

Loops in anti de Sitter space

Sergio L. Cacciatori^{a,b}, Henri Epstein^b and Ugo Moschella^{c,b}

^a*Dipartimento di Scienza e Alta Tecnologia, Università dell'Insubria,
via Valleggio 11, Como 22100, Italy*

^b*INFN, via Celoria 16, Milano 20133 Italy*

^c*Institut des Hautes Etudes Scientifiques, Le Bois-Marie,
35 route de Chartres, Bures-sur-Yvette 91440, France*

E-mail: sergio.cacciatori@uninsubria.it, epstein@ihes.fr,
ugo.moschella@uninsubria.it

ABSTRACT: We discuss general one and two-loop banana diagrams and one-loop diagrams with external lines with arbitrary masses on the anti de Sitter spacetime by using methods of AdS quantum field theory in the dimensional regularization approach. The banana diagrams explicitly computed in this paper are indeed the necessary ingredients for the evaluation of the two-loop effective potential of the Standard Model and can be used to extend the flat space results in presence of a negative cosmological constant. In the one-loop case we also compute the effective potential for an $O(N)$ model in $d = 4$ dimension as an explicit function of the cosmological constant Λ , both exactly and perturbatively up to order Λ . In the two-loop case we show the explicit calculation is possible thanks to a remarkable discrete Källén-Lehmann formula which we found and proved sometimes ago and whose domain of applicability we extend in the present paper.

KEYWORDS: de Sitter space, Scattering Amplitudes

ARXIV EPRINT: [2403.13142](https://arxiv.org/abs/2403.13142)

To Jacques Bros and Michel Gaudin.

In loving memory.

Contents

1	Introduction	1
2	Preliminaries	3
2.1	Geometry	3
3	The analytic structure of two-point functions	7
3.1	Klein-Gordon fields	8
4	Banana integrals: one-loop	9
5	The effective potential for the $O(N)$ model	12
6	Two and three lines (bulk-to-bulk) and a general formula for the bubble and the triangle in a general Euclidean background	14
7	A remark about bulk diagrams versus Witten diagrams	18
8	The Källén-Lehmann expansion of the bubble with two independent masses	20
8.1	An almost trivial example: \widetilde{AdS}_3	21
8.2	A less elementary example: \widetilde{AdS}_5	22
8.3	Cases of \widetilde{AdS}_1 and \widetilde{AdS}_{-1}	24
9	The tadpole and the sunset	25
10	Two-loop banana integrals: the watermelon	26
11	Conclusions	29
A	Remarks on the various spectral representations of dS and AdS propagators	29
B	Proof of theorem 8.3	36

1 Introduction

The outcome of string theory having the largest impact in science is probably the AdS/CFT correspondence [1–4]. Though still conjectural, its robustness is evident in several situations, going far beyond the realm of string theory and supersymmetry [5]. Trying to explore and understand the conjecture has generated an impressive literature dealing with various geometrical and physical aspects of (asymptotically) anti-de Sitter (AdS) spacetimes [6]; in particular, AdS Quantum Field Theory (QFT) has come again to the general attention.

On the AdS side, the conjecture is often checked only at tree-level: Feynman diagrams on that manifold are difficult already at one loop. Sometimes, loop integrals are evaluated through the bootstrap, a strategy introduced by G. Chew in the seventies of the last century, abandoned shortly after the successes of QCD and the Standard Model, and revitalized in the last fifteen years [7]. Bootstrap is essentially based on the interplay between symmetries, unitarity and causality. However, it is not completely clear how to implement it outside the context of conformal field theories and difficulties may arise in presence of anomalies or

non-invertible symmetries [8]. Combining bootstrap methods in CFT with the AdS/CFT conjecture has proven to be useful to guess the result of loop integrals in the AdS (*bulk*) manifold [9–14]; there are however other possibilities [15, 16].

Generally speaking, the absence of a global linear momentum space renders the calculation of Feynman or Schwinger-Keldysh diagrams in a curved background more difficult than usual; to avoid that difficulty, working in position space from scratch is the cleanest and clearest option that may be tremendously effective also in flat space [17]. That is the strategy that we adopt here and in a companion paper [18]. By taking into full account the analyticity properties of the correlation functions in the complexified universal covering of the real AdS manifold, we show how to compute a class of one and two-loop integrals in a simple and direct way, avoiding every detour whatsoever.

Let us focus on one-loop integrals first: sometimes, things are rendered more difficult than they really are by the habit of writing the two-point functions and the propagators as special instances of the Gaussian hypergeometric function ${}_2F_1$ instead of using the more specific expressions in terms Legendre functions of the second kind (Q) for AdS quantum fields and Legendre functions of the first kind (P) for dS fields. This might seem just an irrelevant matter of taste but the contrary happens to be true: the geometrical peculiarities of either the AdS or the dS manifolds that are well captured by the respective Legendre functions Q and P remain hidden in the generic hypergeometric expression.

The superiority of using Q instead of ${}_2F_1$ is evident in our first derivation of the one-loop diagram which is based on the Wronskian relations between solutions of the Legendre differential equation: the integration is immediate. In section 6 we present also a second very simple derivation of the same one-loop diagram as a special instance of a new general formula which is generally valid in curved Euclidean backgrounds.

The explicit knowledge of the one-loop diagram in turn opens the way to the calculation of two-loop banana integrals relative to three independent AdS scalar fields having arbitrary values of their mass parameters. The literature on AdS *two-loop* integrals is scarce, see e.g. [13, 16, 19]. In general the relevant articles do not contain explicit formulas: some of them calculate corrections to the dispersion laws but do so determining only the divergent part [20].

The explicit calculation in the present paper is made possible by the AdS Källén-Lehmann expansion formula which two of us and collaborators proved sometimes ago [29, eqs. (7.8) and (7.9)] (eqs. (8.19) and (8.20) in this paper). A formula identical to our own appeared without an explicit proof in a later paper the same year [30, eq. (53)].¹

Remarkably, this beautifully simple formula is not an integral but a discrete sum. One could have expected this for the uncovered AdS manifold, due to its time periodicity; however, the discreteness of the formula is not affected by moving to the universal covering. Here we extend the validity of Källén-Lehmann expansion to more general conditions in theorem 8.3: the formula remains of course the same as in [29], only the domain of applicability becomes wider w.r.t. the original proof given in [29]. Then we use it together with our 1-loop calculation to write explicit formulae for the two-loop banana integral with three independent masses in arbitrary (complex) dimension d (eqs. (10.2), (10.7) and (10.8)). Obtaining these formulae would be quite challenging using other methods without further assumptions (for example,

¹Numbered as eq. (52) in the first version [arXiv:1111.6972v1](https://arxiv.org/abs/1111.6972v1) of the preprint.

without assuming the validity of the AdS/CFT conjecture and/or doing suitable guesses in bootstrap methods). The Källén-Lehmann expansion renders the calculation is next to elementary by linearization.

We would like to stress also that computing these integrals goes well beyond the aim of testing the power of the methods used in the present paper: these are exactly the diagrams necessary to compute the two-loop effective potential for the Standard Model and can be used to generalize the results in [31] in presence of a negative cosmological constant. The simple explicit formulas we have found are new and we think that they can lead to some simplifications also in the flat limit.

The plan of the paper is as follows: in section 2 we recall a few preliminary notions about the geometry of AdS and about AdS QFT. In section 3 we discuss in more detail the analyticity properties of the scalar two-point functions and recall how to write the propagators in terms of Legendre functions of the second kind. A simple application of this representation is presented in section 4 where we compute the 1-loop banana integral with two arbitrary masses in any complex dimensions by using the Wronskian associated with the Legendre differential equation.

In section 5 we apply the results of section 4 to compute the 1-loop effective potential for an AdS $O(N)$ scalar field model in a four dimensions. We provide an exact expression (at 1-loop) of the potential as a function of the cosmological constant Λ and also an expansion in $\sqrt{-\Lambda}$ to order Λ ; in the flat limit $\Lambda \rightarrow 0$ the standard flat potential is correctly reproduced. The latter fact is indeed nontrivial.

The bubble is reconsidered again in section 6 where we prove the general formula (6.8) valid in any (curved) Euclidean background. In the same vein we determine also a formula the triangular loop that may be easily generalized to N -line case.

Section 7 contains a curious fact we found on the relation between bulk diagram vs. Witten diagrams which is worth of further investigation. We show that in at least one possibly pathological case, Witten diagrams are not the boundary limit of the corresponding bulk diagram.

In section 8 we discuss the Källén-Lehmann formula for AdS scalar quantum fields with two distinct masses and we extend the original proof given in [29] to more general conditions. We also discuss instructive elementary deductions of the same formula in odd dimensions $d = -1, d = 1, d = 3, d = 5$. In sections 9 and 10 we use the KL expansion formula to compute explicit expressions for the 1-loop tadpole and the one loop sunset and for the 2-loop banana integral with arbitrary masses and dimensions.

The text is followed by two appendices. In the first we provide some further details about the usage of Legendre functions of first and second kind in dS and AdS QFT's and also clarify some points that have been discussed in the recent literature. In the second we give the details about the proof of the expansion theorem of [29] in general conditions.

2 Preliminaries

2.1 Geometry

The d -dimensional real AdS spacetime with radius $R > 0$ may be visualized as the manifold

$$AdS_d = \{x \in \mathbf{R}^{d+1} : x^2 = x \cdot x = R^2\} \quad (2.1)$$

where the scalar product $x \cdot x$ is intended in the sense of the ambient space \mathbf{R}^{d+1} with two timelike directions and metric mostly minus as follows:

$$x \cdot y = x^0 y^0 - x^1 y^1 - \dots - x^{d-1} y^{d-1} + x^d y^d. \quad (2.2)$$

The complexification of the AdS manifold is defined analogously

$$AdS_d^{(c)} = \{z = x + iy \in \mathbf{C}^{d+1} : z^2 = R^2\}; \quad (2.3)$$

$z \in AdS^{(c)}$ if and only if $x^2 - y^2 = R^2$ and $x \cdot y = 0$, i.e. the real and imaginary parts of z are orthogonal w.r.t. the scalar product (2.2). The symmetry group of the anti de Sitter spacetime is the pseudo-orthogonal group of the ambient space $SO(2, d - 1)$. This group may also be regarded as the conformal group of transformations of the boundary, represented as the null cone of the ambient space

$$C_d = \{\xi \in \mathbf{R}^{d+1} : \xi^2 = \xi \cdot \xi = 0\}. \quad (2.4)$$

This simple geometrical fact lies at basis of the AdS/CFT correspondence. The null cone of the ambient space plays also the role of giving a causal order to the AdS spacetime which is however only *local*, due to the existence of closed timelike curves; two events are spacelike separated if

$$(x_1 - x_2)^2 = 2 - 2x_1 \cdot x_2 < 0. \quad (2.5)$$

While $AdS_d^{(c)}$ is simply connected,² the real manifold AdS_d is not and admits a nontrivial universal covering space \widetilde{AdS}_d . The covering manifold is globally causal but remains non-globally hyperbolic, because of the boundary at spacelike infinity. However, even though time-loops are not present in the covering space, there remains an unavoidable trace of the periodicity of the uncovered (otherwise called pure or true) AdS manifold: timelike geodesics issued at any point of \widetilde{AdS}_d focus infinitely many times every half period, exactly as it happen on the true AdS manifold. This fact will be conspicuous in section 8 where we discuss the Källén-Lehmann representation of the product of two scalar two-point functions with mass parameters λ and ν , in position space with fixed end-points (sometimes called the bubble diagram).

Coordinates

Except for the considerations regarding the flat limit we shall always take $R = 1$ throughout this paper. A concrete way of representing both AdS_d and its covering \widetilde{AdS}_d by global coordinates is to introduce the diffeomorphism χ of $S_1 \times \mathbf{R}^{d-1}$ onto AdS_d given by

$$x(t, \vec{x}) = \left(\sqrt{1 + \vec{x}^2} \cos t, \vec{x}, \sqrt{1 + \vec{x}^2} \sin t \right) \quad (2.6)$$

where S_1 is identified to $\mathbf{R}/2\pi\mathbf{Z}$. The diffeomorphism $\tilde{\chi}$, defined by lifting χ on the covering \mathbf{R}^d of $S_1 \times \mathbf{R}^{d-1}$ provides a global coordinate system on \widetilde{AdS}_d .

²This may be seen by changing z^μ to iz^μ for $\mu = 1, \dots, d - 1$, the complex AdS manifold becomes the complex unit sphere in \mathbf{C}^{d+1} , which has the same homotopy type as the real unit sphere S_d . It follows that for $d \geq 2$ the covering space of $AdS_d^{(c)}$ is $AdS_d^{(c)}$ itself.

It is possible to identify in the complex manifold $AdS_d^{(c)}$ an analog of the Euclidean subspace of the complex Minkowski spacetime: here we choose the connected real submanifold \mathbf{H}_d of $AdS_d^{(c)}$ — that we call the Euclidean anti de Sitter spacetime — defined by

$$\mathbf{H}_d = \{z \in AdS_d^{(c)} : z(y) = (y^0, \dots, y^{d-1}, iy^d), y^\mu \in \mathbf{R}, y^0 > 0\}. \quad (2.7)$$

This is indeed a Lobachevsky space modeled as the upper sheet of a two-sheeted hyperboloid embedded in a $(d + 1)$ -dimensional Minkowski spacetime:

$$H_d = \{y \in \mathbf{R}^{d+1} : y^{0^2} - y^{1^2} - \dots - y^{d^2} = R^2, y^0 > 0\}. \quad (2.8)$$

Coordinates for \mathbf{H}_d are obtained by changing t into is in eq. (2.6); this yields the following parametrization:

$$z(is, \vec{x}) = \left(\sqrt{1 + \vec{x}^2} \operatorname{ch} s, \vec{x}, i\sqrt{1 + \vec{x}^2} \operatorname{sh} s \right). \quad (2.9)$$

There also exists an extension of $\tilde{\chi}$ to $\mathbf{C} \times \mathbf{R}^{d-1}$, whose image is a partial complexification of the real covering manifold \widetilde{AdS}_d ; the so-obtained complexified covering contains the same Euclidean spacetime \mathbf{H}_d .

We will make use of the following spherical parametrization of H_d :

$$y(u, \omega) = \left(u, \omega^1 \sqrt{u^2 - 1}, \dots, \omega^d \sqrt{u^2 - 1} \right) \quad (2.10)$$

where $u \geq 1$ and ω are coordinates on the sphere \mathbf{S}_{d-1} ; in these coordinates the Lorentz-invariant measure dy is written

$$\sqrt{g} dy = (u^2 - 1)^{\frac{d-2}{2}} du d\omega \quad (2.11)$$

where $d\omega$ denotes the rotation-invariant measure on the sphere S_{d-1} normalized as follows

$$\omega_d = \int_{S_{d-1}} d\omega = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \quad (2.12)$$

Quantum fields and the energy spectrum condition

A general approach to AdS QFT has been discussed in [25, 32]. Here we summarize only the main consequences of the spectral condition; we refer the reader to [25, 32] for more details and other results.

We focus on a scalar quantum field on \widetilde{AdS}_d which is fully determined by the set of its n -point vacuum expectation values:

$$\mathcal{W}_n(x_1, \dots, x_n) = \langle \Omega, \phi(x_1) \dots \phi(x_n) \Omega \rangle. \quad (2.13)$$

The field can be restricted to the uncovered spacetime AdS_d if and only if it 2π -periodic in the time-parameter t . The properties of AdS-invariance, positive-definiteness and hermiticity are formulated exactly as for Minkowskian scalar QFT [25, 33] and we do not spell them out.

Even though the AdS manifold is not globally hyperbolic, it is however natural to ask for *local commutativity* by requiring that the fields commute at spacelike separation:

$$\begin{aligned} \mathcal{W}(x_1, \dots, x_i, x_{i+1}, \dots, x_n) &= \mathcal{W}(x_1, \dots, x_{i+1}, x_i, \dots, x_n) \\ &\text{for all } x_i, x_{i+1} \text{ space-like separated in the sense of } \widetilde{AdS}_d. \end{aligned} \quad (2.14)$$

For theories well-defined in the uncovered AdS_d itself, this amounts at least to commutativity under the condition $(x_i - x_{i+1})^2 < 0$.

The above properties are more or less obvious but do not select a unique solution for a given model and some extra criterion is needed. Since the very beginning of AdS quantum field theory the correlation functions and the propagators have been constructed and selected by studying the behaviour of the modes at the boundary located at spacelike infinity [34]. This is very natural from the viewpoint of the Cauchy problem because of the lack of global hyperbolicity.

On the other hand, the fact that the field equations do not uniquely select the propagators is not a peculiarity of AdS quantum fields theory. For infinite quantum systems there are always infinitely many inequivalent representations of the commutation relations and to select a physically meaningful quantization one always needs to impose some extra physical requirement. This is normally done in terms of a *spectral condition*, i.e. a condition on the energy spectrum of the states of the theory, and we act in this way also in the AdS case. Whatever is the choice it will also silently select the behaviour at spacelike infinity of the modes and the correlators.

Since the parameter of the (covering) group of rotations in the $(0, d)$ -plane is interpreted as a time variable, we require that the corresponding generator M_{0d} be represented in the Hilbert space of the theory by a self-adjoint operator whose spectrum is bounded from below. By a standard Laplace transform argument [33] in the corresponding time-variables, this requirement is equivalent to the following analyticity property of the n -point functions:

Spectral condition. *Each tempered distribution*

$$\mathcal{W}_n(x_1(t_1, \vec{x}_1), \dots, x_n(t_n, \vec{x}_n)) \tag{2.15}$$

is the boundary value of a holomorphic function $W_n(z_1, \dots, z_n)$ which is defined in a complex neighborhood of the set

$$\left\{ z = (z_1, \dots, z_n); z_j \in \widetilde{AdS}_d^{(c)}; z_j = z_j(\tau_j, \vec{x}_j); \text{Im } \tau_1 < \text{Im } \tau_2 < \dots < \text{Im } \tau_n \right\}. \tag{2.16}$$

As a by-product, the *Schwinger functions*

$$S_n(x_1(is_1, \vec{x}_1), \dots, x_n(is_n, \vec{x}_n)) \tag{2.17}$$

are well-defined and are obtained as the restrictions of the holomorphic functions W_n 's to the Euclidean submanifold

$$\{ z = (z_1, \dots, z_n), z_j \in \mathbf{H}_d, s_1 < s_2 < \dots < s_n \} \tag{2.18}$$

at non coinciding points. Sometimes natural distributional extensions exist: this is the case if the singularities of the \mathcal{W}_n are weak, for example in lower dimensions or in the presence of regularization such as dimensional regularization.

3 The analytic structure of two-point functions

From now on we consider only fields completely determined by their two-point function $\mathcal{W}(x_1, x_2)$ as we do in perturbation theory i.e. we assume that the truncated n -point functions vanish.

There are two distinguished complex domains [25, 32] of $AdS_d^{(c)}$, invariant under real AdS transformations, which are of crucial importance for a full understanding of the structures associated with two-point functions:

$$\begin{aligned} \mathcal{Z}_{1+} &= \{z = x + iy \in AdS_d^{(c)}; y^2 > 0, \epsilon(z) = +1\}, \\ \mathcal{Z}_{1-} &= \{z = x + iy \in AdS_d^{(c)}; y^2 > 0, \epsilon(z) = -1\}, \end{aligned} \tag{3.1}$$

where

$$\epsilon(z) = \text{sign}(y^0 x^d - x^0 y^d). \tag{3.2}$$

\mathcal{Z}_{1+} and \mathcal{Z}_{1-} are the AdS analogues of the usual forward and backward tubes of complex Minkowski spacetime, obtained in correspondence with the energy-momentum spectrum condition [33]. The spaces $\mathcal{Z}_{1\pm}$ and AdS_d have the same homotopy type. Their universal coverings are denoted $\tilde{\mathcal{Z}}_{1\pm}$. The AdS spectral condition implies that a general two-point function satisfies the following [25]

Normal analyticity condition for two-point functions. $\mathcal{W}(x_1, x_2)$ is the boundary value of a function $W(z_1, z_2)$ holomorphic in the domain $\tilde{\mathcal{Z}}_{1-} \times \tilde{\mathcal{Z}}_{1+}$

$$\mathcal{W}(x_1, x_2) = (\Omega, \phi(x_1)\phi(x_2)\Omega) = \lim_{\substack{z_1 \in \tilde{\mathcal{Z}}_{1-}, z_2 \in \tilde{\mathcal{Z}}_{1+} \\ z_1 \rightarrow x_1, z_2 \rightarrow x_2}} W(z_1, z_2). \tag{3.3}$$

AdS invariance then implies that to $W(z_1, z_2)$ there corresponds a function $w(\zeta)$ of a single complex variable ζ that can be identified with $z_1 \cdot z_2$ when z_1 and z_2 are both in $AdS_{d+1}^{(c)}$; the function $w(\zeta)$ is called *the reduced two-point function*. Complex AdS invariance and normal analyticity imply the following

Maximal analyticity property. *The reduced two-point function $w(\zeta)$ extends to a function analytic in the covering $\hat{\Delta}_1$ of the cut-plane*

$$\Delta_1 = \{\mathbf{C} \setminus [-1, 1]\}. \tag{3.4}$$

For theories periodic in the time coordinate $w(\zeta)$ is analytic in Δ_1 . For later use we introduce also

$$\hat{\Delta}_1 = \Delta_1 \cup \{\infty\}, \tag{3.5}$$

Therefore, the two-point function of any field satisfying locality AdS invariance and the spectral condition enjoys *maximal analyticity*, as it happens in the Minkowski [33] and de Sitter cases [24]. Maximal analyticity, in turn, completely determines the two-point functions for Klein-Gordon fields and, as a consequence, also selects the boundary behaviour of the modes.

3.1 Klein-Gordon fields

Klein-Gordon fields display the simplest example of the previous analytic structure. For a given mass m the two-point function $\mathcal{W}(x_1, x_2)$ must satisfy the equation

$$(\square_{x_i} + m^2)\mathcal{W}(x_1, x_2) = 0, \quad i = 1, 2, \quad (3.6)$$

w.r.t. both variables, where \square_{x_i} is the Laplace-Beltrami operator relative to the AdS metric. Such two-point functions are labelled by the (complex) dimension d and a (complex) parameter ν as follows

$$W_\nu^{(\text{AdS})}(z_1, z_2) = w_\nu^d(\zeta) = \frac{1}{(2\pi)^{\frac{d}{2}}} (\zeta^2 - 1)^{-\frac{d-2}{4}} e^{-i\pi\frac{d-2}{2}} Q_{-\frac{1}{2}+\nu}^{\frac{d-2}{2}}(\zeta) \quad (3.7)$$

$$= \frac{\Gamma\left(\frac{d-1}{2}\right)}{2\pi^{\frac{d+1}{2}}} D_{\nu-\frac{d-1}{2}}^{\frac{d-1}{2}}(\zeta) \quad (3.8)$$

$$= \frac{\Gamma\left(\frac{d-1}{2} + \nu\right)}{2\pi^{\frac{d-1}{2}} (2\zeta)^{\frac{d-1}{2}+\nu} \Gamma(\nu+1)} {}_2F_1\left(\frac{d-1}{4} + \frac{\nu}{2}, \frac{d+1}{4} + \frac{\nu}{2}; \nu+1; \frac{1}{\zeta^2}\right) \quad (3.9)$$

where the various parameters are related as follows:

$$m^2 = \nu^2 - \frac{(d-1)^2}{4}. \quad (3.10)$$

At the r.h.s. of (3.7) $Q_\beta^\alpha(\zeta)$ denotes an associated Legendre function of the second kind [35]; at the r.h.s. of eq. (3.8) $D_\sigma^\delta(\zeta)$ is an ultraspherical (Gegenbauer) function of the second kind [36]; they are meromorphic functions of α and β and, respectively, δ and σ [35, 36]. As functions of the complex variable ζ , they are analytic in the cut-plane

$$\Delta_2 = \{\mathbf{C} \setminus [-\infty, 1]\}. \quad (3.11)$$

Their definitions in terms of the Gauss hypergeometric function is provided by the last equality (3.9). Note in particular that $(\zeta)^{\sigma+2\delta} D_\sigma^\delta(\zeta)$ and $(\zeta)^{\kappa+\frac{d-1}{2}} w_\kappa^d(\zeta)$ are even and analytic in $\hat{\Delta}_1$.

Since for each value of the mass squared there are two possible two-point functions, the question arises whether they are both acceptable or not. The spectral condition gives the answer (Breitenlohner and Freedmann phenomenon) [38]: the two-point functions give rise to a representation of M_{0d} having positive spectrum only when ν is real and bigger than -1 . There are two possible cases

1. for $\nu > 1$ the spectrum condition (3.3) uniquely select one field theory for each given value of mass parameter ν ;
2. for $|\nu| < 1$ there are two acceptable theories for each given mass. The difference between the two theories is in their large distance behavior; more precisely, in view of [35, eq. (3.3.1.4)] one has that

$$w_{-\nu}^d(\zeta) = w_\nu^d(\zeta) + \frac{\sin \pi\nu \Gamma\left(\frac{d-1}{2} - \nu\right) \Gamma\left(\frac{d-1}{2} + \nu\right)}{2(2\pi)^{\frac{d}{2}}} (\zeta^2 - 1)^{-\frac{d-2}{4}} P_{-\frac{1}{2}-\nu}^{-\frac{d-2}{2}}(\zeta). \quad (3.12)$$

The last term in this relation is *regular on the cut* $\zeta \in [-1, 1]$ and therefore does not contribute to the commutator. By consequence the two theories represent the same algebra of local observables at short distances. But since the second term at the r.h.s. grows the faster the larger is $|\nu|$ (see [35, eqs. (3.9.2)]) the two theories have drastically different long range behaviors.

The Schwinger function (otherwise called the Euclidean propagator) is the restriction of the maximally analytic two-point function to the Euclidean Lobachevsky manifold. Choosing the points in eq. (3.7) as follows

$$z_0 = (1, 0, \dots, 0, 0), \quad z(u, \omega) = \left(u, \omega^1 \sqrt{u^2 - 1}, \dots, \omega^{d-1} \sqrt{u^2 - 1}, i\omega^d \sqrt{u^2 - 1}\right), \quad u > 1 \tag{3.13}$$

so that $\zeta = z_0 \cdot z(u, \omega) = u > 1$, we write the propagator as

$$G_\nu^{(\text{AdS})d}(z_0 \cdot z(u, \omega)) = G_\nu^{(\text{AdS})d}(u) = G_\nu^d(u) = w_\nu(u) = \frac{e^{-i\pi \frac{d-2}{2}}}{(2\pi)^{\frac{d}{2}}} (u^2 - 1)^{-\frac{d-2}{4}} Q_{-\frac{1}{2}+\nu}^{\frac{d-2}{2}}(u). \tag{3.14}$$

Remark 3.1. We stress again that forgetting, as is frequently done in the literature, that the two-point function is a Legendre function and not just a much less specific (although equivalent) hypergeometric function, implicates a significant loss of information and a consequent increase in the difficulties in solving even simple problems.

This oblivion is even more curious when one thinks that in one of the first (if not the very first) papers dealing with AdS QFT [37] the correlation functions are written exclusively in terms of Legendre functions Q .

4 Banana integrals: one-loop

After this preparation we can now write the n -loop banana integral on the Lobachevsky Euclidean manifold \mathbf{H}_d with $n + 1$ lines:

$$I_{n+1}(\nu_1, \dots, \nu_{n+1}, d) = \int_{\mathbf{H}_d} G_{\nu_1}^d(x \cdot z) G_{\nu_2}^d(x \cdot z) \dots G_{\nu_{n+1}}^d(x \cdot z) \sqrt{g(z)} dz, \tag{4.1}$$

where y varies on H_d and x is a fixed reference point. The above definition has to be intended as a dimensional regularization of an expression that in general is divergent.

Using the coordinates (3.13) and integrating over the angles (4.1) reduces to

$$I_{n+1}(\nu_1, \dots, \nu_{n+1}, d) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_1^\infty G_{\nu_1}^d(u) G_{\nu_2}^d(u) \dots G_{\nu_{n+1}}^d(u) (u^2 - 1)^{\frac{d-2}{2}} du. \tag{4.2}$$

The zero-loop case gives

$$I_1(\nu, d) = \frac{2^{\frac{2-d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_1^\infty e^{-i\pi \frac{d-2}{2}} Q_{-\frac{1}{2}+\nu}^{\frac{d-2}{2}}(u) (u^2 - 1)^{\frac{d-2}{4}} du. \tag{4.3}$$

The value of this integral might be deduced from [35, section 3.12, eq. (29), p. 172] also reproduced in [45, ET II324(23), p. 809], but, unfortunately, those formulae are flawed by the same typo; we take this opportunity to write here the correct general formula:

$$\int_1^\infty Q_\nu^\mu(u)(u^2 - 1)^{\lambda-1} du = \frac{\Gamma\left(\frac{1}{2} + \frac{\nu}{2} + \frac{\mu}{2}\right)\Gamma\left(1 - \lambda + \frac{\nu}{2}\right)\Gamma\left(\lambda + \frac{\mu}{2}\right)\Gamma\left(\lambda - \frac{\mu}{2}\right)}{2^{2-\mu}\Gamma\left(1 + \frac{\nu}{2} - \frac{\mu}{2}\right)\Gamma\left(\frac{1}{2} + \lambda + \frac{\nu}{2}\right)}, \quad (4.4)$$

valid for $|\mu| < 2 \operatorname{Re} \lambda < \operatorname{Re} \nu + 2$; this gives the same result as in flat space:

$$I_1(\nu, d) = \frac{1}{\nu^2 - \frac{(d-1)^2}{4}} = \frac{1}{m^2}. \quad (4.5)$$

Let us now focus on the already nontrivial two-line case. The above expression looks particularly clean when expressed in terms of the Legendre functions Q :

$$I_2(\lambda, \nu, d) = \int_{H_d} G_\lambda^d(x \cdot z) G_\nu^d(x \cdot z) \sqrt{g(z)} dz = \frac{e^{-i\pi(d-2)}}{2^{d-1} \pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} \int_1^\infty Q_{-\frac{1}{2}+\lambda}^{\frac{d-2}{2}}(u) Q_{-\frac{1}{2}+\nu}^{\frac{d-2}{2}}(u) du. \quad (4.6)$$

As simple as it may look, the integral at the r.h.s. of eq. (4.6) is not available anywhere in the literature accessible to us.

Here the advantage of expressing the propagator in terms of Legendre functions is manifest, as the above integral can be evaluated by using the Wronskian relations associated with Legendre's differential equation.

Recall that the function Q is holomorphic in the domain Δ_2 ; the function $z \mapsto (z^2 - 1)^{\frac{1}{2}}$ is also understood as holomorphic in Δ_2 and equal to $|u^2 - 1|^{\frac{1}{2}}$ when $z = u > 1$. The known recursion relations

$$(1 - z^2) \frac{dQ_\rho^\mu(z)}{dz} = -\rho z Q_\rho^\mu(z) + (\rho + \mu) Q_{\rho-1}^\mu(z), \quad (4.7)$$

$$Q_{\rho-1}^\mu(z) = z Q_\rho^\mu(z) - (\rho - \mu + 1)(z^2 - 1)^{\frac{1}{2}} Q_\rho^{\mu-1}(z), \quad (4.8)$$

(see [35, section 3.8, eqs. (10), (5), p. 161]) give

$$(1 - z^2) \frac{dQ_\rho^\mu(z)}{dz} = \mu z Q_\rho^\mu(z) - (\rho + \mu)(\rho - \mu + 1)(z^2 - 1)^{\frac{1}{2}} Q_\rho^{\mu-1}(z). \quad (4.9)$$

These identities allow us to evaluate the relevant integral over any arc, with extremities a and b , entirely contained in Δ_2 :

$$\int_a^b Q_\rho^\mu(z) Q_\sigma^\mu(z) (\rho - \sigma)(\sigma + \rho + 1) dz = \left[-(z^2 - 1)^{\frac{1}{2}} (\sigma + \mu)(\sigma - \mu + 1) Q_\rho^\mu(z) Q_\sigma^{\mu-1}(z) + (z^2 - 1)^{\frac{1}{2}} (\rho + \mu)(\rho - \mu + 1) Q_\rho^{\mu-1}(z) Q_\sigma^\mu(z) \right]_a^b. \quad (4.10)$$

We need also the asymptotic behaviour of the Legendre function of the second kind at $z \sim 1$ and $z \rightarrow \infty$ [35, section 3.9.2, eqs. (21), (5), (6), pp. 163–164]:

$$\text{as } z \rightarrow +\infty, \quad Q_\rho^\mu(z) \sim \text{const. } z^{-\rho-1}; \quad (4.11)$$

$$\text{as } z \rightarrow 1, \quad \text{if } \operatorname{Re} \mu > 0, \quad Q_\rho^\mu(z) \sim e^{i\pi\mu} 2^{\frac{\mu}{2}-1} \Gamma(\mu) (z-1)^{-\frac{\mu}{2}}; \quad (4.12)$$

$$\text{as } z \rightarrow 1, \quad \text{if } \operatorname{Re} \mu < 0, \quad Q_\rho^\mu(z) \sim \frac{e^{i\pi\mu} 2^{-\frac{\mu}{2}-1} \Gamma(-\mu) \Gamma(\rho + \mu + 1) (z-1)^{\frac{\mu}{2}}}{\Gamma(\rho - \mu + 1)}. \quad (4.13)$$

We deduce that the integral

$$\int_1^\infty Q_\rho^\mu(z) Q_\sigma^\mu(z) dz \tag{4.14}$$

converges if $\text{Re}(\rho + \sigma) > -1$ and $|\text{Re} \mu| < 1$. Let us suppose that $0 < \text{Re} \mu < 1$. It follows that

$$\int_1^\infty Q_\rho^\mu(z) Q_\sigma^\mu(z) dz = \frac{e^{2i\pi\mu} \Gamma(\mu) \Gamma(1 - \mu)}{2(\rho - \sigma)(\sigma + \rho + 1)} \left[\frac{\Gamma(\rho + \mu + 1)}{\Gamma(\rho - \mu + 1)} - \frac{\Gamma(\sigma + \mu + 1)}{\Gamma(\sigma - \mu + 1)} \right]. \tag{4.15}$$

This equation remains valid, by analytic continuation, when $|\text{Re} \mu| < 1$ and $\text{Re}(\rho + \sigma) > -1$ (note that the term in brackets vanishes for $\mu = 0$). Letting ρ tend to σ (4.15) tends to

$$\int_1^\infty Q_\sigma^\mu(z) Q_\sigma^\mu(z) dz = \frac{e^{2i\pi\mu} \Gamma(\mu) \Gamma(1 - \mu) \Gamma(\sigma + \mu + 1)}{2(2\sigma + 1) \Gamma(\sigma - \mu + 1)} \left[\psi(\sigma + \mu + 1) - \psi(\sigma - \mu + 1) \right]. \tag{4.16}$$

Again this equation is valid when $|\text{Re} \mu| < 1$ and $\text{Re} 2\sigma > -1$ but the r.h.s. can be continued outside of this region.

In the end, the above argument provides a nice formula for the one-loop banana integral with two independent masses; here it is:

$$I_2(\lambda, \nu, d) = \frac{\Gamma\left(1 - \frac{d}{2}\right)}{2^d \pi^{\frac{d}{2}} (\lambda^2 - \nu^2)} \left(\frac{\Gamma\left(\frac{d-1}{2} + \nu\right)}{\Gamma\left(\frac{3-d}{2} + \nu\right)} - \frac{\Gamma\left(\frac{d-1}{2} + \lambda\right)}{\Gamma\left(\frac{3-d}{2} + \lambda\right)} \right). \tag{4.17}$$

A few remarks are in order:

1. At $d = 2$ we may extract from the general formula the following finite result:

$$I_2(\lambda, \nu, 2) = \frac{\psi\left(\lambda + \frac{1}{2}\right) - \psi\left(\nu + \frac{1}{2}\right)}{2\pi\lambda^2 - 2\pi\nu^2}. \tag{4.18}$$

2. In odd spacetime dimension the formula becomes very simple; for instance at $d = 1, 3$

$$I_2(\lambda, \nu, 1) = \frac{1}{2\lambda^2\nu + 2\lambda\nu^2}, \quad I_2(\lambda, \nu, 3) = \frac{1}{4\pi\lambda + 4\pi\nu}. \tag{4.19}$$

3. At $d = 4$ we encounter the first divergence. The Laurent expansion of the formula near $d = 4$ gives

$$\begin{aligned} I_2(\lambda, \nu, d)|_{d \approx 4} &= -\frac{1}{8\pi^2(d-4)} + \frac{1 - \gamma + \log(4\pi)}{16\pi^2} - \frac{1}{8\pi^2(\lambda + \nu)} \\ &+ \frac{(1 - 4\lambda^2) \psi\left(\lambda - \frac{1}{2}\right) - (1 - 4\nu^2) \psi\left(\nu - \frac{1}{2}\right)}{32\pi^2(\lambda - \nu)(\lambda + \nu)} + \text{O}(d - 4). \end{aligned} \tag{4.20}$$

4. In the limit when the two masses are equal the general formula reduces to

$$I_2(\nu, \nu, d) = \frac{\Gamma\left(1 - \frac{d}{2}\right) \Gamma\left(\frac{d-1}{2} + \nu\right)}{2^{d+1} \pi^{\frac{d}{2}} \nu \Gamma\left(\frac{3-d}{2} + \nu\right)} \left(\psi\left(\frac{3-d}{2} + \nu\right) - \psi\left(\frac{d-1}{2} + \nu\right) \right). \tag{4.21}$$

This expression coincides with a result of [14].

5 The effective potential for the $O(N)$ model

In this section, as an application of our results, we compute the effective potential for the $O(N)$ scalar model on anti de Sitter manifold in four dimensions. For generalities about the model we refer to the companion paper [18]. As shown there, the 1-loop effective potential has the form $V_{\text{eff}}(\nu_0, \nu_1) = W(\nu_0) + W(\nu_1)$, where here

$$\nu_0 = \sqrt{m_R^2 + 3c_g\varphi_R^2 + \frac{9}{4}}, \tag{5.1}$$

$$\nu_1 = \sqrt{m_R^2 + c_g\varphi_R^2 + \frac{9}{4}}, \tag{5.2}$$

and $W(\nu)$ satisfies

$$\frac{d}{d\nu^2} \frac{d}{d\nu^2} W(\nu) = -\frac{1}{2} I_2(\nu, \nu, d), \tag{5.3}$$

by definition (notice that differently from the de Sitter case, here I depends on ν^2 through ν , so the derivative in ν^2 has to be intended as $\frac{1}{2\nu} \frac{d}{d\nu}$). Here we have already introduced the renormalized mass m_R and field φ_R . Integrating twice in $d\nu^2$ with I given by (4.20), and $d = 4 - 2\varepsilon$, we get

$$\begin{aligned} W(\nu) = & C\nu^2 + D - \frac{1}{64\pi^2} \left(\frac{1}{\varepsilon} + 1 - \gamma + \log(4\pi) \right) \nu^4 + \frac{(\nu^2 - 1/4)^2}{32\pi^2} \psi \left(\nu - \frac{1}{2} \right) \\ & - \frac{1}{32\pi^2} \int \left(\nu^2 - \frac{1}{4} \right)^2 \psi' \left(\nu - \frac{1}{2} \right) d\nu, \end{aligned} \tag{5.4}$$

where we have also included an integration by part, and C and D are integration constants. To evaluate the last integral, we use the Abel-Plana formula [46] to write

$$\psi' \left(\nu - \frac{1}{2} \right) = \frac{1}{2} \frac{1}{\left(\nu - \frac{1}{2} \right)^2} + \frac{1}{\nu - \frac{1}{2}} + 4 \int_0^\infty \frac{\nu - \frac{1}{2}}{\left(\left(\nu - \frac{1}{2} \right)^2 + t^2 \right)^2} \frac{tdt}{e^{2\pi t} - 1}, \tag{5.5}$$

which allows us to write

$$\int \left(\nu^2 - \frac{1}{4} \right)^2 \psi' \left(\nu - \frac{1}{2} \right) d\nu = \frac{\nu^4}{4} + \frac{\nu^3}{3} + \frac{5}{24}\nu^2 + \frac{\nu}{4} - \frac{7}{48} + \frac{1 - 2\nu^2}{12} \log \left(\nu - \frac{1}{2} \right)^2 - \mathcal{B}(\nu), \tag{5.6}$$

where

$$\begin{aligned} \mathcal{B}(\nu) := & 2 \int_0^\infty \frac{t^3 dt}{e^{2\pi t} - 1} \frac{t^2 - 2\nu}{t^2 + \left(\nu - \frac{1}{2} \right)^2} + 12 \int_0^\infty \frac{t^2 dt}{e^{2\pi t} - 1} \arctan \frac{\nu - \frac{1}{2}}{t} \\ & - 2(1 - 2\nu^2) \int_0^\infty \frac{tdt}{e^{2\pi t} - 1} \log \left(1 + \frac{t^2}{\left(\nu - \frac{1}{2} \right)^2} \right). \end{aligned} \tag{5.7}$$

Notice that for $\nu \rightarrow \infty$ we have

$$\mathcal{B}(\nu) = \frac{3}{2\pi^2} \zeta(3) + \frac{1}{60} + \dots, \tag{5.8}$$

where the dots stay for terms that vanish for $\nu \rightarrow \infty$.

Adding the three level bare potential finally gives the renormalized potential

$$\begin{aligned}
V_R = & \frac{m_R^2}{2}\varphi_R^2 + \frac{c_g}{4}\varphi_R^4 + \frac{1}{32\pi^2}(m_R^2 + 3c_g\varphi_R^2 + 2)^2\psi\left(\nu_0 - \frac{1}{2}\right) \\
& + \frac{N-1}{32\pi^2}(m_R^2 + c_g\varphi_R^2 + 2)^2\psi\left(\nu_1 - \frac{1}{2}\right) + \frac{1}{32\pi^2}\mathcal{B}(\nu_0) + \frac{N-1}{32\pi^2}\mathcal{B}(\nu_1) \\
& - \frac{1}{32\pi^2}\left(\frac{\nu_0^3}{3} + \frac{5}{24}\nu_0^2 + \frac{\nu_0}{4} - \frac{7}{48} + \frac{1-2\nu_0^2}{12}\log\left(\nu_0 - \frac{1}{2}\right)^2\right) \\
& - \frac{N-1}{32\pi^2}\left(\frac{\nu_1^3}{3} + \frac{5}{24}\nu_1^2 + \frac{\nu_1}{4} - \frac{7}{48} + \frac{1-2\nu_1^2}{12}\log\left(\nu_1 - \frac{1}{2}\right)^2\right) \\
& - \frac{1}{64\pi^2}\left[(m_R^2 + 3c_g\varphi_R^2 + 2)^2 + (N-1)(m_R^2 + c_g\varphi_R^2 + 2)^2\right]\log\mu_R^2. \tag{5.9}
\end{aligned}$$

We can now restore full dimensions by reintroducing the curvature radius R , such that $\Lambda = -\frac{3}{R^2}$ is the cosmological constant and

$$V = \frac{V}{R^4}, \quad m_R = Rm, \quad \varphi_R = R\varphi, \quad \mu_R = R\mu. \tag{5.10}$$

This gives for the effective potential in $d = 4$:

$$\begin{aligned}
V = & \frac{m^2}{2}\varphi^2 + \frac{c_g}{4}\varphi^4 + \frac{1}{32\pi^2}\left(m^2 + 3c_g\varphi^2 - \frac{2}{3}\Lambda\right)^2\psi\left(\nu_0 - \frac{1}{2}\right) \\
& + \frac{N-1}{32\pi^2}\left(m^2 + c_g\varphi^2 - \frac{2}{3}\Lambda\right)^2\psi\left(\nu_1 - \frac{1}{2}\right) + \frac{\Lambda^2}{288\pi^2}\mathcal{B}(\nu_0) + \frac{(N-1)\Lambda^2}{288\pi^2}\mathcal{B}(\nu_1) \\
& - \frac{\Lambda^2}{288\pi^2}\left(\frac{\nu_0^3}{3} + \frac{5}{24}\nu_0^2 + \frac{\nu_0}{4} - \frac{7}{48} + \frac{1-2\nu_0^2}{12}\log\left(\nu_0 - \frac{1}{2}\right)^2\right) \\
& - \frac{(N-1)\Lambda^2}{288\pi^2}\left(\frac{\nu_1^3}{3} + \frac{5}{24}\nu_1^2 + \frac{\nu_1}{4} - \frac{7}{48} + \frac{1-2\nu_1^2}{12}\log\left(\nu_1 - \frac{1}{2}\right)^2\right) \\
& - \frac{1}{64\pi^2}\left[\left(m^2 + 3c_g\varphi^2 - \frac{2}{3}\Lambda\right)^2 + (N-1)\left(m^2 + c_g\varphi^2 - \frac{2}{3}\Lambda\right)^2\right]\log\frac{3\mu^2}{|\Lambda|}, \tag{5.11}
\end{aligned}$$

with

$$\nu_0 = 3\sqrt{\frac{1}{4} - \frac{m^2}{3\Lambda} - c_g\frac{\varphi}{\Lambda}}, \tag{5.12}$$

$$\nu_1 = 3\sqrt{\frac{1}{4} - \frac{m^2}{3\Lambda} - c_g\frac{\varphi}{3\Lambda}}. \tag{5.13}$$

For small cosmological constant we get the expansion

$$\begin{aligned}
V = & \frac{m^2}{2}\varphi^2 + \frac{c_g}{4}\varphi^4 + \frac{1}{64\pi^2}\left(m^2 + 3c_g\varphi^2\right)^2\log\frac{m^2 + 3c_g\varphi^2}{\mu^2} \\
& + \frac{N-1}{64\pi^2}\left(m^2 + c_g\varphi^2\right)^2\log\frac{m^2 + c_g\varphi^2}{\mu^2}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8\pi^2} \left(\frac{m^2 + 3c_g\varphi^2}{3} \right)^{\frac{3}{2}} \sqrt{-\Lambda} - \frac{N-1}{8\pi^2} \left(\frac{m^2 + c_g\varphi^2}{3} \right)^{\frac{3}{2}} \sqrt{-\Lambda} \\
& - \frac{\Lambda(m^2 + 3c_g\varphi^2)}{288\pi^2} \left(\frac{11}{8} + \frac{1}{2} \log \frac{3m^2 + 9c_g\varphi^2}{-\Lambda} + 6 \log \frac{m^2 + 3c_g\varphi^2}{\mu^2} \right) \\
& - \frac{\Lambda(N-1)(m^2 + c_g\varphi^2)}{288\pi^2} \left(\frac{11}{8} + \frac{1}{2} \log \frac{3m^2 + 3c_g\varphi^2}{-\Lambda} + 6 \log \frac{m^2 + c_g\varphi^2}{\mu^2} \right) \\
& + O(|\Lambda|^{\frac{3}{2}}).
\end{aligned} \tag{5.14}$$

Notice that there is an important difference as compared to the effective potential in de Sitter computed in [18]: here the cosmological function appears as the square root $\sqrt{-\Lambda}$ in place of Λ . This is due to the symmetry $\nu \rightarrow -\nu$ of the Wightman function in the de Sitter case (that is a symmetry of Legendre functions of the first kind) a symmetry that AdS quantum fields do not share.

6 Two and three lines (bulk-to-bulk) and a general formula for the bubble and the triangle in a general Euclidean background

Generalizing eq. (4.6), we now consider the two-line integral where the external points do not coincide:

$$F_{\lambda\nu}(x, y) = \int_{\mathbf{H}_d} G_\lambda^d(x \cdot z) G_\nu^d(y \cdot z) \sqrt{g(z)} dz. \tag{6.1}$$

The evaluation of the analogous of eq. (6.1) in flat space is elementary by Fourier transform. Here we do not have Fourier transform but we may proceed by observing that $F_{\lambda\nu}(x, y)$ solves the following equations:

$$(-\nabla_x^2 + m_\lambda^2) F_{\lambda\nu}(x, y) = \int \delta(z, x) G_\nu^d(z, y) \sqrt{g(z)} dz = G_\nu^d(x \cdot y), \tag{6.2}$$

$$(-\nabla_y^2 + m_\nu^2) F_{\lambda\nu}(x, y) = \int G_\lambda^d(x \cdot z) \delta(y, z) \sqrt{g(z)} dz = G_\lambda^d(x \cdot y). \tag{6.3}$$

Lorentz invariance implies that $F_{\lambda\nu}(x, y) = F_{\lambda\nu}(x \cdot y)$ and therefore

$$\nabla_x^2 F_{\lambda\nu}(x, y) = \nabla_y^2 F_{\lambda\nu}(x, y). \tag{6.4}$$

Subtracting eq. (6.3) from eq. (6.2) it follows that³

$$F_{\lambda\nu}(x, y) = -\frac{G_\lambda^d(x \cdot y) - G_\nu^d(x \cdot y)}{m_\lambda^2 - m_\nu^2}. \tag{6.5}$$

³Let us check how this remarkable formula works in the one-dimensional AdS_1 . Here

$$\begin{aligned}
G_\nu^1(\text{ch } v) &= \frac{e^{-\nu|v|}}{2\nu}, \\
F_{\lambda\nu}^1(a, b) &= \int_{-\infty}^{\infty} \frac{e^{-\lambda|v-a|}}{2\lambda} \frac{e^{-\nu|v-b|}}{2\nu} dv.
\end{aligned}$$

It is evident from the above elementary derivation that eq. (6.5) is valid for the Schwinger propagator of a Klein-Gordon field in a generic curved background \mathcal{E}_d with Euclidean signature whenever eq. (6.4) is verified:

$$F_{m_1 m_2}(x, y) = \int_{\mathcal{E}_d} G_{m_1}^{(\mathcal{E})}(x, z) G_{m_2}^{(\mathcal{E})}(z, y) \sqrt{g(z)} dz = -\frac{G_{m_1}^{(\mathcal{E})}(x, y) - G_{m_2}^{(\mathcal{E})}(x, y)}{m_1^2 - m_2^2}. \quad (6.6)$$

In the limiting case when the two masses coincide this becomes

$$F_{mm}(x, y) = -\frac{\partial}{\partial m^2} G_m^{(\mathcal{E})}(x, y). \quad (6.7)$$

Taking the limit $y \rightarrow x$ we deduce a general formula for the bubble in a curved background with Euclidean signature (similar remarks apply to the chronological propagator in Lorentzian signature):

$$\text{bubble}(m_1, m_2) = \int_{\mathcal{E}_d} G_{m_1}^{(\mathcal{E})}(x, z) G_{m_2}^{(\mathcal{E})}(z, x) \sqrt{g(z)} dz = -\lim_{y \rightarrow x} \frac{G_{m_1}^{(\mathcal{E})}(x, y) - G_{m_2}^{(\mathcal{E})}(x, y)}{m_1^2 - m_2^2}. \quad (6.8)$$

Let us apply the above formula to flat Euclidean space: here the propagator has the following short distance behaviour

$$G_m^{(M)}(r) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{r}{m}\right)^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(mr) \simeq \frac{r^{2-d}}{4\pi^{\frac{d}{2}}} \Gamma\left(\frac{d}{2}-1\right) + \frac{m^{d-2}}{(4\pi)^{\frac{d}{2}}} \Gamma\left(1-\frac{d}{2}\right) \quad (6.9)$$

where M stands for Minkowski and $r^2 = (x_1 - x_2)^2$ is the Euclidean distance. Inserting the expansion (6.9) in eq. (6.6) the diverging terms drop and the standard flat space result [17] is recovered in the limit without any calculation:

$$\text{bubble}_M(m_1, m_2) = -\lim_{r \rightarrow 0} \frac{G_{m_1}^{(M)}(x, y) - G_{m_2}^{(M)}(x, y)}{m_1^2 - m_2^2} = -\frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \frac{(m_1^2)^{\frac{d}{2}-1} - (m_2^2)^{\frac{d}{2}-1}}{m_1^2 - m_2^2}. \quad (6.10)$$

Similarly, in the AdS case, we may evaluate (6.6) with the help of eq. (3.9); here $y \rightarrow x$ means $\zeta \rightarrow 1$; with the condition $d < 2$ at $\zeta = 1$ that formula gives

$$\lim_{\zeta \rightarrow 1} \frac{\Gamma\left(\frac{d-1}{2} + \nu\right) {}_2F_1\left(\frac{d-1}{4} + \frac{\nu}{2}, \frac{d+1}{4} + \frac{\nu}{2}; \nu + 1; \frac{1}{\zeta^2}\right)}{2\pi^{\frac{d-1}{2}} (2\zeta)^{\frac{d-1}{2} + \nu} \Gamma(\nu + 1)} = \frac{\Gamma\left(1-\frac{d}{2}\right) \Gamma\left(\frac{d-1}{2} + \nu\right)}{2^d \pi^{\frac{d}{2}} \Gamma\left(\frac{3-d}{2} + \nu\right)} \quad (d < 2). \quad (6.11)$$

Suppose to fix the ideas that $a < b$. It follows that

$$\begin{aligned} F_{\lambda\nu}(a, b) &= \int_b^\infty \frac{e^{-\lambda(v-a)}}{2\lambda} \frac{e^{-\nu(v-b)}}{2\nu} dv + \int_a^b \frac{e^{-\lambda(v-a)}}{2\lambda} \frac{e^{-\nu(b-v)}}{2\nu} dv + \int_b^a \frac{e^{-\lambda(a-v)}}{2\lambda} \frac{e^{-\nu(b-v)}}{2\nu} dv \\ &= \frac{1}{\lambda^2 - \nu^2} \left(\frac{e^{-\nu(b-a)}}{2\nu} - \frac{e^{-\lambda(b-a)}}{2\lambda} \right). \end{aligned}$$

Inserting this into eq. (6.6) we immediately recover eq. (4.17) again without calculation:

$$\text{bubble}_{\text{AdS}} = F_{\lambda\nu}(x, x) = \frac{\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}(\lambda^2 - \nu^2)} \left(\frac{\Gamma\left(\frac{d+1}{2} + \nu\right)}{\Gamma\left(\frac{3-d}{2} + \nu\right)} - \frac{\Gamma\left(\frac{d+1}{2} + \lambda\right)}{\Gamma\left(\frac{3-d}{2} + \lambda\right)} \right). \quad (6.12)$$

Removing the restriction $d < 2$, this formula defines the AdS bubble by analytic continuation in d as a meromorphic function of d .

Finally, in the de Sitter case (see appendix A) we may use eq. (A.5) at $\zeta = 1$, always under the condition $d < 2$:

$$\begin{aligned} \lim_{\zeta \rightarrow -1} \frac{\Gamma\left(\frac{d-1}{2} + i\kappa\right) \Gamma\left(\frac{d-1}{2} - i\kappa\right)}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} {}_2F_1\left(\frac{d-1}{2} + i\kappa, \frac{d-1}{2} - i\kappa; \frac{d}{2}; \frac{1-\zeta}{2}\right) \\ = \frac{\Gamma\left(1 - \frac{d}{2}\right) \Gamma\left(\frac{d-1}{2} - i\kappa\right) \Gamma\left(\frac{d-1}{2} + i\kappa\right)}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{1}{2} - i\kappa\right) \Gamma\left(\frac{1}{2} + i\kappa\right)} \end{aligned} \quad (6.13)$$

so that

$$\text{bubble}_{\text{AdS}} = \frac{\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}(\lambda^2 - \nu^2)} \left(\frac{\Gamma\left(\frac{d-1}{2} - i\nu\right) \Gamma\left(\frac{d-1}{2} + i\nu\right)}{\Gamma\left(\frac{1}{2} - i\nu\right) \Gamma\left(\frac{1}{2} + i\nu\right)} - \frac{\Gamma\left(\frac{d-1}{2} - i\lambda\right) \Gamma\left(\frac{d-1}{2} + i\lambda\right)}{\Gamma\left(\frac{1}{2} - i\lambda\right) \Gamma\left(\frac{1}{2} + i\lambda\right)} \right). \quad (6.14)$$

Again, this formula defines the AdS bubble by analytic continuation in d as a meromorphic function of d .

Iterating once the above construction, under the same condition of validity of eq. (6.6) we may compute the 3-line diagram with two convolutions on a Euclidean manifold as follows:

$$\begin{aligned} F_{m_1 m_2 m_3}(x, y) &= \int_{\mathcal{E}_d} G_{m_1}^{(\mathcal{E})}(x, w) G_{m_2}^{(\mathcal{E})}(w, z) G_{m_3}^{(\mathcal{E})}(z, y) \sqrt{g(w)} dw \sqrt{g(z)} dz \\ &= - \int_{\mathcal{E}_d} \frac{G_{m_1}^{(\mathcal{E})}(x, z) - G_{m_2}^{(\mathcal{E})}(x, z)}{m_1^2 - m_2^2} G_{m_3}^{(\mathcal{E})}(z, y) \sqrt{g(z)} dz \\ &= \frac{G_{m_1}^{(\mathcal{E})}(x, y)}{(m_1^2 - m_2^2)(m_1^2 - m_3^2)} + \frac{G_{m_2}^{(\mathcal{E})}(x, y)}{(m_2^2 - m_3^2)(m_2^2 - m_1^2)} + \frac{G_{m_3}^{(\mathcal{E})}(x, y)}{(m_3^2 - m_1^2)(m_3^2 - m_2^2)} \end{aligned} \quad (6.15)$$

a formula which is readily generalized to the $(n + 1)$ -line case. In a completely analogous way we may fully evaluate the triangular loop by taking the limit $w \rightarrow x$ in eq. (6.15).

In the concrete cases under study here, we do it as before assuming at first that $d < 2$ and then generalizing the formula by analytic continuation. We get:

$$\text{Triangle}_M(m_1, m_2, m_3) = - \frac{\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \frac{m_1^{d-2}(m_2^2 - m_3^2) + m_2^{d-2}(m_3^2 - m_1^2) + m_3^{d-2}(m_1^2 - m_2^2)}{(m_1^2 - m_2^2)(m_2^2 - m_3^2)(m_3^2 - m_1^2)}, \quad (6.16)$$

$$\text{Triangle}_{\text{AdS}}(\kappa, \nu, \lambda) = \frac{\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \left(\frac{\Gamma\left(\frac{d-1}{2} + \kappa\right)}{(\kappa^2 - \nu^2)(\kappa^2 - \lambda^2)\Gamma\left(\frac{3-d}{2} + \kappa\right)} + \frac{\Gamma\left(\frac{d-1}{2} + \nu\right)}{(\nu^2 - \kappa^2)(\nu^2 - \lambda^2)\Gamma\left(\frac{3-d}{2} + \nu\right)} + \frac{\Gamma\left(\frac{d-1}{2} + \lambda\right)}{(\lambda^2 - \kappa^2)(\lambda^2 - \nu^2)\Gamma\left(\frac{3-d}{2} + \lambda\right)} \right). \quad (6.17)$$

$$\text{Triangle}_{\text{dS}}(\kappa, \nu, \lambda) = \frac{\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \left(\frac{\Gamma\left(\frac{d-1}{2} - i\kappa\right)\Gamma\left(\frac{d-1}{2} + i\kappa\right)}{(\kappa^2 - \nu^2)(\kappa^2 - \lambda^2)\Gamma\left(\frac{1}{2} - i\kappa\right)\Gamma\left(\frac{1}{2} + i\kappa\right)} + \frac{\Gamma\left(\frac{d-1}{2} - i\nu\right)\Gamma\left(\frac{d-1}{2} + i\nu\right)}{(\nu^2 - \kappa^2)(\nu^2 - \lambda^2)\Gamma\left(\frac{1}{2} - i\nu\right)\Gamma\left(\frac{1}{2} + i\nu\right)} + \frac{\Gamma\left(\frac{d-1}{2} - i\lambda\right)\Gamma\left(\frac{d-1}{2} + i\lambda\right)}{(\lambda^2 - \kappa^2)(\lambda^2 - \nu^2)\Gamma\left(\frac{1}{2} - i\lambda\right)\Gamma\left(\frac{1}{2} + i\lambda\right)} \right). \quad (6.18)$$

The above formulae allow to easily compute the one-loop renormalization of the coupling constant in a scalar ϕ^3 theory. For example, let us consider three massive scalar fields, ϕ_j with mass m_j , $j = 1, 2, 3$, with interaction $g_0\phi_1\phi_2\phi_3$. After including a counterterm δg_0 , at zero external momenta we get for the physical coupling in D dimensions

$$g = g_0 + \delta g + g_0^3 \text{Triangle}_{\mathcal{M}}(m_1, m_2, m_3)_{d \sim D} \quad (6.19)$$

where \mathcal{M} is the given Euclidean spacetime and $d \sim D$ means $d = D - 2\epsilon$. In $D = 4$ the renormalization of g is finite. In the flat case we get

$$g = g_0 + \delta g + g_0^3 \frac{m_1^2(m_2^2 - m_3^2) \log m_1^2 + m_2^2(m_3^2 - m_1^2) \log m_2^2 + m_3^2(m_1^2 - m_2^2) \log m_3^2}{(4\pi)^2(m_1^2 - m_2^2)(m_2^2 - m_3^2)(m_3^2 - m_1^2)}. \quad (6.20)$$

Notice that the sum of the coefficients of the logarithms is zero, so we can replace $\log m_j^2$ with $\log m_j^2/\mu^2$ for an arbitrary mass scale μ .

For AdS we get

$$g = g_0 + \delta g + g_0^3 \frac{(m_2^2 - m_3^2)(\nu_1 + (m_1^2 + 2)\psi(\nu_1 - 1/2)) + \text{cyc}}{2\pi^2(m_1^2 - m_2^2)(m_2^2 - m_3^2)(m_3^2 - m_1^2)}, \quad (6.21)$$

where $\nu_j = \sqrt{m_j^2 + \frac{9}{4}}$, and $f(m_1, m_2, m_3) + \text{cyc} = f(m_1, m_2, m_3) + f(m_2, m_3, m_1) + f(m_3, m_1, m_2)$.

Finally, it is worth to underline a remarkable property of eqs. (6.5) and (6.15): they can be used also when the Schwinger function is known only approximately. A simple example is an application to Quantum Field Theory on the Rindler space. We refer to [40–43]. Beyond the Minkowski vacuum $|M\rangle$ one can consider the Fulling-Rindler vacuum $|R\rangle$. Let us call

$G_A^{\beta,d}(x, y; m)$ the Schwinger function at inverse temperature β in d spacetime dimensions and for the vacuum state $|A\rangle$, for a scalar field of mass m . In coordinates $(\tau, \xi, x^1, \dots, x^{d-2})$ such that the Euclidean Rindler metric is

$$ds^2 = \xi^2 d\tau^2 + d\xi^2 + \delta_{ij} dx^i dx^j, \tag{6.22}$$

it is manifest that the Euclidean Rindler's wedge has the topology of a cylinder with period normalized to 2π for the Euclidean time. As it is well known, this leads to the identity

$$G_M^{\infty,d}(x, y; m) = G_R^{2\pi,d}(x, y; m), \tag{6.23}$$

which corresponds to the fact that the zero temperature Minkowski vacuum appears as a thermal state at temperature $\frac{1}{2\pi}$ to the Rindler vacuum (see [40], and reference therein). It is interesting to consider also the Schwinger function $G_R^{\infty,4}(x, y)$. Assuming $|\tau_x - \tau_y| < \pi$, one can prove (see [43]) that

$$G_R^{\infty,4}(x, y; m) = G_R^{2\pi,4}(x, y; m) + \frac{m}{8\pi^3} \int_0^\infty du \frac{K_1(mR(u))}{R(u)} F_\infty(u, \tau_x - \tau_y), \tag{6.24}$$

$$R(u)^2 = \xi_x^2 + \xi_y^2 + 2\xi_x \xi_y \operatorname{ch} u + (x_1 - y_1)^2 + (x_2 - y_2)^2, \tag{6.25}$$

$$F_\infty(u, v) = -2 \frac{v + \pi}{(v + \pi)^2 + u^2} + 2 \frac{v - \pi}{(v - \pi)^2 + u^2}. \tag{6.26}$$

In this way, the UV divergences are confined in $G_R^{2\pi,4}(x, y; m) = G_M^{\infty,4}(x, y; m)$. At this stage, an immediate application of the above general strategy shows that

$$\text{bubble}_{\text{Rindler}}(m_1, m_2) = \text{bubble}_M(m_1, m_2). \tag{6.27}$$

Another possible application, which we don't include here, could be to compute the bubble integral (and the effective potential), for a massive scalar field in the far region of a Schwarzschild black hole of mass M (t.i., for M/R), at the first order in M , starting from the results in [44].

7 A remark about bulk diagrams versus Witten diagrams

Now we want to consider the limit of the expression (6.5) when x and/or y tend to the boundary. To this aim it is useful to give coordinates to the Lobachevsky space as follows:

$$x^0 = \operatorname{ch} v + \frac{1}{2} e^v r^2, \quad x^i = e^v r^i, \quad x^{d-1} = \operatorname{sh} v - \frac{1}{2} e^v r^2 \tag{7.1}$$

$$y^0 = \operatorname{ch} v' + \frac{1}{2} e^{v'} r'^2, \quad y^i = e^{v'} r'^i, \quad x^{d-1} = \operatorname{sh} v' - \frac{1}{2} e^{v'} r'^2 \tag{7.2}$$

so that

$$x \cdot y = \operatorname{ch}(v - v') + \frac{1}{2} e^{v+v'} (r - r')^2. \tag{7.3}$$

Let v tend to plus infinity while v' is held fixed:

$$x \cdot y \simeq \frac{e^v}{2} (e^{-v'} + e^{v'} (r - r')^2) = e^v (\xi \cdot y); \tag{7.4}$$

in the second equality we have introduced the lightlike vector

$$\xi^0 = \frac{1+r^2}{2}, \quad \xi^i = r^i, \quad \xi^{d-1} = \frac{1-r^2}{2}. \quad (7.5)$$

Eq. (3.9) provides the behaviour of the propagator for large values of ζ ; at leading order

$$w_\nu^d(\zeta) \simeq \frac{\Gamma\left(\frac{d-1}{2} + \nu\right)}{2\pi^{\frac{d-1}{2}}(2\zeta)^{\frac{d-1}{2} + \nu}\Gamma(\nu+1)}. \quad (7.6)$$

It follows that

$$\begin{aligned} F_{\lambda\nu}(x, y) &\simeq \frac{G_\nu^d(e^v(\xi \cdot y)) - G_\lambda^d(e^v(\xi \cdot y))}{\lambda^2 - \nu^2} \\ &\simeq \frac{1}{\lambda^2 - \nu^2} \left(\frac{e^{-(\frac{d-1}{2} + \nu)v}\Gamma\left(\frac{d-1}{2} + \nu\right)}{2\pi^{\frac{d-1}{2}}(2\xi \cdot y)^{\frac{d-1}{2} + \nu}\Gamma(\nu+1)} - \frac{e^{-(\frac{d-1}{2} + \lambda)v}\Gamma\left(\frac{d-1}{2} + \lambda\right)}{2\pi^{\frac{d-1}{2}}(2\xi \cdot y)^{\frac{d-1}{2} + \lambda}\Gamma(\lambda+1)} \right). \end{aligned} \quad (7.7)$$

To extract a finite limit we now need to specify something about the masses. Let us suppose that $\lambda > \nu$; in this case, a standard rescaling procedure [4, 32] leads us to

$$e^{(\frac{d-1}{2} + \nu)v} F_{\lambda\nu}(x, y) \rightarrow F_{\lambda\nu}(\xi, y) = \frac{1}{\lambda^2 - \nu^2} \left(\frac{\Gamma\left(\frac{d-1}{2} + \nu\right)}{2\pi^{\frac{d-1}{2}}(2\xi \cdot y)^{\frac{d-1}{2} + \nu}\Gamma(\nu+1)} \right). \quad (7.8)$$

Let now v' tend to plus infinity:

$$\xi \cdot y = e^{v'}(\xi \cdot \eta); \quad (7.9)$$

where we have introduced the lightlike vector

$$\eta^0 = \frac{1+r'^2}{2}, \quad \eta^i = r'^i, \quad \eta^{d-1} = \frac{1-r'^2}{2}. \quad (7.10)$$

Proceeding as before we get ($\lambda > \nu$)

$$e^{(\frac{d-1}{2} + \nu)v'} F_{\lambda\nu}(\xi, y) \rightarrow F_{\lambda\nu}(\xi, \eta) = \frac{1}{\lambda^2 - \nu^2} \left(\frac{\Gamma\left(\frac{d-1}{2} + \nu\right)}{2\pi^{\frac{d-1}{2}}(2\xi \cdot \eta)^{\frac{d-1}{2} + \nu}\Gamma(\nu+1)} \right). \quad (7.11)$$

Had we performed the limit in the opposite order or else had we put $v = v'$ and taken the limit together, the situation would not have changed: when $\lambda > \nu$ the finite result extracted from eq. (6.5) is (7.11).

Let us now revert the order of the operations and take the dominant terms in eq. (6.1) before integration. We are led to the following Witten diagram which is the simplest among the ones considered in [9]:

$$\tilde{F}_{\lambda\nu}(\xi, \eta) = \int_{\mathbf{H}_d} \frac{\Gamma\left(\frac{d-1}{2} + \lambda\right)}{2\pi^{\frac{d-1}{2}}(2(\xi \cdot z))^{\frac{d-1}{2} + \lambda}\Gamma(\lambda+1)} \frac{\Gamma\left(\frac{d-1}{2} + \nu\right)}{2\pi^{\frac{d-1}{2}}(2(\eta \cdot z))^{\frac{d-1}{2} + \nu}\Gamma(\nu+1)} \sqrt{g(z)} dz. \quad (7.12)$$

We see that this Witten diagram cannot be finite. Otherwise it should be function of $(\xi \cdot \eta)$ homogeneous of degree $-\frac{d-1}{2} - \lambda$ and, at same time, of degree $-\frac{d-1}{2} - \nu$, a thing which is logically impossible. This is also what happens in [9], eq. (24), where for $n = 2$ the integral is proportional to

$$\int_0^\infty \frac{dt_1}{t_1} t_1^{\Delta_1} \int_0^\infty \frac{dt_2}{t_2} t_2^{\Delta_2} e^{-t_1 t_2 (x_1 - x_2)^2} = \Gamma(\Delta_2) ((x_1 - x_2)^2)^{2\Delta_2} \int_0^\infty \frac{dt_1}{t_1} t_1^{\Delta_1 - \Delta_2}, \quad (7.13)$$

which is divergent.

We are led to the conclusion that, at least in this example, this Witten diagram does not arise as the limit of corresponding diagrams where the endpoints lie in the bulk. This example may however be pathological.

In the three-line case the Witten diagram is completely determined by Lorentz invariance and by the homogeneity in the conical variables ξ_1, ξ_2 and ξ_3 ; its simple formula that the structure of a star-triangle relation [27]:

$$\begin{aligned} \tilde{G}_{\nu_1 \nu_2 \nu_3}(\xi_1, \xi_2, \xi_3) &= c_0(\nu_1, \nu_2, \nu_3) \int_{\mathbf{H}_d} \frac{1}{(\xi_1 \cdot z)^{\frac{d-1}{2} + \nu_1} (\xi_2 \cdot z)^{\frac{d-1}{2} + \nu_2} (\xi_3 \cdot z)^{\frac{d-1}{2} + \nu_2}} \sqrt{g(z)} dz \\ &= c(\nu_1, \nu_2, \nu_3) (\xi_1 \cdot \xi_2)^{\frac{1-d}{4} + \frac{\nu_3}{2} - \frac{\nu_1}{2} - \frac{\nu_2}{2}} (\xi_2 \cdot \xi_3)^{\frac{1-d}{4} + \frac{\nu_1}{2} - \frac{\nu_2}{2} - \frac{\nu_3}{2}} (\xi_1 \cdot \xi_3)^{\frac{1-d}{4} + \frac{\nu_2}{2} - \frac{\nu_1}{2} - \frac{\nu_3}{2}} \end{aligned} \quad (7.14)$$

where

$$c(\nu_1, \nu_2, \nu_3) = \frac{\Gamma\left(\frac{d-1+2\nu_1+2\nu_2-2\nu_3}{4}\right) \Gamma\left(\frac{d-1+2\nu_1-2\nu_2+2\nu_3}{4}\right) \Gamma\left(\frac{d-1-2\nu_1+2\nu_2+2\nu_3}{4}\right) \Gamma\left(\frac{d-1+2\nu_1+2\nu_2+2\nu_3}{4}\right)}{2^{\frac{3d+2\nu_1+2\nu_2+2\nu_3+13}{4}} \pi^{d-1} \Gamma(\nu_1+1) \Gamma(\nu_2+1) \Gamma(\nu_3+1)}. \quad (7.15)$$

This also coincides with formula (24) in [9] once expanded by using the formula

$$\int_0^\infty \frac{dt_1}{t_1} t_1^{\Delta_1} \int_0^\infty \frac{dt_2}{t_2} t_2^{\Delta_2} \int_0^\infty \frac{dt_3}{t_3} t_3^{\Delta_3} e^{-t_1 t_2 a_3^2 - t_1 t_3 a_2^2 - t_2 t_3 a_1^2} = \frac{1}{2} \prod_{j=1}^3 \Gamma(-\sigma_j) a_j^{2\sigma_j} \quad (7.16)$$

with $\sigma_j = \Delta_j - \frac{\Delta_1 + \Delta_2 + \Delta_3}{2}$.

The corresponding 3-line bulk diagram is given by

$$G_{\lambda \nu \kappa}(x, y, w) = \int_{\mathbf{H}_d} G_\lambda^d(x \cdot z) G_\nu^d(y \cdot z) G_\kappa^d(w \cdot z) \sqrt{g(z)} dz = \mathbf{G}_{\lambda \nu \kappa}(a, b, c). \quad (7.17)$$

$$a = x \cdot y, \quad b = y \cdot w, \quad c = w \cdot x. \quad (7.18)$$

At the moment we are unable to determine whether, contrary to what happens in the two-line case, the Witten diagram (7.14) may be obtained as boundary limit of (7.17) or not.

8 The Källén-Lehmann expansion of the bubble with two independent masses

In this section we come back and elaborate on the AdS Källén-Lehmann expansion formula which we found and proved in [29] and that appeared also in a later paper [30].

Recall that for any given analytic two-point function $W(z_1, z_2)$ of a local quantum field in Minkowski space satisfying the Wightman axioms [33] there exists a tempered weight ρ with support in the positive real axis such that

$$W(z_1, z_2) = \int_0^\infty \rho(m^2)W_m(z_1, z_2)dm^2; \tag{8.1}$$

here

$$W_m(z_1, z_2) = \frac{1}{(2\pi)^d} \int e^{-ik(z_1-z_2)}\theta(k^0)\delta(k^2 - m^2)dk \tag{8.2}$$

is the analytic continuation of the two-point function of a Klein-Gordon field with mass m ; z_1 belongs to the past tube and z_2 to the forward tube if the complex Minkowski space [33]. The Källén-Lehmann weight $\rho(m^2)$ is a positive measure if and only if the boundary value $W(x_1, x_2)$ is a distribution of positive type.

In particular, for any two given masses m_1 and m_2 the weight for the bubble

$$W_{m_1}(z_1, z_2)W_{m_2}(z_1, z_2) = \int_{(m_1+m_2)^2}^\infty \rho(m^2 : m_1, m_2)W_m(z_1, z_2)dm^2 \tag{8.3}$$

is easily obtained by Fourier transform. The above relation can be extended to Schwinger functions at non-coincident Euclidean points. A distributional extension to coincident Euclidean points is also possible if no renormalization is necessary, e.g. for low dimension.

As regards the general case of AdS fields on the covering manifold \widetilde{AdS}_d , it is of course possible to consider a general superposition *à la* Källén-Lehmann as follows:

$$w(\zeta) = \int_{-1}^\infty \rho_1(\kappa)w_\nu^d(\zeta)d\nu + \int_{-1}^1 \rho_2(\kappa)w_{-\nu}^d(\zeta)d\nu. \tag{8.4}$$

We expressed the above relation using the reduced maximally analytic two-point function; under suitable condition this implies the same representation for the Euclidean propagator:

$$G(u) = \int_{-1}^\infty \rho_1(\kappa)G_\nu^d(u)d\nu + \int_{-1}^1 \rho_2(\kappa)G_{-\nu}^d(u)d\nu. \tag{8.5}$$

However, when we restrict our attention to the bubble, we do not expect an integral as in flat space; even though we consider fields with non integer masses that are well-defined only on the covering manifold, their correlation functions are quasi-periodic. *This calls for a Källén-Lehmann series: the weight has to be a series of pure point measures.*

Before addressing the general case, discussing two concrete examples will clarify the point.

8.1 An almost trivial example: \widetilde{AdS}_3

In odd spacetime dimension the Schwinger function can be written in terms of elementary functions. To this aim, it is useful to parametrize the invariant variable by an hyperbolic angle: $u = \text{ch } v$. In $d = 3$ this gives the following expressions: [35, eq. (12), p. 150]:

$$Q_{\kappa-\frac{1}{2}}^{\frac{1}{2}}(\text{ch } v) = e^{\frac{i\pi}{2}-\nu v} \sqrt{\frac{\pi}{2 \text{sh } v}}, \tag{8.6}$$

$$G_\kappa^3(\text{ch } v) = \frac{e^{-\frac{i\pi}{2}} Q_{\kappa-\frac{1}{2}}^{\frac{1}{2}}(\text{ch } v)}{2\pi\sqrt{2\pi \text{sh } v}} = \frac{e^{-\nu v}}{4\pi \text{sh } v}. \tag{8.7}$$

The bubble with two independent mass parameters λ and ν is readily decomposed into its Källén-Lehmann series by an elementary manipulation:

$$\begin{aligned} G_\lambda^3(\text{ch } v)G_\nu^3(\text{ch } v) &= \frac{e^{-(\lambda+\nu)v}}{16\pi^2 \text{sh}^2 v} = \frac{e^{-(\lambda+\nu+1)v}}{8\pi^2 \text{sh } v(1 - e^{-2v})} = \sum_{k=0}^{\infty} \frac{e^{-(\lambda+\nu+1+2k)v}}{8\pi^2 \text{sh } v} \\ &= \frac{1}{2\pi} \sum_{k=0}^{\infty} G_{2k+1+\lambda+\nu}^3(\text{ch } v). \end{aligned} \tag{8.8}$$

So, we get a Källén-Lehmann series as opposed to an integral, with a weight that in this simple case is just a constant.

8.2 A less elementary example: \widetilde{AdS}_5

In view of its importance in the AdS/CFT correspondence, let us discuss now the less trivial five-dimensional case: here

$$G_\nu^5(\text{ch } v) = \frac{1}{(2\pi)^{\frac{5}{2}}} (\text{sh } v)^{-\frac{3}{2}} e^{-\frac{3\pi i}{2}} Q_{\kappa-\frac{1}{2}}^{\frac{3}{2}}(\text{ch } v) = \frac{(\nu+1)e^{-(\nu-1)v} - (\nu-1)e^{-(\nu+1)v}}{16\pi^2 \text{sh}^3 v}; \tag{8.9}$$

to derive the above formula we used eq. (8.6) and the recurrence relation [35, eq. (4), p. 160]

$$Q_\rho^{\mu+1}(z) = \frac{(z^2 - 1)^{-\frac{1}{2}}}{(\rho - \mu + 1)} \left((\rho - \mu)(2\rho + 1)Q_{\rho+1}^\mu(z) - (\rho + \mu)(\rho + \mu + 1)Q_{\rho-1}^\mu(z) \right). \tag{8.10}$$

Now, taking into account the identity

$$\frac{1}{4 \text{sh}^3 v} = \sum_{k=0}^{\infty} k(k-1)e^{-(2k-1)v}, \quad \text{Re } v > 0 \tag{8.11}$$

we get a series expansion which is not yet a Källén-Lehmann expansion:

$$G_\lambda^5(\text{ch } v)G_\nu^5(\text{ch } v) = \sum_{k=0}^{\infty} \frac{(\lambda+1)(\nu+1) + 2k^2 + 2k(\lambda+\nu+1)}{8\pi^4 \text{sh}^3 v} e^{-(2k+\lambda+\nu+1)v}. \tag{8.12}$$

For λ , ν and $\lambda + \nu$ greater than -1 all the terms at the r.h.s. are positive. We may therefore rearrange the series as it is absolutely convergent.

The construction goes as follows: in the first term of the series ($k = 0$) we replace the exponential $e^{-(\lambda+\nu+1)v}$ with the Schwinger function $G_{\lambda+\nu+2}^5(\text{ch } v)$ and subtract the corresponding term proportional to $e^{-(\lambda+\nu+3)v}$ from the second term ($k = 1$). Then we iterate the procedure. It results the following Källén-Lehmann series for the bubble in \widetilde{AdS}_5 :

$$G_\lambda^5(\text{ch } v)G_\nu^5(\text{ch } v) = \sum_{k=0}^{\infty} \frac{(k+1)(\lambda+k+1)(\nu+k+1)(\lambda+\nu+k+1)}{2\pi^2(\lambda+\nu+2k+1)(\lambda+\nu+2k+3)} G_{\lambda+\nu+2k+2}^5(\text{ch } v). \tag{8.13}$$

Now let us consider the general problem. The following two theorems were proved in [29] and solve the general problem of finding the weight for the Källén-Lehmann series of the bubble in \widetilde{AdS}_d . The first theorem concern the ultraspherical Gegenbauer functions; it holds for integer values of the parameters:

Theorem 8.1 ([29]). *Let $c_\delta(m, n|l)$ be given by*

$$c_\delta(m, n|l) = \frac{\pi \alpha_\delta \left(\frac{l+m-n}{2}\right) \alpha_\delta \left(\frac{l-m+n}{2}\right) \alpha_\delta \left(\frac{l+m+n}{2} + \delta\right) \alpha_\delta \left(\frac{l-m-n}{2} - \delta\right)}{\alpha_\delta(l) \alpha_\delta(l + \delta)}, \quad (8.14)$$

where

$$\alpha_\delta(s) = \frac{\Gamma(s + \delta)}{\Gamma(\delta)\Gamma(s + 1)}, \quad (8.15)$$

(i) *Let m and n be non-negative integers and suppose that one of the two following conditions is satisfied*

- (a) $\delta = \frac{1}{2} - r$, $r \geq 1$ is an integer such that $m \geq 2r$ and $n \geq 2r$;
- (b) 2δ is a strictly positive integer;

then

$$D_m^\delta(z) D_n^\delta(z) = \sum_{\substack{l=m+n+2\delta+2k \\ k \in \mathbf{Z}, k \geq 0}} c_\delta(m, n|l) D_l^\delta(z) \quad (8.16)$$

holds with uniform convergence for z in any compact subset of $\mathbf{C} \setminus [-1, 1]$.

(ii) *If m , n and 2δ are not necessarily integers but the following conditions are satisfied:*

$$\delta > 0, \quad m+2\delta > 0, \quad n+2\delta > 0, \quad m+\delta+1 > 0, \quad n+\delta+1 > 0, \quad m+n+2\delta+1 \geq 0, \quad (8.17)$$

then (8.16) holds with uniform convergence for z in any compact subset of $\mathbf{C} \setminus (-\infty, 1]$.

The proof of this theorem can be found in [29]; it actually holds in the sense of functions with tempered behavior in $\tilde{\Delta}_1$, so that the conclusion also holds for the boundary values of both sides in (8.16).

Remark 8.1. If in eq. (8.16) both sides are multiplied by $z^{m+n+4\delta}$, both sides become formal power series in z^{-2} (with coefficients meromorphic in the parameters). It is proved in [29] that these two formal power series are equal without any restriction on the parameters. Moreover the series converges for $|z^2| > 1$, as the l.h.s. is holomorphic there. This does not prove that the series (in k) on the r.h.s. converges: the theorem gives sufficient conditions for that.

The second theorem gives the Källén-Lehmann expansion for AdS quantum fields on the covering manifold with two distinct mass parameters. It was shown under rather restrictive conditions but it holds for continuous values of the real mass parameters.

Theorem 8.2 ([29]). *Let $\delta = \frac{d-1}{2}$. Under the conditions*

$$\delta > 0, \quad \lambda + \delta > 0, \quad \nu + \delta > 0, \quad \lambda + 1 > 0, \quad \nu + 1 > 0, \quad \lambda + \nu + 1 \geq 0, \quad (8.18)$$

$$w_\lambda^d(\zeta) w_\nu^d(\zeta) = \sum_{k=0}^{\infty} \rho_d(k; \lambda, \nu) w_{\delta+2k+\lambda+\nu}^d(\zeta), \quad \zeta \in \tilde{\Delta}_1 \quad (8.19)$$

where the weight is given by

$$\begin{aligned} \rho_d(k; \lambda, \nu) &= \frac{(2\delta + 4k + 2\lambda + 2\nu)\Gamma(1 + 2k + \lambda + \nu)}{4\pi^\delta \Gamma(\delta)\Gamma(2\delta + 2k + \lambda + \nu)} \\ &\quad \times \frac{\Gamma(\delta + k)\Gamma(\delta + \lambda + k)\Gamma(\delta + \nu + k)\Gamma(\delta + \lambda + \nu + k)}{\Gamma(1 + k)\Gamma(1 + \lambda + k)\Gamma(1 + \nu + k)\Gamma(1 + \lambda + \nu + k)}. \end{aligned} \quad (8.20)$$

Remark 8.2. As anticipated, for any given two masses λ and ν the KL expansion of the bubble $w_\lambda^d(\zeta) w_\nu^d(\zeta)$ is a series and not an integral. This is a leftover of the periodicity of the true AdS manifold.

In fact the statement of the theorem holds under much weaker conditions as we explicitly show in the following examples. Theorem 8.3 below gives weaker sufficient conditions for the validity of the statement of theorem 8.2.

8.3 Cases of \widetilde{AdS}_1 and \widetilde{AdS}_{-1}

The proof of the above formula is valid only for $d > 1$. This does not mean that there is no Källén-Lehmann expansion for $d \leq 1$. Let us consider indeed the simplest case $d = 1$; the Schwinger function now is really elementary:

$$G_\nu^1(\text{ch } v) = \frac{e^{-\nu|v|}}{2\nu} \quad (8.21)$$

and the Källén-Lehmann series has only one nonzero term:

$$G_\lambda^1(\text{ch } v) G_\nu^1(\text{ch } v) = \frac{e^{-(\lambda+\nu)|v|}}{4\lambda\nu} = \frac{\lambda + \nu}{2\lambda\nu} G_{\lambda+\nu}^1(\text{ch } v). \quad (8.22)$$

This is perfectly coherent with the above formula (8.20); indeed, only the first coefficient has a non zero limit at $d = 1$:

$$\lim_{d \rightarrow 1} \rho_d(0; \lambda, \nu) = \frac{\lambda + \nu}{2\lambda\nu}, \quad \lim_{d \rightarrow 1} \rho_d(k \neq 0; \lambda, \nu) = 0. \quad (8.23)$$

The above feature is shared by every odd negative spacetime dimension. Let us consider as a further example \widetilde{AdS}_{-1} : the general formula for the propagator (3.14) here gives

$$G_\nu^{-1}(\text{ch } v) = (2\pi)^{\frac{1}{2}} (\text{sh } v)^{\frac{3}{2}} e^{i\pi\frac{3}{2}} Q_{-\frac{1}{2}+\kappa}^{-\frac{3}{2}}(\text{ch } v). \quad (8.24)$$

Taking into account eq. (8.9) and the relation [35, section 3.2, eq. (2), p. 140]

$$e^{i\pi\mu} \Gamma(\rho + \mu + 1) Q_\rho^{-\mu}(z) = e^{-i\pi\mu} \Gamma(\rho - \mu + 1) Q_\rho^\mu(z) \quad (8.25)$$

we get

$$G_\nu^{-1}(\text{ch } v) = \frac{\pi \Gamma(\nu - 1)}{2 \Gamma(\nu + 2)} \left((\nu + 1) e^{-(\nu-1)v} - (\nu - 1) e^{-(\nu+1)v} \right). \quad (8.26)$$

A direct verification shows that the Källén-Lehmann series here consists only of two terms

$$\begin{aligned} G_\lambda^{-1}(\text{ch } v) G_\nu^{-1}(\text{ch } v) &= \frac{\pi(\lambda + \nu - 2)(\lambda + \nu - 1)}{2\lambda\nu(\lambda - 1)(\nu - 1)} G_{\lambda+\nu-1}^{-1}(\text{ch } v) \\ &\quad - \frac{\pi(\lambda + \nu + 1)(\lambda + \nu + 2)}{2\lambda\nu(\lambda + 1)(\nu + 1)} G_{\lambda+\nu+1}^{-1}(\text{ch } v) \end{aligned} \quad (8.27)$$

in accordance with our general result (8.20).

Theorem 8.3. Let $\delta = \frac{d-1}{2}$. Let V be the open complex set:⁴

$$V = \{(\delta, \lambda, \nu) \in \mathbf{C}^3 : \delta + \lambda \notin \mathbf{Z}_-, \delta + \nu \notin \mathbf{Z}_-, \delta + \lambda + \nu \notin \mathbf{Z}_-, \lambda + \nu \notin \mathbf{Z}_-\}. \quad (8.28)$$

For $(\delta, \lambda, \nu) \in V$, $\zeta \in \widehat{\Delta}_1$ and any integer $k \geq 0$ let

$$S_k(\delta, \lambda, \nu, \zeta) = \rho_d(k; \lambda, \nu) \zeta^{2\delta+\lambda+\nu} w_{\delta+\lambda+\nu+2k}^d(\zeta), \quad (8.29)$$

where $\rho_d(k; \lambda, \nu)$ is given by (8.20). Then for all $(\delta, \lambda, \nu) \in V$ and $\zeta \in \widehat{\Delta}_1$,

$$\zeta^{2\delta+\lambda+\nu} w_\lambda^d(\zeta) w_\nu^d(\zeta) = \sum_{k=0}^{\infty} S_k(\delta, \lambda, \nu, \zeta). \quad (8.30)$$

Here the series on the r.h.s. converges absolutely and uniformly on every compact subset of $V \times \widehat{\Delta}_1$. Moreover (8.30) also holds for the boundary values of both sides from $\mathcal{Z}_{1-} \times \mathcal{Z}_{1+}$ in the sense of tempered distributions.

Recall that (see (3.9)), for any integer $p \geq 0$, $\zeta \mapsto \zeta^{\delta+\nu-2p} w_\nu^d(\zeta)$ is even and holomorphic in $\widehat{\Delta}_1$. This makes it possible to express the result (8.30) as an equality of functions analytic in $\widehat{\Delta}_1$.

The proof of this theorem is given in appendix B.

9 The tadpole and the sunset

The tadpole (in a trilinear interaction or cubic self-interaction) is readily evaluated using the Källén-Lehmann expansion (8.19):

$$\begin{aligned} F_{\lambda, \kappa \sigma}(x) &= \int_{\mathbf{H}_d} G_\lambda^d(x \cdot z) G_\kappa^d(z \cdot w) G_\sigma^d(z \cdot w) \sqrt{g(w)} dw \sqrt{g(z)} dz \\ &= \sum_{k=0}^{\infty} \rho(k; \kappa, \sigma) \int_{\mathbf{H}_d} G_\lambda^d(x \cdot z) G_{\delta+2k+\kappa+\sigma}(z \cdot w) \sqrt{g(w)} dw \sqrt{g(z)} dz. \end{aligned} \quad (9.1)$$

By using eqs. (4.5) and (6.5) this becomes

$$\begin{aligned} F_{\lambda, \kappa \sigma}(x) &= - \sum_{k=0}^{\infty} \frac{\rho(k; \kappa, \sigma)}{\lambda^2 - (\delta + 2k + \kappa + \sigma)^2} \int_{\mathbf{H}_d} \left(G_\lambda^d(x \cdot w) - G_{\delta+2k+\kappa+\sigma}(x \cdot w) \right) \sqrt{g(w)} dw \\ &= \frac{f(\lambda, \kappa, \sigma)}{\delta^2 - \lambda^2} + g(\lambda, \kappa, \sigma), \end{aligned} \quad (9.2)$$

where

$$f(\lambda, \kappa, \sigma) = \sum_{k=0}^{\infty} \left(\frac{\rho(k; \kappa, \sigma)}{2\lambda(2k + \delta + \kappa + \lambda + \sigma)} - \frac{\rho(k; \kappa, \sigma)}{2\lambda(2k + \delta + \kappa - \lambda + \sigma)} \right) \quad (9.3)$$

$$\begin{aligned} g(\lambda, \kappa, \sigma) &= \sum_{k=0}^{\infty} \left(- \frac{\rho(k; \kappa, \sigma)}{2\delta(\delta^2 - \lambda^2)(2k + \kappa + \sigma)} + \frac{\rho(k; \kappa, \sigma)}{2\delta(\delta^2 - \lambda^2)(2k + 2\delta + \kappa + \sigma)} \right. \\ &\quad \left. + \frac{\rho(k; \kappa, \sigma)}{2\lambda(\delta^2 - \lambda^2)(2k + \delta + \kappa - \lambda + \sigma)} - \frac{\rho(k; \kappa, \sigma)}{2\lambda(\delta^2 - \lambda^2)(2k + \delta + \kappa + \lambda + \sigma)} \right). \end{aligned} \quad (9.4)$$

⁴Recall that \mathbf{Z}_- is the set of non-positive integers and \mathbf{Z}_+ is the set of non-negative integers.

Defining

$$\begin{aligned}
 F_{\text{tad}}(x) &= \sum_{k=0}^{\infty} \frac{\rho(k; \kappa, \sigma)}{2k+x} \\
 &= \frac{\pi^{-\delta} \delta (2+\kappa+\sigma) \Gamma(1+\delta+\kappa) \Gamma(1+\delta+\sigma) \Gamma(1+\delta+\kappa+\sigma)}{(2+x) \Gamma(2+\kappa) \Gamma(2+\sigma) \Gamma(2+2\delta+\kappa+\sigma)} \\
 &\quad \times {}_7F_6 \left(\begin{matrix} 1+\frac{x}{2}, 1+\delta, 1+\delta+\kappa, \frac{3}{2}+\frac{\kappa}{2}+\frac{\sigma}{2}, 2+\frac{\kappa}{2}+\frac{\sigma}{2}, 1+\delta+\sigma, 1+\delta+\kappa+\sigma \\ 2+\frac{x}{2}, 2+\kappa, 1+\delta+\frac{\kappa}{2}+\frac{\sigma}{2}, \frac{3}{2}+\delta+\frac{\kappa}{2}+\frac{\sigma}{2}, 2+\sigma, 2+\kappa+\sigma \end{matrix} ; 1 \right) \\
 &\quad + \frac{\pi^{-\delta} \Gamma(\delta+\kappa) \Gamma(\delta+\sigma) \Gamma(1+\delta+\kappa+\sigma)}{2x \Gamma(1+\kappa) \Gamma(1+\sigma) \Gamma(2\delta+\kappa+\sigma)} \\
 &\quad \times {}_7F_6 \left(\begin{matrix} \frac{x}{2}, \delta, \delta+\kappa, \frac{1}{2}+\frac{\kappa}{2}+\frac{\sigma}{2}, 1+\frac{\kappa}{2}+\frac{\sigma}{2}, \delta+\sigma, \delta+\kappa+\sigma \\ 1+\frac{x}{2}, 1+\kappa, \delta+\frac{\kappa}{2}+\frac{\sigma}{2}, \frac{1}{2}+\delta+\frac{\kappa}{2}+\frac{\sigma}{2}, 1+\sigma, 1+\kappa+\sigma \end{matrix} ; 1 \right) \tag{9.5}
 \end{aligned}$$

we get

$$F_{\lambda, \kappa \sigma}(x) = \frac{F_{\text{tad}}(2\delta+\kappa+\sigma) - F_{\text{tad}}(\kappa+\sigma)}{2\delta(\delta^2 - \lambda^2)} \tag{9.6}$$

which corresponds to (4.5) with a multiplicative renormalization of the mass.

We may also compute in a similar way the general one-loop correction to the (bulk-to-bulk) diagram (6.1) for arbitrary masses in the loop:

$$F_{\lambda \nu}^{(1)}(x, y) = \int_{\mathbf{H}_d} G_{\lambda}^d(x \cdot z) G_{\kappa}^d(z \cdot w) G_{\sigma}^d(z \cdot w) G_{\nu}^d(w \cdot y) \sqrt{g(w)} dw \sqrt{g(z)} dz. \tag{9.7}$$

First, we linearize (9.7) using the Källén-Lehmann expansion (8.19):

$$F_{\lambda \nu}^{(1)}(x, y) = \sum_{k=0}^{\infty} \rho(k; \kappa, \sigma) \int_{\mathbf{H}_d} G_{\lambda}^d(x \cdot z) G_{\delta+2k+\kappa+\sigma}^d(z \cdot w) G_{\nu}^d(w \cdot y) \sqrt{g(w)} dw \sqrt{g(z)} dz. \tag{9.8}$$

Next, to perform the integrals we use eq. (6.15):

$$\begin{aligned}
 F_{\lambda \nu}^{(1)}(x, y) &= \frac{f(\nu, \kappa, \sigma) G_{\nu}^d(x \cdot y)}{(\nu^2 - \lambda^2)} + \frac{f(\lambda, \kappa, \sigma) G_{\lambda}^d(x \cdot y)}{(\lambda^2 - \nu^2)} \\
 &\quad + \sum_{k=0}^{\infty} \frac{\rho(k; \kappa, \sigma) G_{\delta+2k+\kappa+\sigma}^d(x \cdot y)}{((\delta+2k+\kappa+\sigma)^2 - \nu^2)((\delta+2k+\kappa+\sigma)^2 - \lambda^2)}, \tag{9.9}
 \end{aligned}$$

where

$$f(\lambda, \kappa, \sigma) = \frac{1}{2\lambda} (F_{\text{tad}}(\delta+\kappa+\sigma+\lambda) - F_{\text{tad}}(\delta+\kappa+\sigma-\lambda)). \tag{9.10}$$

10 Two-loop banana integrals: the watermelon

Here we provide a general formula for the watermelon with three lines and three independent mass parameters

$$I_3(\lambda, \nu, \kappa, d) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_1^{\infty} G_{\lambda}^d(u) G_{\nu}^d(u) G_{\kappa}^d(u) (u^2 - 1)^{\frac{d-2}{2}} du. \tag{10.1}$$

The evaluation is made possible and actually rendered elementary by the Källén-Lehmann formula (8.20); we use it to replace the first two factors in the integral with a series (10.1):

$$\begin{aligned}
 I_3(\lambda, \nu, \kappa, d) &= \sum_{k=0}^{\infty} \rho(k; \lambda, \nu) \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_1^{\infty} G_{\delta+2k+\lambda+\nu}^d(u) G_{\kappa}^d(u) du \\
 &= \sum_{k=0}^{\infty} \rho(k; \lambda, \nu) I_2((\delta + 2k + \lambda + \nu), \kappa, d) \\
 &= I_3^{(1)}(\lambda, \nu, \kappa, d) - I_3^{(2)}(\lambda, \nu, \kappa, d),
 \end{aligned} \tag{10.2}$$

where we used again the notation $\delta = \frac{d-1}{2}$. By inserting the 1-loop integral (4.17) $I_3^{(1)}$ and $I_3^{(2)}$ are determined by summing the relevant hypergeometric series:

$$\begin{aligned}
 I_3^{(1)}(\lambda, \nu, \kappa, d) &= \frac{\Gamma\left(1 - \frac{d}{2}\right) \Gamma\left(\frac{d-1}{2} + \kappa\right)}{2^d \pi^{\frac{d}{2}} \Gamma\left(\frac{3-d}{2} + \kappa\right)} \sum_{k=0}^{\infty} \frac{\rho(k; \lambda, \nu)}{\left(\left(\frac{d-1}{2} + 2k + \lambda + \nu\right)^2 - \kappa^2\right)} \\
 &= \frac{2^{-2\delta-3} \pi^{-2\delta-\frac{1}{2}} \Gamma\left(\frac{1}{2} - \delta\right) \Gamma(\delta + \kappa) \Gamma(\delta + \lambda) \Gamma(\delta + \nu) \Gamma(\delta + \lambda + \nu) \Gamma(2(\delta + \lambda + \nu) + 1)}{\Gamma(\lambda + 1) \Gamma(\nu + 1) \Gamma(1 - \delta + \kappa) ((\delta + \lambda + \nu)^2 - \kappa^2) \Gamma(2(\delta + \lambda + \nu)) \Gamma(2\delta + \lambda + \nu)} \\
 &\quad \times {}_9F_8 \left(\begin{matrix} \delta, \delta + \lambda, \delta + \nu, \delta + \lambda + \nu, \frac{1+\lambda+\nu}{2}, \frac{2+\lambda+\nu}{2}, \frac{2+\delta+\lambda+\nu}{2}, \frac{\delta-\kappa+\lambda+\nu}{2}, \frac{\delta+\kappa+\lambda+\nu}{2} \\ 1 + \lambda, 1 + \nu, 1 + \lambda + \nu, \frac{\delta+\lambda+\nu}{2}, \frac{2\delta+\lambda+\nu}{2}, \frac{1+2\delta+\lambda+\nu}{2}, \frac{2+\delta-\kappa+\lambda+\nu}{2}, \frac{2+\delta+\kappa+\lambda+\nu}{2} \end{matrix}; 1 \right),
 \end{aligned} \tag{10.3}$$

and

$$\begin{aligned}
 I_3^{(2)}(\lambda, \nu, \kappa, d) &= \frac{\Gamma\left(1 - \frac{d}{2}\right)}{2^d \pi^{\frac{d}{2}}} \sum_{k=0}^{\infty} \frac{\rho(k; \lambda, \nu)}{\left(\left(\frac{d-1}{2} + 2k + \lambda + \nu\right)^2 - \kappa^2\right)} \frac{\Gamma(d - 1 + 2k + \lambda + \nu)}{\Gamma(1 + 2k + \lambda + \nu)} \\
 &= \frac{4^{-\delta-1} \pi^{-2\delta-\frac{1}{2}} \Gamma\left(\frac{1}{2} - \delta\right) \Gamma(\delta + \lambda) \Gamma(\delta + \nu) \Gamma(\delta + \lambda + \nu + 1)}{\Gamma(\lambda + 1) \Gamma(\nu + 1) \Gamma(\lambda + \nu + 1) (\delta - \kappa + \lambda + \nu) (\delta + \kappa + \lambda + \nu)} \\
 &\quad \times {}_7F_6 \left(\begin{matrix} \delta, \delta + \lambda, \delta + \nu, \delta + \lambda + \nu, \frac{2+\delta+\lambda+\nu}{2}, \frac{\delta-\kappa+\lambda+\nu}{2}, \frac{\delta+\kappa+\lambda+\nu}{2} \\ 1 + \lambda, 1 + \nu, 1 + \lambda + \nu, \frac{\delta+\lambda+\nu}{2}, \frac{2+\delta-\kappa+\lambda+\nu}{2}, \frac{2+\delta+\kappa+\lambda+\nu}{2} \end{matrix}; 1 \right).
 \end{aligned} \tag{10.4}$$

These expressions can be simplified by using a simple hypergeometric identity which, however, does not seem to appear in the literature; we provide a derivation here. By using the following identity for Pochhammer symbols

$$\frac{(a)_n (b)_n ((a+b)/2 + 1)_n}{(a+1)_n (b+1)_n ((a+b)/2)_n} = \frac{ab}{a+b} \left(\frac{1}{a+n} + \frac{1}{b+n} \right), \tag{10.5}$$

we can write

$$\begin{aligned}
 & {}_{p+1}F_p \left(\begin{matrix} a_1, \dots, a_{p-2}, \frac{a+b}{2} + 1, a, b \\ b_1, \dots, b_{p-3}, \frac{a+b}{2}, a+1, b+1 \end{matrix} ; 1 \right) \\
 &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p-2})_n}{(b_1)_n \cdots (b_{p-3})_n} \frac{1}{n!} \frac{(a)_n (b)_n ((a+b)/2 + 1)_n}{(a+1)_n (b+1)_n ((a+b)/2)_n} \\
 &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p-2})_n}{(b_1)_n \cdots (b_{p-3})_n} \frac{1}{n!} \frac{ab}{a+b} \left(\frac{1}{a+n} + \frac{1}{b+n} \right) \\
 &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p-2})_n}{(b_1)_n \cdots (b_{p-3})_n} \frac{1}{n!} \frac{1}{a+b} \left(b \frac{(a)_n}{(a+1)_n} + a \frac{(b)_n}{(b+1)_n} \right),
 \end{aligned}$$

which gives the identity

$$\begin{aligned}
 & {}_{p+1}F_p \left(\begin{matrix} a_1, \dots, a_{p-2}, \frac{a+b}{2} + 1, a, b \\ b_1, \dots, b_{p-3}, \frac{a+b}{2}, a+1, b+1 \end{matrix} ; 1 \right) \\
 &= \frac{b}{a+b} {}_{p-1}F_{p-2} \left(\begin{matrix} a_1, \dots, a_{p-2}, a \\ b_1, \dots, b_{p-3}, a+1 \end{matrix} ; 1 \right) \\
 &\quad + \frac{a}{a+b} {}_{p-1}F_{p-2} \left(\begin{matrix} a_1, \dots, a_{p-2}, b \\ b_1, \dots, b_{p-3}, b+1 \end{matrix} ; 1 \right).
 \end{aligned} \tag{10.6}$$

By using (10.6) with $a = \frac{\delta+\lambda+\nu-\kappa}{2}$ and $b = \frac{\delta+\lambda+\nu+\kappa}{2}$ we get

$$\begin{aligned}
 I_3^{(1)}(\lambda, \nu, \kappa, d) &= \frac{2^{-2\delta-3} \pi^{-2\delta-\frac{1}{2}} \Gamma\left(\frac{1}{2} - \delta\right) \Gamma(\delta + \kappa) \Gamma(\delta + \lambda) \Gamma(\delta + \nu) \Gamma(\delta + \lambda + \nu)}{\Gamma(\lambda + 1) \Gamma(\nu + 1) \Gamma(1 - \delta + \kappa) \Gamma(2\delta + \lambda + \nu)} \\
 &\times \sum_{\epsilon=\pm} \frac{1}{\delta + \lambda + \nu + \epsilon\kappa} {}_7F_6 \left(\begin{matrix} \delta, \delta + \lambda, \delta + \nu, \delta + \lambda + \nu, \frac{1+\lambda+\nu}{2}, \frac{2+\lambda+\nu}{2}, \frac{\delta+\epsilon\kappa+\lambda+\nu}{2} \\ 1 + \lambda, 1 + \nu, 1 + \lambda + \nu, \frac{2\delta+\lambda+\nu}{2}, \frac{1+2\delta+\lambda+\nu}{2}, \frac{2+\delta+\epsilon\kappa+\lambda+\nu}{2} \end{matrix} ; 1 \right),
 \end{aligned} \tag{10.7}$$

and

$$\begin{aligned}
 I_3^{(2)}(\lambda, \nu, \kappa, d) &= \frac{4^{-\delta-1} \pi^{-2\delta-\frac{1}{2}} \Gamma\left(\frac{1}{2} - \delta\right) \Gamma(\delta + \lambda) \Gamma(\delta + \nu) \Gamma(\delta + \lambda + \nu)}{\Gamma(\lambda + 1) \Gamma(\nu + 1) \Gamma(\lambda + \nu + 1)} \\
 &\times \sum_{\epsilon=\pm} \frac{1}{\delta + \epsilon\kappa + \lambda + \nu} {}_5F_4 \left(\begin{matrix} \delta, \delta + \lambda, \delta + \nu, \delta + \lambda + \nu, \frac{\delta+\epsilon\kappa+\lambda+\nu}{2} \\ 1 + \lambda, 1 + \nu, 1 + \lambda + \nu, \frac{2+\delta+\epsilon\kappa+\lambda+\nu}{2} \end{matrix} ; 1 \right).
 \end{aligned} \tag{10.8}$$

In the one-dimensional case (and, in general, in odd negative dimension) the above formulae become extremely simple: for example

$$\begin{aligned}
 I_3^{(1)}(\lambda, \nu, \kappa, 1) &= \frac{\lambda + \nu}{4\kappa\lambda\nu(\lambda + \nu - \kappa)(\lambda + \nu + \kappa)}, \\
 I_3^{(2)}(\lambda, \nu, \kappa, 1) &= \frac{1}{4\lambda\nu(\lambda + \nu - \kappa)(\lambda + \nu + \kappa)}, \\
 I_3(\lambda, \nu, \kappa, 1) &= \int_0^\infty \frac{e^{-t\lambda} e^{-t\nu} e^{-t\kappa}}{(2\lambda)(2\nu)(2\kappa)} dt = I_3^{(1)} - I_3^{(2)} = \frac{1}{4\kappa\lambda\nu(\kappa + \lambda + \nu)}.
 \end{aligned} \tag{10.9}$$

The watermelon diverges, as in flat space, starting at $d = 3$. Actually, $I_3^{(2)}$ diverges at $d = 3$ while the first divergence of $I_3^{(1)}$ is at $d = 4$. At $d = 2$ the difference $I_3^{(1)} - I_3^{(2)}$ has a finite limit.

11 Conclusions

In the present paper, we have revisited methods and tools for quantum field theory on the anti de Sitter spacetime already available in the literature [25, 29] but whose potential seems to have gone unnoticed. In particular, we have shown here their power in computing loop integrals directly, showing no need of bootstrapping at this level.

We have also shown the technical superiority of conceiving the correlation functions of anti de Sitter quantum fields as Legendre functions of the second kind rather than the less specific, although equivalent, hypergeometric functions.

The methods used in this paper include theorems that render almost elementary the computation of certain integrals which otherwise quite complicated even in flat Minkowski spacetime: one example is the two-loop banana integral, the watermelon, with three arbitrary masses. The study of the watermelon where it diverges at $d = 3, 4, 5$ dimensions has been left for future study where we will also show how to deal with loop Witten diagrams, i.e. loop integrals with external legs going to spacelike infinity.

Acknowledgments

We thank the anonymous referee for his careful reading and his suggestions to improve the paper. U.M. gratefully thanks the Department of Theoretical Physics of CERN for hospitality and support while writing this paper.

A Remarks on the various spectral representations of dS and AdS propagators

In this section we would like to discuss some points on the ongoing literature on harmonic analysis on dS and AdS, which, in our opinion, need clarification.

In section 3 we have shown that *Legendre's associate functions of the second kind*⁵ have analyticity properties that are tailor-made for the geometry of the anti de Sitter manifold and its covering: the maximal analyticity property of the two-point functions (3.7) in the covering cut-plane $\tilde{\Delta}_1$ is a consequence of AdS invariance, local commutativity and the positivity of the spectrum of the energy operator M_{0d} ; points of the cut of $\tilde{\Delta}_1$ correspond to pair of timelike separated events on the real manifold \tilde{AdS}_d .

On the other hand *Legendre's functions of the first kind*⁶ have analyticity properties that are tailor-made for the geometry of the de Sitter manifold. A short summary will help in making clear what follows. The complex de Sitter universe may be represented as the one-sheeted hyperboloid

$$dS_d^{(c)} = \{z \in M_{d+1}^{(c)} : z \cdot z = -R^2 = -1\}, \tag{A.1}$$

⁵More precisely, Legendre's functions of the second kind multiplied by the factor $(\zeta^2 - 1)^{-\frac{d-2}{4}}$.

⁶See footnote 5.

immersed in a $(d + 1)$ -dimensional complex Minkowski spacetime. The same letter ζ used before here denotes the complex de Sitter invariant variable

$$\zeta = z_1 \cdot z_2 = z_1^0 z_2^0 - z_1^1 z_2^1 - \dots - z_1^d z_2^d, \tag{A.2}$$

i.e. the scalar product in the ambient spacetime sense of two complex events $z_1, z_2 \in dS_d^{(c)}$. Two real events x_1 and x_2 are timelike separated if and only if

$$(x_1 - x_2)^2 = -2 - 2x_1 \cdot x_2 > 0. \tag{A.3}$$

The canonically normalized (Bunch-Davis) maximally analytic two-point function of a de Sitter Klein-Gordon field is given by

$$W_\kappa^{(\text{dS})}(z_1, z_2) = w_\kappa^{(\text{dS})}(\zeta) = \frac{\Gamma\left(\frac{d-1}{2} + i\kappa\right) \Gamma\left(\frac{d-1}{2} - i\kappa\right)}{2(2\pi)^{d/2}} (\zeta^2 - 1)^{-\frac{d-2}{4}} P_{-\frac{1}{2}+i\kappa}^{-\frac{d-2}{2}}(\zeta) \tag{A.4}$$

$$= \frac{\Gamma\left(\frac{d-1}{2} + i\kappa\right) \Gamma\left(\frac{d-1}{2} - i\kappa\right)}{(4\pi)^{d/2} \Gamma\left(\frac{d}{2}\right)} {}_2F_1\left(\frac{d-1}{2} + i\kappa, \frac{d-1}{2} - i\kappa; \frac{d}{2}; \frac{1-\zeta}{2}\right) \tag{A.5}$$

where the (complex) mass squared is related to the parameter κ and the spacetime dimension d as follows (we set $R = 1$):

$$m^2 = \frac{(d-1)^2}{4} + \kappa^2. \tag{A.6}$$

The Legendre function [35] of the first kind $P_\beta^\alpha(\zeta)$ is holomorphic in the cut-plane Δ_2 — the same as $Q_\beta^\alpha(\zeta)$ — but the reduced two-point function $w_\kappa^{(\text{dS})}(\zeta)$ is holomorphic in the larger domain

$$\zeta = z_1 \cdot z_2 \in \Delta = \mathbf{C} \setminus (-\infty, -1], \tag{A.7}$$

i.e. everywhere except on the causality cut (A.3) cut (*maximal analyticity property*, [24]). The factor $(\zeta^2 - 1)^{-\frac{d-2}{4}}$ exactly compensates the singularity of $P_{-\frac{1}{2}+i\kappa}^{-\frac{d-2}{2}}(\zeta)$ at $\zeta = 1$ making the reduced two-point function regular there.

All the above being said and having specified the differences between the de Sitter and the anti-de Sitter two-point functions, let us focus now on the mathematical possibility to express the anti-de Sitter *reduced* two-point function (3.7) as a superposition of *reduced* de Sitter two-point functions (A.4) i.e. as a superposition of Legendre functions of the first kind.

The first thing we would like to recall here is that a Fourier-like representation of the two-point function (A.4) (equivalently (A.5)) in terms of *plane waves* is available since 1994 [23, 24]:

$$W_\nu^{(\text{dS})}(z_1, z_2) = \frac{\Gamma\left(\frac{d-1}{2} + i\kappa\right) \Gamma\left(\frac{d-1}{2} - i\kappa\right) e^{\pi\kappa}}{2^{d+1} \pi^d} \int_\gamma (\xi \cdot z_1)^{-\frac{d-1}{2}-i\kappa} (\xi \cdot z_2)^{-\frac{d-1}{2}+i\kappa} \alpha(\xi). \tag{A.8}$$

Here $\xi \in C^+ = \partial V^+ = \{\xi \in M_{d+1}, \xi^2 = 0, \xi^0 > 0\}$ is a forward light-like vector in the ambient spacetime; it can be also thought as a point of the boundary of the de Sitter manifold

at timelike infinity. A natural physical interpretation for such a ξ is that of the asymptotic direction of the momentum of a particle [39].

The plane waves $\psi_\lambda(z, \xi) = (\xi \cdot z)^\lambda$ composing the above formula solve individually the Klein-Gordon equation in the respective tuboids:

$$\square_{\text{dS}}\psi(z) + m^2\psi(z) = 0, \tag{A.9}$$

where

$$m^2 = -\lambda(\lambda + d - 1) = \frac{(d-1)^2}{4} + \kappa^2, \quad \lambda = -\frac{d-1}{2} + i\kappa, \tag{A.10}$$

(d is the spacetime dimension).

The complex de Sitter events z_1 and z_2 belong to the backward and, respectively, the forward de Sitter tuboids \mathcal{T}_- and \mathcal{T}_+ ; these are the intersections of the ambient tubes T_- and T_+ (which are related to the positivity of the energy momentum operator [33]) with the complex de Sitter manifold:

$$\mathcal{T}_\pm = \{x + iy \in X_d^{(c)} : x \cdot y = 0, y^2 > 0, \text{sign } y^0 = \pm\}. \tag{A.11}$$

In standard coordinates, the $(d-1)$ -form $\alpha(\xi)$ in eq. (A.8) is written

$$\alpha(\xi) = (\xi^0)^{-1} \sum_{j=1}^d (-1)^{j+1} \xi^j d\xi^1 \dots \widehat{d\xi^j} \dots d\xi^d, \tag{A.12}$$

γ denotes any $(d-1)$ -cycle in the forward light-cone C^+ ; the integrand in eq. (A.8) is a closed differential form and therefore the result does not depend on the choice of the manifold γ . In particular we may choose the unit spherical section of the cone \mathbf{S}_{d-1} (equipped with its canonical orientation):

$$\gamma_0 = \mathbf{S}_{d-1} = C^+ \cap \{\xi : \xi^0 = 1\} = \{\xi \in C^+ : \xi^{1^2} + \dots + \xi^{d^2} = 1\}. \tag{A.13}$$

With this choice $\alpha(\xi)$ coincides with the rotation invariant measure $d\xi$ on \mathbf{S}_{d-1} normalized as usual:

$$\omega_d = \int_{\gamma_0} \alpha(\xi) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{A.14}$$

Eq. (A.8) is very flexible and may be used to produce many distinct integral representation of the Legendre function of the first kind. One possible construction goes as follows: the tubes \mathcal{T}_- and \mathcal{T}_+ are bordered by two Riemannian (as opposed to Euclidean) manifolds \mathcal{H}_- and \mathcal{H}_+ whose points are purely imaginary:

$$\mathcal{H}_\pm = \{x + iy \in X_d^{(c)} : x = 0, y^2 = R^2 = 1, \text{sign } y^0 = \pm\}. \tag{A.15}$$

Both \mathcal{H}_- and \mathcal{H}_+ are equivalent to the d -dimensional Lobachevsky space H_d which is also equivalent to the Euclidean AdS manifold \mathbf{H}_d , see eq. (2.7).

We now choose the two points in eqs. (A.8) to be purely imaginary: $z_1 = -iY_1 \in \mathcal{H}_-$ and $z_2 = iY_2 \in \mathcal{H}_+$; we choose to integrate over the parabolic basis of the cone

$$\gamma_p = C^+ \cap \{\xi : \xi^0 + \xi^d = 1\}, \tag{A.16}$$

which we parametrize as follows:

$$\xi^0 = P^0 = \frac{1}{2} + x^2, \quad \xi^i = P^i = x^i, \quad \xi^d = P^d = \frac{1}{2} - x^2, \quad dP = dx \quad (\text{A.17})$$

and the result is⁷

$$\Omega_\kappa(Y_1, Y_2) = W_\kappa^{(\text{dS})}(-iY_1, iY_2) = N(\kappa) \int_\gamma (P \cdot Y_1)^{-\frac{d-1}{2}-i\kappa} (P \cdot Y_2)^{-\frac{d-1}{2}+i\kappa} dP \quad (\text{A.18})$$

(see [9, eq. (113)]).

Even though it is a stretch, we might now insist in identifying both \mathcal{H}_- and \mathcal{H}_+ with the same copy of a Euclidean AdS manifold; in the same vein we may also, identify the light-cone C^+ with the boundary of the Euclidean AdS manifold; with these premises $\Omega_\kappa(Y_1, Y_2)$ may be taken as a basis of harmonic functions (proportional to Legendre functions of the first kind, see eq. (A.4)) that can be used to develop the AdS propagators. While this may be a useful technical tool, as for instance in the calculations of Witten diagrams in [9], it is somehow an unnatural move as we try to explain now.

Expressing the dS two-point function in as a superposition of AdS correlators we already did in eq. (3.12) which we now rewrite in a more suggestive way:

$$w_{-\nu}^{\text{AdS}}(\zeta) - w_\nu^{\text{AdS}}(\zeta) = \sin \pi \nu w_{i\nu}^{\text{dS}}(\zeta). \quad (\text{A.19})$$

Expressing the scalar AdS propagator (which is a Legendre functions of the second kind) as a superposition of Legendre functions of the first kind is also simple; it can be done in a one-line calculation by taking a generalized Mehler-Fock inverse transform [35, eq. (3.15.8)]. In the two-dimensional case $d = 2$ the formula we are looking for is in books since a very long time [45, eq. (7.213)]:

$$Q_{-\frac{1}{2}+\nu}(u) = \int_0^\infty \kappa \text{th}(\pi\kappa) P_{-\frac{1}{2}+i\kappa}(u) (x^2 + \nu^2)^{-1} dx, \quad \text{Re } \kappa > 0, \quad u \in \Delta_2; \quad (\text{A.20})$$

the above identity holds in particular for real $u > 1$. It is however clear that something goes wrong for $-1 < u < 1$: the integrand is holomorphic for $u \in \Delta$ while the Legendre function at the l.h.s. is defined only for $u \in \Delta_2 \subset \Delta$; the cut of Q is longer and the integral ill-defined on the cut.

To find the extension of the above formula to general dimension d , let us take the Riemann-Liouville transform of both sides of eq. (A.20):

$$\begin{aligned} & \int_z^\infty (u-z)^{\mu-1} P_{-\frac{1}{2}+i\kappa}(u) du \\ &= \frac{1}{\pi} \text{ch}(\pi x) \Gamma(\mu) \Gamma\left(\frac{1}{2} - \mu - i\kappa\right) \Gamma\left(\frac{1}{2} - \mu + i\kappa\right) (z^2 - 1)^{\mu/2} P_{-\frac{1}{2}+i\kappa}^\mu(z), \end{aligned}$$

⁷In a portion of the recent literature concerning harmonic analysis on the anti-de Sitter manifold a formula like (A.18) is called a “split representation” (see e.g. [28] and reference therein). The name has been introduced to denote “bulk-to-bulk propagators” that are represented as integrals of products of two scalar “bulk-to-boundary propagators”. Our simple construction shows that “bulk-to-boundary propagators” are nothing but plane waves and a split representation is a special instance of the general analytic formula (A.8). This remains true also in the more complicated tensorial cases treated in [28]. We believe that our old denomination [24] “Fourier-like” or “plane-waves representation” makes more justice to eq. (A.8) and to any other split representation.

$$\begin{aligned} & \int_z^\infty (u-z)^{\mu-1} Q_{-\frac{1}{2}+\nu}(u) du = \\ & = \frac{1}{\pi} e^{-i\pi\mu} \cos(\pi\mu + \pi\nu) \Gamma(\mu) \Gamma\left(\frac{1}{2} - \mu - \nu\right) \Gamma\left(\frac{1}{2} - \mu + \nu\right) (z^2 - 1)^{\mu/2} Q_{-\frac{1}{2}+\nu}^\mu(z). \end{aligned} \tag{A.21}$$

The sought extension immediately follows:

$$Q_{-\frac{1}{2}+\nu}^\mu(z) = e^{i\pi\mu} \int_0^\infty \frac{\kappa \operatorname{sh}(\pi\kappa)}{(\kappa^2 + \nu^2)} \frac{\Gamma\left(\frac{1}{2} - \mu - i\kappa\right) \Gamma\left(\frac{1}{2} - \mu + i\kappa\right)}{\cos(\pi\mu + \pi\nu) \Gamma\left(\frac{1}{2} - \mu - \nu\right) \Gamma\left(\frac{1}{2} - \mu + \nu\right)} P_{-\frac{1}{2}+i\kappa}^\mu(z) d\kappa. \tag{A.22}$$

Taking into account the identity (8.25) we get our final formula generalizing (A.20):

$$e^{-i\pi\frac{d-2}{2}} Q_{-\frac{1}{2}+\nu}^{\frac{d-2}{2}}(z) = \int_0^\infty \frac{\kappa \operatorname{sh}(\pi\kappa)}{\pi(\kappa^2 + \nu^2)} \Gamma\left(\frac{d-1}{2} - i\kappa\right) \Gamma\left(\frac{d-1}{2} + i\kappa\right) P_{-\frac{1}{2}+i\kappa}^{\frac{d-2}{2}}(z) d\kappa. \tag{A.23}$$

Eqs. (3.7) and (A.4) allow to rewrite the above identity (A.23) in a suggestive way:

$$w_\nu^{(\text{AdS})}(\zeta) = \frac{2}{\pi} \int_0^\infty \frac{x \operatorname{sh}(\pi\kappa)}{(\kappa^2 + \nu^2)} w_\kappa^{(\text{dS})}(\zeta) d\kappa, \quad \zeta \in \Delta_2. \tag{A.24}$$

One should never forget that $w_\nu^{(\text{AdS})}(\zeta)$ and $w_\kappa^{(\text{dS})}(\zeta)$ have different domains of analyticity. Even the variable ζ has different meanings at the r.h.s. and at the l.h.s. In particular the locus of AdS coincident Euclidean points projects on $\zeta = 1$ while the coincident dS Euclidean points project on $\zeta = -1$. In these respective places the Schwinger functions have distributional singularities that in integer dimensions cause the appearance of a δ source at the r.h.s. of the Klein-Gordon equation.

Inserting eq. (A.18) into eq. (A.24) one gets the so-called “split representation” [28] of the AdS Schwinger function:

$$w_\kappa^{(\text{AdS})}(Y_1 \cdot Y_2) = \frac{2}{\pi} \int_0^\infty \frac{\kappa \operatorname{sh}(\pi\kappa)}{(\kappa^2 + \nu^2)} N(\kappa) \int_\gamma (P \cdot Y_1)^{-\frac{d-1}{2} - i\kappa} (P \cdot Y_2)^{-\frac{d-1}{2} + i\kappa} dP. \tag{A.25}$$

This representation has proven to be useful in computing Witten diagrams on the Euclidean anti de Sitter space [9]. However, *it cannot be extended to the real anti de Sitter manifold.*

Finally we may use eq. (3.12) and write

$$w_\nu^{(\text{AdS})}(\zeta) = \frac{i}{\pi} \int_0^\infty \frac{\kappa}{(\kappa^2 + \nu^2)} \left(w_{i\kappa}^{(\text{AdS})}(\zeta) - w_{-i\kappa}^{(\text{AdS})}(\zeta) \right) d\kappa. \tag{A.26}$$

But the important formula is eq. (A.23). The equivalent expressions (A.24) and (A.26) are just ornamental.

The idea of expanding the AdS time-ordered propagator in Legendre functions of the first kind has been introduced long ago by Düsedau and Freedman [47] as an intermediate step in computing the Källén-Lehmann expansion of a propagator. Even in the title of their paper, Düsedau and Freedman reserve the name of *spectral representation* to the Källén-Lehmann expansion, the spectral parameter being the mass of the two-point functions entering in there.

Their eq. (31) corresponds to our eq. (A.23) specialized to $d = 4$ and is described as a special instance of an integral transform of Gelfand and Graev [47].

The possibility of representing AdS propagators in terms of Legendre functions of the first kind has been recently rediscovered and exploited. In this recent literature this is no more intended as an intermediate step towards a true Källén-Lehmann representation of the propagator but as a “spectral representation” *per se*.

Eq. (A.24) is called in [9, appendix B.1] “the harmonic space representation of the bulk to bulk propagator”. Eq. (A.26) in [14, appendix B.2] is the “spectral representation” of the propagator. These denominations are, in our opinion, rather misleading because the Legendre functions of the first and second kind (obviously) solve the same Legendre differential equation (the delta terms arising only at coincident Euclidean points) and also because that representation has already a name: generalized inverse Mehler-Fock transformation (A.23).

The “spectral representation” is sometimes rather inconvenient: a key result of [14] consists in determining (through a quite involved bootstrapping and a certain amount of clever guesswork) the “spectral representation” of the bubble $w_\lambda(\zeta)w_\nu(\zeta)$ in the particular case of equal mass parameters $\lambda = \nu$.

It is indeed quite elementary to compute such “spectral representation” knowing the Källén-Lehmann representation (8.19) and the Mehler-Fock antitransform of the propagator (A.24); here we perform the calculation in the general case of two different masses λ and ν and for the maximally analytic functions which include as a special case the Schwinger function at non-coinciding points:

$$w_\lambda^{(\text{AdS})}(\zeta) w_\nu^{(\text{AdS})}(\zeta) = \frac{2}{\pi} \int_0^\infty \sum_{k=0}^\infty \frac{\rho(k; \lambda, \nu)}{\left(\kappa^2 + \left(\frac{d-1}{2} + 2k + \lambda + \nu\right)^2\right)} \kappa \text{sh}(\pi\kappa) w_\kappa^{(\text{dS})}(\zeta) d\kappa. \tag{A.27}$$

By taking the series first, the weight is written as a generalized hypergeometric series:

$$B_d(\lambda, \nu, \kappa) = \sum_{k=0}^\infty \frac{\rho(k; \lambda, \nu)}{\left(\kappa^2 + \left(2k + \frac{d-1}{2} + \lambda + \nu\right)^2\right)}. \tag{A.28}$$

We have already encountered this series when computing the watermelon in section 10 (see eq. (10.3)):

$$B_d(\lambda, \nu, x) = \frac{\pi^{-\delta} \Gamma(\delta + \lambda) \Gamma(\delta + \nu) \Gamma(\delta + \lambda + \nu + 1)}{2\Gamma(\lambda + 1) \Gamma(\nu + 1) \Gamma(2\delta + \lambda + \nu) ((\delta + \lambda + \nu)^2 + \kappa^2)} \times {}_9F_8 \left(\begin{matrix} \delta, \delta + \lambda, \delta + \nu, \delta + \lambda + \nu, \frac{1+\lambda+\nu}{2}, \frac{2+\lambda+\nu}{2}, \frac{2+\delta+\lambda+\nu}{2}, \frac{\delta-i\kappa+\lambda+\nu}{2}, \frac{\delta+i\kappa+\lambda+\nu}{2} \\ 1 + \lambda, 1 + \nu, 1 + \lambda + \nu, \frac{\delta+\lambda+\nu}{2}, \frac{2\delta+\lambda+\nu}{2}, \frac{1+2\delta+\lambda+\nu}{2}, \frac{2+\delta-i\kappa+\lambda+\nu}{2}, \frac{2+\delta+i\kappa+\lambda+\nu}{2} \end{matrix}; 1 \right). \tag{A.29}$$

In the simplest case where the two masses are equal the above formula reduces to

$$\begin{aligned}
 B_d(\nu, \nu, \kappa) &= \frac{\pi^{-\delta} \Gamma(\delta + \nu)^2 \Gamma(\delta + 2\nu + 1)}{2\Gamma(\nu + 1)^2 \Gamma(2(\delta + \nu)) ((\delta + 2\nu)^2 + \kappa^2)} \\
 &\times {}_7F_6 \left(\begin{matrix} \delta, \nu + \frac{1}{2}, \frac{\delta}{2} + \nu + 1, \frac{\delta}{2} + \nu - \frac{i\kappa}{2}, \frac{\delta}{2} + \nu + \frac{i\kappa}{2}, \delta + \nu, \delta + 2\nu \\ \nu + 1, \frac{\delta}{2} + \nu, \frac{\delta}{2} + \nu - \frac{i\kappa}{2} + 1, \frac{\delta}{2} + \nu + \frac{i\kappa}{2} + 1, \delta + \nu + \frac{1}{2}, 2\nu + 1 \end{matrix} ; 1 \right).
 \end{aligned}
 \tag{A.30}$$

By using formula (10.6) to simplify eq. (A.30) (with $a = \frac{\delta+2\nu-i\kappa}{2}$ and $b = \frac{\delta+2\nu+i\kappa}{2}$) we see that eq. (4.7) of [14] is an immediate consequence of the Källén-Lehmann formula.

The weight $B_d(\lambda, \nu, \kappa)$ of the “spectral representation” is proportional to $I_3^{(1)}(\lambda, \nu, i\kappa, d)$ which is one of the two terms of the 2-loop vacuum diagram studied in section 10: this explains why $B_d(\lambda, \nu, x)$ may diverge at $d = 3, 4 \dots$; actually, $B_d(\lambda, \nu, \kappa)$ does not diverge at $d = 3$ because the divergences in $I_3^{(1)}(\lambda, \nu, i\kappa, d)$ are milder than the divergences in $I_3^{(2)}(\lambda, \nu, i\kappa, d)$; the latter does diverge also at $d = 3$.

Whatever is the case, $B_d(\lambda, \nu, \kappa)$ is to be compared with the Källén-Lehmann weight (8.20), which is splendidly regular at $d = 3, 4 \dots$

In our opinion, this is already a sufficient reason to avoid the use of the so-called “spectral representation” and keep using the good old Källén-Lehmann representation.

Note also that in the “spectral representation” the bubble is expanded as an integral of Legendre functions of the first kind as opposed to the series of Legendre functions of the second kind exhibited at the r.h.s. of eq. (8.19). The quasi-periodical character of the bubble is now completely hidden in a weird representation.

From a mathematical viewpoint, the Mehler-Fock representation (A.23) of the Legendre functions may however be useful in other situations, as for the calculation of Euclidean Witten diagrams in [9]. For example, using the known integral of three Legendre functions of the first kind [27]

$$\begin{aligned}
 h_d(\lambda, \nu, \kappa) &= \int_1^\infty P_{-\frac{1}{2}+i\lambda}^{-\frac{d-2}{2}}(u) P_{-\frac{1}{2}+i\nu}^{-\frac{d-2}{2}}(u) P_{-\frac{1}{2}+i\kappa}^{-\frac{d-2}{2}}(u) (u^2 - 1)^{-\frac{d-2}{4}} du \\
 &= \frac{2^{\frac{d}{2}}}{(4\pi)^{\frac{3}{2}} \Gamma\left(\frac{d-1}{2}\right)} \frac{\prod_{\epsilon, \epsilon', \epsilon'' = \pm 1} \Gamma\left(\frac{d-1}{4} + \frac{i\epsilon\lambda + i\epsilon'\nu + i\epsilon''\kappa}{2}\right)}{\prod_{\epsilon, \epsilon', \epsilon'' = \pm 1} \Gamma\left(\frac{d-1}{2} + i\epsilon\lambda\right) \Gamma\left(\frac{d-1}{2} + i\epsilon'\nu\right) \Gamma\left(\frac{d-1}{2} + i\epsilon''\kappa\right)}
 \end{aligned}
 \tag{A.31}$$

we may compute the following related integral that may play a role in either dS or AdS two-loop calculations (details are omitted):

$$\begin{aligned}
 &\int_1^\infty Q_{-\frac{1}{2}+a}^{-\frac{d-2}{2}}(u) P_{-\frac{1}{2}+i\nu}^{-\frac{d-2}{2}}(u) P_{-\frac{1}{2}+i\kappa}^{-\frac{d-2}{2}}(u) (u^2 - 1)^{-\frac{d-2}{4}} du \\
 &= \frac{\pi^{3/2} G_{8,8}^{5,5} \left(1 \left| \begin{matrix} 1 - \frac{a}{2}, \frac{5-d}{4} + \frac{i\kappa - i\nu}{2}, \frac{5-d}{4} - \frac{i\kappa - i\nu}{2}, \frac{5-d}{4} + \frac{i\kappa + i\nu}{2}, \frac{5-d}{4} - \frac{i\kappa + i\nu}{2}, 0, \frac{1}{2}, 1 + \frac{a}{2} \\ \frac{a}{2}, \frac{d-1}{4} + \frac{i\kappa}{2} - \frac{i\nu}{2}, \frac{d-1}{4} - \frac{i\kappa}{2} + \frac{i\nu}{2}, \frac{d-1}{4} + \frac{i\kappa}{2} + \frac{i\nu}{2}, \frac{d-1}{4} - \frac{i\kappa}{2} - \frac{i\nu}{2}, \frac{1}{2}, 1, -\frac{a}{2} \end{matrix} \right)}{2^{2-\frac{d}{2}} e^{\frac{1}{2}id\pi} \cos\left(\pi a - \frac{\pi d}{2}\right) \Gamma\left(\frac{d-1}{2}\right) \prod_{\epsilon = \pm} \Gamma\left(\frac{d-1}{2} + i\epsilon\kappa\right) \Gamma\left(\frac{d-1}{2} + i\epsilon\nu\right) \Gamma\left(\frac{d-1}{2} + \epsilon a\right)}
 \end{aligned}
 \tag{A.32}$$

where G is Meijer's function. The limit $a \rightarrow i\lambda$ may be expressed as a combination of hypergeometric series ${}_7F_6$ which we do not reproduce here.

B Proof of theorem 8.3

Let $(\delta, \lambda, \nu) \in V$, $\zeta \in \widehat{\Delta}_1$, $k \in \mathbf{Z}_+$. Using (3.7) and [35, section 3.3, eq. (44), pp. 136–137], we get

$$S_k(\delta, \lambda, \nu, \zeta) = A_k(\delta, \lambda, \nu)B_k(\delta, \lambda, \nu, \zeta), \quad (\text{B.1})$$

$$A_k(\delta, \lambda, \nu) = \frac{\Gamma(\delta + k)}{2^{\delta+2}\pi^{2\delta}\Gamma(\delta)}(\delta + \lambda + \nu + 2k) \\ \times \frac{\Gamma(1 + \lambda + \nu + 2k)\Gamma(\delta + \lambda + k)\Gamma(\delta + \nu + k)\Gamma(\delta + \lambda + \nu + k)}{\Gamma(1 + k)\Gamma(1 + \lambda + k)\Gamma(1 + \nu + k)\Gamma(1 + \lambda + \nu + k)\Gamma(1 + \delta + \lambda + \nu + 2k)}, \quad (\text{B.2})$$

$$B_k(\delta, \lambda, \nu, \zeta) = (1 - \zeta^{-2})^{-\frac{\delta}{2}} \left[1 + (1 - \zeta^{-2})^{\frac{1}{2}}\right]^{-\delta - \lambda - \nu} \left[\zeta + (\zeta^2 - 1)^{\frac{1}{2}}\right]^{-2k} \\ \times F\left(\delta, 1 - \delta; \delta + 1 + \sigma; \frac{-\zeta + (\zeta^2 - 1)^{\frac{1}{2}}}{2(\zeta^2 - 1)^{\frac{1}{2}}}\right), \quad \sigma = \lambda + \nu + 2k. \quad (\text{B.3})$$

The conditions $(\delta, \lambda, \nu) \in V$, $\zeta \in \widehat{\Delta}_1$, ensure that S_k is well-defined for every $k \in \mathbf{Z}_+$. The main point of the proof is the convergence of $\sum_k S_k$. Once this is established the analytic functions on both sides of (8.30) are known to coincide on $U_0 \times \Delta_2$, where U_0 is defined by the conditions (8.18), hence they coincide wherever they are both defined.

For any κ , $\zeta \mapsto (1 - \zeta^{-2})^\kappa$ is even and analytic in $\widehat{\Delta}_1$. For $\zeta \in \Delta_1$, $(\zeta^2 - 1)^{\frac{1}{2}} = \zeta(1 - \zeta^{-2})^{\frac{1}{2}}$. Let

$$z = \frac{-\zeta + (\zeta^2 - 1)^{\frac{1}{2}}}{2(\zeta^2 - 1)^{\frac{1}{2}}} = \frac{-1 + (1 - \zeta^{-2})^{\frac{1}{2}}}{2(1 - \zeta^{-2})^{\frac{1}{2}}}. \quad (\text{B.4})$$

The map $\zeta \mapsto z$ is even and holomorphic in $\widehat{\Delta}_1$, and it maps $\widehat{\Delta}_1$ onto $\{u \in \mathbf{C} : \text{Re } u < \frac{1}{2}\}$.

For $\zeta \in \Delta_1$, we set

$$\zeta = \text{ch}(\omega), \quad \omega = \tau + i\theta, \quad \tau > 0, \quad |\theta| \leq \pi. \quad (\text{B.5})$$

Then

$$\left[\zeta + (\zeta^2 - 1)^{\frac{1}{2}}\right] = e^\omega, \quad (\text{B.6})$$

$$z = \frac{1}{1 - e^{2\omega}}, \quad |z| \leq \frac{1}{e^{2\tau} - 1}, \quad 1 + |z| \leq \frac{1}{1 - e^{-2\tau}}, \quad (\text{B.7})$$

$$(1 - \zeta^{-2})^{\frac{1}{2}} = \frac{\text{sh}(\omega)}{\text{ch}(\omega)}, \quad \text{sh}(\tau) \leq |\text{sh}(\omega)| \leq \text{ch}(\tau), \quad \text{sh}(\tau) \leq |\text{ch}(\omega)| \leq \text{ch}(\tau). \quad (\text{B.8})$$

In order to estimate the hypergeometric function appearing in (B.3), we use [35, section 2.3.2, pp. 76–77].⁸ This provides the following estimate. Let $a = \alpha + i\alpha'$, $b = \beta + i\beta'$, $c = \gamma + i\gamma'$,

⁸We use formula (11), p. 76 (as corrected in the Errata) in the case $n = 0$.

$w \in \mathbf{C}$, and suppose:

$$\beta \geq \frac{1}{2}, \quad \gamma - \beta \geq \frac{1}{2}, \quad |1 - w| > \varepsilon, \quad |\arg(1 - w)| < \pi(1 - \varepsilon) \quad \text{for some } \varepsilon > 0. \quad (\text{B.9})$$

Then

$$|F(a, b, ; c; w)| \leq 1 + \frac{|a||w|M(w)|\beta|\sqrt{\text{ch}(\pi\beta') \text{ch} \pi(\gamma' - \beta')}}{\gamma}, \quad (\text{B.10})$$

with

$$M(w) = \sup_{0 \leq u \leq 1} |(1 - uw)^{-a-1}|. \quad (\text{B.11})$$

To apply this to the hypergeometric function appearing in (B.3), we treat separately the cases $\text{Re } \delta \geq \frac{1}{2}$ and $\text{Re } \delta \leq \frac{1}{2}$.

(1) Case $\text{Re } \delta \geq \frac{1}{2}$. In this case we set $a = 1 - \delta$, $b = \delta$, $c = \delta + 1 + \sigma = \delta + 1 + \lambda + \nu + 2k$, and $w = z$ defined as above, and satisfying (B.7) as well as $\text{Re } z < \frac{1}{2}$. For sufficiently large k , the conditions (B.9) are satisfied and $\gamma > 1$. We have $-\alpha - 1 - i\alpha' = \text{Re } \delta - 2 + i \text{Im } \delta$ and

$$M(z) \leq \begin{cases} e^{2\pi|\text{Im } \delta|2^{-\text{Re } \delta}} & \text{if } \text{Re } \delta - 2 < 0, \\ e^{2\pi|\text{Im } \delta|(1 + |z|)^{\text{Re } \delta - 2}} & \text{if } \text{Re } \delta - 2 \geq 0. \end{cases} \quad (\text{B.12})$$

Hence, with (B.5) and (B.7),

$$\begin{aligned} |z|M(z) &\leq 4e^{2\pi|\text{Im } \delta|}(e^{2\tau} - 1)^{-1}(1 - e^{-2\tau})^{-(\text{Re } \delta - 2)_+}, \quad (\text{B.13}) \\ |B_k(\delta, \lambda, \nu, \zeta)| &\leq e^{2\pi|\text{Im } \delta|} \left(\frac{\text{ch}(\tau)}{\text{sh}(\tau)} \right)^{\text{Re } \delta} \\ &\times e^{2\pi|\text{Im}(\delta + \lambda + \nu)|} \left(\frac{e^\tau}{\text{sh}(\tau)} \right)^{-\text{Re}(\delta + \lambda + \nu)_-} \left(\frac{e^\tau}{\text{ch}(\tau)} \right)^{-\text{Re}(\delta + \lambda + \nu)_+} \\ &\times \left[1 + 4|1 - \delta|e^{2\pi|\text{Im } \delta|}(e^{2\tau} - 1)^{-1}(1 - e^{-2\tau})^{-(\text{Re } \delta - 2)_+} \right. \\ &\quad \left. \times \text{ch}(\pi \text{Im } \delta)^{\frac{1}{2}} \text{ch}(\pi \text{Im } \sigma)^{\frac{1}{2}} \right] \\ &\times e^{-2k\tau}. \quad (\text{B.14}) \end{aligned}$$

(2) Case $\delta \leq \frac{1}{2}$. In this case we set $a = \delta$, $b = 1 - \delta$, $c = \delta + 1 + \sigma = \delta + 1 + \lambda + \nu + 2k$, and $w = z$ defined as above. For sufficiently large k , the conditions (B.9) are satisfied and $\gamma > 1$. We have $-\alpha - 1 - i\alpha' = -\text{Re } \delta - 1 - i \text{Im } \delta$ and

$$\begin{aligned} |z|M(z) &\leq 4e^{2\pi|\text{Im } \delta|}(e^{2\tau} - 1)^{-1}(1 - e^{-2\tau})^{(\text{Re } \delta + 1)_-}, \quad (\text{B.15}) \\ |B_k(\delta, \lambda, \nu, \zeta)| &\leq e^{2\pi|\text{Im } \delta|} \left(\frac{\text{ch}(\tau)}{\text{sh}(\tau)} \right)^{|\text{Re } \delta|} \\ &\times e^{2\pi|\text{Im}(\delta + \lambda + \nu)|} \left(\frac{e^\tau}{\text{sh}(\tau)} \right)^{-\text{Re}(\delta + \lambda + \nu)_-} \left(\frac{e^\tau}{\text{ch}(\tau)} \right)^{-\text{Re}(\delta + \lambda + \nu)_+} \\ &\times \left[1 + 4|\delta|e^{2\pi|\text{Im } \delta|}(e^{2\tau} - 1)^{-1}(1 - e^{-2\tau})^{(\text{Re } \delta + 1)_-} \right. \\ &\quad \left. \times \text{ch}(\pi \text{Im } \delta)^{\frac{1}{2}} \text{ch}(\pi \text{Im}(\sigma + 2\delta))^{\frac{1}{2}} \right] \\ &\times e^{-2k\tau}. \quad (\text{B.16}) \end{aligned}$$

The estimates (B.14) and (B.16) hold when (δ, λ, ν) remains in a compact subset K of V intersected with $\{\operatorname{Re} \delta \geq \frac{1}{2}\}$ or $\{\operatorname{Re} \delta \leq \frac{1}{2}\}$ respectively, and for all $k \geq k_0(K)$. With the same restrictions on (δ, λ, ν) , we can estimate $A_k(\delta, \lambda, \nu)$ by the Erdélyi-Tricomi theorem ([48, eq. (5.11.13)], [49], [50, pp. 118 ff]) and we find that there is a $k_1(K) > k_0(K)$ such that, for every $k \geq k_1(K)$,

$$|S_k| \leq H \operatorname{ch}(\tau)^P \tau^{-Q} k^N e^{-2\tau k}, \tag{B.17}$$

where $H > 0$, $P > 0$, $Q > 0$ and $N > 0$ may depend on K (but not on τ). Therefore $\sum_k S_k$ (resp. $\sum_{k=k_1}^\infty S_k$) converges absolutely and uniformly, when all the parameters remain in $K \times \{\zeta = \operatorname{ch}(\tau + i\theta) : \varepsilon < \tau < \varepsilon^{-1}\}$, ($\varepsilon > 0$), to a function $h(\delta, \lambda, \nu, \zeta)$ (resp. $h_{k_1}(\delta, \lambda, \nu, \zeta)$) holomorphic in these variables. Since $\sum_{k=0}^L S_k$ is holomorphic in $\widehat{\Delta}_1$ it follows, by the maximum principle, that the convergence extends to $K \times \{\zeta = \operatorname{ch}(\tau + i\theta) : \varepsilon < \tau\} \cup \{\infty\}$, i.e. the functions h and h_{k_1} are holomorphic there.

For any function f holomorphic in Δ_1 , and $P > 0$, $Q > 0$, let

$$\|f\|_{P,Q} = \sup_{\zeta \in \Delta_2} e^{-P\tau} \tau^Q |f(\zeta)|, \quad \zeta = \operatorname{ch}(\tau + i\theta), \quad \tau > 0, \quad |\theta| < \pi. \tag{B.18}$$

It then follows from our estimates that if (δ, λ, ν) remains in the compact K , there exist constants $P > 0$, $Q > 0$, and $M > 0$ such that

$$\left\| \sum_{k=k_1}^L S_k \right\|_{P,Q} \leq M \quad \forall L > 0, \tag{B.19}$$

for any $\varepsilon > 0$ there is an $L > 0$ such that

$$\left\| \sum_{k=L}^\infty S_k \right\|_{P,Q} < \varepsilon, \tag{B.20}$$

$$\|h_{k_1}\|_{P,Q} \leq M. \tag{B.21}$$

(The functions appearing in the l.h.s. of these inequalities are considered as functions of $\zeta \in \Delta_2$ at fixed δ, λ, ν .) Restricting these estimates to the tuboid $\mathcal{Z}_{1-} \times \mathcal{Z}_{1+}$ it is possible to show that (8.30) remains true for the boundary values of both sides in the sense of tempered distributions. We omit the lengthy but straightforward details.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] J.M. Maldacena, *The AdS/CFT Correspondence*, in *Handbook of Quantum Gravity*, Springer, Singapore (2024) [DOI:10.1007/978-981-19-3079-9_65-1] [INSPIRE].
- [2] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200] [INSPIRE].

- [3] O. DeWolfe, *TASI Lectures on Applications of Gauge/Gravity Duality*, *PoS TASI2017* (2018) 014 [[arXiv:1802.08267](#)] [[INSPIRE](#)].
- [4] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [5] J. Babington, D.E. Crooks and N.J. Evans, *A Nonsupersymmetric deformation of the AdS/CFT correspondence*, *JHEP* **02** (2003) 024 [[hep-th/0207076](#)] [[INSPIRE](#)].
- [6] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rep.* **323** (2000) 183 [[hep-th/9905111](#)] [[INSPIRE](#)].
- [7] D. Poland, S. Rychkov and A. Vichi, *The Conformal Bootstrap: Theory, Numerical Techniques, and Applications*, *Rev. Mod. Phys.* **91** (2019) 015002 [[arXiv:1805.04405](#)] [[INSPIRE](#)].
- [8] C. Copetti, L. Cordova and S. Komatsu, *Non-Invertible Symmetries, Anomalies and Scattering Amplitudes*, [arXiv:2403.04835](#) [[INSPIRE](#)].
- [9] J. Penedones, *Writing CFT correlation functions as AdS scattering amplitudes*, *JHEP* **03** (2011) 025 [[arXiv:1011.1485](#)] [[INSPIRE](#)].
- [10] D. Carmi, *Loops in AdS: From the Spectral Representation to Position Space. Part III*, [arXiv:2402.02481](#) [[INSPIRE](#)].
- [11] D. Carmi, *Loops in AdS: from the spectral representation to position space. Part II*, *JHEP* **07** (2021) 186 [[arXiv:2104.10500](#)] [[INSPIRE](#)].
- [12] D. Carmi, *Loops in AdS: From the Spectral Representation to Position Space*, *JHEP* **06** (2020) 049 [[arXiv:1910.14340](#)] [[INSPIRE](#)].
- [13] J.M. Drummond and H. Paul, *Two-loop supergravity on $AdS_5 \times S^5$ from CFT*, *JHEP* **08** (2022) 275 [[arXiv:2204.01829](#)] [[INSPIRE](#)].
- [14] D. Carmi, L. Di Pietro and S. Komatsu, *A Study of Quantum Field Theories in AdS at Finite Coupling*, *JHEP* **01** (2019) 200 [[arXiv:1810.04185](#)] [[INSPIRE](#)].
- [15] E.T. Akhmedov, U. Moschella and F.K. Popov, *Ultraviolet phenomena in AdS self-interacting quantum field theory*, *JHEP* **03** (2018) 183 [[arXiv:1802.02955](#)] [[INSPIRE](#)].
- [16] I. Bertan and I. Sachs, *Loops in Anti-de Sitter Space*, *Phys. Rev. Lett.* **121** (2018) 101601 [[arXiv:1804.01880](#)] [[INSPIRE](#)].
- [17] S.L. Cacciatori, H. Epstein and U. Moschella, *Banana integrals in configuration space*, *Nucl. Phys. B* **995** (2023) 116343 [[arXiv:2304.00624](#)] [[INSPIRE](#)].
- [18] S.L. Cacciatori, H. Epstein and U. Moschella, *Loops in de Sitter space*, *JHEP* **07** (2024) 182 [[arXiv:2403.13145](#)] [[INSPIRE](#)].
- [19] R. Roiban, A. Tirziu and A.A. Tseytlin, *Two-loop world-sheet corrections in $AdS_5 \times S^5$ superstring*, *JHEP* **07** (2007) 056 [[arXiv:0704.3638](#)] [[INSPIRE](#)].
- [20] P. Sundin and L. Wulff, *One- and two-loop checks for the $AdS_3 \times S^3 \times T^4$ superstring with mixed flux*, *J. Phys. A* **48** (2015) 105402 [[arXiv:1411.4662](#)] [[INSPIRE](#)].
- [21] L.F. Alday, A. Bissi and X. Zhou, *One-loop gluon amplitudes in AdS*, *JHEP* **02** (2022) 105 [[arXiv:2110.09861](#)] [[INSPIRE](#)].
- [22] A. Herderschee, *A new framework for higher loop Witten diagrams*, *JHEP* **06** (2024) 008 [[arXiv:2112.08226](#)] [[INSPIRE](#)].
- [23] J. Bros, U. Moschella and J.P. Gazeau, *Quantum field theory in the de Sitter universe*, *Phys. Rev. Lett.* **73** (1994) 1746 [[INSPIRE](#)].

- [24] J. Bros and U. Moschella, *Two point functions and quantum fields in de Sitter universe*, *Rev. Math. Phys.* **8** (1996) 327 [[gr-qc/9511019](#)] [[INSPIRE](#)].
- [25] J. Bros, H. Epstein and U. Moschella, *Towards a general theory of quantized fields on the anti-de Sitter space-time*, *Commun. Math. Phys.* **231** (2002) 481 [[hep-th/0111255](#)] [[INSPIRE](#)].
- [26] J. Bros, *Complexified de Sitter space: Analytic causal kernels and Kallen-Lehmann type representation*, *Nucl. Phys. B Proc. Suppl.* **18** (1991) 22 [[INSPIRE](#)].
- [27] J. Bros, H. Epstein, M. Gaudin, U. Moschella and V. Pasquier, *Triangular invariants, three-point functions and particle stability on the de Sitter universe*, *Commun. Math. Phys.* **295** (2010) 261 [[arXiv:0901.4223](#)] [[INSPIRE](#)].
- [28] M.S. Costa, V. Gonçalves and J. Penedones, *Spinning AdS Propagators*, *JHEP* **09** (2014) 064 [[arXiv:1404.5625](#)] [[INSPIRE](#)].
- [29] J. Bros, H. Epstein, M. Gaudin, U. Moschella and V. Pasquier, *Anti de Sitter quantum field theory and a new class of hypergeometric identities*, *Commun. Math. Phys.* **309** (2012) 255 [[arXiv:1107.5161](#)] [[INSPIRE](#)].
- [30] A.L. Fitzpatrick and J. Kaplan, *Analyticity and the Holographic S-Matrix*, *JHEP* **10** (2012) 127 [[arXiv:1111.6972](#)] [[INSPIRE](#)].
- [31] C. Ford, I. Jack and D.R.T. Jones, *The Standard model effective potential at two loops*, *Nucl. Phys. B* **387** (1992) 373 [[hep-ph/0111190](#)] [[INSPIRE](#)].
- [32] M. Bertola, J. Bros, U. Moschella and R. Schaeffer, *A general construction of conformal field theories from scalar anti-de Sitter quantum field theories*, *Nucl. Phys. B* **587** (2000) 619 [[INSPIRE](#)].
- [33] R.F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*, Princeton University Press (2000).
- [34] S.J. Avis, C.J. Isham and D. Storey, *Quantum Field Theory in anti-de Sitter Space-Time*, *Phys. Rev. D* **18** (1978) 3565 [[INSPIRE](#)].
- [35] A. Erdélyi ed., *The Bateman project: Higher Transcendental Functions. Volume I*, McGraw-Hill, New York, NY, U.S.A. (1953).
- [36] G. Szegő, *Orthogonal Polynomials*, in *Colloquium Publications*, 4th edition, American Mathematical Society, Providence, RI, U.S.A. (1975).
- [37] C. Fronsdal, *Elementary particles in a curved space. Ii*, *Phys. Rev. D* **10** (1974) 589 [[INSPIRE](#)].
- [38] P. Breitenlohner and D.Z. Freedman, *Stability in Gauged Extended Supergravity*, *Ann. Phys.* **144** (1982) 249 [[INSPIRE](#)].
- [39] S. Cacciatori, V. Gorini, A. Kamenshchik and U. Moschella, *Conservation laws and scattering for de Sitter classical particles*, *Class. Quant. Grav.* **25** (2008) 075008 [[arXiv:0710.0315](#)] [[INSPIRE](#)].
- [40] N.F. Svaiter and C.A.D. Zarro, *A comment on Schwinger functions in Euclidean Rindler space*, *Class. Quant. Grav.* **25** (2008) 095008 [[INSPIRE](#)].
- [41] M.E.X. Guimaraes and B. Linet, *Selfinteraction and quantum effects near a point mass in three-dimensional gravitation*, *Class. Quant. Grav.* **10** (1993) 1665 [[INSPIRE](#)].
- [42] M.E.X. Guimaraes and B. Linet, *Scalar Green's functions in an Euclidean space with a conical-type line singularity*, *Commun. Math. Phys.* **165** (1994) 297 [[INSPIRE](#)].
- [43] B. Linet, *Euclidean scalar and spinor Green's functions in Rindler space*, [gr-qc/9505033](#) [[INSPIRE](#)].

- [44] C. García-Recio and L.L. Salcedo, *The perturbative scalar massless propagator in Schwarzschild spacetime*, *Class. Quant. Grav.* **30** (2013) 097001 [[arXiv:1303.6620](#)] [[INSPIRE](#)].
- [45] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press (1996).
- [46] P.L. Butzer, P.J.S.G. Ferreira, G. Schmeisser and R.L. Stens, *The Summation Formulae of Euler-Maclaurin, Abel-Plana, Poisson, and their Interconnections with the Approximate Sampling Formula of Signal Analysis*, *Results Math.* **59** (2011) 359.
- [47] D.W. Düsedau and D.Z. Freedman, *Lehmann Spectral Representation for Anti-de Sitter Quantum Field Theory*, *Phys. Rev. D* **33** (1986) 389 [[INSPIRE](#)].
- [48] National Institute of Standards and Technology, *NIST Digital Library of Mathematical Functions*, <https://dlmf.nist.gov/>.
- [49] F.G. Tricomi and A. Erdélyi, *The asymptotic expansion of a ratio of Gamma functions*, *Pacific J. Math.* **1** (1951) 133.
- [50] F.W.J. Olver, *Asymptotics and Special Functions*, A K Peters, Wellesley, MA, U.S.A. (1997).