbb{R}^N} U^2 dy - \tilde{C}_{N,s} S \left[ \frac{2}{(2_s^*)^2} \ln S^{\frac{N}{2s}} - \frac{2}{2_s^*} S^{-\frac{N}{2s}} \int_{\mathbb{R}^N} U^{2_s^*} \ln U dx \right] \right\} \\ &+ o(\mu_j^{2s}), \end{aligned} \tag{4.1}$$

where  $\hat{x}_0$  is a global minimum point of  $V(x)$  and

$$\tilde{C}_{N,s} = \Psi_{N,s} \left[ V(x_0) + \frac{1}{2s} x_0 \cdot \nabla V(x_0) \right].$$

*Proof* Let

$$\phi_j(x) = U\left(\frac{x - \hat{x}_0}{\mu_j}\right).$$

Then by inspection

$$[\phi_j]_s^2 = \mu_j^{N-2s} [U]_s^2 = \mu_j^{N-2s} S^{\frac{N}{2s}}, \tag{4.2}$$

and by dominated convergence, we have

$$\begin{aligned} \int_{\mathbb{R}^N} V(x)\phi^2 dx &= \mu_j^N \int_{\mathbb{R}^N} V(\hat{x}_0 + \mu_j y) U^2 dy \\ &= V(\hat{x}_0) \mu_j^N \int_{\mathbb{R}^N} U^2 dy + o(\mu_j^N). \end{aligned} \tag{4.3}$$

By (3.79), we also get

$$\epsilon_j = \tilde{C}_{N,s} \mu_j^{2s} + o(\mu_j^{2s}). \tag{4.4}$$

Thus, by using Taylor’s formula, we get

$$\begin{aligned} &\mu_j^{\frac{-2N}{2s^*-\epsilon_j}} \left( \int_{\mathbb{R}^N} |\phi|^{2s^*-\epsilon_j} dx \right)^{\frac{2}{2s^*-\epsilon_j}} \\ &= \left( \int_{\mathbb{R}^N} U^{2s^*-\epsilon_j} dx \right)^{\frac{2}{2s^*-\epsilon_j}} \\ &= \left[ \int_{\mathbb{R}^N} (U^{2s^*} - \epsilon_j U^{2s^*} \ln U) dx + o(\epsilon_j) \right]^{\frac{2}{2s^*-\epsilon_j}} \\ &= \left[ \left( S^{\frac{N}{2s}} - \epsilon_j \int_{\mathbb{R}^N} U^{2s^*} \ln U dx \right)^{\frac{2}{2s^*-\epsilon_j}} + o(\epsilon_j) \right] \\ &= \left[ S^{\frac{N-2s}{2s}} + \epsilon_j S^{\frac{N-2s}{2s}} \left( \frac{2}{(2s^*)^2} \ln S^{\frac{N}{2s}} - \frac{2}{2s^*} S^{-\frac{N}{2s}} \int_{\mathbb{R}^N} U^{2s^*} \ln U dx \right) + o(\epsilon_j) \right] \\ &= S^{\frac{N-2s}{2s}} \left[ 1 + \epsilon_j \left( \frac{2}{(2s^*)^2} \ln S^{\frac{N}{2s}} - \frac{2}{2s^*} S^{-\frac{N}{2s}} \int_{\mathbb{R}^N} U^{2s^*} \ln U dx \right) + o(\epsilon_j) \right]. \end{aligned} \tag{4.5}$$

So, by (4.4),  $\mu_j^{\frac{2N}{2s^*-\epsilon_j}} = \mu_j^{N-2s} + o(\epsilon_j)$ , we have

$$\begin{aligned} &\left( \int_{\mathbb{R}^N} |\phi|^{2s^*-\epsilon_j} dx \right)^{\frac{2}{2s^*-\epsilon_j}} \\ &= \mu_j^{N-2s} S^{\frac{N-2s}{2s}} \left[ 1 + \tilde{C}_{N,s} \mu_j^{2s} \left( \frac{2}{(2s^*)^2} \ln S^{\frac{N}{2s}} - \frac{2}{2s^*} S^{-\frac{N}{2s}} \int_{\mathbb{R}^N} U^{2s^*} \ln U dx \right) + o(\mu_j^{2s}) \right]. \end{aligned} \tag{4.6}$$

By the very definition of  $S_{2_s^*-\epsilon_j}^V$ , we have

$$\begin{aligned}
 S_{2_s^*-\epsilon_j}^V &\leq \frac{\|u_j\|_{s,V}^2}{\|u_j\|_{2_s^*-\epsilon_j}^2} \\
 &\leq \frac{\mu_j^{N-2s} S^{\frac{N}{2s}} + V(\hat{x}_0)\mu_j^N \int_{\mathbb{R}^N} U^2 dy + o(\mu_j^N)}{\mu_j^{N-2s} S^{\frac{N-2s}{2s}} \left[ 1 + \tilde{C}_{N,s} \mu_j^{2s} \left( \frac{2}{(2_s^*)^2} \ln S^{\frac{N}{2s}} - \frac{2}{2_s^*} S^{-\frac{N}{2s}} \int_{\mathbb{R}^N} U^{2_s^*} \ln U dx \right) + o(\mu_j^{2s}) \right]} \\
 &= \left[ S + \mu_j^{2s} S^{\frac{2s-N}{2s}} V(\hat{x}_0) \int_{\mathbb{R}^N} U^2 dy + o(\mu_j^{2s}) \right] \\
 &\quad \cdot \left[ 1 - \tilde{C}_{N,s} \mu_j^{2s} \left( \frac{2}{(2_s^*)^2} \ln S^{\frac{N}{2s}} - \frac{2}{2_s^*} S^{-\frac{N}{2s}} \int_{\mathbb{R}^N} U^{2_s^*} \ln U dx \right) + o(\mu_j^{2s}) \right]. \tag{4.7}
 \end{aligned}$$

This concludes the proof. □

For simplicity, set  $\mu_j := \mu_{\epsilon_j}$ ,  $x_j := x_{\epsilon_j}$ . For  $v_j(x) = \mu_j^{\frac{2s}{2_s^*-2-\epsilon}} u_j(x_j + \mu_j x)$ , let  $v_j = U + \mu_j^{2s} w_j$ , then by (3.39) we have

$$(-\Delta)^s w_j - (2_s^* - 1)U^{2_s^*-2} w_j + V(x_j + \mu_j x)v_j = F(w_j) \text{ in } \mathbb{R}^N, \tag{4.8}$$

where

$$F(w_j) = \mu_j^{-2s} \left[ (U + \mu_j^{2s} w_j)^{2_s^*-1-\epsilon} - (2_s^* - 1)\mu_j^{2s} U^{2_s^*-2} w_j - U^{2_s^*-1} \right].$$

Define the operator  $L$  as follows:

$$L := (-\Delta)^s - (2_s^* - 1)U^{2_s^*-2}.$$

Then (4.8) can be rewritten as

$$Lw_j + V(x_j + \mu_j x)v_j = F(w_j).$$

**Proposition 5** Assume  $N > 6s$ . Then  $w_j \rightarrow w$  in  $L^\infty(\mathbb{R}^N)$  as  $j \rightarrow \infty$ , where  $w$  is a bounded solution of

$$(-\Delta)^s w - (2_s^* - 1)U^{2_s^*-2} w + V(x_0)U + \tilde{C}(N, s)U^{2_s^*-1} \ln U = 0 \text{ in } \mathbb{R}^N. \tag{4.9}$$

In order to prove Proposition 30, we need the following result from [17]

**Lemma 13** (Nondegeneracy) The solution  $U$  is nondegenerate in the sense that all bounded solutions of equation

$$(-\Delta)^s \phi = (2_s^* - 1)U^{2_s^*-2} \phi \text{ in } \mathbb{R}^N$$

are linear combinations of the functions

$$\frac{N-2s}{2} U + x \cdot \nabla U, \quad \frac{\partial U}{\partial x_i}, \quad i = 1, 2, \dots, N.$$

Let

$$X = \text{span} \left\{ \frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_N}, \frac{N-2s}{2} U + x \cdot \nabla U \right\}$$

Clearly,  $X \subset L^p(\mathbb{R}^N)$  with  $\frac{N}{N-2s} < p < +\infty$ . For  $1 < r < \frac{N}{2s}$ , define

$$Y_r := \left\{ u \in L^r(\mathbb{R}^N) : \int_{\mathbb{R}^N} uv dx = 0 \text{ for all } v \in X \right\}.$$

Then  $L^r(\mathbb{R}^N) = X \oplus Y_q$ , where  $\frac{N}{N-2s} < r < \frac{N}{2s}$ .

**Lemma 14** *Suppose  $N > 4s$ . Then for any  $1 < q < \frac{N}{4s}$ , there exists a constant  $C > 0$  such that*

$$\|u\|_{\mathcal{W}^{2s,r}} \leq C(\|Lu\|_r + \|Lu\|_q), \tag{4.10}$$

for all  $u \in Y_r \cap \mathcal{W}^{2s,r}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$  with  $Lu \in L^q(\mathbb{R}^N)$ ,  $\frac{1}{q} - \frac{2s}{N} = \frac{1}{r}$ .

*Proof* It is enough to prove

$$\|u\|_r \leq C(\|Lu\|_r + \|Lu\|_q).$$

In fact, by

$$(-\Delta)^s u + u = Lu + [1 - (2_s^* - 1)U^{2_s^*-2}]u$$

we get

$$\|u\|_{\mathcal{W}^{2s,r}} \leq \|Lu\|_r + C\|u\|_r \leq C(\|Lu\|_r + \|Lu\|_q).$$

Assume that  $u \neq 0$ . Otherwise, we are done. By homogeneity, we can replace  $u$  by  $\frac{u}{\max\{\|u\|_{C^2}, \|u\|_r\}}$  in (4.10). Thus, assume that there exists a sequence  $\{u_n\} \subset Y_r \cap \mathcal{W}^{2s,r}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$  such that

$$\text{either } \|u_n\|_{C^2} = 1, \|u_n\|_r < 1, \text{ or } \|u_n\|_{C^2} < 1, \|u_n\|_r = 1, \tag{4.11}$$

and

$$\|Lu_n\|_q + \|Lu_n\|_r \rightarrow 0. \tag{4.12}$$

Then, there exists  $u_\infty \in C^2(\mathbb{R}^N)$  such that after passing to a subsequence if necessary,  $u_n \rightarrow u_\infty$  in  $C^2_{loc}(\mathbb{R}^N)$  and in particular,  $u_n \rightarrow u_\infty$  in  $L^t_{loc}(\mathbb{R}^N)$ ,  $r \leq t < 2_s^*$ . Let  $I = (-\Delta)^{-s}$  the Riesz potentials defined by

$$(I * f)(x) = \frac{1}{\gamma(s)} \int_{\mathbb{R}^N} \frac{f(y)}{|x - y|^{N-2s}} dy$$

with

$$\gamma(s) = \pi^{\frac{N}{2}} 2^{2s} \Gamma(s) / \Gamma(N/2 - s).$$

See Chapter V in [37]. Then, we have

$$u_n - I * [(2_s^* - 1)U^{2_s^*-2}u_n] = I * Lu_n.$$

By Hardy-Littlewood-Sobolev inequality [26, 37], we have

$$\|I * Lu_n\|_r \leq C\|Lu_n\|_q \rightarrow 0$$

and Hölder’s inequality yields

$$\begin{aligned} & \|I * [(2_s^* - 1)U^{2_s^*-2}(u_n - u_m)]\|_r \\ &= (2_s^* - 1) \left[ \int_{B_R(0)} |I * U^{2_s^*-2}(u_n - u_m)|^r dx + \int_{B_R^c(0)} |I * U^{2_s^*-2}(u_n - u_m)|^r dx \right]^{\frac{1}{r}} \\ &\leq C\|U^{2_s^*-2}(u_n - u_m)\|_{L^q(B_R(0))} + C\|u_n - u_m\|_r \left( \int_{|x| \geq R} U^{\frac{2N}{N-2s}} dx \right)^{\frac{2s}{N}} \\ &\leq C\|u_n - u_m\|_{L^r(B_R(0))} + C\|u_n - u_m\|_r \left( \int_{|x| \geq R} U^{\frac{2N}{N-2s}} dx \right)^{\frac{2s}{N}}, \end{aligned} \tag{4.13}$$

where  $\frac{1}{q} - \frac{2s}{N} = \frac{1}{r}$ .

Thus,  $\{I * [(2_s^* - 1)U^{2_s^* - 2}u_n]\}$  is a Cauchy sequence in  $L^r(\mathbb{R}^N)$  and then  $\{u_n\}$  is a Cauchy sequence in  $L^r(\mathbb{R}^N)$ . So,  $u_\infty \in L^r(\mathbb{R}^N)$ ,  $u_\infty \in Y_r$  and

$$(-\Delta)^s u_\infty - (2_s^* - 1)U^{2_s^* - 2}u_\infty = 0 \text{ in } \mathbb{R}^N. \tag{4.14}$$

By (4.11),  $u_\infty \in L^\infty(\mathbb{R}^N)$  and  $u_\infty \in X$  by (4.14). But since  $u_\infty \in Y_r$ , we get  $u_\infty \equiv 0$ , which is a contradiction from (4.11).  $\square$

For fixed  $j = 1, 2, \dots$ , we have

$$(-\Delta)^s v_j + v_j = (1 + \mu_j^{2s} V(x_j + \mu_j x))v_j + v_j^{2_s^* - 1 - \epsilon_j} \text{ in } \mathbb{R}^N. \tag{4.15}$$

Note that  $0 \leq v_j \leq 1$ . Thus,  $v_j \in \mathcal{W}^{2s,p}(\mathbb{R}^N) \cap C^{2,\beta}(\mathbb{R}^N)$  for  $p \in [2, +\infty)$  and  $w_j \in \mathcal{W}^{2s,p}(\mathbb{R}^N) \cap C^{2,\beta}(\mathbb{R}^N)$  for  $2 \leq p < +\infty$ .

Let

$$w_j = \sum_{i=1}^{N+1} a_{ij} e_i + z_j, \quad j = 1, 2, \dots,$$

where  $e_1 = \frac{\partial U}{\partial x_i}$ ,  $i = 1, 2, \dots, N$ ,  $e_{N+1} = \frac{N-2s}{2}U + x \cdot \nabla U$ ,  $z_j \in Y_q \cap C^{2,\beta}(\mathbb{R}^N)$ ,  $\frac{1}{q} + \frac{1}{p} = 1$ .

**Lemma 15** *Assume  $N > 6s$  and let  $M_j = \max\{|a_{1j}|, |a_{2j}|, \dots, |a_{(N+1)j}|\}$ . Then  $M_j$  and  $\|z_j\|_{\mathcal{W}^{2s,r}}$  are bounded as  $j \rightarrow \infty$ .*

*Proof* We may assume without loss of generality,  $M_j \rightarrow +\infty$  as  $j \rightarrow \infty$  and

$$\frac{1}{M_j}(a_{1j}, \dots, a_{(N+1)j}) \rightarrow (b_1, \dots, b_{N+1}) \neq 0, \text{ as } j \rightarrow \infty.$$

Then

$$(-\Delta)^s \frac{z_j}{M_j} = (2_s^* - 1)U^{2_s^* - 2} \frac{z_j}{M_j} + \frac{1}{M_j} [F(w_j) - V(x_j + \mu_j x)v_j]. \tag{4.16}$$

Let us now now estimate the three terms in the right hand side of equation (4.16). We have

$$\begin{aligned} |\mu_j^{2s} F(w_j)| &= |(U + \mu_j^{2s} w_j)^{2_s^* - 1 - \epsilon_j} - (2_s^* - 1)\mu_j^{2s} U^{2_s^* - 2} w_j - U^{2_s^* - 1}| \\ &\leq |U^{2_s^* - 1} - U^{2_s^* - 1 - \epsilon_j}| + |(2_s^* - 1)\mu_j^{2s} U^{2_s^* - 2} w_j - (2_s^* - 1 - \epsilon_j)\mu_j^{2s} U^{2_s^* - 2 - \epsilon_j} w_j| \\ &\quad + |(U + \mu_j^{2s} w_j)^{2_s^* - 1 - \epsilon_j} - U^{2_s^* - 1 - \epsilon_j} - (2_s^* - 1 - \epsilon_j)\mu_j^{2s} U^{2_s^* - 2 - \epsilon_j} w_j| \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4.17}$$

Hence

$$I_1 = U^{2_s^* - 1 - \epsilon_j} |U^{\epsilon_j} - 1| = U^{2_s^* - 1 - \epsilon_j} |\epsilon_j \ln U + o(\epsilon_j)| \leq \epsilon_j U^{2_s^* - 1 - \epsilon_j} (|\ln U| + 1) \tag{4.18}$$

and since  $v_j \leq CU$  and  $\mu_j^{2s} w_j = v_j - U$ , we get

$$\begin{aligned}
 I_2 &= \mu_j^{2s} U^{2s-2-\epsilon_j} |w_j| |(2_s^* - 1)U^{\epsilon_j} - (2_s^* - 1 - \epsilon_j)| \\
 &\leq C \epsilon_j \mu_j^{2s} |w_j| U^{2s-2-\epsilon_j} (|\ln U| + 1) \\
 &\leq C \epsilon_j U^{2s-1-\epsilon_j} (|\ln U| + 1).
 \end{aligned}
 \tag{4.19}$$

Set  $g(t) = (U + t\mu_j^{2s} w_j)^{2s-1-\epsilon_j}$ . Then, we obtain

$$\begin{aligned}
 I_3 &= |g(1) - g(0) - g'(0)| \\
 &\leq \int_0^1 t |g''(1-t)| dt \\
 &\leq C \int_0^1 t [U + (1-t)\mu_j^{2s} w_j]^{2s-3-\epsilon_j} \mu_j^{4s} w_j^2 dt \\
 &\leq C \mu_j^{2s} |w_j| |v_j - U| \int_0^1 t U^{2s-3-\epsilon_j} dt \\
 &\leq C \mu_j^{2s} |w_j| |v_j - U| U^{2s-3-\epsilon_j}.
 \end{aligned}
 \tag{4.20}$$

Thus, by (4.17)–(4.20) and Remark 27, we get

$$|F(w_j)| \leq C \left[ U^{2s-1-\epsilon_j} (|\ln U| + 1) + |w_j| |v_j - U| U^{2s-3-\epsilon_j} \right].
 \tag{4.21}$$

So, by dominated convergence we obtain

$$\begin{aligned}
 \|F(w_j)\|_q &\leq C \left[ \|U^{2s-1-\epsilon_j} (|\ln U| + 1)\|_q + \|w_j|v_j - U| U^{2s-3-\epsilon_j}\|_q \right] \\
 &\leq C \left[ \|U^{2s-1-\epsilon_j} (|\ln U| + 1)\|_q + \|w_j\|_r \| |v_j - U| U^{2s-3-\epsilon_j} \|_{\frac{N}{2s}} \right] \\
 &\leq C [1 + o(1) \|w_j\|_r].
 \end{aligned}
 \tag{4.22}$$

Thus, we get

$$\frac{1}{M_j} \|F(w_j)\|_q \leq C \left[ o(1) + o(1) \left\| \frac{z_j}{M_j} \right\|_r \right].
 \tag{4.23}$$

Again by dominated convergence we get

$$\begin{aligned}
 \|F(w_j)\|_r &\leq C \left[ \|U^{2s-1-\epsilon_j} (|\ln U| + 1)\|_r + \|w_j|v_j - U| U^{2s-3-\epsilon_j}\|_r \right] \\
 &\leq C [1 + o(1) \|w_j\|_r],
 \end{aligned}
 \tag{4.24}$$

which yields

$$\frac{1}{M_j} \|F(w_j)\|_r \leq C \left[ o(1) + o(1) \left\| \frac{z_j}{M_j} \right\|_r \right].
 \tag{4.25}$$

By Lemma 32, for  $\frac{N}{N-2s} < q < \frac{N}{4s}$  with  $\frac{1}{q} - \frac{2s}{N} = \frac{1}{r}$  (Note that  $N > 6s$  is needed), we get

$$\begin{aligned}
 \left\| \frac{z_j}{M_j} \right\|_{\mathcal{W}^{2s,r}} &\leq \frac{C}{M_j} \left[ \|V_j v_j\|_q + \|V_j v_j\|_r + \|F(w_j)\|_q + \|F(w_j)\|_r \right] \\
 &\leq \frac{C}{M_j} [1 + \|F(w_j)\|_q + \|F(w_j)\|_r] \\
 &\leq C \left[ o(1) + o(1) \left\| \frac{z_j}{M_j} \right\|_r \right].
 \end{aligned}
 \tag{4.26}$$

Thus, we have

$$\left\| \frac{z_j}{M_j} \right\|_{\mathcal{W}^{2s,r}} = o(1). \tag{4.27}$$

By Proposition 28, we get

$$\left\| \frac{z_j}{M_j} \right\|_t = o(1), \quad r \leq t \leq \frac{Nr}{N - 2sr}. \tag{4.28}$$

By choosing  $r$  close to  $\frac{N}{2s}$ ,  $t$  can be arbitrarily large. Besides, from (4.24), we have

$$\begin{aligned} \left| L \left( \frac{z_j}{M_j} \right) \right| &\leq C \frac{1}{M_j} \left[ U + U^{2s^*-1-\epsilon_j} (|\ln U| + 1) + |w_j| |v_j - U| U^{2s^*-2-\epsilon_j} \right] \\ &\leq o(1) \left[ U + U^{2s^*-1-\epsilon_j} (|\ln U| + 1) + U^{2s^*-2-\epsilon_j} \sum_{i=1}^{N+1} |e_i| \right] + o(1) \left| \frac{z_j}{M_j} \right|, \end{aligned} \tag{4.29}$$

which yields that  $L \left( \frac{z_j}{M_j} \right) \in L^t(\mathbb{R}^N)$ . Thus, from (4.28), we get

$$\left\| \frac{z_j}{M_j} \right\|_{\mathcal{W}^{2s,t}} = o(1). \tag{4.30}$$

By Lemma 28, we have

$$\left\| \frac{z_j}{M_j} \right\|_{C^{0,\mu}} = o(1), \tag{4.31}$$

for some  $0 < \mu < 1$ . In particular we have  $\left\| \frac{z_j}{M_j} \right\|_\infty \leq C$  and from (4.29),  $\|(-\Delta)^s \frac{z_j}{M_j}\|_\infty \leq C$ . From Lemma 4.4 in [8],  $\left\| \frac{z_j}{M_j} \right\|_{C^{2,\beta}} \leq C$ . So,  $\frac{z_j}{M_j} \rightarrow 0$  in  $L^\infty(\mathbb{R}^N)$  and  $C^1_{loc}(\mathbb{R}^N)$  as  $j \rightarrow \infty$ . Since  $v_j(0) = U(0) = 1$  and both they achieve their maximum at 0, we get

$$\begin{aligned} 0 &= w_j(0) = M_j \left( \sum_{i=1}^{N+1} b_i e_i(0) + o(1) \right), \\ 0 &= \nabla w_j(0) = M_j \left( \sum_{i=1}^{N+1} b_i \nabla e_i(0) + o(1) \right). \end{aligned} \tag{4.32}$$

By direct calculations, it follows  $(b_1, b_2, \dots, b_{N+1}) = 0$ , which is a contradiction.

Similarly, we can prove the remaining part of the Lemma. □

**Lemma 16** *Assume  $N > 6s$ . Then  $z_j \rightarrow z$  in  $C^1_{loc}(\mathbb{R}^N)$ , where  $z$  is radial and satisfies*

$$(-\Delta)^s z - (2_s^* - 1)U^{2_s^*-2}z + V(x_0)U - \tilde{C}(N, s)U^{2_s^*-1} \ln U = 0 \text{ in } \mathbb{R}^N. \tag{4.33}$$

*Proof* By Lemma 33, there exists a subsequence  $\{z_{j_k}\}$  such that  $z_{j_k} \rightharpoonup z$  in  $\mathcal{W}^{2s,r}$  and  $z_{j_k} \rightarrow z$  in  $C^1_{loc}(\mathbb{R}^N)$ , see also [8]. Since  $\|z_j\|_\infty$  is bounded, from (4.19) and (4.20), we get

$$\frac{I_2 + I_3}{\mu_j^{2s}} = o(1) \tag{4.34}$$



and

$$\begin{aligned} \frac{1}{\mu_j^{2s}}(U^{2s^*-1} - U^{2s^*-1-\epsilon_j}) &= \frac{\epsilon_j \ln U + o(\epsilon_j)}{\mu_j^{2s}} U^{2s^*-1} \\ &= \frac{\mu_j^{2s} \tilde{C}(N, s) \ln U + o(\mu_j^{2s})}{\mu_j^{2s}} U^{2s^*-1} \\ &= \tilde{C}(N, s) U^{2s^*-1} \ln U + o(1). \end{aligned} \tag{4.35}$$

Thus,  $z$  satisfies (4.33).

Since  $z_{jk} \in Y_r$ , we get  $z \in Y_r$ . Thus, (4.33) has at most one such solution, and  $z_j \rightarrow z$  in  $\mathcal{W}^{2s,r}$ . Moreover, since  $(-\Delta)^s$  is invariant with respect to the action of the orthogonal group  $O(n)$  on  $\mathbb{R}^N$  (see [15]), if  $T$  denotes a rotation in  $\mathbb{R}^N$ , since (4.33) is invariant under rotation, then  $z(Tx) - z(x) \in X$ . Consequently,  $z(Tx) = z(x)$ . This proves that  $z$  is radial.  $\square$

**Lemma 17** Assume  $N > 6s$ . Then  $|a_{ij}| \rightarrow 0, i = 1, 2, \dots, N$  and  $a_{(N+1)j} \rightarrow -\frac{2}{N-2s}z(0)$  as  $j \rightarrow \infty$ .

*Proof* Note that

$$\begin{aligned} 0 &= \sum_{i=1}^{N+1} a_{ij}e_i(0) + z_j(0), \\ 0 &= \sum_{i=1}^{N+1} a_{ij}\nabla e_i(0) + \nabla z_j(0), \end{aligned} \tag{4.36}$$

which gives

$$\begin{aligned} 0 &= \frac{N - 2s}{2}a_{(N+1)j} + z_j(0), \\ 0 &= \sum_{i=1}^N b_i\nabla e_i(0) + \nabla z_j(0). \end{aligned} \tag{4.37}$$

Since  $\nabla z(0) = 0$ , we get the result.  $\square$

**Lemma 18** Assume  $N > 6s$ . Then  $w_j \rightarrow w$  in  $L^\infty(\mathbb{R}^N)$  as  $j \rightarrow \infty$ , where

$$w = z - \frac{2}{N - 2s}z(0) \left( \frac{N - 2s}{2}U + x \cdot \nabla U \right).$$

*Proof* It sufficient to prove  $z_j \rightarrow z$  in  $L^\infty(\mathbb{R}^N)$  as  $j \rightarrow \infty$ . In fact, by consider  $L(z_j - z)$ , the proof is analogous to the proof of Lemma 33.  $\square$

**Theorem 8** Assume  $N > 6s, u_{\epsilon_j}$  is a ground state of (1.1) satisfying (1.6) which has a maximum point  $x_{\epsilon_j}$  satisfying  $x_{\epsilon_j} \rightarrow x_0$  as  $j \rightarrow \infty$ . Then

$$\begin{aligned} S_{2s^*-\epsilon_j}^V &= S + S^{-\frac{N-2s}{2s}} \mu_j^{2s} \int_{\mathbb{R}^n} \left[ \frac{2}{2_s^*} \tilde{C}_{N,s} U^{2s^*} \ln U + V(x_0)U^2 \right] dx \\ &\quad - \mu_j^{2s} \tilde{C}_{N,s} \frac{2}{(2_s^*)^2} S \ln S^{\frac{N}{2s}} + o(\mu_j^{2s}). \end{aligned} \tag{4.38}$$

*Proof* By the very definition of  $S_{2_s^*-\epsilon_j}^V$ , we have

$$\begin{aligned}
 \left(S_{2_s^*-\epsilon_j}^V\right)^{\frac{2_s^*-\epsilon_j}{2_s^*-2-\epsilon_j}} &= \int_{\mathbb{R}^n} v_j^{2_s^*-\epsilon_j} dx \\
 &= \int_{\mathbb{R}^n} \left(U + \mu_j^{2_s} w_j\right)^{2_s^*-\epsilon_j} dx \\
 &= \int_{\mathbb{R}^n} \left(U^{2_s^*-\epsilon_j} + (2_s^* - \epsilon_j)U^{2_s^*-1-\epsilon_j} \mu_j^{2_s} w_j\right. \\
 &\quad \left. + \frac{1}{2}(2_s^* - \epsilon_j)(2_s^* - 1 - \epsilon_j)(U + t\mu_j^{2_s} w_j)^{2_s^*-2-\epsilon_j} \mu_j^{4_s} w_j^2\right) dx, \quad t \in (0, 1).
 \end{aligned}
 \tag{4.39}$$

Since  $v_j \leq CU$  and  $v_j = U + \mu_j^{2_s} w_j$ , then by Lemma 18,

$$\begin{aligned}
 &\int_{\mathbb{R}^n} (U + t\mu_j^{2_s} w_j)^{2_s^*-2-\epsilon_j} \mu_j^{4_s} w_j^2 dx \\
 &\leq C \|w_j\|_\infty \mu_j^{2_s} \int_{\mathbb{R}^n} U^{2_s^*-2-\epsilon_j} |v_j - U| dx \\
 &\leq C \|w_j\|_\infty \mu_j^{2_s} \left(\int_{\mathbb{R}^n} U^{\frac{2_s^*(2_s^*-2-\epsilon_j)}{2_s^*-1}} dx\right)^{\frac{2_s^*-1}{2_s^*}} \left(\int_{\mathbb{R}^n} |v_j - U|^{2_s^*} dx\right)^{\frac{1}{2_s^*}} \\
 &= o(\mu_j^{2_s}).
 \end{aligned}
 \tag{4.40}$$

By (4.39), we get

$$\begin{aligned}
 \left(S_{2_s^*-\epsilon_j}^V\right)^{\frac{2_s^*-\epsilon_j}{2_s^*-2-\epsilon_j}} &= \int_{\mathbb{R}^n} \left[U^{2_s^*-\epsilon_j} + (2_s^* - \epsilon_j)U^{2_s^*-1-\epsilon_j} \mu_j^{2_s} w_j\right] dx + o(\mu_j^{2_s}) \\
 &= \int_{\mathbb{R}^n} \left[U^{2_s^*-\epsilon_j} - \epsilon_j U^{2_s^*} \ln U + (2_s^* - \epsilon_j)U^{2_s^*-1-\epsilon_j} \mu_j^{2_s} w_j\right] dx + o(\epsilon_j) + o(\mu_j^{2_s}) \\
 &= \int_{\mathbb{R}^n} \left[U^{2_s^*-\epsilon_j} - \epsilon_j U^{2_s^*} \ln U + 2_s^* U^{2_s^*-1} \mu_j^{2_s} w_j\right] dx + o(\epsilon_j) + o(\mu_j^{2_s}) \\
 &= S^{\frac{N}{2_s^*}} + \mu_j^{2_s} \int_{\mathbb{R}^n} \left[-\tilde{C}_{N,s} U^{2_s^*} \ln U + 2_s^* U^{2_s^*-1} w\right] dx + o(\mu_j^{2_s})
 \end{aligned}
 \tag{4.41}$$

By (4.15), we get

$$\begin{aligned}
 \int_{\mathbb{R}^n} U^{2_s^*-1} w dx &= \int_{\mathbb{R}^n} (-\Delta)^s U w dx \\
 &= \int_{\mathbb{R}^n} (-\Delta)^s w U dx \\
 &= \int_{\mathbb{R}^n} [(2_s^* - 1)U^{2_s^*-2} w - V(x_0)U - \tilde{C}_{N,s} U^{2_s^*-1} \ln U] U dx.
 \end{aligned}
 \tag{4.42}$$

Thus,

$$(2_s^* - 2) \int_{\mathbb{R}^n} U^{2_s^*-1} w dx = \int_{\mathbb{R}^n} [V(x_0)U + \tilde{C}_{N,s} U^{2_s^*-1} \ln U] U dx.
 \tag{4.43}$$

So,

$$\begin{aligned} \left(S_{2_s^* - \epsilon_j}^V\right)^{\frac{2_s^* - \epsilon_j}{2_s^* - 2 - \epsilon_j}} &= S^{\frac{N}{2s}} + \mu_j^{2s} \int_{\mathbb{R}^n} \left[-\tilde{C}_{N,s} U^{2_s^*} \ln U + \frac{2_s^*}{2_s^* - 2} \left(V(x_0)U^2 + \tilde{C}_{N,s} U \ln U\right)\right] dx + o(\mu_j^{2s}) \\ &= S^{\frac{N}{2s}} + \mu_j^{2s} \int_{\mathbb{R}^n} \left[\frac{2}{2_s^* - 2} \tilde{C}_{N,s} U^{2_s^*} \ln U + \frac{2_s^*}{2_s^* - 2} V(x_0)U^2\right] dx + o(\mu_j^{2s}). \end{aligned} \tag{4.44}$$

Thus, we have

$$\begin{aligned} S_{2_s^* - \epsilon_j}^V &= S + \frac{2s}{N} S^{-\frac{N-2s}{2s}} \mu_j^{2s} \int_{\mathbb{R}^n} \left[\frac{2}{2_s^* - 2} \tilde{C}_{N,s} U^{2_s^*} \ln U + \frac{2_s^*}{2_s^* - 2} V(x_0)U^2\right] dx \\ &\quad - \epsilon_j \frac{2}{(2_s^*)^2} S \ln S^{\frac{N}{2s}} + o(\mu_j^{2s}), \end{aligned} \tag{4.45}$$

which yields

$$\begin{aligned} S_{2_s^* - \epsilon_j}^V &= S + S^{-\frac{N-2s}{2s}} \mu_j^{2s} \int_{\mathbb{R}^n} \left[\frac{2}{2_s^*} \tilde{C}_{N,s} U^{2_s^*} \ln U + V(x_0)U^2\right] dx \\ &\quad - \mu_j^{2s} \tilde{C}_{N,s} \frac{2}{(2_s^*)^2} S \ln S^{\frac{N}{2s}} + o(\mu_j^{2s}). \end{aligned} \tag{4.46}$$

The proof is complete. □

**Theorem 9** Assume  $(V_1)$ ,  $(V_2)$  with  $\inf_{x \in \mathbb{R}^N} V(x) < \sup_{x \in \mathbb{R}^N} V(x)$ ,  $N > 6s$  and let  $u_\epsilon$  be the ground state of (1.1) which has a maximum point at  $x_\epsilon$ . Then, up to a subsequence,  $V(x_{\epsilon_j}) \rightarrow \min_{x \in \mathbb{R}^N} V(x)$  as  $\epsilon_j \rightarrow 0^+$ .

*Proof* By Theorems 29 and 37, it is sufficient to prove that  $x_\epsilon$  remains bounded. We argue by contradiction. Assume that there exists a sequence  $x_j \rightarrow \infty$  such that

$$\lim_{j \rightarrow \infty} V(x_j) = V_\infty > \inf_{x \in \mathbb{R}^N} V(x). \tag{4.47}$$

By Remark 27,

$$\epsilon_j = A\mu_j^{2s} + o(\mu_j^{2s})$$

for some  $A > 0$ . Analogous to the proof of Theorems 29 and 37 with  $\tilde{C}_{N,s}$  and  $V(x_0)$  replaced by  $A$  and  $V_\infty$ , we get  $V_\infty \leq \inf_{x \in \mathbb{R}^N} V(x)$ , which contradicts (4.47). □

*Proof of Theorem 3.* It follows from Theorems 1, 29, 37 and 38.

### 5 Local Uniqueness: Proof of Theorem 4

Let us argue by contradiction. Suppose that there exists a sequence  $\epsilon_j \rightarrow 0$  and two ground states far apart, namely  $u_j^1 := u_{\epsilon_j}^1$  and  $u_j^2 := u_{\epsilon_j}^2$ . Set

$$v_j^i(x) := (\mu_j^i)^{\frac{2s}{2_s^* - 2 - \epsilon_j}} u_j^i(\mu_j^i x), \quad i = 1, 2.$$

Then  $v_j^i \rightarrow U$  in  $C_{loc}^{2,\beta}(\mathbb{R}^N)$  for  $i = 1, 2$  as  $j \rightarrow \infty$ .

Assume further that  $v_j^1 \neq v_j^2$ . Set

$$\theta_j := v_j^1 - v_j^2, \quad \psi_j := \frac{v_j^1 - v_j^2}{\|v_j^1 - v_j^2\|_\infty}.$$

Then

$$(-\Delta)^s \psi_j + (\mu_j^1)^{2s} V(\mu_j^1 x) v_j^1 - (\mu_j^2)^{2s} V(\mu_j^2 x) v_j^2 = \Phi_n \psi_n \text{ in } \mathbb{R}^N, \tag{5.1}$$

where

$$\Phi_n = (2_s^* - 1 - \epsilon_j) \int_0^1 [t v_j^1(x) + (1-t) v_j^2(x)]^{2_s^* - 2 - \epsilon_j} dt. \tag{5.2}$$

Since  $\|v_j^i\|_\infty = 1, i = 1, 2$ , by standard regularity we have  $\psi_j \rightarrow \psi$  in  $C_{loc}^{2,\beta}(\mathbb{R}^N)$ . By Lemma 18, we have that  $\{\psi_j\}$  is uniformly bounded in  $H_V^s(\mathbb{R}^N)$ . Without loss of generality, we may assume that  $\psi_n \rightarrow \psi$  in  $H_V^s(\mathbb{R}^N)$ .

From (5.1), we have

$$\begin{aligned} \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \psi_j (-\Delta)^{\frac{s}{2}} \varphi dx &= -(\mu_j^1)^{2s} \int_{\mathbb{R}^N} V(\mu_j^1 x) v_j^1 \varphi dx + (\mu_j^2)^{2s} \int_{\mathbb{R}^N} V(\mu_j^2 x) v_j^2 \varphi dx \\ &\quad + \int_{\mathbb{R}^N} \Phi_n \psi_n \varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \end{aligned} \tag{5.3}$$

Taking  $j \rightarrow \infty$  in (5.3), we get

$$(-\Delta)^s \psi = (2_s^* - 1) U^{2_s^* - 2} \psi \text{ in } \mathbb{R}^N. \tag{5.4}$$

Note that  $\|\psi_j\|_\infty = 1$  implies  $\|\psi\|_\infty = 1$ . By Lemma 31,

$$\psi \in X = \text{span} \left\{ \frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_N}, \frac{N - 2s}{2} U + x \cdot \nabla U \right\}.$$

On the other hand, since  $v_j^i$  is radially symmetric,  $\psi$  is a radial function as well. Thus,

$$\psi(x) = c \frac{\lambda^2 - |x|^2}{(\lambda^2 + |x|^2)^{\frac{N-2s+2}{2}}}$$

for some constant  $c \in \mathbb{R}$ .

We next actually prove that  $c = 0$ . Indeed, otherwise assume for simplicity  $c = 1$ . By Pohozaev’s identity, we have

$$\frac{1}{N} (\mu_j^i)^{2s} \int_{\mathbb{R}^N} V(\mu_j^i x) |v_j^i|^2 dx + \frac{1}{2N} (\mu_j^i)^{2s+1} \int_{\mathbb{R}^N} x \cdot \nabla V(\mu_j^i x) |v_j^i|^2 dx = \frac{\epsilon_j}{2_s^* (2_s^* - \epsilon_j)} \int_{\mathbb{R}^N} |v_j^i|^{2_s^* - \epsilon_j} dx, \quad i = 1, 2. \tag{5.5}$$

By Remark 27, we get  $(\mu_j^i)^{2s} \sim \epsilon_j$ . Thus, from (5.5), we have

$$\frac{V_\infty}{N} \int_{\mathbb{R}^N} \psi_j (v_j^1 + v_j^2) dx + o(1) \geq \frac{2_s^* - \epsilon_j}{2_s^* (2_s^* - \epsilon_j)} \int_{\mathbb{R}^N} \psi_j \int_0^1 [t v_j^1 + (1-t) v_j^2]^{2_s^* - 1 - \epsilon_j} dt dx. \tag{5.6}$$

Notice that

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} \psi_j (v_j^1 + v_j^2) dx = 2 \int_{\mathbb{R}^N} \psi U dx = 2\lambda^{N-2s} \int_{\mathbb{R}^N} \frac{\lambda^2 - |x|^2}{(\lambda^2 + |x|^2)^{N-2s+1}} dx. \tag{5.7}$$

Direct calculations show that

$$\int_{\mathbb{R}^N} \frac{\lambda^2 - |x|^2}{(\lambda^2 + |x|^2)^{N-2s+1}} dx < 0. \tag{5.8}$$

Thus, from (5.7), we obtain

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} \psi_j (v_j^1 + v_j^2) dx < 0. \tag{5.9}$$

On the other hand, we have

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} \psi_j \int_0^1 [t v_j^1 + (1-t)v_j^2]^{2_s^* - 1 - \epsilon_j} dt dx = \int_{\mathbb{R}^N} \psi U^{2_s^* - 1}. \tag{5.10}$$

By a suitable scaling we end up with

$$\int_{\mathbb{R}^N} \psi U^{2_s^* - 1} \sim \int_0^{+\infty} \frac{(1-r^2)r^{N-1}}{(1+r^2)^{1+N}} dr = 0. \tag{5.11}$$

By combining (5.6)–(5.11), we get a contradiction.

Thus,  $c = 0$  and  $\psi_j \rightarrow 0$  in  $\Omega \subset \subset \mathbb{R}^N$ . If we let  $y_j \in \mathbb{R}^N$  such that  $\psi_j(y_j) = \|\psi_j\|_\infty = 1$ , then  $y_j \rightarrow +\infty$  as  $j \rightarrow \infty$ . However, by Lemma 25, we get  $v_j^i(x) \leq C \frac{1}{|x|^{N-2s}}$ ,  $i = 1, 2$  and thus  $|\psi_j(x)| \leq C \frac{1}{|x|^{N-2s}}$ , which implies  $\psi_j(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . This is a contradiction since  $\psi_j(y_j) = 1$ .

The proof of Theorem 4 is now complete.

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