ALMOST ALL SETS OF NONNEGATIVE INTEGERS AND THEIR SMALL PERTURBATIONS ARE NOT SUMSETS

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Abstract. Fix $\alpha \in (0, 1/3)$. We show that, from a topological point of view, almost all sets $A \subseteq \mathbb{N}$ have the property that, if $A' = A$ for all but $o(n^\alpha)$ elements, then A' is not a nontrivial sumset $B + C$. In particular, almost all A are totally irreducible. In addition, we prove that the measure analogue holds with $\alpha = 1$.

1. Introduction

A subset A of the nonnegative integers $\mathbb{N}$ is said to be *irreducible* if there do not exist $B, C \subseteq \mathbb{N}$ such that $|B|, |C| \geq 2$ and

$$A = B + C,$$

where $B + C := \{b + c : b \in B, c \in C\}$. In addition, we say that $A \subseteq \mathbb{N}$ is *totally irreducible* (or *totally primitive*) if there are no sets $B, C \subseteq \mathbb{N}$ such that $|B|, |C| \geq 2$ and

$$A =_\star B + C$$

meaning that the symmetric difference $A \triangle (B + C)$ is finite, cf. [4, 24]; equivalently, A and $B + C$ belong to the same equivalence class in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$, where Fin stands for the family of finite subsets of $\mathbb{N}$. An old conjecture of Ostmann [18, p. 13], which is still open, states that the set of primes is totally irreducible, cf. [7, 8] for partial results. See also [4, 19] for sumsets of symmetric sets of integers $S \subseteq \mathbb{Z}$ and references therein.

Let Σ and $\Sigma_\star$ be the family of irreducible and totally irreducible sets, respectively, so that $\Sigma_\star \subseteq \Sigma$. Also, identify the family of infinite sets $S \subseteq \mathbb{N}$ with the set of reals in $(0, 1]$ through their unique nonterminating dyadic expansions. Also, denote by $\lambda : \mathscr{M} \to \mathbb{R}$ the Lebesgue measure, where $\mathscr{M}$ stands for the completion of the Borel σ-algebra on $(0, 1]$. Accordingly, a theorem of Wirsing [24] states that almost all infinite sets are totally irreducible, in the measure theoretic sense:

Theorem 1.1. $\lambda(\Sigma_\star) = 1$. *In particular, also* $\lambda(\Sigma) = 1$.

Results on the same spirit have been studied by Sárközy [21, 22] for the Hausdorff dimension. Another related result has been recently obtained by Bienvenu and Geroldinger in [5, Theorem 6.2]. Here, once we identify $\mathcal{P}(\mathbb{N})$ with the Cantor space $\{0, 1\}^{\mathbb{N}}$, we show that the analogue of Theorem 1.1 holds in the category sense, so that almost all sets are totally primitive also topologically (recall that a set is said to be comeager if its complement is of the first Baire category, namely, its complement is contained in a countable union of closed sets with empty interior):

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Theorem 1.2. *$\Sigma_\star$ is a comeager subset of $\mathcal{P}(\mathbb{N})$. In particular, also Σ is comeager.*

Our Theorem 1.2 will be obtained as a consequence of a stronger result, which need some additional notation.

Definition 1.3. Given a family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ closed under finite unions and subsets, a set $A \subseteq \mathbb{N}$ is said *$\mathcal{I}$-irreducible* if there are no $B, C \subseteq \mathbb{N}$ such that $|B|, |C| \geq 2$ and

$$A =_{\mathcal{I}} B + C,$$

meaning that the symmetric difference $A \bigtriangleup (B + C) \in \mathcal{I}$. The family of $\mathcal{I}$-irreducible sets is denoted by $\Sigma(\mathcal{I})$.

Note that $\emptyset$-irreducible and Fin-irreducible are simply the classical irreducible and totally irreducible sets, respectively; similarly, also $\Sigma(\emptyset) = \Sigma$ and $\Sigma(\mathrm{Fin}) = \Sigma_\star$. It is clear that $\Sigma(\mathcal{I}) \subseteq \Sigma(\mathcal{J})$ whenever $\mathcal{J} \subseteq \mathcal{I}$. Given $\alpha \in (0, 1]$, define the family

$$\mathcal{Z}_\alpha := \left\{ S \subseteq \mathbb{N} : \lim_{n \to \infty} \frac{|S \cap [0, n]|}{n^\alpha} = 0 \right\}.$$

Hereafter, we will use the shorter notations $S(n) := S \cap [0, n]$ and $\mathcal{Z} := \mathcal{Z}_1$. Of course, $\mathcal{Z}$ is simply the zero set of the upper asymptotic density on $\mathbb{N}$, cf. [16]. Structural properties of the families $\mathcal{Z}_\alpha$ are studied, e.g., in [2, 11, 12]. With this notation, Erdős conjectured that the set $Q := \{n^2 : n \in \mathbb{N}\}$ of nonnegative squares belongs to $\Sigma(\mathcal{Z}_{1/2})$. Then, Sárközy and Szemerédi proved in [23] a slightly weaker statement, namely, $Q \in \Sigma(\mathcal{Z}_\alpha)$ for all $\alpha \in (0, \frac{1}{2})$.

With the same spirit of Erdős' conjecture, Sárközy and Szemerédi's result, and the claimed analogue stated in Theorem 1.2, we show that, from a topological point of view, almost all sets belong to $\mathcal{Z}_\alpha$, provided that α is sufficiently small:

Theorem 1.4. *$\Sigma(\mathcal{Z}_\alpha)$ is a comeager subset of $\mathcal{P}(\mathbb{N})$ for each $\alpha \in (0, \frac{1}{3})$.*

Observe that Theorem 1.2 is now immediate since $\Sigma(\mathcal{Z}_{1/4}) \subseteq \Sigma_\star$. Results in the same spirit, but completely different contexts, appeared, e.g., in [1, 3, 13, 14, 15].

In addition, we prove that the measure analogue Theorem 1.4 holds, hence providing a generalization of Wirsing's Theorem 1.1:

Theorem 1.5. $\lambda(\Sigma(\mathcal{Z})) = 1$.

It is remarkable that Sárközy and Szemerédi proved in [23] a general criterion for a set $S \subseteq \mathbb{N}$ and all its small perturbations to be totally irreducible. However, the hypotheses of such result do not seem to apply in our case for a proof of Theorem 1.5.

Before we proceed to the proofs of Theorem 1.4 and Theorem 1.5, some remarks are in order. First, suppose that the family $\mathcal{I}$ contains the finite sets Fin. Since $\{0, 1\} + \{0, 1\} = \{0, 1, 2\} =_{\mathcal{I}} A$ for all $A \in \mathcal{I}$ then $\mathcal{I} \cap \Sigma(\mathcal{I}) = \emptyset$. In particular, if $\mathcal{I}$ is a maximal ideal, that is, the complement of a free ultrafilter on $\mathbb{N}$, then $\mathcal{I}$ is not a meager subset of $\mathcal{P}(\mathbb{N})$, hence $\Sigma(\mathcal{I})$ is not comeager. However, the families $\mathcal{Z}_\alpha$ are $F_{\sigma\delta}$-subsets of $\mathcal{P}(\mathbb{N})$ for each $\alpha \in (0, 1]$, hence they are meager, cf. [2, Proposition 1.1].

Second, Sárközy proved in [20] that there exists a constant $c > 0$ such that, if $A \subseteq \mathbb{N}$ is infinite, then there exist a totally irreducible $B \in \Sigma_\star$ and $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \quad |(A \bigtriangleup B)(n)| \leq c \frac{|A(n)|}{\sqrt{\log \log |A(n)|}}.$$

In particular, since the function $t \mapsto t/\sqrt{\log\log t}$ is definitively increasing, it follows that $A =_{\mathcal{Z}} B$, therefore every equivalence class of $\mathcal{P}(\mathbb{N})/\mathcal{Z}$ contains a totally irreducible set.

Lastly, solving another conjecture of Erdős, it has been shown in [9, 17] that, if $A \subseteq \mathbb{N}$ has positive upper asymptotic density, i.e., $A \notin \mathcal{Z}$, then there exist two infinite sets $B, C \subseteq \mathbb{N}$ such that $B + C \subseteq A$. Hence, even in the optimistic case that one could prove the comeagerness of $\Sigma(\mathcal{Z})$, cf. Section 3, there is no hope to show that also the smaller family of sets $A \subseteq \mathbb{N}$ such that, if $A' =_{\mathcal{Z}} A$, then A' does not contain a nontrivial sumset is comeager. (Note that the same remark holds also in the measure sense: indeed, since almost all numbers are normal, then almost all subsets of $\mathbb{N}$ of them have asymptotic density $\frac{1}{2}$, hence almost all of them contains a sumset between two infinite sets.)

We conclude with an easy consequence of Theorem 1.2:

Corollary 1.6. *Fix two families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$ containing $\{0\}$. Then*

$$\mathcal{A} + \mathcal{B} := \{A + B : A \in \mathcal{A}, B \in \mathcal{B}\}$$

is meager if and only if both $\mathcal{A}$ and $\mathcal{B}$ are meager.

2. PROOFS

Proof of Theorem 1.4. Fix $\alpha \in (0, \frac{1}{3})$. We are going to use the Banach–Mazur game defined as follows, see [10, Theorem 8.33]: Players I and II choose alternatively nonempty open subsets of $\{0,1\}^{\mathbb{N}}$ as a nonincreasing chain

$$U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \cdots,$$

where Player I chooses the sets $U_0, U_1, \ldots$; Player II is declared to be the winner of the game if

$$\bigcap\nolimits_{m \geq 0} V_m \cap \Sigma(\mathcal{Z}_\alpha) \neq \emptyset. \tag{2.1}$$

Then Player II has a winning strategy (that is, he is always able to choose suitable sets $V_0, V_1, \ldots$ so that (2.1) holds at the end of the game) if and only if $\Sigma(\mathcal{Z}_\alpha)$ is a comeager set in the Cantor space $\{0,1\}^{\mathbb{N}}$. Since $\mathcal{P}(\mathbb{N})$ is identified with $\{0,1\}^{\mathbb{N}}$, a basic open set in $\mathcal{P}(\mathbb{N})$ will be a cylinder of the type $\{A \subseteq \mathbb{N} : A(k) = F\}$ for some integer $k \in \mathbb{N}$ and some (possibly empty) finite set $F \subseteq [0, k]$.

At this point, we define define the strategy of player II recursively as it follows. Suppose that the nonempty open sets $U_0 \supseteq V_0 \supseteq \cdots \supseteq U_m$ have been already chosen, for some $m \in \mathbb{N}$. Then there exists a finite set $F_m \in \mathrm{Fin}$ and an integer $k_m \geq \max(F_m \cup \{0\})$ such that

$$\forall A \subseteq \mathbb{N}, \quad A(k_m) = F_m \implies A \in U_m.$$

Without loss of generality we can assume that $|F_m| \geq 2$. Using the continuity of the map $\beta \mapsto \alpha\beta$ and recalling that $\alpha < \frac{1}{3}$, we can fix a real $\beta \in (\frac{3}{4}, 1)$ such that $\alpha\beta < \frac{1}{4}$.

Thus, define $t_m := \lfloor k_m^\beta \rfloor$ and

$$V_m := \left\{ A \subseteq \mathbb{N} : A(7k_m + t_m^2) = F_m \cup (k_m, 2k_m] \cup \bigcup_{i=1}^{t_m} \{5k_m + it_m\} \right\}.$$

In other words, a set $A \in U_m$ belongs to V_m if it contains the block of integers $[k_m + 1, 2k_m]$ and, then, it is followed by an arithmetic progression of t_m elements and distance t_m; note that each $A \in V_m$ ends with a further gap of lenght $2k_m$.

Hence, by construction, V_m is a nonempty open set contained in U_m. Finally, observe that there exists a unique set $A \subseteq \mathbb{N}$ such that

$$\{A\} = \bigcap\nolimits_{m \geq 0} V_m.$$

Indeed, since the sequence $(k_m)_{m \geq 0}$ is strictly increasing, the definition of the sets V_m gives us, in particular, all the finite truncations $A(k_m)$.

To complete the proof, we have to show that $A \in \Sigma(\mathcal{Z}_\alpha)$. For, let us suppose for the sake of contradiction that there exist $A', B, C \subseteq \mathbb{N}$ such that $|B|, |C| \geq 2$ and

$$A =_{\mathcal{Z}_\alpha} A' = B + C.$$

Let m be a sufficiently large integer with the property that $|(A \triangle A')(n)| \leq \frac{1}{2}n^\alpha$ for all $n \geq 2k_m$, which is possible since $A \triangle A' \in \mathcal{Z}_\alpha$ (further properties will be specified in the course of the proof recalling simply that "m is large"). Since $(k_m, 2k_m] \subseteq A$ by construction, then

$$\begin{aligned} |A'(2k_m)| &\geq \left| A\left(2k_m - \frac{1}{2}(2k_m)^\alpha\right)\right| \\ &\geq \left|(A \cap (k_m, 2k_m - \frac{k_m^\alpha}{2^{1-\alpha}}]\right| = \left\lfloor k_m \left(1 - \frac{1}{(2k_m)^{1-\alpha}}\right)\right\rfloor \geq \frac{k_m}{2}, \end{aligned}$$

where the last inequality holds since m is large. On the other hand, $A'(n)$ is contained in $B(n) + C(n)$ for all $n \in \mathbb{N}$, so that

$$|A'(2k_m)| \leq |B(2k_m)| \cdot |C(2k_m)|,$$

which implies that

$$\max\{|B(2k_m)|, |C(2k_m)|\} \geq \sqrt{\frac{k_m}{2}}.$$

Up to relabeling of the sets B and C, we can assume without loss of generality that $|B(2k_m)| \geq |C(2k_m)|$.

Now, observe that, since m is large,

$$|(A \triangle A')(7k_m + t_m^2)| \leq \frac{1}{2}(7k_m + t_m^2)^\alpha \leq t_m^{2\alpha}, \tag{2.2}$$

which is smaller than $\frac{1}{2}t_m$. This implies that there exists a subset $S_m \subseteq \{1, \ldots, t_m\}$ such that $|S_m| \geq \frac{1}{2}t_m$ and

$$\{5k_m + it_m : i \in S_m\} \subseteq A'.$$

Hence, for each $i \in S_m$ there exist $b_i \in B$ and $c_i \in C$ such that $5k_m + it_m = b_i + c_i$. Therefore

$$\forall i \in S_m, \quad \max\{b_i, c_i\} \geq \frac{5k_m + it_m}{2} > 2k_m.$$

At this point, let us suppose that there exists $i \in S_m$ such that $c_i > 2k_m$. It follows that

$$B(2k_m) + \{c_i\} \subseteq (B + C) \cap (2k_m, 7k_m + it_m] \subseteq A' \cap (2k_m, 7k_m + t_m^2],$$

so that, since m is large, we have

$$
\begin{aligned}
|(A \triangle A')(7k_m + t_m^2)| &\ge |B(2k_m)| - \frac{2k_m}{t_m} \\
&\ge \sqrt{\frac{k_m}{2}} - \sqrt{\frac{k_m}{8}} = \sqrt{\frac{k_m}{8}}.
\end{aligned}
$$

However, this contradicts (2.2): indeed, since $2\alpha\beta < \frac{1}{2}$ and m is large, we have

$$
|(A \triangle A')(7k_m + t_m^2)| \le t_m^{2\alpha} \le k_m^{2\alpha\beta} \le \sqrt{\frac{k_m}{16}}. \tag{2.3}
$$

This means that $c_i \le 2k_m < b_i$ for all $i \in S_m$. Let $i, j \in [1, t_m]$ such that $i-j \ge 4k_m^{1-\beta}$. Since m is large, then

$$
\begin{aligned}
b_i - b_j = (i-j)t_m - c_i + c_j &\ge (i-j)t_m - 2k_m \\
&\ge 4k_m^{1-\beta}t_m - 2k_m \ge 3k_m - 2k_m = k_m.
\end{aligned}
$$

Hence there exist integers $1 = i_1 < i_2 < \cdots < i_{q_m} \le t_m$ such that

$$
\forall j = 1, \ldots, q_m - 1, \quad i_{j+1} - i_j \ge 4k_m^{1-\beta} \quad \text{ and } \quad b_{i_{j+1}} - b_{i_j} \ge k_m
$$

and, since m is large,

$$
q_m \ge \frac{t_m}{5k_m^{1-\beta}} \ge \frac{1}{6}k_m^{2\beta-1}.
$$

Let us call $c' := \min C$ and $c'' := \min C \setminus \{c'\}$. Since m is large, we can assume that $t_m \ge 2c''$. It follows that

$$
b_{i_1} + c' < b_{i_1} + c'' < b_{i_2} + c' < b_{i_2} + c'' < \cdots < b_{i_{q_m}} + c' < b_{i_{q_m}} + c''.
$$

Therefore we have $2q_m$ distinct elements in $(B+C)\cap(2k_m, 7k_m+t_m^2) = A'\cap(2k_m, 7k_m+t_m^2)$ and at most half of them are nor equal to any $5k_m + ht_m$, $1 \le h \le t_m$. Indeed since $c'' - c' \le \frac{1}{2}t_m$, $b_{i_j} + c'$ and $b_{i_j} + c''$ cannot be together written as $5k_m + h't_m$ and $5k_m + h''t_m$, respectively. It follows that

$$
|(A \triangle A')(7k_m + t_m^2)| \ge q_m \ge \frac{1}{6}k_m^{2\beta-1}
$$

which contradicts again (2.3), since m is large and $2\beta - 1 > \frac{1}{2}$. □

Proof of Theorem 1.5. Hereafter, denote explicitly by $h : \mathrm{Fin}^+ \to (0,1]$ the bijection between the family $\mathrm{Fin}^+ := \mathcal{P}(\mathbb{N}) \setminus \mathrm{Fin}$ of all infinite subsets of $\mathbb{N}$ and the set of reals in $(0,1]$ through their unique nonterminating dyadic expansions. Also, let Ω be the set of normal numbers in $(0,1]$. It follows by Borel's normal number theorem that $\Omega \in \mathscr{M}$ and $\lambda(\Omega) = 1$, see e.g. [6, Theorem 1.2]. Observe that, if a set A belongs to $\widehat{\Omega} := h^{-1}[\Omega]$, then, by the definition of normal numbers,

$$
|\{j \in [0,n] : A(n + |F|) \cap (I + j) = F + j\}| = 2^{-|I|}n + o(n) \tag{2.4}
$$

as $n \to \infty$, for all nonempty finite sets $F \subseteq I$ such that $I \subseteq \mathbb{N}$ is an interval containing 0. Then the claim can be rewritten equivalently as

$$
\lambda\left(h\left[\widehat{\Omega} \setminus \Sigma(\mathcal{Z})\right]\right) = 0,
$$

and note that, by definition,

$$\widehat{\Omega} \setminus \Sigma(\mathcal{Z}) = \left\{A \in \widehat{\Omega} : \exists A', B, C \subseteq \mathbb{N}, A =_{\mathcal{Z}} A' = B + C \text{ and } |B|, |C| \geq 2\right\}.$$

First, we claim that, if $A \in \widehat{\Omega}$ and $A = B + C$ for some $B, C \subseteq \mathbb{N}$ with $|B|, |C| \geq 2$, then both B and C need to be infinite sets. Indeed, suppose for the sake of contradiction that B is a finite set and define $m := 1 + \max B$. Since $A \in \widehat{\Omega}$ there exists an integer $a \in A$ bigger than m such that $A \cap [a - m, a + m] = \{a\}$. However, since $a \in A$ there exist $b \in B$ and $c \in C$ such that $a = b + c$. Since $|B| \geq 2$ there exists $b' \in B \setminus \{b\}$. This is a contradiction because $a' := b' + c$ would be an integer in $A \cap [a - m, a + m]$ which is different from a. By symmetry, also C needs to be infinite.

Second, we claim that, if $A \in \widehat{\Omega}$ and $A = B + C$ for some infinite sets $B, C \subseteq \mathbb{N}$, then both B and C belong to $\mathcal{Z}$. For, let $(b_n : n \in \mathbb{N})$ be the increasing enumeration of the integers in B and define $I_k := [0, b_{k+1} - 1]$ for all $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and note that, if $c \in C(n)$ then $c + b_i \in (B + C)(n + b_{k+1}) = A(n + b_{k+1})$ for all $i \in [0, k]$ and all $n \in \mathbb{N}$. Letting $\mathscr{S}_k$ be the family $\{S \subseteq \mathbb{N} : \{b_0, b_1, \ldots, b_k\} \subseteq S \subseteq I_k\}$, we obtain

$$|C(n)| \leq \sum_{S \in \mathscr{S}_k} |\{j \in [0, n + b_{k+1}] : A(n + b_{k+1}) \cap (I_k + j) = S + j\}|$$

for all $n \in \mathbb{N}$. At this point, since $A \in \widehat{\Omega}$ and $|\mathscr{S}_k| = 2^{|I_k| - (k+1)}$, it follows by (2.4) that

$$|C(n)| \leq |\mathscr{S}_k| \cdot \left(2^{-|I_k|} n + o(n)\right) \leq 2^{-k} n$$

for all sufficiently large $n \in \mathbb{N}$. By the arbitrariness of $k \in \mathbb{N}$, we conclude that $C \in \mathcal{Z}$ and, by symmetry, $B \in \mathcal{Z}$ as well.

Third, note that, if $A \in \widehat{\Omega}$ and $A' =_{\mathcal{Z}} A$, then, by the definition of normal numbers, $A' \in \widehat{\Omega}$ as well. Putting everything together it follows the set $\widehat{\Omega} \setminus \Sigma(\mathcal{Z})$ can be rewritten equivalently as the family of all $A \in \widehat{\Omega}$ such that $A =_{\mathcal{Z}} A' = B + C$ for some $A' \in \widehat{\Omega}$ and some infinite $B, C \in \mathcal{Z}$. Therefore, by monotonicity, it is enough to check that $\lambda(h[\mathscr{A}]) = 0$, where

$$\mathscr{A} := \left\{A \subseteq \mathbb{N} : \exists A' \subseteq \mathbb{N}, \exists B, C \in \mathcal{Z} \cap \mathrm{Fin}^+,\ A =_{\mathcal{Z}} A' = B + C\right\}.$$

Let k be a sufficiently large integer that will be chosen later (it will be enough to set $k = 17$). Suppose that $A \in \mathscr{A}$ and pick $A' \subseteq \mathbb{N}$ and infinite sets $B, C \in \mathcal{Z}$ such that $A =_{\mathcal{Z}} A' = B + C$. Then there exists $n_0 = n_0(k) \in \mathbb{N}$ such that $A'(n) = (B(n) + C(n)) \cap [0, n]$ and

$$\max\{|(A \bigtriangleup A')(n)|, |B(n)|, |C(n)|\} \leq n/k$$

for all $n \geq n_0$. At this point, for each $n \in \mathbb{N}$, let $\mathcal{E}_n$ be the family of all $X \subseteq [0, n]$ such that $\max\{|X \bigtriangleup X'|, |Y|, |Z|\} \leq n/k$ and $X' = (Y + Z) \cap [0, n]$ for some $X', Y, Z \subseteq [0, n]$. Hence $A(n) \in \mathcal{E}_n$ for all $n \geq n_0$, which implies that

$$\mathscr{A} \subseteq \bigcap_{n \geq n_0} \mathcal{E}_n \subseteq \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \mathcal{E}_n \subseteq \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \mathcal{E}_n. \tag{2.5}$$

To conclude the proof, let us compute the probability $P(\mathcal{E}_n)$ of the event $\mathcal{E}_n$ with respect to the uniform probability measure P on $[0, n]$. Observe that both Y and Z can be chosen

in at most
$$w_{n,k} := \sum_{i=0}^{n/k} \binom{n+1}{i}$$
ways and, for each $X' := (Y+Z) \cap [0,n]$, the set X can be obtained with at most $w_{n,k}$ modifications. Hence X' can be chosen in at most $w_{n,k}^2$ possibilities and, for each such X', its modification X will be obtained in at most $w_{n,k}$ ways. Using Stirling's approximation, it follows that
$$\begin{aligned} P(\mathcal{E}_n) &\le \frac{1}{2^{n+1}} \cdot w_{n,k}^2 \cdot w_{n,k} \ll \frac{n^3}{2^n} \cdot \binom{n}{n/k}^3 \\ &\ll \frac{n^3}{2^n} \cdot \left(\frac{n^{n+\frac{1}{2}}}{(\frac{n}{k})^{\frac{n}{k}+\frac{1}{2}} \cdot (\frac{k-1}{k}n)^{\frac{k-1}{k}n+\frac{1}{2}}} \right)^3 = \frac{k^3}{(k-1)^{3/2}} \cdot \frac{n^2}{2^n} \cdot \alpha_k^{3n}, \end{aligned}$$
as $n \to \infty$, where
$$\alpha_k := k^{1/k} \cdot \left(\frac{k}{k-1} \right)^{(k-1)/k}.$$
Since $\alpha_t \to 1^+$ as $t \to \infty$, we can fix an integer $k \in \mathbb{N}$ for which $\alpha_k \in (1, 2^{1/3})$. Hence there exists $c \in (1/2, 1)$ for which
$$P(\mathcal{E}_n) \le c^n$$
for all sufficiently large n. Since $\sum_n P(\mathcal{E}_n) < \infty$, it follows by Borel–Cantelli lemma and inclusion (2.5) that $\lambda(h[\mathscr{A}]) = 0$, which concludes the proof. $\square$

Proof of Corollary 1.6. First, suppose that $\mathcal{A}$ is not meager (the case $\mathcal{B}$ not meager is analogous). Then $\mathcal{A}+\mathcal{B}$ contains $\mathcal{A}+\{\{0\}\} = \mathcal{A}$, so that $\mathcal{A}+\mathcal{B}$ is not meager.

Conversely, suppose that both $\mathcal{A}$ and $\mathcal{B}$ are meager, and note that
$$\mathcal{A}+\mathcal{B} \subseteq \bigcup\nolimits_{a \in \mathbb{N}: \{a\} \in \mathcal{A}} (\{a\}+\mathcal{B}) \cup \bigcup\nolimits_{b \in \mathbb{N}: \{b\} \in \mathcal{B}} (\mathcal{A}+\{b\}) \cup (\mathcal{P}(\mathbb{N}) \setminus \Sigma).$$
The claim follows by the fact that all sets $\{a\}+\mathcal{B}$ and $\mathcal{A}+\{b\}$ are meager, and that $\mathcal{P}(\mathbb{N}) \setminus \Sigma$ is meager as well by Theorem 1.2. $\square$

3. Concluding Remarks and Open Questions

In the same spirit of [5, Section 6], the statement of Theorem 1.4 holds also replacing $\mathbb{N}$ with a numerical submonoid of $\mathbb{N}$, that is, a pair $(M,+)$ where M is a cofinite subset of $\mathbb{N}$ closed under sum. Indeed, the very same proof holds substituting the definition of $k_0 = \max F_0$ with $k_0 = \max(F_0 \cup M^c)$.

We leave as an open question for the interested reader to check whether Theorem 1.4 can be strenghtened to prove the comeagerness of $\Sigma(\mathcal{Z}_{1/2})$, on the same lines of Erdős' conjecture, or even of the smaller subset $\Sigma(\mathcal{Z})$, in analogy with Theorem 1.5.

Lastly, we conclude with an evocative question: is it true that every (set identified with a) normal number is not a nontrivial sumset? With the notation of the proof of Theorem 1.5, this amounts to ask whether the inclusion $\widehat{\Omega} \subseteq \Sigma$ holds.

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