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Positive solutions to the planar logarithmic Choquard equation with exponential nonlinearity ☆

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ABSTRACT

In this paper we study the following nonlinear Choquard equation

$$-\Delta u+u=\left(\ln\frac{1}{|x|}*F(u)\right)f(u),\text{in }\mathbb{R}^2,$$

where $f\in C^1(\mathbb{R},\mathbb{R})$ and F is the primitive of the nonlinearity f vanishing at zero. We use an asymptotic approximation approach to establish the existence of positive solutions to the above problem in the standard Sobolev space $H^1(\mathbb{R}^2)$. We give a new proof and at the same time extend part of the results established in (Cassani-Tarsi, Calc.Var.PDE, 2021) [11].

1. Introduction

Let us consider the following class of nonlocal equations

$$-\Delta u+u=\left(\ln\frac{1}{|x|}*F(u)\right)f(u),\quad \text{in }\mathbb{R}^2, \tag{1.1}$$

where $F\in C^1(\mathbb{R},\mathbb{R})$ is the primitive function of the nonlinearity f vanishing at zero. This two-dimensional problem has remained open for a long time because of the sign-changing nature of the Coulomb interaction, given by the convolution, for which variational methods do not straightforward apply.

Indeed, on the one hand, Eq. (1.1) has a variational structure, in the sense that at least formally, it turns out to be the Euler–Lagrange equation related to the energy functional

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u|^2+u^2\right)\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(|x-y|)F(u(y))F(u(x))\mathrm{d}x\mathrm{d}y. \tag{1.2}$$

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On the other hand, the second term in (1.2) is not well-defined on the natural Sobolev space $H^1(\mathbb{R}^2)$. A few attempts to overcome this difficulty have been done, in particular for power-like nonlinearities, see [13] and references therein, where the finiteness of such logarithmic convolution term is required as additional condition, which settles the problem in a proper subspace of $H^1(\mathbb{R}^2)$. However, it is well known how the smaller the space the more difficult is to get energy estimates in order to prove existence results. Recently, in [11] a different approach has been developed by means of a new weighted version of the Trudinger–Moser inequality, for which the problem is well defined in a log-weighted Sobolev space where variational methods can be applied up to cover the maximal possible nonlinear growth, which in dimension two it is of exponential type. See also [6] for further extensions in this direction.

Let us make the following assumptions on the nonlinearity $f \in C^1(\mathbb{R})$:

(f_1) $f(t) \geq 0$ for all $t \geq 0$ and $f(t) \equiv 0$ for all $t \leq 0$. $f(t) \leq Ct^p e^{4\pi t^2}$ as $t \to +\infty$ for some $p > 0$ and $f(t) = o(t)$ as $t \to 0^+$.
(f_2) there exist $C > 1 > \tau > 0$ such that $\tau \leq \frac{F(t)f'(t)}{f^2(t)} \leq C$ for any $t > 0$;
(f_3) $\lim_{t\to+\infty} \frac{F(t)f'(t)}{f^2(t)} = 1$ or equivalently $\lim_{t\to+\infty} \frac{d}{dt}\frac{F(t)}{f(t)} = 0$;
(f_4) there exist $\beta > 0$ and $0 < \rho < 1/4$ such that

$$\lim_{t\to+\infty} \frac{tF(t)}{e^{4\pi t^2}} \geq \beta > \frac{1}{\rho^2\sqrt{\ln 2}\cdot \pi^{\frac{3}{2}}}.$$

Notice that from (f_3), we deduce that there exist $\tau_0 \in (0,1)$ and $t_0 > 0$ such that $\frac{F(t)f'(t)}{f^2(t)} \geq \tau_0$ for any $t \geq t_0$, which in turn implies that

$$\int_{t_0}^{t} \frac{f'(s)}{f(s)}\mathrm{d}s \geq \tau_0 \int_{t_0}^{t} \frac{f(s)}{F(s)}\mathrm{d}s.$$

So, from (f_1) we have that there exist $M_0 > 0$ and $t_0 > 0$ such that

$$F(t) \leq M_0 f(t), \quad t \geq t_0. \tag{1.3}$$

Condition (f_2) implies

$$F(t) \leq (1-\tau)f(t)t, \quad t \geq 0, \tag{1.4}$$

which can be seen in formula (1.5) of [11]. Condition (f_1) is the usual exponential growth assumption as prescribed by the Trudinger–Moser inequality, which gives the following

$$0 \leq F(t) \leq C \cdot \begin{cases} t^2, & t \leq \bar{t}, \\ t^{p-1}e^{4\pi t^2}, & t \geq \tilde{t} \end{cases} \tag{1.5}$$

for some $\tilde{t} > \bar{t} > 0$. Assumptions (f_2) and (f_3) have been introduced in [11] and turn out to be the key ingredients in order to prove boundedness of Palais–Smale sequences. In fact, the difficulty here to obtain the boundedness of Palais–Smale sequences is due to the presence of a non-homogeneous nonlinearity f for which the Ambrosetti–Rabinowitz condition fails to work. Assumption (f_4) is in the spirit of de Figueiredo–Miyagaki–Ruf condition [15] which in dimension two turns out to be the key ingredient in order to have compactness. (It is somehow the equivalent of the Brezis–Nirenberg condition [5] for the mountain pass level $c < \frac{1}{N}S^{N/2}$ in higher dimensions $N \geq 3$, where S is the optimal constant for the critical Sobolev embedding $H^1 \hookrightarrow L^{2^*}$.)

Remark 1.1. For the sake of clarity, let us make some explicit examples of functions F in [11] satisfying our set of assumptions, for instance, $F_i(t) \in C^2(\mathbb{R})$ satisfying

$$F_1(t) = \begin{cases} 0, & t \leq 0, \\ t^q, & 0 \leq t \leq t_0, \\ a(t), & t_0 \leq t \leq t_1, \\ e^{4\pi t^2} - 1, & t \geq t_1, \end{cases} \qquad F_2(t) = \begin{cases} 0, & t \leq 0, \\ t^q e^{4\pi t^2}, & t \geq 0, \end{cases}$$

where $q > 2$, $t_1 > t_0$ and $a(t)$ is a positive function such that $F_1 \in C^2(\mathbb{R})$. Now, let us further consider the functions

$$F_3(t) = \begin{cases} 0, & t \leq 0, \\ \frac{t^2}{\ln\frac{1}{\ln(1+t)}}, & 0 \leq t \leq t_0, \\ b(t), & t_0 \leq t \leq t_1, \\ t^p e^{4\pi t^2}, & t \geq t_1, \end{cases} \qquad F_4(t) = \begin{cases} 0, & t \leq 0, \\ \frac{t^2}{\ln(1+\frac{1}{t})}e^{4\pi t^2}, & t \geq 0, \end{cases}$$

where $p > 0$, $t_1 > e - 1 > t_0$ and $b(t)$ is a positive function such that $F_3 \in C^2(\mathbb{R})$. Observe that F_3 and F_4 stand for a class of nonlinearities that fit our assumptions but which are not covered in [11]. In this respect, the approach developed here is an improvement of [11] by allowing for more general nonlinearities. Indeed, it is easy to verify that F_3 satisfies (f_1)–(f_4). So, let us only check F_4. A direct computation yields

$$f_4(t) = \begin{cases} 0, & t \le 0, \\ \left[2t\left(1+4\pi t^2\right)a + \left(\frac{t}{1+t}\right)a^2\right]e^{4\pi t^2}, & t \ge 0, \end{cases}$$

and

$$f_4'(t) = \begin{cases} 0, & t \le 0, \\ \left\{2\left[1+8\pi t^2 + \left(1+4\pi t^2\right)(8\pi t^2 + \frac{1}{t+1})\right]a + \left[\frac{1}{(1+t)^2} + \frac{8\pi t^2}{1+t}\right]a^2 + \frac{2}{(1+t)^2}a^3\right\}e^{4\pi t^2}, & t \ge 0, \end{cases}$$

where $a = \frac{1}{\ln(1+\frac{1}{t})}$. Moreover,

$$\lim_{t\to+\infty} \frac{F(t)f'(t)}{f^2(t)} = \frac{\left\{2\left[1+8\pi t^2 + \left(1+4\pi t^2\right)(8\pi t^2 + \frac{1}{t+1})\right] + \left[\frac{1}{(1+t)^2} + \frac{8\pi t^2}{1+t}\right]a + \frac{2}{(1+t)^2}a^2\right\}}{\left[2\left(1+4\pi t^2\right) + \left(\frac{1}{1+t}\right)a\right]^2} = 1,$$

and

$$\lim_{t\to 0^+} \frac{F(t)f'(t)}{f^2(t)} = \frac{\left\{2\left[1+8\pi t^2 + \left(1+4\pi t^2\right)(8\pi t^2 + \frac{1}{t+1})\right] + \left[\frac{1}{(1+t)^2} + \frac{8\pi t^2}{1+t}\right]a + \frac{2}{(1+t)^2}a^2\right\}}{\left[2\left(1+4\pi t^2\right) + \left(\frac{1}{1+t}\right)a\right]^2} = 1,$$

thus F_4 satisfies (f_2)–(f_3), whereas (f_1) and (f_4) clearly hold.

Our main result is the following:

Theorem 1.2. *Suppose the nonlinearity f satisfies (f_1)–(f_4). Then, problem (1.1) possesses a positive solution $u \in H^1(\mathbb{R}^2)$.*

We observe that as a consequence of [14], the solution obtained in Theorem 1.2 is actually radially symmetric, up to translations, and strictly decreasing.

From the point of view of applications, Eq. (1.1) boasts a longstanding presence in quite different nonlinear contexts, from the early studies on Polarons by Fröhlich in the '30 s, to more recent applications to quantum gravity and plasma physics; see [23] and references therein. The mathematical success, initiated by Lieb in the 70 s [18] (let us mention that in the last two years more than 200 papers have been devoted to this topic) is due to the richness of the framework with challenges which range form Functional Analysis, to local and nonlocal systems of PDEs and Potential Theory, see [1,6,8,12,14,22] and references therein. Let us emphasize that a major difficulty in studying the so-called limiting case (1.1), is the lack of a proper function space setting. The approach developed in [11] in dimension two and the extension to any dimension in [6], consists of introducing a logarithmic weight on the mass part of the Sobolev norm, namely

$$\|u\|_V^2 := \|u\|_{H^1}^2 + \left(\int_{\mathbb{R}^2} |u|^q \log(1+|x|)\mathrm{d}x\right)^{\frac{2}{q}}, \quad q > 2, \tag{1.6}$$

and completing smooth compactly supported functions with respect to this norm. This, together with a suitable version of the Trudinger–Moser inequality yields a weighted Sobolev space in which the energy functional is well-defined and of class C^1. This opens the way to variational methods which provide mountain pass type solutions to (1.1) which have finite energy in terms of (1.6). However, observe that the norm (1.6) is not invariant under translations whereas the energy functional I does. Moreover, the quadratic part of the energy functional is never coercive in that context. Because of the above reasons, the authors in [6,11] can consider nonlinearities f whose asymptotic behavior near the origin is given by s^q, with $q > 1$, which plays a crucial role in proving the boundedness of PS sequences. A major purpose of this paper is to remove this restriction and allow to consider the wider range of super-linear nonlinearities, that is $f(t) = o(t)$, as $t \to 0$. Here we exploit an asymptotic approximation procedure developed in [20,21] to set problem (1.1) in the standard Sobolev space $H^1(\mathbb{R}^2)$. This approximation approach was also used earlier in [25], where the authors studied some scalar field equations with logarithmic nonlinearities. Let us mention also [2] for further penalization methods in the case of power-like nonlinearities. Here, in order to overcome the difficulty of the sign-changing Newtonian kernel, we modify Eq. (1.1) as follows

$$-\Delta u + u - \left[G_\alpha(x) * F(u)\right]f(u) = 0, \quad \text{in } \mathbb{R}^2, \tag{1.7}$$

where $\alpha \in (0,1)$ and

$$\lim_{\alpha\to 0^+} G_\alpha(x) := \lim_{\alpha\to 0^+} \frac{|x|^{-\alpha}-1}{\alpha} = -\ln|x|$$

for $x \in \mathbb{R}^2 \setminus \{0\}$. The corresponding energy functional to (1.7) is well defined in $H^1(\mathbb{R}^2)$ for fixed $\alpha \in (0,1)$, which enables us to use minimax methods to establish the existence of positive solutions for (1.7).

By passing to the limit as $\alpha \to 0^+$, a convergence argument within $H^1(\mathbb{R}^2)$ allows us to prove that the limit solution turns out to be a positive solution to the original problem (1.1). Here the main difficulty is the balance between the too loose sign-changing logarithm kernel and the exponential growth rate of a fairly general nonlinearity. Some uniform bounds with respect to the asymptotic approximation parameter α turn out to be the key in order to get compactness and then existence of a finite energy (in the H^1 sense) solution. In a nutshell, the advantage of this method is that one can deal with the approximating functionals in the natural space $H^1(\mathbb{R}^2)$ simply by using the Hardy–Littlewood–Sobolev inequality. The price to pay is that one needs estimates for the family of approximated positive solutions of (1.7), which are uniform with respect to the parameter α. We are confident that the methods developed here might reveal useful also in other contexts. Recently in [9] the method has been applied to study a class of planar Schrödinger-Poisson systems in the fractional Sobolev limiting case, see also [24] for extensions to the zero-mass case.

This paper is organized as follows: In Section 2, we give some preliminaries which will be useful in the sequel. In Section 3, we prove the compactness of Cerami's sequences for the modified energy functional at the mountain pass level. In Section 4, we obtain a positive solution to the original equation by passing to the limit as $\alpha \to 0^+$, which in turn proves the main Theorem 1.2.

2. Preliminaries

For $1 \le s \le +\infty$, we denote by $\|\cdot\|_s$ the usual norm of the Lebesgue space $L^s(\mathbb{R}^2)$ as well as $H^1(\mathbb{R}^2) := \{u \in L^2(\mathbb{R}^2) : \nabla u \in L^2(\mathbb{R}^2)\}$ is the usual Sobolev space endowed with the norm $\|u\| := \left(\int_{\mathbb{R}^2}(|\nabla u|^2 + u^2)\mathrm{d}x\right)^{\frac{1}{2}}$. In the sequel, when ti is not misleading, constants C may change value from line to line.

Let us next recall some basic facts starting with the Hardy–Littlewood–Sobolev inequality, see for instance [3], which will be frequently used throughout this paper.

Lemma 2.1 (*Hardy–Littlewood–Sobolev inequality*). *Let $s, r > 1$ and $\alpha \in (0, N)$ with $1/s + \alpha/N + 1/r = 2$, $f \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C_{s,N,\alpha,r}$ independent of f, h such that*

$$\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\alpha} * f(x)\right]h(x)\mathrm{d}x \le C_{s,N,\alpha,r}\|f\|_s\|h\|_r.$$

If $t = s = \frac{2N}{2N-\alpha}$, then

$$C_{s,N,\alpha,r} = C_{N,\alpha} = \pi^{\alpha/2}\frac{\Gamma(\frac{N}{2}-\frac{\alpha}{2})}{\Gamma(N-\frac{\alpha}{2})}\left\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right\}^{-1+\frac{\alpha}{N}},$$

and if $N = 2, \alpha \in (0, 1]$, then $C_{2,\alpha} \le 2\sqrt{\pi}$.

As aforementioned, in dimension two the maximal degree of summability for functions with membership in $H^1(\mathbb{R}^2)$ is of exponential type, for which we recall the first result available in the whole plane due to [7]: if $\theta > 0$ and $u \in H^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2}\left[e^{\theta|u|^2} - 1\right] < \infty, \tag{2.1}$$

and

$$\sup_{\|\nabla u\|_2\le 1,\ \|u\|_2\le M}\int_{\mathbb{R}^2}\left(e^{\alpha|u|^2} - 1\right)\mathrm{d}x \le C(\alpha)\|u\|_2 < \infty,\ \text{if } \alpha < 4\pi, 0 < M < \infty, \tag{2.2}$$

where the positive constant M is independent of $u \in H^1(\mathbb{R}^2)$. (See also [10] for a wide context on this topic.)

Lemma 2.2. *Assume that $u_n \to u$ in $H^1(\mathbb{R}^2)$, then there exists $C > 0$ independent of n such that*

$$\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|x-y|^\nu}\mathrm{d}y \le C$$

uniformly for $x \in \mathbb{R}^2$, where $\nu \in (0, 1)$.

Proof. By Hölder's inequality, (f_1) and (f_2), one has

$$\int_{\mathbb{R}^2}F(u_n(y))\mathrm{d}y < +\infty$$

uniformly for n, since $u_n \to u$ in $H^1(\mathbb{R}^2)$ (see [16]). It then follows from (f_1), (f_2) and (2.2) that

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{F(u_n(y))}{|x-y|^\nu}\mathrm{d}y &= \int_{|x-y|\le 1}\frac{F(u_n(y))}{|x-y|^\nu}\mathrm{d}y + \int_{|x-y|\ge 1}\frac{F(u_n(y))}{|x-y|^\nu}\mathrm{d}y\\ &\le C + \left(\int_{|x-y|\le 1}\frac{1}{|x-y|^{2\nu}}\mathrm{d}y\right)^{1/2}\cdot\left(\int_{\mathbb{R}^2}F^2(u_n(y))\mathrm{d}y\right)^{1/2}\\ &\le C + C\left(\int_{\mathbb{R}^2}|u_n|^4 + (e^{9\pi|u_n|^2} - 1)\mathrm{d}y\right)^{1/2}\\ &\le C,\end{aligned} \tag{2.3}$$

where the last inequality follows from the fact that $u_n \to u$ in $H^1(\mathbb{R}^2)$. □

The proof of the following Lemma is standard and we omit it.

Lemma 2.3. *For any $\alpha \in (0,1]$, there exists $C_\beta > 0$ such that*

$$\frac{s^{-\alpha}-1}{\alpha} \le C_\beta s^{-\beta}$$

holds for all $\beta \in (\alpha, +\infty)$ and $s > 0$.

3. The asymptotic approximation method

3.1. The modified equation

Recently in [14], the authors used the method of moving planes to prove that positive solutions of (1.1) are radially symmetric. Based on this fact, it is natural to restrict ourself to the space $H_r^1(\mathbb{R}^2)$ of radially symmetric functions.

Set $G_\alpha(x) = \frac{|x|^{-\alpha}-1}{\alpha}$, $\alpha \in (0,1)$, $x \in \mathbb{R}^2$ and consider the following equation

$$-\Delta u + u = \left[G_\alpha(x) * F(u)\right] f(u) \tag{3.1}$$

which has the associated energy functional

$$I_\alpha(u) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u|^2 + u^2\right)\mathrm{d}x + \frac{1}{2\alpha}\left[\int_{\mathbb{R}^2} F(u)\mathrm{d}x\right]^2 - \frac{1}{2\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u)\right]F(u)\mathrm{d}x.$$

3.2. Regularity of the modified energy functional

According to the definition of G_α, by means of the Hardy–Littlewood–Sobolev inequality, let us prove the following

Lemma 3.1. *Let $0 < \alpha < 1$, then $I_\alpha \in C^1(H_r^1(\mathbb{R}^2), \mathbb{R})$ and*

$$I'_\alpha(u)v = \int_{\mathbb{R}^2}\left(\nabla u \nabla v + uv\right)\mathrm{d}x + \frac{1}{\alpha}\int_{\mathbb{R}^2} F(u)\mathrm{d}x \cdot \int_{\mathbb{R}^2} f(u)v\mathrm{d}x - \frac{1}{\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u)\right] f(u)v\mathrm{d}x$$

for any $u, v \in H_r^1(\mathbb{R}^2)$.

Proof. It is standard to check that for fixed $0 < \alpha < 1$, one has $I_\alpha \in C(H_r^1(\mathbb{R}^2), \mathbb{R})$. By straightforward calculations, the Gâteaux derivative of I_α at $u \in H_r^1(\mathbb{R}^2)$ is given by

$$I'_\alpha(u)v = \int_{\mathbb{R}^2}\left(\nabla u \nabla v + uv\right)\mathrm{d}x + \frac{1}{\alpha}\int_{\mathbb{R}^2} F(u)\mathrm{d}x \cdot \int_{\mathbb{R}^2} f(u)v\mathrm{d}x - \frac{1}{\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u)\right] f(u)v\mathrm{d}x,$$

for any $v \in H_r^1(\mathbb{R}^2)$. It remains to prove that $I'_\alpha(u_n) \to I'_\alpha(u)$ if $u_n \to u$ in $H_r^1(\mathbb{R}^2)$. Now on the one hand, observe that uniformly in $v \in H_r^1(\mathbb{R}^2)$, with $\|v\| \le 1$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x \cdot \int_{\mathbb{R}^2} f(u_n)v\mathrm{d}x - \int_{\mathbb{R}^2} F(u)\mathrm{d}x \cdot \int_{\mathbb{R}^2} f(u)v\mathrm{d}x\right| \\
&\le \int_{\mathbb{R}^2} F(u_n)\mathrm{d}x \int_{\mathbb{R}^2} |f(u_n)v - f(u)v|\mathrm{d}x + \int_{\mathbb{R}^2}\left|F(u_n) - F(u)\right|\mathrm{d}x \cdot \int_{\mathbb{R}^2} f(u)v\mathrm{d}x \\
&\le C\int_{\mathbb{R}^2} |f(u_n)v - f(u)v|\mathrm{d}x + o_n(1) \\
&\le C\left(\int_{\mathbb{R}^2} |f(u_n) - f(u)|^2\mathrm{d}x\right)^{1/2} \cdot \|v\| + o_n(1)\|v\|.
\end{aligned} \tag{3.2}$$

On the other hand, by (f_1) and (f_2), one has for any $x \in \mathbb{R}^2$ and any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(u_n)|^2 \le C_\varepsilon\left(|u_n|^2 + e^{(8\pi+\varepsilon)|u_n|^2}\right) =: g(u_n).$$

It is easy to check that $g(u_n) \to g(u)$ in $L^1(\mathbb{R}^2)$ due to the fact that $u_n \to u$ in $H_r^1(\mathbb{R}^2)$. Hence, by the Lebesgue dominated convergence theorem, we get $\|f(u_n)\|_2^2 \to \|f(u)\|_2^2$ as $n \to \infty$. Therefore, from (3.2) one has that, uniformly in $v \in H_r^1(\mathbb{R}^2)$, with $\|v\| \le 1$,

$$\left|\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x \cdot \int_{\mathbb{R}^2} f(u_n)v\mathrm{d}x - \int_{\mathbb{R}^2} F(u)\mathrm{d}x \cdot \int_{\mathbb{R}^2} f(u)v\mathrm{d}x\right| = o_n(1). \tag{3.3}$$

Moreover, by Lemma 2.2 and the Hardy–Littlewood–Sobolev inequality, we have for any $v \in H^1_r(\mathbb{R}^2)$, with $\|v\| \le 1$,

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^2}\Big[\frac{1}{|x|^\alpha} * F(u_n)\Big] f(u_n)v\mathrm{d}x - \int_{\mathbb{R}^2}\Big[\frac{1}{|x|^\alpha} * F(u)\Big] f(u)v\mathrm{d}x\Big| \\
&\le \int_{\mathbb{R}^2}\Big[\frac{1}{|x|^\alpha} * F(u_n)\Big] |f(u_n)-f(u)|v\mathrm{d}x + \int_{\mathbb{R}^2}\Big[\frac{1}{|x|^\alpha} * |F(u_n)-F(u)|\Big] f(u)v\mathrm{d}x \\
&\le C\int_{\mathbb{R}^2} |f(u_n)-f(u)|v\mathrm{d}x + \|F(u_n)-F(u)\|_{\frac{4}{4-\alpha}}\|f(u)v\|_{\frac{4}{4-\alpha}} \\
&\le Co_n(1)\|v\| + C\|F(u_n)-F(u)\|_{\frac{4}{4-\alpha}}\|v\|.
\end{aligned}
\tag{3.4}
$$

Observe that by the mean value theorem, Hölder's inequality, (f_1), (f_2), (2.1) and (2.2), there exists function $\theta(x) \in (0,1)$ such that

$$
\begin{aligned}
&\|F(u_n)-F(u)\|^{\frac{4}{4-\alpha}}_{\frac{4}{4-\alpha}} \\
&\le \int_{\mathbb{R}^2} |f(u_n+\theta(x)(u_n-u))(u_n-u)|^{\frac{4}{4-\alpha}}\mathrm{d}x \\
&\le C\int_{\mathbb{R}^2} |(|u_n|+|u|)(u_n-u)|^{\frac{4}{4-\alpha}}\mathrm{d}x + C\int_{\mathbb{R}^2} (|u_n|+|u|)^{\frac{4p}{4-\alpha}}|u_n-u|^{\frac{4}{4-\alpha}} e^{\frac{16\pi}{4-\alpha}(|u_n|+|u_0|)^2}\mathrm{d}x \\
&\le o_n(1) + C\Big(\int_{\mathbb{R}^2} (|u_n|+|u|)^{\frac{4pt}{4-\alpha}}|(u_n-u)|^{\frac{4t}{4-\alpha}}\mathrm{d}x\Big)^{\frac{1}{t}}\Big(\int_{\mathbb{R}^2} e^{\frac{16\pi t'}{4-\alpha}(|u_n|+|u_0|)^2}\mathrm{d}x\Big)^{\frac{1}{t'}} \\
&\le o_n(1) + Co_n(1)\cdot\Big(\int_{\mathbb{R}^2} e^{\frac{64\pi t'}{4-\alpha}(|u_n-u_0|^2+|u_0|^2)}\mathrm{d}x\Big)^{\frac{1}{t'}} \\
&\le Co_n(1),
\end{aligned}
$$

where $\frac{1}{t}+\frac{1}{t'}=1$. Then, from (3.4) we have

$$
\Big|\int_{\mathbb{R}^2}\Big[\frac{1}{|x|^\alpha} * F(u_n)\Big] f(u_n)v\mathrm{d}x - \int_{\mathbb{R}^2}\Big[\frac{1}{|x|^\alpha} * F(u)\Big] f(u)v\mathrm{d}x\Big| = o_n(1)\|v\|,
$$

which together with (3.3) completes the proof.

3.3. Critical points of the modified energy functional

Lemma 3.2. *Assume that (f_1)–(f_4) hold, then we have:*

(i) there exist constants $\rho, \eta > 0$ such that $I_\alpha|_{S_\rho} \ge \eta > 0$ for all

$$
u \in S_\rho = \{u \in H^1_r(\mathbb{R}^2) : \ \|u\| = \rho\};
$$

(ii) there exists $e \in H^1_r(\mathbb{R}^2)$ with $\|e\| > \rho$ such that $I_\alpha(e) < 0$.

Proof. We first assume that $u \in H^1_r(\mathbb{R}^2)$ and $\|u\| \le \theta$ for some $\theta \in (0,1)$. Obviously, $\int_{\mathbb{R}^2}|\nabla u|^2 < 1$. Moreover, by (f_1)-(f_2) and (1.5), there is $C_\theta > 0$ such that

$$
|F(u)|^{\frac{4}{3}} \le C\Big(|u|^{\frac{8}{3}} + |u|^{\frac{4}{3}(p+1)}e^{\frac{8\pi}{3}|u|^2}\Big).
$$

So by (f_1), the Hardy–Littlewood–Sobolev inequality yields

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{|x-y|}F(u(y))F(u(x))\mathrm{d}x\mathrm{d}y \\
&\le C\Big(\int_{\mathbb{R}^2}|F(u)|^{\frac{4}{3}}\mathrm{d}x\Big)^{\frac{3}{2}} \\
&\le C\Big(\int_{\mathbb{R}^2}|u|^{\frac{8}{3}} + |u|^{\frac{4}{3}(p+1)}e^{\frac{8\pi}{3}|u|^2}\mathrm{d}x\Big)^{\frac{3}{2}} \\
&\le C\Big(\|u\|^4_{\frac{8}{3}} + \|u\|^{2(p+1)}_{\frac{4\varsigma}{3}(p+1)}\Big), \quad \varsigma > 1.
\end{aligned}
\tag{3.5}
$$

From (3.5) and Lemma 2.3 we deduce that

$$
\begin{aligned}
I_\alpha(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|x-y|^{-\alpha}-1}{\alpha}F(u(y))F(u(x))\mathrm{d}x\mathrm{d}y \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\int\int_{|x-y|\leq 1}\frac{|x-y|^{-\alpha}-1}{\alpha}F(u(y))F(u(x))\mathrm{d}x\mathrm{d}y \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\int\int_{|x-y|\leq 1}\frac{F(u(y))F(u(x))}{|x-y|}\mathrm{d}x\mathrm{d}y \\
&\geq \frac{1}{2}\|u\|^2 - C\left(\|u\|^4_{\frac{8}{3}} + \|u\|^{2(p+1)}_{\frac{4\varsigma}{3}(p+1)}\right).
\end{aligned} \tag{3.6}
$$

So, let $\|u\| = \rho > 0$ be small enough, it is easy to check that there exists $\eta > 0$ such that $I_\alpha(u) \geq \eta$ for any $\alpha \in (0,1)$: this completes the proof of (i).

For the proof of (ii), let us take $e_0 \in H^1_r(\mathbb{R}^2)$ satisfies $e_0(x) \equiv 1$ for $x \in B_{\frac{1}{8}}(0)$, $e_0(x) \equiv 0$ for $x \in \mathbb{R}^2 \setminus B_{\frac{1}{4}}(0)$ and $|\nabla e_0(x)| \leq C$. Set

$$
\Psi(t) := \frac{1}{2}\left(\int_{\mathbb{R}^2} F\left(te_0\right)\mathrm{d}x\right)^2, \tag{3.7}
$$

then by (f_2) we have $F(t) \leq (1-\tau)f(t)t$ for $t \geq 0, \tau \in (0,1)$, and then

$$
\frac{\Psi'(t)}{\Psi(t)} \geq \frac{2}{(1-\tau)t} \quad \text{for all } t > 0.
$$

Integrating this over $[1, s]$, we find

$$
\Psi(s) = \frac{1}{2}\left(\int_{\mathbb{R}^2} F\left(se_0\right)\mathrm{d}x\right)^2 \geq \Psi(1)s^{\frac{2}{1-\tau}}. \tag{3.8}
$$

Note that

$$
\frac{s^{-\alpha}-1}{\alpha} \geq \ln\frac{1}{s}, \quad \text{for } s \in (0,1].
$$

From (3.7) and (3.8) we have

$$
\begin{aligned}
I_\alpha(te_0) &= \frac{t^2}{2}\|e_0\|^2 - \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|x-y|^{-\alpha}-1}{\alpha}F\left(te_0(y)\right)F\left(te_0(x)\right)\mathrm{d}x\mathrm{d}y \\
&= \frac{t^2}{2}\|e_0\|^2 - \frac{1}{2}\int\int_{|x-y|\leq\frac{1}{2}}\frac{|x-y|^{-\alpha}-1}{\alpha}F\left(te_0(y)\right)F\left(te_0(x)\right)\mathrm{d}x\mathrm{d}y \\
&\leq \frac{t^2}{2}\|e_0\|^2 - \frac{1}{2}\int\int_{|x-y|\leq\frac{1}{2}}\ln\frac{1}{|x-y|}F\left(te_0(y)\right)F\left(te_0(x)\right)\mathrm{d}x\mathrm{d}y \\
&\leq \frac{t^2}{2}\|e_0\|^2 - \frac{\ln 2}{2}\left(\int_{\mathbb{R}^2}F(te_0)\mathrm{d}x\right)^2 \\
&\leq \frac{t^2}{2}\|e_0\|^2 - \frac{\ln 2}{2}\left(\int_{\mathbb{R}^2}F(e_0)\mathrm{d}x\right)^2 t^{\frac{2}{1-\tau}},
\end{aligned} \tag{3.9}
$$

which implies that there exists $t_0 > 0$ large enough such that $I_\alpha(t_0e_0) < 0$. □

So I_α has a mountain pass geometry, with mountain pass value

$$
c_\alpha = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\alpha(\gamma(t)),
$$

where

$$
\Gamma := \{\gamma \in C([0,1], H^1_r(\mathbb{R}^2)) : \gamma(0) = 0, \gamma(1) = e\}.
$$

The existence of a Cerami sequence for I_α, namely $\{u_n\} \subset H^1_r(\mathbb{R}^2)$ such that

$$
I_\alpha(u_n) \to c_\alpha, \qquad (1 + \|u_n\|)I'_\alpha(u_n) \to 0, \qquad \text{as } n \to +\infty,
$$

is given by the variant of the Mountain Pass Theorem in [17].

Remark 3.3. Observe from Lemma 3.2 that there exist two constants $a, b > 0$ independent of α such that $a < c_\alpha < b$.

3.4. Level estimates for the modified energy

Let us now define Moser's type functions $w_n(x)$ supported in $B_\rho(0)$ as follows:

$$w_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\ln n}, & 0 \le |x| \le \rho/n, \\ \frac{\ln(\rho/|x|)}{\sqrt{\ln n}}, & \rho/n \le |x| \le \rho, \\ 0, & |x| \ge \rho, \end{cases}$$

where $\rho < \frac{1}{4}$ is given in (f_4). We have

$$\begin{aligned} \|w_n\|^2 &= \int_{B_\rho(0)} \left(|\nabla w_n|^2 + w_n^2 \right) \mathrm{d}x \\ &= 1 + \rho^2 \left(\frac{1}{4\ln n} - \frac{1}{4n^2 \ln n} - \frac{1}{2n^2}\right) \\ &=: 1 + \rho^2 \delta_n. \end{aligned} \tag{3.10}$$

From

$$\frac{s^{-\alpha} - 1}{\alpha} \ge \ln \frac{1}{s}, \quad \text{for } s, \alpha \in (0, 1],$$

the following holds

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^2} \left[G_\alpha(x) * F(w_n) \right] F(w_n) \mathrm{d}x \\ &= \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|x-y|^{-\alpha} - 1}{\alpha} F(w_n(x)) F(w_n(y)) \mathrm{d}x \mathrm{d}y \\ &= \frac{1}{2} \int_{B_\rho(0)} \int_{B_\rho(0)} \frac{|x-y|^{-\alpha} - 1}{\alpha} F(w_n(x)) F(w_n(y)) \mathrm{d}x \mathrm{d}y \\ &\ge \frac{1}{2} \int_{B_\rho(0)} \int_{B_\rho(0)} \ln \frac{1}{|x-y|} F(w_n(x)) F(w_n(y)) \mathrm{d}x \mathrm{d}y \ge 0. \end{aligned} \tag{3.11}$$

Lemma 3.4. *The mountain pass level c_α satisfies* $\sup_{\alpha \in (0,1)} c_\alpha < \frac{1}{2}$.

Proof. Recalling (f_4), for

$$\varepsilon \in \left(0, \beta - \frac{1}{\rho^2 \sqrt{\ln 2} \cdot \pi^{3/2}} \right), \tag{3.12}$$

there exists $t_\varepsilon > 0$ such that

$$tF(t) \ge (\beta - \varepsilon) e^{4\pi t^2}, \quad \text{for } t \ge t_\varepsilon. \tag{3.13}$$

We divide the proof into three cases:

Case 1. Let $t \in \left[0, \sqrt{\frac{1}{2}} \right]$, then by (3.10)–(3.11), we have for large n,

$$\begin{aligned} I_\alpha(tw_n) &= \frac{t^2}{2} \|w_n\|^2 - \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|x-y|^{-\alpha} - 1}{\alpha} F(tw_n(y)) F(tw_n(x)) \mathrm{d}y \mathrm{d}x \\ &< \frac{t^2}{2} \|w_n\|^2 < \frac{1}{2}. \end{aligned} \tag{3.14}$$

Case 2. Let $t \in \left(\sqrt{2}, +\infty \right)$. According to the definition of w_n, we have for large n, $tw_n(x) \ge t_\varepsilon$ for $x \in B_{\rho/n}$. From (3.10)–(3.13), we deduce that for large n,

$$\begin{aligned} I_\alpha(tw_n) &\le \frac{t^2}{2} \|w_n\|^2 - \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} F(tw_n(y)) F(tw_n(x)) \mathrm{d}y \mathrm{d}x \\ &\le \frac{t^2}{2} \|w_n\|^2 - \frac{\ln 2}{2} \left(\int_{\mathbb{R}^2} F(tw_n) \mathrm{d}x \right)^2 \\ &< \frac{1 + \delta_n \rho^2}{2} t^2 - \frac{(\ln 2) \pi^3 (\beta - \varepsilon)^2 \rho^4}{n^4 t^2 \ln n} e^{4t^2 \ln n} \\ &< t^2 \left(1 - \frac{1}{t^4 \ln n} n^{4(t^2 - 1)} \right). \end{aligned} \tag{3.15}$$

Let

$$g(n, t) := \frac{1}{t^4 \ln n} n^{4(t^2 - 1)}, t \ge \sqrt{2}, n \ge 2,$$

then there exists $n_0 > 0$ such that $g(n,t) \geq 1$ for $n \geq n_0$ and $t \geq \sqrt{2}$. Hence, when $t \in \left(\sqrt{2}, +\infty\right)$ and n is large enough,

$$I_\alpha(tw_n) < \frac{1}{2}. \tag{3.16}$$

Case 3. Let $t \in \left[\sqrt{\frac{1}{2}}, \sqrt{2}\right]$. According to the definition of w_n, we have for sufficiently large n, $tw_n(x) \geq t_\varepsilon$ for $x \in B_{\rho/n}$. From (3.10)–(3.13) we have

$$\begin{aligned} I_\alpha(tw_n) \leq& \frac{t^2}{2}\|w_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\frac{1}{|x-y|} F(tw_n(y))F(tw_n(x))\mathrm{d}y\mathrm{d}x \\ <& \frac{1+\delta_n\rho^2}{2}t^2 - \frac{(\ln 2)\pi^3(\beta-\varepsilon)^2\rho^4}{n^4 t^2 \ln n} e^{4t^2\ln n} \\ \leq& \frac{1+\delta_n\rho^2}{2}t^2 - \frac{1}{2\ln n} n^{4(t^2-1)} := \psi_n(t). \end{aligned} \tag{3.17}$$

Then, there exists $t_n > 0$ such that $\psi_n(t_n) = \max_{t>0}\psi_n(t)$ and

$$t_n^2 = \left[1 + \frac{\ln(1+\rho^2\delta_n) - \ln 4}{4\ln n}\right]. \tag{3.18}$$

Clearly, $t_n \in \left[\sqrt{\frac{1}{2}}, \sqrt{2}\right]$ for large n. Then, by (3.17) and (3.18) we have

$$\begin{aligned} \psi_n(t) \leq& \psi_n(t_n) \\ =& t_n^2\frac{1+\delta_n\rho^2}{2} - \frac{1}{8\ln n}(1+\delta_n\rho^2) \\ =& (1+\delta_n\rho^2)\left[\frac{1}{2} + \frac{\ln(1+\rho^2\delta_n) - \ln 4}{8\ln n} - \frac{1}{8\ln n}\right] \\ \leq& \left(1 + \frac{\rho^2}{4\ln n}\right)\left[\frac{1}{2} + \frac{\ln(1+\rho^2\delta_n)}{8\ln n} - \frac{1+\ln 4}{8\ln n}\right], \end{aligned} \tag{3.19}$$

which, together with the definition of δ_n and the fact $\rho^2 < \ln 4$, implies for sufficiently large n

$$\psi_n(t) < \frac{1}{2}. \tag{3.20}$$

Combining (3.14), (3.16) and (3.20), we have $I_\alpha(tw_n) < \frac{1}{2}$. Indeed, for fixed n large enough, there exists $t_0 > 0$ such that $I_\alpha(t_0 w_n) < 0$. Define $\gamma(t) = tt_0 w_n$ for $t \in [0,1]$, then $\gamma \in \Gamma$ and the conclusion follows. □

3.5. Compactness

In this Section we analyze the behavior of Cerami's sequences. Let us begin with the following

Lemma 3.5. *Assume (f_1)–(f_4) and let $\{u_n\} \subset H_r^1(\mathbb{R}^2)$ be an arbitrary Cerami sequence for I_α at level c_α. Then, $\{u_n\}$ stays bounded in $H_r^1(\mathbb{R}^2)$ as well as*

$$\left|\int_{\mathbb{R}^2}\left[G_\alpha(x) * F(u_n)\right]F(u_n)\mathrm{d}x\right| < C, \qquad \left|\int_{\mathbb{R}^2}\left[G_\alpha(x) * F(u_n)\right]f(u_n)u_n\mathrm{d}x\right| < C.$$

Proof. Since $\{u_n\} \subset H_r^1(\mathbb{R}^2)$ is a Cerami sequence, as $n \to \infty$ we have

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^2}\left[G_\alpha(x) * F(u_n)\right]F(u_n)\mathrm{d}x \to c_\alpha \tag{3.21}$$

and for all $v \in H_r^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2}\nabla u_n\nabla v\mathrm{d}x + \int_{\mathbb{R}^2}u_n v\mathrm{d}x - \int_{\mathbb{R}^2}\left[G_\alpha(x) * F(u_n)\right]f(u_n)v\mathrm{d}x = o_n(1)\|v\|. \tag{3.22}$$

Now take $v = u_n$ to get

$$\|u_n\|^2 - \int_{\mathbb{R}^2}\left[G_\alpha(x) * F(u_n)\right]f(u_n)u_n\mathrm{d}x = o_n(1). \tag{3.23}$$

In order to verify the boundedness of $\{u_n\}$, let us introduce a suitable test function as follows

$$v_n := \begin{cases} \frac{F(u_n)}{f(u_n)}, & u_n > 0, \\ (1-\tau)u_n, & u_n \leq 0, \end{cases}$$

with τ as (f_2). We have $|v_n| \le C|u_n|$ since (1.4) holds and $f(t) = 0$ if and only if $t \le 0$. Moreover, the following inequality holds (see Lemma 6.1 of [11] for further details):

$$|\nabla v_n|^2 \le C|\nabla u_n|^2.$$

Thus v_n is well defined in $H^1_r(\mathbb{R}^2)$. Taking $v = v_n$ in (3.22) and recalling that $f(t), F(t) = 0$ for $t \le 0$ we have

$$\begin{aligned}(1-\tau)\int_{u_n\le 0}|\nabla u_n|^2 dx + \int_{u_n>0}|\nabla u_n|^2\left(1-\frac{F(u_n)f'(u_n)}{f^2(u_n)}\right)dx + (1-\tau)\int_{u_n\le 0}u_n^2 dx \\ + \int_{\mathbb{R}^2} u_n\frac{F(u_n)}{f(u_n)}dx - \int_{\mathbb{R}^2}\Big[G_\alpha(|x|) * F(u_n)\Big]F(u_n)dx = o_n(1)\|v_n\|.\end{aligned} \tag{3.24}$$

and by recalling (3.21) we also have

$$(1-\tau)\int_{\mathbb{R}^2}|\nabla u_n|dx + (1-\tau)\int_{\mathbb{R}^2}u_n^2 dx + 2c_\alpha - \|u_n\|^2 \ge o_n(1)\|v_n\|,$$

which implies

$$\tau\|u_n\|^2 \le o_n(1)\|u_n\| + 2c_\alpha. \tag{3.25}$$

As a consequence, we have proved that $\|u_n\| \le C$ for some $C > 0$ independent of n. Moreover, we immediately have from (3.21) and (3.23)

$$\left|\int_{\mathbb{R}^2}\Big[G_\alpha(x) * F(u_n)\Big]F(u_n)dx\right| < C, \qquad \left|\int_{\mathbb{R}^2}\Big[G_\alpha(x) * F(u_n)\Big]f(u_n)u_n dx\right| < C. \quad \square$$

In view of Remark 6.2 in [11], from now on we will always assume positivity of Cerami sequences.

Lemma 3.6. *Let $\{u_n\}$ be a bounded Cerami sequence for I_α at level c_α. Then, there exists $C > 0$ independent of n such that*

$$\int_{\mathbb{R}^2} f(u_n)u_n dx \le C \quad \text{and} \quad \int_{\mathbb{R}^2} F(u_n)^\kappa dx \le C,$$

where $\kappa \in [1, \frac{1}{2a})$ with $0 < a < \frac{1}{2}$ as in Remark 3.3.

Proof. Since $\{u_n\}$ is bounded in $H^1_r(\mathbb{R}^2)$, we may assume up to a subsequence $u_n \rightharpoonup u$ in $H^1_r(\mathbb{R}^2)$, $u_n \to u$ in $L^s_{loc}(\mathbb{R}^2)$ for any $2 \le s < +\infty$ and $u_n \to u$ a.e. in $\mathbb{R}^2$, for which

$$\lim_{n\to+\infty}\|u_n\|^2 = A^2 \ge \|u\|^2.$$

Let us introduce the following auxiliary function

$$H(t) = \int_0^t \frac{\sqrt{F(s)f'(s)}}{f(s)}ds,$$

and define $v_n := H(u_n)$. Let us show that

$$\|v_n\|^2 \le 1. \tag{3.26}$$

From

$$\int_{\mathbb{R}^2}\Big[G_\alpha(|x|) * F(u_n)\Big]F(u_n)dx = A^2 - 2c_\alpha$$

and

$$\int_{\mathbb{R}^2}|\nabla u_n|^2\left(1-\frac{F(u_n)f'(u_n)}{f^2(u_n)}\right) - \int_{\mathbb{R}^2}\Big[G_\alpha(|x|) * F(u_n)\Big]F(u_n)dx + \int_{\mathbb{R}^2}u_n\frac{F(u_n)}{f(u_n)}dx = o_n(1),$$

we have

$$\int_{\mathbb{R}^2}|\nabla u_n|^2\left(1-\frac{F(u_n)f'(u_n)}{f^2(u_n)}\right)dx + \int_{\mathbb{R}^2}u_n\frac{F(u_n)}{f(u_n)}dx + 2c_\alpha - \|u_n\|^2 = o_n(1),$$

and in turn

$$\begin{aligned}\|v_n\|^2 &= \int_{\mathbb{R}^2}|\nabla H(u_n)|^2 dx + \int_{\mathbb{R}^2}H^2(u_n)dx \\ &= 2c_\alpha + \int_{\mathbb{R}^2}\left(u_n\frac{F(u_n)}{f(u_n)} - u_n^2 + H^2(u_n)\right) + o_n(1) \\ &\le 2c_\alpha < 1\end{aligned}$$

for n large enough. Next we give an L^1-estimate of the sequence $\{f(u_n)u_n\}$ by using the estimate for $\{\|v_n\|\}$. By (f_3), for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$\frac{\sqrt{F(t)f'(t)}}{f(t)} \in [1-\varepsilon, 1+\varepsilon], \quad \text{for all } t \ge t_\varepsilon.$$

By (f_2) we have

$$v_n \geq \int_0^{t_\varepsilon} \tau \mathrm{d}t + \int_{t_\varepsilon}^{u_n} (1-\varepsilon)\mathrm{d}t \geq (1-\varepsilon)(u_n - t_\varepsilon), \tag{3.27}$$

which implies

$$u_n \leq t_\varepsilon + \frac{v_n}{1-\varepsilon}$$

for any $x \in \mathbb{R}^2$. Hence, by (f_1) we have that for any given $\varepsilon > 0$ above, there exists C_ε such that

$$\begin{aligned} \int_{\mathbb{R}^2} f(u_n)u_n \mathrm{d}x &\leq \int_{u_n \leq t_\varepsilon} f(u_n)u_n \mathrm{d}x + \int_{u_n \geq t_\varepsilon} f(u_n)u_n \mathrm{d}x \\ &\leq C_\varepsilon \|u_n\|^2 + \int_{u_n \geq t_\varepsilon} f\left(t_\varepsilon + \frac{v_n}{1-\varepsilon}\right)\left(t_\varepsilon + \frac{v_n}{1-\varepsilon}\right)\mathrm{d}x \\ &\leq C_\varepsilon \|u_n\|^2 + C_\varepsilon \int_{u_n \geq t_\varepsilon} e^{4\pi\left(t_\varepsilon + \frac{v_n}{1-\varepsilon}\right)^2}\left(t_\varepsilon + \frac{v_n}{1-\varepsilon}\right)^{p+1}\mathrm{d}x \\ &\leq C_\varepsilon \|u_n\|^2 + C_\varepsilon \int_{u_n \geq t_\varepsilon} e^{4\pi(1+\varepsilon)\left(t_\varepsilon + \frac{v_n}{1-\varepsilon}\right)^2}\mathrm{d}x, \end{aligned} \tag{3.28}$$

where in the last inequality we use the fact that for large values of u_n, also v_n is large such that

$$\left(t_\varepsilon + \frac{v_n}{1-\varepsilon}\right)^{p+1} \leq C_\varepsilon e^{4\pi\varepsilon\left(t_\varepsilon + \frac{v_n}{1-\varepsilon}\right)^2}.$$

In view of (3.27), $v_n \geq \tau t_\varepsilon$ if $u_n \geq t_\varepsilon$, and then it follows from (3.28) that

$$\begin{aligned} \int_{\mathbb{R}^2} f(u_n)u_n \mathrm{d}x &\leq C_\varepsilon \|u_n\|^2 + C_\varepsilon \int_{u_n \geq t_\varepsilon} e^{4\pi(1+\varepsilon)^2 \frac{v_n^2}{(1-\varepsilon)^2}}\mathrm{d}x \\ &\leq C_\varepsilon \|u_n\|^2 + C_\varepsilon \int_{\mathbb{R}^2} e^{4\pi(1+\varepsilon)^2 \frac{v_n^2}{(1-\varepsilon)^2}} - 1\mathrm{d}x \\ &\leq C_\varepsilon \|u_n\|^2 + C_\varepsilon \int_{\mathbb{R}^2} e^{4\pi(1+\varepsilon)^2 \|v_n\|^2 \frac{v_n^2}{\|v_n\|^2(1-\varepsilon)^2}} - 1\mathrm{d}x. \end{aligned} \tag{3.29}$$

Since $\|v_n\|^2 \leq 2c_\alpha + o_n(1)$, $\|v_n\|^2 \leq 2c_\alpha + \sigma < 1$ for n large enough and $\sigma > 0$ is sufficiently small which is also independent of α. Finally, the following holds

$$\frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}\|v_n\|^2 \leq \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}(2c_\alpha + \sigma) < 1$$

for $\varepsilon > 0$ small enough. As a consequence, from (3.29) and (f_2) we get

$$\int_{\mathbb{R}^2} f(u_n)u_n \mathrm{d}x \leq C \quad \text{and then} \quad \int_{\mathbb{R}^2} F(u_n)\mathrm{d}x \leq C$$

for some C independent of n. The remain case $\kappa \in (1, \frac{1}{2a})$ has been proven in [11], that is,

$$\int_{\mathbb{R}^2} F(u_n)^\kappa \mathrm{d}x \leq C.$$

Combining the above facts yields the proof. □

Since $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^2)$, up to a subsequence still denoted by $\{u_n\}$, there exists $u_\alpha \in H_r^1(\mathbb{R}^2)$ such that

$$u_n \rightharpoonup u_\alpha \ \text{ weakly in } H_r^1(\mathbb{R}^2), \qquad u_n \to u_\alpha \ \text{ strongly in } L^p(\mathbb{R}^2),\ p \in (2, +\infty). \tag{3.30}$$

Lemma 3.7. *Assume (f_1)–(f_4) and let $\{u_n\}$ be a bounded Cerami sequence for I_α at level c_α. Then,*

$$\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u_n)\right]F(u_n)\mathrm{d}x \to \int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u_\alpha)\right]F(u_\alpha)\mathrm{d}x \tag{3.31}$$

and

$$\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x \to \int_{\mathbb{R}^2} F(u_\alpha)\mathrm{d}x, \quad \textit{as } n \to \infty. \tag{3.32}$$

Proof. From $I_\alpha'(u_n)u_n = o_n(1)$ we obtain the following

$$\begin{aligned} &\|u_n\|^2 + \frac{1}{\alpha}\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x \int_{\mathbb{R}^2} f(u_n)u_n \mathrm{d}x \\ &= 2c_\alpha + \frac{1}{\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u_n)\right]f(u_n)u_n \mathrm{d}x, \end{aligned}$$

which implies immediately by Lemma 3.5 and Lemma 3.4 that there exists $C > 0$ independent of n such that

$$\frac{1}{\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u_n)\right] f(u_n)u_n\mathrm{d}x \le C. \tag{3.33}$$

From (1.3) and (3.33) we deduce that for any $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$\int_{u_n\ge M_\varepsilon}\left[\frac{1}{|x|^\alpha} * F(u_n)\right] F(u_n)\mathrm{d}x \le \frac{M_0}{M_\varepsilon}\int_{u_n\ge M_\varepsilon}\left[\frac{1}{|x|^\alpha} * F(u_n)\right] f(u_n)u_n\mathrm{d}x \le \varepsilon. \tag{3.34}$$

and

$$\int_{u_\alpha\ge M_\varepsilon}\left[\frac{1}{|x|^\alpha} * F(u_\alpha)\right] F(u_\alpha)\mathrm{d}x \le \varepsilon. \tag{3.35}$$

Observe that from Lemma 3.6 and Hölder's inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}^2}\frac{1}{|x-y|^\alpha}F(u_n(y))\mathrm{d}y\\
&\le \int_{|x-y|\le 1}\frac{1}{|x-y|^\alpha}F(u_n(y))\mathrm{d}y + \int_{|x-y|\ge 1}\frac{1}{|x-y|^\alpha}F(u_n(y))\mathrm{d}y\\
&\le C + \left(\int_{|x-y|\le 1}\frac{1}{|x-y|}dy\right)^\alpha\cdot\left(\int_{\mathbb{R}^2}F(u_n(y))^{\frac{1}{1-\alpha}}\mathrm{d}y\right)^{1-\alpha}\\
&\le C + C\left(\int_{\mathbb{R}^2}F(u_n(y))^{\frac{1}{1-\alpha}}\mathrm{d}y\right)^{1-\alpha} \le C,
\end{aligned} \tag{3.36}$$

where in the above inequality we have taken $\alpha \in \left(0, 1-2a\right)$. Let us define the following sequence of functions

$$\begin{aligned}
G(x,u_n) :=& \left[\frac{1}{|x|^\alpha} * F(u_n)\right]\left(\varepsilon|u_n|^2 + C_\varepsilon|u_n|^q\right)\\
&\ge \left[\frac{1}{|x|^\alpha} * F(u_n)\right]\left(f(u_n)u_n\chi_{u_n<M_\varepsilon}\right),
\end{aligned} \tag{3.37}$$

where $C_\varepsilon > 0$ depends only on ε and $q > 2$. Moreover, using the Hardy–Littlewood–Sobolev inequality and (3.36) we deduce that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}G(x,u_n) - G(x,u_\alpha)\mathrm{d}x\right|\\
\le& C\varepsilon + C_\varepsilon\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u_n)\right]\left(|u_n|^q - |u_\alpha|^q\right)\mathrm{d}x\right|\\
&+ C_\varepsilon\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * \left(F(u_n) - F(u_\alpha)\right)\right]|u_\alpha|^q\mathrm{d}x\right|\\
\le& C\varepsilon + C_\varepsilon C\|F(u_n)\|_{\frac{4}{4-\alpha}}\||u_n|^q - |u_\alpha|^q\|_{\frac{4}{4-\alpha}}\\
&+ C_\varepsilon\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * \left(F(u_n) - F(u_\alpha)\right)\right]|u_\alpha|^q\mathrm{d}x\\
\le& C\varepsilon + C_\varepsilon C\cdot o_n(1) + C_\varepsilon\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * \left(F(u_n) - F(u_\alpha)\right)\right]|u_\alpha|^q\mathrm{d}x\\
:=& C\varepsilon + C_\varepsilon C\cdot o_n(1) + D_1,
\end{aligned} \tag{3.38}$$

where in the above inequality we have the fact that $\|F(u_n)\|_{\frac{4}{4-\alpha}}$ is uniformly bounded by taking α small enough. There exists $R > 0$ such that

$$\begin{aligned}
D_1 := E_1 + E_2 :=& C_\varepsilon\int_{|x|\ge R_\varepsilon}\left[\frac{1}{|x|^\alpha} * \left(F(u_n) - F(u_\alpha)\right)\right]|u_\alpha|^q\mathrm{d}x\\
&+ C_\varepsilon\int_{|x|\le R_\varepsilon}\left[\frac{1}{|x|^\alpha} * \left(F(u_n) - F(u_\alpha)\right)\right]|u_\alpha|^q\mathrm{d}x,
\end{aligned} \tag{3.39}$$

where for any fixed $\varepsilon > 0$, by taking $R = R_\varepsilon > 0$ large enough, it follows from (3.36) that

$$|E_1| \le C_\varepsilon C\int_{|x|\ge R_\varepsilon}|u_\alpha|^q\mathrm{d}x < \varepsilon.$$

Moreover, by virtue of (3.36), we employ the Lebesgue dominated convergence theorem to deduce that

$$E_2 = C_\varepsilon\int_{|x|\le R_\varepsilon}\left[\frac{1}{|x|^\alpha} * \left(F(u_n) - F(u_\alpha)\right)\right]|u_\alpha|^q\mathrm{d}x = C_\varepsilon\cdot o_n(1).$$

Based on the above facts, combining (3.38) and (3.39), one has

$$\left|\int_{\mathbb{R}^2}G(x,u_n) - G(x,u_\alpha)\mathrm{d}x\right| = o_n(1),$$

that is, the control function sequence $\{G(x,u_n)\}$ has a strong convergence subsequence in $L^1(\mathbb{R}^2)$. Hence, using the Lebesgue dominated convergence theorem once again, from (3.37) we immediately obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha}*F(u_n)\right]f(u_n)u_n\chi_{u_n<M_\varepsilon}\mathrm{d}x\\&\to\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha}*F(u_\alpha)\right]f(u_\alpha)u_\alpha\chi_{u_\alpha<M_\varepsilon}\mathrm{d}x,\quad \text{as } n\to\infty,\end{aligned}$$

which, together with (3.34) and (3.35), implies (3.31). Analogously we also have

$$\int_{\mathbb{R}^2}F(u_n)\mathrm{d}x\to\int_{\mathbb{R}^2}F(u_\alpha)\mathrm{d}x,\quad as\ n\to\infty.\quad\square$$

Lemma 3.8. *Assume that (f_1)–(f_4) and let $\{u_n\}$ be a bounded Cerami sequence for I_α at level c_α. Then, there exists a nontrivial $u_\alpha\in H_r^1(\mathbb{R}^2)$ such that $u_n\to u_\alpha$ in $H_r^1(\mathbb{R}^2)$, as $n\to\infty$.*

Proof. We first *claim* that for any $\varphi\in C_0^\infty(\mathbb{R}^2)$

$$\frac{1}{\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha}*F(u_n)\right]f(u_n)\varphi\mathrm{d}x\to\frac{1}{\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha}*F(u_\alpha)\right]f(u_\alpha)\varphi\mathrm{d}x,\quad \text{as } n\to\infty.\tag{3.40}$$

Indeed, let us define the sequence of functions

$$g(x,u_n):=\left[\frac{1}{|x|^\alpha}*F(u_n)\right]f(u_n)$$

restricted to any compact domain Ω. Hence, (3.36) implies $g(x,u_n)\le Cf(u_n)$ for $x\in\Omega$. Moreover, using (f_1)–(f_2) and Lemma 3.6 we have that there exists $M>0$ such that

$$\begin{aligned}&\int_\Omega g(x,u_n)\mathrm{d}x\le\int_\Omega Cf(u_n)\mathrm{d}x\\&\le\int_{\Omega\cap\{x|\,u_n\le M\}}Cf(u_n)\mathrm{d}x+\int_{\Omega\cap\{x|\,u_n\ge M\}}Cf(u_n)\mathrm{d}x\\&\le C+\frac{C}{M}\int_{\Omega\cap\{x|\,u_n\ge M\}}f(u_n)u_n\mathrm{d}x\le C.\end{aligned}$$

Then $g\in L^1(\Omega)$. Thanks to (3.33), using similar arguments as that in Lemma 2.1 of [15], the claim follows. Similarly, we can also prove

$$\frac{1}{\alpha}\int_{\mathbb{R}^2}f(u_n)\mathrm{d}x\to\frac{1}{\alpha}\int_{\mathbb{R}^2}f(u_\alpha)\mathrm{d}x,\quad \text{in } L^1_{loc}(\mathbb{R}^2),\quad \text{as } n\to\infty.\tag{3.41}$$

It follows from (3.40), (3.41), and Lemma 3.7 that for any $\varphi\in C_0^\infty(\mathbb{R}^2)$,

$$I'_\alpha(u_n)\varphi\to I'_\alpha(u_\alpha)\varphi,\quad \text{as } n\to\infty.\tag{3.42}$$

That is, $I'_\alpha(u_\alpha)=0$. Let us use the following

$$v_\alpha=\begin{cases}\frac{F(u_\alpha)}{f(u_\alpha)}, & u_\alpha>0,\\(1-\tau)u_\alpha, & u_\alpha\le 0,\end{cases}$$

in place of v_n in Lemma 3.5, to obtain in a similar fashion

$$2I_\alpha(u_\alpha)\ge\tau\|u_\alpha\|^2\ge 0.\tag{3.43}$$

Next we distinguish two cases:

Case 1. $u_\alpha\equiv 0$. Recalling Lemma 3.7, one has

$$\frac{1}{2}>c_\alpha=I_\alpha(u_n)+o_n(1)=\frac{1}{2}\|u_n\|^2+o_n(1).$$

Then there exists $\varepsilon_0>0$ sufficiently small such that for large n

$$\|u_n\|^2<(1-\varepsilon_0),\tag{3.44}$$

and then there exists $\theta\in(1,2)$ such that

$$(1+\varepsilon_0)(1-\varepsilon_0)\theta<1.$$

In conclusion, it follows from (f_1)–(f_3) and (3.44) that for any $\xi > 0$, there exists $C_\xi > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^2} f(u_n)u_n \mathrm{d}x \\ &\le \int_{\mathbb{R}^2} \left(\xi u_n^2 + C_\xi |u_n|^{p+1} e^{4\pi\|u_n\|^2 \frac{|u_n|^2}{\|u_n\|^2}}\right)\mathrm{d}x \\ &\le \xi\|u_n\|^2 + C_\xi \left(\int_{\mathbb{R}^2} |u_n|^{\theta'(p+1)}\mathrm{d}x\right)^{\frac{1}{\theta'}} \left(\int_{\mathbb{R}^2} e^{4\pi(1-\epsilon_0)\theta \frac{|u_n|^2}{\|u_n\|^2}}\mathrm{d}x\right)^{\frac{1}{\theta}} \\ &\le \xi\|u_n\|^2 + C_\xi C \left(\int_{\mathbb{R}^2} |u_n|^{\theta'(p+1)}\mathrm{d}x\right)^{\frac{1}{\theta'}} \left(\int_{\mathbb{R}^2} e^{4\pi(1+\epsilon_0)(1-\epsilon_0)\theta \frac{|u_n|^2}{\|u_n\|^2}} - 1\mathrm{d}x\right)^{\frac{1}{\theta}}, \end{aligned} \tag{3.45}$$

where $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. Hence, combining (3.30) and (3.45), together with the arbitrariness of ξ, yields

$$\int_{\mathbb{R}^2} f(u_n)u_n \mathrm{d}x = o_n(1), \tag{3.46}$$

which implies by (3.36) the following

$$\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\alpha} * F(u_n)\right] f(u_n)u_n \mathrm{d}x = o_n(1).$$

From $I'_\alpha(u_n)u_n = o_n(1)$ we get $u_n \to 0$ in $H^1_r(\mathbb{R}^2)$, as $n \to \infty$. This is a contradiction with the fact $I_\alpha(u_n) \to c_\alpha$, as $n \to \infty$.

Case 2. $u_\alpha \not\equiv 0$. That is, $\|u_\alpha\| > 0$. Next we show that

$$\|u_n\|^2 \to \|u_\alpha\|^2, \quad \text{as} \quad n \to \infty. \tag{3.47}$$

By Fatou's lemma and Lemma 3.7 we have

$$\begin{aligned} I_\alpha(u_\alpha) =& \frac{1}{2}\|u_\alpha\|^2 + \frac{1}{2\alpha}\left(\int_{\mathbb{R}^2} F(u_\alpha)\mathrm{d}x\right)^2 - \frac{1}{2\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u_\alpha)\right]F(u_\alpha)\mathrm{d}x \\ \le& \liminf_{n\to\infty}\left(\frac{1}{2}\|u_n\|^2 + \frac{1}{2\alpha}\left(\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x\right)^2 - \frac{1}{2\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u_n)\right]F(u_n)\mathrm{d}x\right) \\ =& \liminf_{n\to\infty} I_\alpha(u_n) = c_\alpha. \end{aligned} \tag{3.48}$$

If $I_\alpha(u_\alpha) = c_\alpha$, by (3.48) we obtain immediately (3.47). Otherwise, if $I_\alpha(u_\alpha) < c_\alpha$, then

$$\|u_\alpha\|^2 + \frac{1}{\alpha}\left(\int_{\mathbb{R}^2} F(u_\alpha)\mathrm{d}x\right)^2 < 2c_\alpha + \frac{1}{\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u_\alpha)\right]F(u_\alpha)\mathrm{d}x. \tag{3.49}$$

In view of the definition of I_α, we also have

$$\lim_{n\to\infty}\left(\|u_n\|^2 + \frac{1}{\alpha}\left(\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x\right)^2\right) = 2c_\alpha + \frac{1}{\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u_\alpha)\right]F(u_\alpha)\mathrm{d}x. \tag{3.50}$$

Take

$$w_n = \frac{u_n}{\sqrt{\|u_n\|^2 + \frac{1}{\alpha}\left(\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x\right)^2}}$$

and

$$w_\alpha = \frac{u_\alpha}{\sqrt{2c_\alpha + \frac{1}{\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u_\alpha)\right]F(u_\alpha)\mathrm{d}x}}.$$

From (3.49) and (3.50) we have $\|w_n\| \le 1$, $w_n \rightharpoonup w_\alpha$ and $\|w_\alpha\| < 1$. It is not difficult to deduce by (3.49) that $\lim_{n\to\infty}\|u_n\|^2 > \|u_\alpha\|^2$ and $\lim_{n\to\infty}\|w_n\|^2 > \|w_\alpha\|^2$. Following Lions [19], one has

$$\sup_{n\in\mathbb{N}}\left(\int_{\mathbb{R}^2} e^{4\pi r w_n^2} - 1\right)\mathrm{d}x < \infty \tag{3.51}$$

for all

$$r < \bar{r} := \frac{1}{B - \|w_\alpha\|^2} = 2\frac{c_\alpha + \frac{1}{2\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha} * F(u_\alpha)\right]F(u_\alpha)\mathrm{d}x}{\|u_n\|^2 - \|u_\alpha\|^2} + o_n(1),$$

where $B = \lim_{n\to\infty}\|w_n\|^2$. By Lemma 3.7 and (3.43), the Brezis–Lieb lemma yields

$$I_\alpha(u_n) - I_\alpha(u_\alpha) = \frac{1}{2}\left(\|u_n\|^2 - \|u_\alpha\|^2\right) + o_n(1) < c_\alpha < \frac{1}{2}.$$

Then, recalling (3.50), we can always choose $s > 1$ sufficiently close to 1 such that

$$\begin{aligned} & s\left(\|u_n\|^2+\frac{1}{\alpha}\left(\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x\right)^2\right) \\ \le & r<\frac{1}{B-\|w_\alpha\|^2} \\ = & 2\frac{c_\alpha+\frac{1}{2\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha}*F(u_\alpha)\right]F(u_\alpha)\mathrm{d}x}{\|u_n\|^2-\|u_\alpha\|^2}+o_n(1) \end{aligned} \tag{3.52}$$

for some r satisfying (3.51). Based on the above facts, it follows from (3.41), (3.51), (3.52) and (f_1) that

$$\begin{aligned} & \left|\int_{\mathbb{R}^2} f(u_n)u_n-f(u_\alpha)u_\alpha\mathrm{d}x\right| \\ = & \left|\int_{\mathbb{R}^2} f(u_n)(u_n-u_\alpha)-(f(u_n)-f(u_\alpha))u_\alpha\mathrm{d}x\right| \\ \le & C\left(\int_{\mathbb{R}^2}|u_n|^{ps}e^{4s\pi\|u_n\|^2\frac{|u_n|^2}{\|u_n\|^2}}dx\right)^{\frac{1}{s}}\left(\int_{\mathbb{R}^2}|u_n-u_\alpha|^{\frac{s}{s-1}}dx\right)^{\frac{s-1}{s}} \\ & +C\int_{\mathbb{R}^2}|u_n(u_n-u_\alpha)|\mathrm{d}x+\int_{\mathbb{R}^2}\left|(f(u_n)-f(u_\alpha))u_\alpha\right|\mathrm{d}x\to 0, \quad \text{as} \quad n\to\infty, \end{aligned} \tag{3.53}$$

where we use the fact that $s > 1$ is sufficiently close to 1. On the other hand, from (3.36) and (3.40) we deduce that

$$\begin{aligned} & \left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha}*F(u_n)\right]f(u_n)u_n\mathrm{d}x-\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha}*F(u_\alpha)\right]f(u_\alpha)u_\alpha\mathrm{d}x\right| \\ \le & \left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha}*F(u_n)\right]\left(f(u_n)u_n-f(u_\alpha)u_\alpha\right)\mathrm{d}x\right| \\ & +\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\alpha}*\left(F(u_n)-F(u_\alpha)\right)\right]f(u_\alpha)u_\alpha\mathrm{d}x\right| \\ \le & C\left(\int_{\mathbb{R}^2}|u_n|^{ps}e^{4s\pi\|u_n\|^2\frac{|u_n|^2}{\|u_n\|^2}}dx\right)^{\frac{1}{s}}\left(\int_{\mathbb{R}^2}|u_n-u_\alpha|^{\frac{s}{s-1}}dx\right)^{\frac{s-1}{s}} \\ & +C\int_{\mathbb{R}^2}|u_n(u_n-u_\alpha)|\mathrm{d}x+\int_{\mathbb{R}^2}\left|(f(u_n)-f(u_\alpha))u_\alpha\right|\mathrm{d}x\to 0, \quad \text{as} \quad n\to\infty. \end{aligned} \tag{3.54}$$

Combining (3.53) and (3.54) we obtain $u_n \to u_\alpha$ in $H^1_r(\mathbb{R}^2)$. Then $I(u_\alpha) < c_\alpha$ is not true. Hence, $I'_\alpha(u_\alpha) = 0$ and $I_\alpha(u_\alpha) = c_\alpha$. □

4. Proof of theorem 1.1

By virtue of Lemma 3.8, we have that $u_\alpha \in H^1_r(\mathbb{R}^2)$ is a positive critical point of I_α with $I_\alpha(u_\alpha) = c_\alpha$. Recalling Remark 3.3, we have $a < I_\alpha(u_\alpha) < b$ with $a, b > 0$ independent of α. Similar arguments as in Lemma 3.5 yield u_α bounded in $H^1_r(\mathbb{R}^2)$ uniformly in $\alpha > 0$. In order to study the limit properties of u_α as $\alpha \to 0^+$, we are going to establish some estimates for u_α.

We may assume as $\alpha \to 0^+$, up to a subsequence, the following:

$$\begin{aligned} & u_\alpha \rightharpoonup u_0 \quad \text{in } H^1_r(\mathbb{R}^2), \\ & u_\alpha \to u_0 \quad a.e. \text{ in } \mathbb{R}^2, \\ & u_\alpha \to u_0 \quad \text{in } L^s(\mathbb{R}^2) \quad \text{for } s \in (2,+\infty). \end{aligned} \tag{4.1}$$

Lemma 4.1. *For $\omega > 1$, sufficiently close to 1, there exists $C > 0$ independent of $\alpha \in \left(0, \frac{4(\omega-1)}{3\omega}\right)$ such that*

$$\left|\int_{|x-y|\le 1}\frac{F(u_\alpha(y))}{|x-y|^{\frac{4(\omega-1)}{3\omega}}}\mathrm{d}y\right|\le C,$$

and as $|x| \to \infty$

$$\int_{|x-y|\le 1}\frac{F(u_\alpha(y))}{|x-y|^{\frac{4(\omega-1)}{3\omega}}}\mathrm{d}y\to 0.$$

uniformly for $\alpha \in (0, \bar\alpha)$ with $\bar\alpha \in \left(0, \frac{4(\omega-1)}{3\omega}\right)$.

Proof. Arguing as in Lemma 3.6, by taking $v_\alpha := H(u_\alpha)$ and by (3.26), we deduce

$$\sup_{\alpha\in(0,\frac{4(\omega-1)}{3\omega})}\|v_\alpha\|<1$$

and there exists $C > 0$ independent of $\alpha > 0$ such that

$$\int_{\mathbb{R}^2} F(u_\alpha)^\omega \mathrm{d}x < C. \tag{4.2}$$

Moreover, we know that for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$u_\alpha(x) \leq t_\varepsilon + \frac{v_\alpha(x)}{1-\varepsilon} \quad \text{for any } x \in \mathbb{R}^2,$$

which implies by Young's inequality that there exists $C_\varepsilon > 0$ such that

$$u_\alpha^2 \leq C_\varepsilon t_\varepsilon^2 + (1+\varepsilon)\frac{v_\alpha^2}{(1-\varepsilon)^2}.$$

From (f_1)–(f_3) and using the Hölder inequality, one has

$$\begin{aligned}
&\int_{|x-y|\leq 1} \frac{F(u_\alpha(y))}{|x-y|^{\frac{4(\omega-1)}{3\omega}}}\mathrm{d}y \\
&\leq \left(\int_{|x-y|\leq 1} \frac{1}{|x-y|^{\frac{4}{3}}}\mathrm{d}y\right)^{\frac{\omega}{\omega-1}} \cdot \left(\int_{|x-y|\leq 1} |F(u_\alpha)|^\omega \mathrm{d}y\right)^{\frac{1}{\omega}} \\
&\leq C\left(\int_{|x-y|\leq 1} |u_\alpha|^{2\omega} + |u_\alpha|^{p\omega} e^{4\pi\omega(t_\varepsilon + \frac{v_\alpha}{1-\varepsilon})^2}\mathrm{d}y\right)^{\frac{1}{\omega}} \\
&\leq CC_\varepsilon\left(\int_{|x-y|\leq 1} |u_\alpha|^{2\omega} + |u_\alpha|^{p\omega} e^{4\pi\omega(1+\varepsilon)\frac{v_\alpha^2}{(1-\varepsilon)^2}}\mathrm{d}y\right)^{\frac{1}{\omega}} \\
&\leq C\left(\int_{|x-y|\leq 1} |u_\alpha|^{2\omega}\mathrm{d}y + \left(\int_{|x-y|\leq 1} |u_\alpha|^{p\omega\zeta}\mathrm{d}y\right)^{1/\zeta} \cdot \left(\int_{\mathbb{R}^2} e^{4\pi\zeta' \frac{(1+\varepsilon)}{(1-\varepsilon)^2}\omega\|v_\alpha\|^2 \frac{v_\alpha^2}{\|v_\alpha\|^2}} - 1\mathrm{d}x\right)^{1/\zeta'}\right)^{\frac{1}{\omega}} \\
&\leq C\left(\int_{|x-y|\leq 1} |u_\alpha|^{2\omega}\mathrm{d}y + C\left(\int_{|x-y|\leq 1} |u_\alpha|^{p\omega\zeta}\mathrm{d}y\right)^{1/\zeta}\right)^{\frac{1}{\omega}},
\end{aligned}$$

where $\zeta > 1$ and $\zeta' = \frac{\zeta}{\zeta-1}$ and in the last inequality we let ζ', ω sufficiently close to 1. Thanks to (4.1), we obtain the desired result. □

Next we establish the exponential decay of u_α at infinity uniformly with respect to α.

Lemma 4.2. *There exist $R, M > 0$ (independent of α) such that*

$$u_\alpha(x) \leq M \exp\left(-\frac{1}{2}|x|\right) \quad \textit{for } |x| \geq R.$$

Proof. Since u_α is a positive function of Eq. (3.1) and by Lemma 2.3 we obtain

$$\begin{aligned}
-\Delta u_\alpha + u_\alpha &\leq \int_{|x-y|\leq 1} \frac{|x-y|^{-\alpha} - 1}{\alpha} F(u_\alpha(y))\mathrm{d}y f(u_\alpha) \\
&\leq C\int_{|x-y|\leq 1} \frac{F(u_\alpha(y))}{|x-y|^{\frac{4(\omega-1)}{3\omega}}}\mathrm{d}y f(u_\alpha),
\end{aligned} \tag{4.3}$$

where $\alpha \in \left(0, \frac{4(\omega-1)}{3\omega}\right)$. In view of Lemma 4.1, there exist $R_1 > 0$ and $\alpha^* \in \left(0, \frac{4(\omega-1)}{3\omega}\right)$ such that

$$\int_{|x-y|\leq 1} \frac{F(u_\alpha(y))}{|x-y|^{\frac{4(\omega-1)}{3\omega}}}\mathrm{d}y \leq \frac{1}{C} \tag{4.4}$$

for $|x| \geq R_1$ and $\alpha \in (0, \alpha^*)$. By recalling the radial Lemma A.IV in [4], there exists $C > 0$ independent of α such that

$$|u_\alpha(x)| \leq C|x|^{-\frac{1}{2}}\|u_\alpha\| \leq C|x|^{-\frac{1}{2}},$$

which implies

$$\lim_{|x|\to\infty} |u_\alpha(x)| = 0 \quad \text{uniformly for } \alpha \in (0, \alpha^*).$$

Thus, by assumption (f_1), we deduce that there exists $R_2 > 0$ such that

$$f(u_\alpha) \leq \frac{3}{4}u_\alpha, \quad |x| \geq R_2. \tag{4.5}$$

Combining (4.3)–(4.5), let $R = \max\{R_1, R_2\}$ to get for $\alpha \in (0, \alpha^*)$

$$-\Delta u_\alpha + \frac{1}{4}u_\alpha \leq 0, \quad |x| \geq R. \tag{4.6}$$

From (4.6) and the comparison principle, there exists a constant $M \geq \frac{C}{R}e^{R/2}$ such that

$$u_\alpha(x) \leq M \exp\left(-\frac{1}{2}|x|\right) \quad \text{for } |x| \geq R.$$

Here R, M are independent of α. □

Proof of Theorem 1.1. We are now in the position to carry out the proof of our main result which we divide into two steps:

Step 1. Let us show that $u_0 \in H^1_r(\mathbb{R}^2)$ satisfies $I'(u_0) = 0$.
For any $\varphi \in C_0^\infty(\mathbb{R}^2)$, we have

$$\begin{aligned} I'_\alpha(u_\alpha)\varphi = &\int_{\mathbb{R}^2} \nabla u_\alpha \nabla \varphi \mathrm{d}x + \int_{\mathbb{R}^2} u_\alpha \varphi \mathrm{d}x \\ &- \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|x-y|^{-\alpha}-1}{\alpha} F(u_\alpha(y))\mathrm{d}y f(u_\alpha)\varphi \mathrm{d}x. \end{aligned} \tag{4.7}$$

Recalling (3.41), we have

$$\int_{\mathbb{R}^2} f(u_\alpha)\varphi \mathrm{d}x \to \int_{\mathbb{R}^2} f(u_0)\varphi \mathrm{d}x, \quad \text{as } \alpha \to 0^+. \tag{4.8}$$

Then it follows from Lemma 2.3 that for any fixed $\varphi \in C_0^\infty(\mathbb{R}^2)$, we have for $\alpha \in (0, \alpha^*)$

$$\begin{aligned} &\left| \frac{|x-y|^{-\alpha}-1}{\alpha} \chi_{|x-y|\leq 1} F(u_\alpha(y)) f(u_\alpha(x))\varphi(x) \right| \\ &\leq \left| \frac{1}{|x-y|^{\frac{4(\omega-1)}{3\omega}}} \chi_{|x-y|\leq 1} F(u_\alpha(y)) f(u_\alpha(x))\varphi(x) \right| := h_\alpha(x,y), \end{aligned} \tag{4.9}$$

which, together with Lemma 4.1 and (4.8), yields

$$\begin{aligned} &\left| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} h_\alpha(x,y) - \frac{1}{|x-y|^{\frac{4(\omega-1)}{3\omega}}} \chi_{|x-y|\leq 1} F(u_0(y)) f(u_0(x))\varphi(x)\mathrm{d}x\mathrm{d}y \right| \\ \leq &\left| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|^{\frac{4(\omega-1)}{3\omega}}} \chi_{|x-y|\leq 1} \big(F(u_\alpha(y)) - F(u_0(y))\big) f(u_\alpha(x))\varphi(x)\mathrm{d}y\mathrm{d}x \right| \\ &+ \left| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|^{\frac{4(\omega-1)}{3\omega}}} \chi_{|x-y|\leq 1} F(u_0(y))\mathrm{d}y \big(f(u_\alpha(x)) - f(u_0(x))\big)\varphi(x)\mathrm{d}x \right| \\ =: &I_1 + o_\alpha(1). \end{aligned}$$

Observe by Lemma 4.1 and (4.8) that there exists $C > 0$ independent of α such that

$$\begin{aligned} I_1 &\leq C \int_{\mathbb{R}^2} |f(u_\alpha(x))\varphi(x)| \,\mathrm{d}x \\ &\leq C \int_{\mathbb{R}^2} |f(u_0(x))\varphi(x)| \,\mathrm{d}x + C \end{aligned}$$

for α small. Thus, the Lebesgue dominated convergence theorem implies $I_1 \to 0$, as $\alpha \to 0^+$. Furthermore, since $\{h_\alpha\}$ has a strongly convergent subsequence in $L^1(\mathbb{R}^2)$, we use the Lebesgue dominated convergence theorem in (4.9) to get

$$\begin{aligned} &\int\int_{|x-y|\leq 1} \frac{|x-y|^{-\alpha}-1}{\alpha} F(u_\alpha(y))\mathrm{d}y f(u_\alpha(x))\varphi(x)\mathrm{d}x \\ &\to -\int\int_{|x-y|\leq 1} \ln(|x-y|) F(u_0(y))\mathrm{d}y f(u_0(x))\varphi(x)\mathrm{d}x. \end{aligned} \tag{4.10}$$

When $|x-y| \geq 1$, there exists $\tau = \tau(|x-y|) \in (0,1)$ such that

$$0 \geq G_\alpha(x-y) = \frac{|x-y|^{-\alpha}-1}{\alpha} = -|x-y|^{-\tau\alpha} \ln|x-y|, \tag{4.11}$$

where τ depends on $|x-y|$. Since φ has a compact support, from Lemma 4.2 and (f_1) we have

$$\begin{aligned} &\left| \frac{|x-y|^{-\alpha}-1}{\alpha} \cdot \chi_{|x-y|\geq 1} F(u_\alpha(y)) f(u_\alpha(x))\varphi(x) \right| \\ &= \left| |x-y|^{-\tau\alpha} \ln|x-y| \cdot \chi_{|x-y|\geq 1} F(u_\alpha(y)) f(u_\alpha(x))\varphi(x) \right| \\ &\leq \left| (|x| + |y|) F(u_\alpha(y)) f(u_\alpha(x))\varphi(x) \right| \\ &\leq \left| C\left(F(u_\alpha(y)) + M|y|e^{-\frac{1}{2}|y|} \right) f(u_\alpha(x))\varphi(x) \right| \\ &=: \bar{h}_\alpha(x,y). \end{aligned} \tag{4.12}$$

Using Lemma 4.2 and Lemma 3.7, we have that $\{\bar{h}_\alpha\}$ has a strongly convergent subsequence in $L^1(\mathbb{R}^2\times\mathbb{R}^2)$. Combining (4.11) with (4.12), similarly to (4.9), by the Lebesgue dominated convergence theorem, one has

$$\begin{aligned}&\int\int_{|x-y|\geq 1}\frac{|x-y|^{-\alpha}-1}{\alpha}F(u_\alpha(y))\mathrm{d}yf(u_\alpha(x))\varphi(x)\mathrm{d}x\\&\to -\int\int_{|x-y|\geq 1}\ln|x-y|F(u_0(y))\mathrm{d}yf(u_0(x))\varphi(x)\mathrm{d}x.\end{aligned}\tag{4.13}$$

Fatou's lemma yields

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|F(u_0(y))\mathrm{d}yF(u_0(x))\mathrm{d}x\right|\\&\leq\liminf_{\alpha\to 0}\left(\int\int_{|x-y|\leq 1}G_\alpha(x-y)F(u_\alpha(y))\mathrm{d}yF(u_\alpha(x))\mathrm{d}x\right.\\&\left.-\int\int_{|x-y|\geq 1}G_\alpha(x-y)F(u_\alpha(y))\mathrm{d}yF(u_\alpha(x))\mathrm{d}x\right).\end{aligned}\tag{4.14}$$

By the Hardy–Littlewood–Sobolev inequality, (4.2) and Lemma 2.3, there exists $C_\omega>0$ such that

$$\begin{aligned}&\int\int_{|x-y|\leq 1}G_\alpha(x-y)F(u_\alpha(y))\mathrm{d}yF(u_\alpha(x))\mathrm{d}x\\&\leq\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(u_\alpha(y))}{|x-y|^{\frac{4(\omega-1)}{\omega}}}\mathrm{d}yF(u_\alpha(x))\mathrm{d}x\\&\leq C_\omega\left(\int_{\mathbb{R}^2}F^\omega(u_\alpha)\mathrm{d}x\right)^2\leq C\end{aligned}\tag{4.15}$$

uniformly for $\alpha\in\left(0,\frac{4(\omega-1)}{3\omega}\right)$ sufficiently small, where ω as in Lemma 4.1. By Remark 3.3 and (4.15), we also have

$$\begin{aligned}&\int\int_{|x-y|\geq 1}G_\alpha(x-y)F(u_\alpha(y))\mathrm{d}yF(u_\alpha(x))\mathrm{d}x\\&\leq I_\alpha(u_\alpha)+\int\int_{|x-y|\leq 1}G_\alpha(x-y)F(u_\alpha(y))\mathrm{d}yF(u_\alpha(x))\mathrm{d}x-\frac{1}{2}\|u_\alpha\|^2\\&<+\infty\end{aligned}\tag{4.16}$$

uniformly for α small enough. Joining (4.14), (4.15) and (4.16) gives the following

$$\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|F(u_0(y))\mathrm{d}yF(u_0(x))\mathrm{d}x\right|<+\infty.\tag{4.17}$$

Based on (4.10), (4.13) and (4.17), by taking the limit in (4.7), we have $I'(u_0)=0$ with $I(u_0)<+\infty$, that is, $u_0\in H_r^1(\mathbb{R}^2)$ solves Eq. (1.1).

Step 2. It remains to show that $u_0\neq 0$ and that $u_\alpha\to u_0$ in $H_r^1(\mathbb{R}^2)$. Assume by contradiction that $u_\alpha\rightharpoonup 0$ in $H_r^1(\mathbb{R}^2)$, as well as $u_\alpha\to 0$ in $L^t(\mathbb{R}^2)$ for $t\in(2,+\infty)$. Similarly to (3.46), we obtain $\int_{\mathbb{R}^2}f(u_\alpha)u_\alpha\mathrm{d}x=o_\alpha(1)$. So, by Lemma 2.3, Lemma 4.1, we have

$$\begin{aligned}I'_\alpha(u_\alpha)u_\alpha&=\|u_\alpha\|^2-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}G_\alpha(x-y)F(u_\alpha(y))f(u_\alpha(x))u_\alpha(x)\mathrm{d}x\mathrm{d}y\\&\geq\|u_\alpha\|^2-\int\int_{|x-y|\leq 1}G_\alpha(x-y)F(u_\alpha(y))f(u_\alpha(x))u_\alpha(x)\mathrm{d}x\mathrm{d}y\\&\geq\|u_\alpha\|^2-\int\int_{|x-y|\leq 1}\frac{1}{|x-y|^{\frac{4(\omega-1)}{\omega}}}F(u_\alpha(y))f(u_\alpha(x))u_\alpha(x)\mathrm{d}x\mathrm{d}y\\&\geq\|u_\alpha\|^2-C\int_{\mathbb{R}^2}f(u_\alpha(x))u_\alpha(x)\mathrm{d}x,\end{aligned}$$

and thus $u_\alpha \to 0$ in $H_r^1(\mathbb{R}^2)$, as $\alpha \to 0^+$. Then, according to Remark 3.3, (4.12), Lemma 4.2 and Lemma 3.7, we have

$$\begin{aligned}
a &\le I_\alpha(u_\alpha)\\
&= \frac{1}{2}\|u_\alpha\|^2 - \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\alpha(x-y)F(u_\alpha(y))F(u_\alpha(x))\mathrm{d}x\mathrm{d}y\\
&\le -\frac{1}{2}\int\int_{|x-y|\ge 1} G_\alpha(x-y)F(u_\alpha(y))F(u_\alpha(x))\mathrm{d}x\mathrm{d}y + o_\alpha(1)\\
&\le C\int_{\mathbb{R}^2}|x|F(u_\alpha(x))\mathrm{d}x\int_{\mathbb{R}^2}F(u_\alpha(y))\mathrm{d}y + o_\alpha(1)\\
&\le C\left(\int_{|x|\le R}|x|F(u_\alpha(x))\mathrm{d}x + \int_{|x|\ge R}|x|F(u_\alpha(x))\mathrm{d}x\right) + o_\alpha(1)\\
&\le C\left(\int_{|x|\le R}F(u_\alpha(x))\mathrm{d}x + \int_{|x|\ge R}\frac{C|x|}{Me^{\frac{|x|}{2}}}\mathrm{d}x\right)^2 + o_\alpha(1)\\
&= o_\alpha(1),
\end{aligned}$$

which yields a contradiction. So, $u_0 \ne 0$. Furthermore, similarly to (4.9), (4.13), by Lemma 4.2 and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|x-y|^{-\alpha}-1}{\alpha}F(u_\alpha(y))f(u_\alpha(x))u_\alpha(x)\mathrm{d}y\mathrm{d}x\\
&\to -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|F(u_0(y))f(u_0(x))u_0(x)\mathrm{d}x\mathrm{d}y,
\end{aligned} \tag{4.18}$$

from which we conclude that $u_\alpha \to u_0$ in $H_r^1(\mathbb{R}^2)$, as $\alpha \to 0^+$. □

Data availability

No data was used for the research described in the article.

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