Bull. Aust. Math. Soc. (First published online 2023), page 1 of 9*
doi:10.1017/S0004972723000394
*Provisional—final page numbers to be inserted when paper edition is published

ON THE ITERATES OF THE SHIFTED EULER'S FUNCTION

PAOLO LEONETTI ✉ and FLORIAN LUCA

(Received 22 February 2023; accepted 3 April 2023)

Abstract

Let φ be Euler's function and fix an integer $k \geq 0$. We show that for every initial value $x_1 \geq 1$, the sequence of positive integers $(x_n)_{n\geq 1}$ defined by $x_{n+1} = \varphi(x_n) + k$ for all $n \geq 1$ is eventually periodic. Similarly, for all initial values $x_1, x_2 \geq 1$, the sequence of positive integers $(x_n)_{n\geq 1}$ defined by $x_{n+2} = \varphi(x_{n+1}) + \varphi(x_n) + k$ for all $n \geq 1$ is eventually periodic, provided that k is even.

2020 *Mathematics Subject Classification*: primary 11B37; secondary 11A25, 97I30.

Keywords and phrases: nonlinear recurrence sequences, periodic sequences, Euler's function, iterates.

1. Introduction and main results

Let $\mathbb{N}$ be the set of positive integers and fix an arithmetic function $f : \mathbb{N}^d \to \mathbb{N}$ for some $d \in \mathbb{N}$. Let $(x_n)_{n\geq 1}$ be a sequence of positive integers which satisfies the recurrence

$$x_{n+d} = f(x_n, \ldots, x_{n+d-2}, x_{n+d-1}) \quad \text{for all } n \in \mathbb{N}, \tag{1.1}$$

with starting values $x_1, \ldots, x_d \in \mathbb{N}$. In the case where $d = 1$, the sequence $(x_n)_{n\geq 1}$ is simply the orbit of x_1 with respect to f. The aim of this note is to study whether certain recurrence sequences $(x_n)_{n\geq 1}$ of the type (1.1) are eventually periodic independent of their starting values, that is, for all $x_1, \ldots, x_d \in \mathbb{N}$, there exists $T \in \mathbb{N}$ such that $x_n = x_{n+T}$ for all sufficiently large n.

We will frequently use the basic observation that a recurrence sequence $(x_n)_{n\geq 1}$, as in (1.1), is eventually periodic if and only if it is bounded (see [5, page 45]).

We start with a simple result for functions f which are not too large.

PROPOSITION 1.1. *Let $f : \mathbb{N}^d \to \mathbb{N}$ be an arithmetical function, with $d \in \mathbb{N}$, and suppose that there exists $C \in \mathbb{N}$ such that*

$$f(n_1, \ldots, n_d) < \max\{n_1, \ldots, n_d\} \tag{1.2}$$

for all $n_1, \dots, n_d \in \mathbb{N}$ *with* $n_i \geq C$ *for some* $i \in \{1, \dots, d\}$. *Let* $(x_n)_{n\geq 1}$ *be a recurrence sequence as in (1.1), with starting values* $x_1, \dots, x_d \in \mathbb{N}$. *Then*

$$\limsup_{n\to\infty} x_n \leq \max\{f(n_1, \dots, n_d) : n_1, \dots, n_d \leq C-1\}. \tag{1.3}$$

In particular, $(x_n)_{n\geq 1}$ *is bounded above and hence eventually periodic.*

Special instances of Proposition 1.1 in the one-dimensional case $d = 1$ have been previously obtained in the literature. For example, Porges [15] considered the case where $f(n)$ is the sum of squares of the digits of n (see also [7, 11, 19]). Note that (1.2) holds if $f(n_1, \dots, n_d) = o(N)$ as $N = \max\{n_1, \dots, n_d\} \to \infty$.

Of course, there exist other functions f which do not satisfy (1.2) and such that every sequence $(x_n)_{n\geq 1}$, as in (1.1), is eventually periodic: as a trivial example, one can consider $f(n_1, \dots, n_d) := \max\{n_1, \dots, n_d\}$ for all $(n_1, \dots, n_d) \in \mathbb{N}^d$. At the opposite extreme, if f is slightly bigger (for instance, $f(n_1, \dots, n_d) := \max\{n_1, \dots, n_d\} + 1$), then there are no eventually periodic sequences $(x_n)_{n\geq 1}$ as in (1.1). This is the starting point for this work, which motivates the heuristic: *if a function f satisfies (1.2) 'on average', then every sequence* $(x_n)_{n\geq 1}$ *as in (1.1) should be eventually periodic, independent of its starting values.*

We are going to confirm the above heuristic in two cases which involve (shifted iterates of) Euler's function φ (recall that $\varphi(n)$ is the number of integers in $\{1, \dots, n\}$ which are coprime with n). Our first main result is the following theorem.

THEOREM 1.2. *Fix an integer* $k \geq 0$ *and let* $(x_n)_{n\geq 1}$ *be the recurrence sequence defined by*

$$x_{n+1} = \varphi(x_n) + k \quad \textit{for all } n \in \mathbb{N},$$

with starting value $x_1 \in \mathbb{N}$. *Then,*

$$\sup_{n\in\mathbb{N}} x_n \leq \max\{x_1, k^4\} + (k+1)^2. \tag{1.4}$$

In particular, $(x_n)_{n\geq 1}$ *is eventually periodic.*

Note that the dependence of the upper bound (1.4) on x_1 cannot be removed: indeed, if $k := x_1 - \varphi(x_1)$ for some $x_1 \in \mathbb{N}$, then the sequence $(x_n)_{n\geq 1}$ is constantly equal to x_1.

The trivial case $k = 0$ in Theorem 1.2 has been already considered in the literature from different viewpoints (and, of course, it follows by Proposition 1.1 since $\varphi(n) \leq n - 1$ for all $n \geq 2$). Indeed, given a starting value $x_1 \in \mathbb{N}$, then $x_{n+1} = \varphi^{(n)}(x_1)$ for all $n \in \mathbb{N}$, where $\varphi^{(m)}$ is the m-fold iteration of φ. For instance, Pillai [12] showed that

$$\left\lfloor \frac{\log x_1 - \log 2}{\log 3} \right\rfloor + 1 \leq N(x_1) \leq \left\lfloor \frac{\log x_1}{\log 2} \right\rfloor + 1 \quad \text{for all } x_1 \in \mathbb{N},$$

where $N(x_1)$ is the minimal integer n for which $x_n = 1$ (see also [17]) and it has been conjectured by Erdős *et al.* [4] that $N(x_1) \sim \alpha \log x_1$ as $x_1 \to \infty$, for some $\alpha \in \mathbb{R}$. It is known that the understanding of the multiplicative structure of φ and its iterates is, in

https://doi.org/10.1017/S0004972723000394 Published online by Cambridge University Press

some sense, equivalent to the study of the behaviour of the integers of the form $p-1$, where p is a prime. See also [8, 10, 14, 18] for related work.

However, if $k \geq 1$, then the function $f(n) := \varphi(n) + k$ does not satisfy (1.2): indeed, $\varphi(p) = p - 1$ for all primes p, and hence $f(p) \geq p$. However, it is well known that

$$\frac{1}{n}\sum_{i=1}^{n}\varphi(i) = \frac{3}{\pi^2}n + O(\log n) \quad \text{as } n \to \infty \tag{1.5}$$

(see, for example, [2, Theorem 8.6]). Hence, approximating roughly $f(n)$ with $cn + k$, where $c = 3/\pi^2 \in (0, 1)$, we expect that (1.2) holds 'on average', which is the heuristic behind Theorem 1.2.

Our second main result is the following theorem.

THEOREM 1.3. *Fix an even integer $k \geq 0$ and let $(x_n)_{n\geq 1}$ be the recurrence sequence defined by*

$$x_{n+2} = \varphi(x_{n+1}) + \varphi(x_n) + k \quad \textit{for all } n \in \mathbb{N},$$

with starting values $x_1, x_2 \in \mathbb{N}$. Then,

$$\sup_{n\in\mathbb{N}} x_n \leq 4^{X^{3^{k+1}}}, \quad \textit{where } X := \frac{3x_1 + 5x_2 + 7k}{2}. \tag{1.6}$$

In particular, $(x_n)_{n\geq 1}$ is eventually periodic.

The heuristic supporting Theorem 1.3 is similar: thanks to (1.5), the value $f(n, m) := \varphi(n) + \varphi(m) + k$ can be roughly bounded above by $2c\max\{n, m\} + k$, which is definitely smaller than $\max\{n, m\}$ since $2c = 6/\pi^2 < 1$.

We end with an open question to check if d is sufficiently large, then there exist starting values $x_1, \ldots, x_d \in \mathbb{N}$ such that the sequence $(x_n)_{n\geq 1}$ defined as in (1.1) with

$$f(n_1, \ldots, n_d) = \varphi(n_1) + \cdots + \varphi(n_d)$$

is *not* eventually periodic. In a sense, this is related to the open question known as Lehmer's totient problem [9], which asks about the existence of a composite $q \geq 2$ such that $\varphi(q)$ divides $q - 1$: indeed, if $r := (q-1)/\varphi(q)$, $d = r$ and $x_1 = \cdots = x_d = q$, then the sequence $(x_n)_{n\geq 1}$ would be constant.

Lastly, as suggested by the referee, it would be interesting to check whether the upper estimates of our main results are, in some sense, sufficiently sharp and whether some nontrivial lower bounds on the size of $(x_n)_{n\geq 1}$ could be obtained. We do not have an answer to these questions. However, we suspect that they are quite difficult. Indeed, in the setting of Theorem 1.2, one could try to obtain a cycle with period $(x_1, \ldots, x_T)$ made by all primes in arithmetic progression but the last one (so that $x_1 = x_{T+1} = \varphi(x_T) + k$). A possible interpretation may amount to asking whether such examples exist with comparable sizes of T and k.

https://doi.org/10.1017/S0004972723000394 Published online by Cambridge University Press

2. Proofs

PROOF OF PROPOSITION 1.1. Let Q be the set of d-uples $(n_1,\dots,n_d)\in\mathbb{N}^d$ with $n_i\le C-1$ for all $i\in\{1,\dots,d\}$. Suppose that $(n_1,\dots,n_d)\in\mathbb{N}\setminus Q$. Hence, we can pick the largest index $i\in\{1,\dots,d\}$ such that $n_i=\max\{n_1,\dots,n_d\}$. In particular, $n_i\ge C$. We claim that there exists $m\in\mathbb{N}$ such that $(n_{m+1},\dots,n_{m+d})\in Q$. In addition, if m is the least such integer, then $\max\{n_j,\dots,n_{j+d-1}\}$ is decreasing for $j\in\{1,\dots,m\}$. To this aim, suppose for the sake of contradiction that the claim does not hold. By the standing hypothesis (1.2), we get $n_{d+1}=f(n_1,\dots,n_d)<n_i$. Repeating this reasoning, we obtain $n_{d+j}<\max\{n_j,\dots,n_{d+j-1}\}=n_i$ for all $j\in\{1,\dots,i\}$. Hence, $\max\{n_{i+1},\dots,n_{i+d}\}\le n_i-1$. Proceeding similarly, it follows that

$$\max\{n_{i+(k-1)d+1},\dots,n_{i+kd}\}\le n_i-k\quad\text{for all } k\in\mathbb{N}.$$

However, if $k=n_i+1-C$, then $(n_{i+(k-1)d+1},\dots,n_{i+kd})\in Q$, which proves the claim.

To complete the proof, fix starting values $x_1,\dots,x_d\in\mathbb{N}$. By the above claim and the finiteness of Q, it follows that the sequence $(x_n)_{n\ge1}$ is bounded above by the constant

$$\max(\{x_1,\dots,x_d\}\cup\{f(n_1,\dots,n_d):(n_1,\dots,n_d)\in Q\})$$

and that the upper limit in (1.3) holds. □

PROOF OF THEOREM 1.2. First, let us suppose $k\le1$ and fix a starting value $x_1\in\mathbb{N}$. Then,

$$x_{n+1}=\varphi(x_n)+k\le\max\{1,x_n-1\}+1=\max\{2,x_n\}$$

for all $n\in\mathbb{N}$, with the consequence that $x_n\le\max\{x_1,2\}$ for all $n\in\mathbb{N}$.

Suppose hereafter that $k\ge2$. Note that for all $n,m\in\mathbb{N}$,

$$\begin{aligned}x_{n+m}&\le\max\{x_{n+m-1}-1,1\}+k\\&=\max\{x_{n+m-1}+k-1,k+1\}\le\max\{x_n+m(k-1),k+1\}.\end{aligned}\tag{2.1}$$

Let us suppose for the sake of contradiction that $(x_n)_{n\ge1}$ is not bounded above. Hence, there exists a minimal $r_1\in\mathbb{N}$ such that $x_{r_1}\ge k^4$ (in particular, $x_{r_1}>4$).

CLAIM 2.1. *There exists $i\in\{1,\dots,k\}$ such that $x_{r_1+i}<x_{r_1}$.*

PROOF. Since $\varphi(n)\le n-\sqrt{n}$ whenever n is composite (by the fact that there exists a divisor of n which is at most $\sqrt{n}$), it follows that, if x_{r_1+i-1} is composite for some $i\in\{1,\dots,k\}$, then

$$x_{r_1+i}=\varphi(x_{r_1+i-1})+k\le x_{r_1+i-1}-\sqrt{x_{r_1+i-1}}+k.$$

Considering that the map $x\mapsto x-\sqrt{x}$ is increasing on $(4,\infty)$ and using (2.1), we obtain

$$\begin{aligned}x_{r_1+i}&\le x_{r_1}+(i-1)(k-1)-\sqrt{x_{r_1}+(i-1)(k-1)}+k\\&\le x_{r_1}+(k-1)^2-\sqrt{k^4}+k<x_{r_1}.\end{aligned}$$

https://doi.org/10.1017/S0004972723000394 Published online by Cambridge University Press

To conclude, we show that there exists some $i \in \{1, \ldots, k\}$ for which x_{r_1+i-1} is composite. Indeed, in the opposite case, these x_{r_1+i-1} terms are all primes (and greater than k), and hence

$$x_{r_1+i-1} = x_{r_1} + (i-1)(k-1) \equiv x_{r_1} - (i-1) \bmod k$$

for all $i \in \{1, \ldots, k\}$. This is impossible, because there would exist $i \in \{1, \ldots, k\}$ such that k divides x_{r_1+i-1}. □

At this point, (2.1) and Claim 2.1 imply that there exists a minimal $i_1 \in \{1, \ldots, k\}$ such that $x_{r_1+i_1} < x_{r_1}$; and hence

$$\max\{x_1, \ldots, x_{r_1+i_1-1}\} \le x_{r_1} + (i_1 - 1)(k-1) < x_{r_1} + k^2. \tag{2.2}$$

With the same reasoning, we can construct recursively sequences of positive integers (r_n) and (i_n) such that for all $n \in \mathbb{N}$:

(i) r_{n+1} is the minimal integer such that $r_{n+1} \ge r_n + i_n$ and $x_{r_{n+1}} \ge k^4$;
(ii) i_{n+1} is the minimal integer in $\{1, \ldots, k\}$ such that $x_{r_{n+1}+i_{n+1}} < x_{r_n}$; and hence,

$$\max\{x_{r_n+i_n}, \ldots, x_{r_{n+1}+i_{n+1}-1}\} \le x_{r_{n+1}} + (i_{n+1} - 1)(k-1) < x_{r_{n+1}} + k^2. \tag{2.3}$$

Lastly, note that

$$x_{r_n} = \varphi(x_{r_n-1}) + k \le x_{r_n-1} + k \le k^4 + k \le x_{r_1} + k \tag{2.4}$$

for all $n \in \mathbb{N}$. Using (2.2), (2.3) and (2.4), we conclude that

$$x_n < x_{r_1} + k^2 + k \le \max\{x_1, k^4 + k\} + k^2 + k < \max\{x_1, k^4\} + (k+1)^2$$

for all $n \in \mathbb{N}$. This proves (1.4), concluding the proof.

REMARK 2.2. A sketch of a shorter proof that the sequence $(x_n)_{n\ge1}$ in Theorem 1.2 is eventually periodic goes as follows. Set $y_n := \varphi(x_n)$ for all $n \in \mathbb{N}$ and note that $y_n \le Cn$ with $C := \max\{x_1, k\}$. Hence, $\{y_1, \ldots, y_n\}$ is contained in $V_n := \varphi(\mathbb{N}) \cap [1, Cn]$. By a classical result of Pillai [13] (see also [6] and references therein), we have $|V_n| = o(n)$ as $n \to \infty$, and hence there exist distinct $i, j \in \mathbb{N}$ with $y_i = y_j$. This implies that $x_{i+1} = x_{j+1}$; therefore, $(x_n)_{n\ge1}$ is eventually periodic. However, this does not lead to an effective upper bound as in (1.4).

In the proof of Theorem 1.3, we will need also the effective version of the third Mertens' theorem given by Rosser and Schoenfeld [16] in 1962 (see also [1, 3]). As usual, hereafter, we reserve the letter p for primes.

PROPOSITION 2.3. *Let* $\gamma := \lim_n(\sum_{i\le n} 1/i - \log n) = 0.57721\ldots$ *be the Euler–Mascheroni constant. Then the following inequality holds for all* $x \ge 2$:

$$\frac{e^{-\gamma}}{\log x}\left(1 - \frac{1}{\log^2 x}\right) < \prod_{p \le x}\left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log x}\left(1 + \frac{1}{2\log^2 x}\right).$$

PROOF. See [16, Theorem 7 and its Corollary]. □

https://doi.org/10.1017/S0004972723000394 Published online by Cambridge University Press

COROLLARY 2.4. *If $x \geq 6$, then*

$$\prod_{x<p\leq x^3}\left(1-\frac{1}{p}\right)<\frac{1}{2}.$$

PROOF. Thanks to Proposition 2.3, for each $r \geq 2$, there exists $c_r \in (-1, \frac{1}{2})$ such that

$$\prod_{p\leq r}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log x}\left(1+\frac{c_r}{\log^2 x}\right).$$

At this point, fix $x \geq 6$. It follows that

$$\begin{aligned}\prod_{x<p\leq x^3}\left(1-\frac{1}{p}\right)&=\frac{\prod_{p\leq x^3}(1-1/p)}{\prod_{p\leq x}(1-1/p)}\\&=\frac{e^{-\gamma}/(\log x^3)}{e^{-\gamma}/\log x}\cdot\frac{(1+c_{x^3}/(\log x^3)^2)}{1+c_x/(\log x)^2}\\&<\frac{1}{3}\cdot\frac{1+0.5/(9\log^2 x)}{1-1/(\log^2 x)}<\frac{1}{2}.\end{aligned}$$

Indeed, the last inequality is equivalent to

$$\frac{2}{3}\left(1+\frac{1}{18\log^2 x}\right)<1-\frac{1}{\log^2 x},$$

which holds if and only if

$$x>e^{\sqrt{3(1+1/27)}}.$$

The conclusion follows since the value of the right-hand side above is smaller than 6. □

PROOF OF THEOREM 1.3. If $\max\{x_1, x_2\} \leq 2$ and $k = 0$, then $x_n = 2$ for all $n \geq 3$, so the claimed (1.6) holds since $4^{X^{3^{k+1}}} \geq 4 > x_n$ for all $n \in \mathbb{N}$.

Suppose now that $\max\{x_1, x_2\} \geq 3$ or $k \geq 1$, and note that $X = \frac{1}{2}(3x_1 + 5x_2 + 7k) \geq 6$. Then, $\min\{x_3, x_4\} \geq 3$; and hence, x_n is even for all $n \geq 5$. In addition, since $\max\{x_1, \ldots, x_6\} \leq 2X$ and

$$4^{X^{3^{k+1}}}\geq 4^X=2^{2X}>2X,$$

it follows that the claimed inequality holds for all $n \leq 6$.

Let $(p_n)_{n\geq 1}$ be the increasing enumeration of the primes greater than X. Since $\prod_{i=1}^{n}(1-1/p_i)$ converges to zero as $n \to \infty$ by Proposition 2.3, one can find integers $1 =: r_0 < r_1 < \cdots < r_{k+1}$ such that

$$\prod_{i=r_j}^{r_{j+1}-1}\left(1-\frac{1}{p_i}\right)<\frac{1}{2}<\prod_{i=r_j}^{r_{j+1}-2}\left(1-\frac{1}{p_i}\right)\quad\text{for all } j\in\{0,1,\ldots,k\}. \tag{2.5}$$

https://doi.org/10.1017/S0004972723000394 Published online by Cambridge University Press

Define also

$$q_j := \prod_{i=r_j}^{r_{j+1}-1} p_i \quad \text{for all } j \in \{0, 1, \ldots, k\}.$$

Since $\{q_0, q_1, \ldots, q_k\}$ are pairwise coprime, the Chinese remainder theorem yields the existence of some $y \in \mathbb{N}$ such that $y \equiv j \bmod q_j$ for all $j \in \{0, 1, \ldots, k\}$. In particular,

$$y \geq q_0 \geq p_{r_0} > X. \tag{2.6}$$

Let us suppose for the sake of contradiction that $(x_n)_{n\geq 1}$ is not bounded. Then, there exists a minimal $v \in \mathbb{N}$ such that $x_v \geq 2y$. Since $\max\{x_1, \ldots, x_6\} \leq 2X$, it follows by (2.9) that $v \geq 7$. In particular, x_{v-1} and x_{v-2} are even.

CLAIM 2.5. $\max\{\varphi(x_{v-1}), \varphi(x_{v-2})\} < y - k$. □

PROOF. If $x_{v-1} < 2y - 2k$ then, by the fact that x_{v-1} is even, $\varphi(x_{v-1}) \leq \frac{1}{2}x_{v-1} < y - k$. Otherwise, recalling that v is the minimal integer such that $x_v \geq 2y$, then $2y - 2k \leq x_{v-1} < 2y$. In addition, x_{v-1} is even, so $x_{v-1} = 2y - 2j$ for some $j \in \{1, \ldots, k\}$. It follows that

$$\varphi(x_{v-1}) = x_{v-1} \prod_{p \mid x_{v-1}} \left(1 - \frac{1}{p}\right) = (y - j) \prod_{p \mid x_{v-1}}{}' \left(1 - \frac{1}{p}\right),$$

where the last product is extended over the odd prime divisors of x_{v-1}. Since, by construction, we have $y \equiv j \bmod q_j$, we obtain by (2.8) that

$$\varphi(x_{v-1}) \leq (y - j) \prod_{i=r_j}^{r_{j+1}-1} \left(1 - \frac{1}{p_i}\right) < \frac{y - j}{2} < y - k.$$

Note that the last inequality holds because $y > 2k$, thanks to (2.9).

The same argument can be repeated for x_{v-2}. □

We conclude by Claim 2.5 that

$$2y \leq x_v = \varphi(x_{v-1}) + \varphi(x_{v-2}) + k < 2(y - k) + k \leq 2y,$$

which is contradiction. It follows that $x_n < 2y$ for all $n \in \mathbb{N}$.

To complete the proof, it will be enough to show that $2y \leq 4^{X^{3^{k+1}}}$. For this, define $X_n := X^{3^n}$ for all $n \geq 0$ and note that, thanks to Corollary 2.4, we have

$$\prod_{X_n < p \leq X_n^3} \left(1 - \frac{1}{p}\right) < \frac{1}{2} \quad \text{for all } n \geq 0.$$

By the definition of r_j, it follows that

$$r_j \leq X_j \quad \text{for all } j \in \{0, 1, \ldots, k + 1\}.$$

https://doi.org/10.1017/S0004972723000394 Published online by Cambridge University Press

Lastly, since $\prod_{p\le x} p < 4^x$ for all $x \ge 1$ (see, for example, [2, Lemma 2.8]), we conclude that

$$2y \le 2\prod_{j=0}^{k} q_j \le \prod_{i=1}^{r_{k+1}} p_i \le \prod_{p\le X_{k+1}} p \le 4^{X_{k+1}}.$$

Therefore, $x_n \le 4^{X_{k+1}}$ for all $n \in \mathbb{N}$.

Acknowledgement

The authors are thankful to the anonymous referee for a careful reading of the manuscript and for suggesting the open questions stated at the end of Section 1.

References

[1] C. Axler, 'New estimates for some functions defined over primes', *Integers* **18** (2018), Article no. A52, 21 pages.

[2] J.-M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, Graduate Studies in Mathematics, 134 (American Mathematical Society, Providence, RI, 2012).

[3] P. Dusart, 'Estimates of ψ, θ for large values of x without the Riemann hypothesis', *Math. Comp.* **85**(298) (2016), 875–888.

[4] P. Erdős, A. Granville, C. Pomerance and C. Spiro, 'On the normal behavior of the iterates of some arithmetic functions', in: *Analytic Number Theory (Allerton Park, IL, 1989)* (eds. B. C. Berndt, H. G. Diamond, H. Halberstam and A. Hildebrand), Progress in Mathematics, 85 (Birkhäuser, Boston, MA, 1990), 165–204.

[5] G. Everest, A. van der Poorten, I. Shparlinski and T. Ward, *Recurrence Sequences*, Mathematical Surveys and Monographs, 104 (American Mathematical Society, Providence, RI, 2003).

[6] K. Ford, 'The distribution of totients', *Ramanujan J.* **2** (1998), 67–151.

[7] R. Isaacs, 'Iterates of fractional order', *Canad. J. Math.* **2** (1950), 409–416.

[8] Y. Lamzouri, 'Smooth values of the iterates of the Euler phi-function', *Canad. J. Math.* **59**(1) (2007), 127–147.

[9] D. H. Lehmer, 'On Euler's totient function', *Bull. Amer. Math. Soc. (N.S.)* **38**(10) (1932), 745–751.

[10] P. Loomis and F. Luca, 'On totient abundant numbers', *Integers* **8** (2008), Article no. A6, 7 pages.

[11] D. Lorenzini, M. Melistas, A. Suresh, M. Suwama and H. Wang, 'Integer dynamics', *Int. J. Number Theory* **18**(2) (2022), 397–415.

[12] S. S. Pillai, 'On a function connected with $\phi(n)$', *Bull. Amer. Math. Soc. (N.S.)* **35**(6) (1929), 837–841.

[13] S. S. Pillai, 'On some functions connected with $\phi(n)$', *Bull. Amer. Math. Soc. (N.S.)* **35**(6) (1929), 832–836.

[14] P. Pollack, 'Two remarks on iterates of Euler's totient function', *Arch. Math. (Basel)* **97**(5) (2011), 443–452.

[15] A. Porges, 'A set of eight numbers', *Amer. Math. Monthly* **52** (1945), 379–382.

[16] J. B. Rosser and L. Schoenfeld, 'Approximate formulas for some functions of prime numbers', *Illinois J. Math.* **6** (1962), 64–94.

[17] H. Shapiro, 'An arithmetic function arising from the ϕ function', *Amer. Math. Monthly* **50** (1943), 18–30.

[18] I. E. Shparlinski, 'On the sum of iterations of the Euler function', *J. Integer Seq.* **9**(1) (2006), Article no. 06.1.6.

[19] B. M. Stewart, 'Sums of functions of digits', *Canad. J. Math.* **12** (1960), 374–389.

PAOLO LEONETTI, Department of Economics,
Università degli Studi dell'Insubria, via Monte Generoso 71, 21100 Varese, Italy
e-mail: leonetti.paolo@gmail.com

FLORIAN LUCA, School of Mathematics,
University of the Witwatersrand, Private Bag 3, Wits 2050, South Africa
e-mail: Florian.Luca@wits.ac.za

https://doi.org/10.1017/S0004972723000394 Published online by Cambridge University Press