# ON THE NUMBER OF DISTINCT PRIME FACTORS OF A SUM OF SUPER-POWERS

PAOLO LEONETTI AND SALVATORE TRINGALI

Abstract. Given $k, \ell \in \mathbf{N}^+$, let $x_{i,j}$ be, for $1 \le i \le k$ and $0 \le j \le \ell$, some fixed integers, and define, for every $n \in \mathbf{N}^+$, $s_n := \sum_{i=1}^k \prod_{j=0}^\ell x_{i,j}^{n^j}$. We prove that the following are equivalent:

(a) There are a real $\theta > 1$ and infinitely many indices $n$ for which the number of distinct prime factors of $s_n$ is greater than the super-logarithm of $n$ to base $\theta$.

(b) There do not exist non-zero integers $a_0, b_0, \ldots, a_\ell, b_\ell$ such that $s_{2n} = \prod_{i=0}^\ell a_i^{(2n)^i}$ and $s_{2n-1} = \prod_{i=0}^\ell b_i^{(2n-1)^i}$ for all $n$.

We will give two different proofs of this result, one based on a theorem of Evertse (yielding, for a fixed finite set of primes $\mathcal{S}$, an effective bound on the number of non-degenerate solutions of an $\mathcal{S}$-unit equation in $k$ variables over the rationals) and the other using only elementary methods.

As a corollary, we find that, for fixed $c_1, x_1, \ldots, c_k, x_k \in \mathbf{N}^+$, the number of distinct prime factors of $c_1 x_1^n + \cdots + c_k x_k^n$ is bounded, as $n$ ranges over $\mathbf{N}^+$, if and only if $x_1 = \cdots = x_k$.

## 1. Introduction

Given $k, \ell \in \mathbf{N}^+$, let $x_{i,j}$ be, for $1 \le i \le k$ and $0 \le j \le \ell$, some fixed rationals. Then, consider the $\mathbf{Q}$-valued sequence $(s_n)_{n\ge1}$ obtained by taking

$$s_n := \sum_{i=1}^k \prod_{j=0}^\ell x_{i,j}^{n^j} \tag{1}$$

for every $n \in \mathbf{N}^+$ (notations and terminology, if not explained, are standard or should be clear from the context); we refer to $s_n$ as a sum of super-powers of degree $\ell$. Notice that $(s_n)_{n\ge1}$ includes as a special case any $\mathbf{Q}$-valued sequence of general term

$$\sum_{i=1}^k \prod_{j=1}^{\ell_i} y_{i,j}^{f_{i,j}(n)}, \tag{2}$$

where, for each $i = 1, \ldots, k$, we let $\ell_i \in \mathbf{N}^+$ and $y_{i,1}, \ldots, y_{1,\ell_i} \in \mathbf{Q} \setminus \{0\}$, while $f_{i,1}, \ldots, f_{i,\ell_i}$ are polynomials in one variable with integral coefficients. Conversely, sequences of the form (1)

2010 *Mathematics Subject Classification.* Primary: 11A05, 11A41, 11A51. Secondary: 11R27, 11D99.

*Key words and phrases.* Integer sequences; number of distinct prime factors; sum of powers; $\mathcal{S}$-unit equations; super-logarithm and tetration.

P.L. was supported by a PhD scholarship from Università "Luigi Bocconi", and S.T. by NPRP grant No. [5-101-1-025] from the Qatar National Research Fund (a member of Qatar Foundation), by the French ANR Project No. ANR-12-BS01-0011, and by the Austrian FWF Project No. M 1900-N39.

can be viewed as sequences of the form (2), the latter being prototypical of scenarios where polynomials are replaced with more general functions $\mathbf{N}^+ \to \mathbf{Z}$ (see also § 4).

Now, let $\omega(x)$ denote, for each $x \in \mathbf{Z} \setminus \{0\}$, the number of distinct prime divisors of $x$, and define $\omega(0) := \infty$. Then, for $x \in \mathbf{Z}$ and $y \in \mathbf{N}^+$ we set $\omega(xy^{-1}) := \omega(\delta^{-1}x) + \omega(\delta^{-1}y)$, where $\delta$ is the greatest common divisor of $x$ and $y$.

In addition, given $n \ge 2$ and $\theta > 1$, we write $\operatorname{slog}_\theta(n)$ for the super-logarithm of $n$ to base $\theta$, that is, the largest integer $\kappa \ge 0$ for which $\theta^{\otimes\kappa} \le n$, where $\theta^{\otimes 0} := 1$ and $\theta^{\otimes\kappa} := \theta^{\theta^{\otimes(\kappa-1)}}$ for $\kappa \ge 1$; note that $\operatorname{slog}_\theta(n) \to \infty$ as $n \to \infty$.

The main goal of this paper is to provide necessary and sufficient conditions for the boundedness of the sequence $(\omega(s_n))_{n\ge1}$. More precisely, we have:

**Theorem 1.** *The following are equivalent:*

(a) *There is a base $\theta > 1$ such that $\omega(s_n) > \operatorname{slog}_\theta(n)$ for infinitely many $n$.*

(b) $\limsup_{n\to\infty} \omega(s_n) = \infty$.

(c) *There do not exist non-zero rationals $a_0, b_0, \ldots, a_\ell, b_\ell$ such that $s_{2n} = \prod_{j=0}^{\ell} a_j^{(2n)^j}$ and $s_{2n-1} = \prod_{j=0}^{\ell} b_j^{(2n-1)^j}$ for all $n$.*

We will give two proofs of Theorem 1 in § 2, one based on a theorem of Evertse (yielding, for a fixed finite set of primes $\mathcal{S}$, an effective bound on the number of non-degenerate solutions of an $\mathcal{S}$-unit equation in $k$ variables over the rationals), and the other using only elementary methods: It is, in fact, in the second proof that there lies, we hope, the added value of this work.

Results in the spirit of Theorem 1 have been obtained by various authors in the special case of $\mathbf{Z}$-valued sequences raising from the solution of non-degenerate linear homogeneous recurrence equations with (constant) integer coefficients of order $\ge 2$, namely, in relation to a sequence $(u_n)_{n\ge1}$ of general term

$$u_n := \alpha_1^n f_1(n) + \cdots + \alpha_h^n f_h(n), \tag{3}$$

where $\alpha_1, \ldots, \alpha_h$ are the non-zero (and pairwise distinct) roots of the characteristic polynomial of the recurrence under consideration, and $f_1, \ldots, f_h$ are non-zero polynomials in one variable with coefficients in the smallest field extension of the rational field containing $\alpha_1, \ldots, \alpha_h$, see [9, Theorem C.1]. (A recurrence is non-degenerate if its characteristic polynomial has at least two distinct non-zero complex roots and the ratio of any two distinct non-zero characteristic roots is not a root of unity.) More specifically, it was shown by van der Poorten and Schlickewei [14] and, independently, by Evertse [4, Corollary 3], using Schlickewei's $p$-adic analogue of Schmidt's Subspace Theorem [7], that the greatest prime factor of $u_n$ tends to $\infty$ as $n \to \infty$.

In a similar note, effective lower bounds on the greatest prime divisor and on the greatest square-free factor of a sequence of type (3) were obtained under mild assumptions by Shparlinski [10] and Stewart [11–13], based on variants of Baker's theorem on linear forms in the logarithms of algebraic numbers [2]. Further results in the same spirit can be found in [3, § 6.2].

On the other hand, Luca has shown in [6] that if $(v_n)_{n\ge1}$ is a sequence of rational numbers satisfying a recurrence of the form

$$g_0(n)v_{n+2} + g_1(n)v_{n+1} + g_2(n)v_n = 0, \quad \text{for all } n \in \mathbf{N}^+,$$

where $g_0$, $g_1$ and $g_2$ are univariate polynomials over the rational field and not all zero, and $(v_n)_{n\ge n_0}$ is not binary recurrent (viz., a solution of a linear homogeneous second-order recurrence equation with integer coefficients) for some $n_0 \in \mathbf{N}^+$, then there exists a real constant $c > 0$ such that the product of the numerators and denominators (in the reduced fraction) of the non-zero rational terms of the finite sequence $(v_i)_{1\le i\le n}$ has at least $c \log n$ prime factors as $n \to \infty$.

Lastly, it seems worth noting that Theorem 1 can be significantly improved in special cases. E.g., given $a, b \in \mathbf{N}^+$ with $a \neq b$, we have by Zsigmondy's theorem [15] that $\omega(n) \ge d(n) - 2$ for all $n$, where $d(n)$ is the number of (positive integer) divisors of $n$. Now, it is known, e.g., from [8] that $\frac{1}{n}\sum_{i=1}^n d(i)$ is asymptotic to $\log n$ as $n \to \infty$. So, it follows that there exist a constant $c \in \mathbf{R}^+$ and infinitely many $n$ for which $\omega(a^n - b^n) > c \log n$.

**Corollary 2.** *The sequence $(\omega(s_n))_{n\ge 1}$ is bounded if and only if there exist non-zero rationals $a_0, b_0, \ldots, a_\ell, b_\ell$ such that $s_{2n} = \prod_{j=0}^{\ell} a_j^{(2n)^j}$ and $s_{2n-1} = \prod_{j=0}^{\ell} b_j^{(2n-1)^j}$ for all $n$.*

**Corollary 3.** *Let $c_1, \ldots, c_k \in \mathbf{Q}^+$ and $x_1, \ldots, x_k \in \mathbf{Q} \setminus \{0\}$. Then, $(\omega(c_1 x_1^n + \cdots + c_k x_k^n))_{n\ge 1}$ is a bounded sequence only if $|x_1| = \cdots = |x_k|$, and this condition is also sufficient provided that $\sum_{i=1}^k \varepsilon_i c_i \neq 0$, where, for each $i \in [\![1, k]\!]$, $\varepsilon_i := x_i \cdot |x_i|^{-1}$ is the sign of $x_i$.*

The proof of Corollary 3 is postponed to § 3, while Corollary 2 is trivial by Theorem 1.

**Notations.** We reserve the letters $h$, $i$, $j$, and $\kappa$ (with or without subscripts) for non-negative integers, the letters $m$ and $n$ for positive integers, the letters $p$ and $q$ for (positive rational) primes, and the letters $A$, $B$, and $\theta$ for real numbers. We denote by $\mathbf{P}$ the set of all (positive rational) primes and by $v_p(x)$, for $p \in \mathbf{P}$ and a non-zero $x \in \mathbf{Z}$, the $p$-adic valuation of $x$, viz., the exponent of the largest power of $p$ dividing $x$. Given $X \subseteq \mathbf{R}$, we take $X^+ := X \cap ]0, \infty[$. Further notations, if not explained, are standard or should be clear from the context.

## 2. Proof of Theorem 1

The implications (a) ⇒ (b) and (b) ⇒ (c) are straightforward, and (c) ⇒ (a) is trivial if at least one of the sequences $(s_{2n})_{n\ge 1}$ and $(s_{2n-1})_{n\ge 1}$ is eventually zero.

Therefore, we can just focus on the two cases below, in each of which we have to prove that there exists a base $\theta > 1$ such that $\omega(s_n) > \mathrm{slog}_\theta(n)$ for infinitely many $n$.

**Case (i):** *There do not exist $a_0, \ldots, a_\ell \in \mathbf{Q}$ such that $s_{2n} = \prod_{j=0}^{\ell} a_j^{(2n)^j}$ for all $n$.* Then $k \ge 2$, $s_n \neq 0$ for infinitely many $n$, and $|x_{i,j}| \neq 1$ for some $(i, j) \in [\![1, k]\!] \times [\![1, \ell]\!]$ (otherwise we would have $s_{2n} = \sum_{i=1}^k x_{i,0}$, a contradiction).

Without loss of generality, we can suppose that $x_{i,j} \neq 0$ for all $(i, j) \in [\![1, k]\!] \times [\![0, \ell]\!]$ (otherwise we end up with a sum of super-powers with fewer than $k$ summands), and actually that $x_{i,j} > 0$ for $j \neq 0$: This is because $\prod_{j=0}^{\ell} x_{i,j}^{(2n)^j} = x_{i,0} \cdot \prod_{j=1}^{\ell} |x_{i,j}|^{(2n)^j}$ for all $n$, and, insofar as we deal with Case (i), we can replace $(s_n)_{n\ge 1}$ with the subsequence $(s_{2n})_{n\ge 1}$, after noticing that $\omega(s_{2n}) > \mathrm{slog}_\theta(n)$, for some $\theta > 1$, only if $\omega(s_{2n}) > \mathrm{slog}_{2\theta}(2n)$, which is easily proved by induction (we omit details). Accordingly, we may also assume

$$(x_{1,1}, \ldots, x_{1,\ell}) \prec \cdots \prec (x_{k,1}, \ldots, x_{k,\ell}), \tag{4}$$

where $\prec$ denotes the binary relation on $\mathbf{R}^\ell$ defined by taking $(u_1, \ldots, u_\ell) \prec (v_1, \ldots, v_\ell)$ if and only if $|u_i| < |v_i|$ for some $i \in [\![1, \ell]\!]$ and $|u_j| = |v_j|$ for $i < j \le \ell$ (the $\ell$-tuples $(x_{i,1}, \ldots, x_{i,\ell})$ cannot be equal to each other for all $i \in [\![1, k]\!]$, and on the other hand, if two of these tuples are equal, then we can add up some terms in (1) so as to obtain a sum of super-powers of degree $\ell$, but again with fewer summands). It follows by (4) that there exists $N \in \mathbf{N}^+$ such that

$$\sum_{i \in I} \prod_{j=0}^{\ell} x_{i,j}^{n^j} \ne 0, \quad \text{for all } n \ge N \text{ and } \emptyset \ne I \subseteq [\![1, k]\!]. \tag{5}$$

Now, for each $(i, j) \in [\![1, k]\!] \times [\![0, \ell]\!]$ pick $\alpha_{i,j}, \beta_{i,j} \in \mathbf{Z}$ such that $\alpha_{i,j} > 0$ and $x_{i,j} = \alpha_{i,j}^{-1}\beta_{i,j}$, and consequently set $\tilde{x}_{i,j} := \alpha_j x_{i,j}$, where $\alpha_j := \alpha_{1,j} \cdots \alpha_{k,j}$; note that $\tilde{x}_{i,j}$ is a non-zero integer, and $\tilde{x}_{i,j} > 0$ for $j \ne 0$. Then, let $u_n := \sum_{i=1}^k \prod_{j=0}^\ell \tilde{x}_{i,j}^{n^j}$ and $v_n := \prod_{j=0}^\ell \alpha_j^{n^j}$, so that $s_n = u_n v_n^{-1}$.

Clearly, $(u_n)_{n\ge1}$ and $(v_n)_{n\ge1}$ are integer sequences, and $(\tilde{x}_{i,1}, \ldots, \tilde{x}_{i,\ell}) \prec (\tilde{x}_{j,1}, \ldots, \tilde{x}_{j,\ell})$ for $1 \le i < j \le k$. Moreover, $\omega(s_n) \ge \omega(u_n) - \omega(v_n) = \omega(u_n) - \omega(v_1)$ for all $n$. This shows that it is sufficient to prove the existence of a base $\theta > 1$ such that $\omega(u_n) > \operatorname{slog}_\theta(n)$ for infinitely many $n$, and it entails, along with the rest, that we can further assume that $x_{i,j}$ is a non-zero *integer* for every $(i, j) \in [\![1, k]\!] \times [\![0, \ell]\!]$.

We claim that it is also enough to assume $\delta_0 = \cdots = \delta_\ell = 1$, where for each $j \in [\![0, \ell]\!]$ we let $\delta_j := \gcd(x_{1,j}, \ldots, x_{k,j})$. In fact, define, for $1 \le i \le k$ and $0 \le j \le \ell$, $\xi_{i,j} := \delta_j^{-1} x_{i,j}$, and let $(w_n)_{n\ge1}$ and $(\tilde{s}_n)_{n\ge1}$ be the integer sequences of general term $\prod_{j=0}^\ell \delta_j^{n^j}$ and $\sum_{i=1}^k \prod_{j=0}^\ell \xi_{i,j}^{n^j}$, respectively. Then $s_n = w_n \tilde{s}_n$, and hence $\omega(s_n) \ge \omega(\tilde{s}_n)$. On the other hand, there cannot exist $\tilde{a}_0, \ldots, \tilde{a}_\ell \in \mathbf{Z}$ such that $\tilde{s}_{2n} = \prod_{j=0}^\ell \tilde{a}_j^{(2n)^j}$ for all $n$, or else we would have $s_{2n} = \prod_{j=0}^\ell (\delta_j \tilde{a}_j)^{(2n)^j}$ for every $n$ (which is impossible). This leads to the claim.

With the above in mind, let $\mathcal{P}$ be the set of all (positive) prime divisors of $\mathfrak{z} := \prod_{i=1}^k \prod_{j=1}^\ell x_{i,j}$; observe that $\mathcal{P}$ is finite and non-empty, as the preceding considerations yield $|\mathfrak{z}| \ge 2$. Then

$$s_n = \sum_{i=1}^k \left( x_{i,0} \prod_{p \in \mathcal{P}} p^{e_p^{(i)}(n)} \right), \quad \text{for every } n \in \mathbf{N}^+, \tag{6}$$

where $e_p^{(i)}$ denotes, for all $p \in \mathbf{P}$ and $i \in [\![1, k]\!]$, the function $\mathbf{N}^+ \to \mathbf{N} : n \mapsto \sum_{j=1}^\ell n^j v_p(x_{i,j})$.

Since $\delta_0 = \cdots = \delta_\ell = 1$, it is easily seen that for every $p \in \mathbf{P}$ there are $i, j \in [\![1, k]\!]$ for which $e_p^{(i)} \ne e_p^{(j)}$, and there exist $i_p \in [\![1, k]\!]$ and $n_p \ge N$ such that $e_p^{(i_p)}(n) < e_p^{(i)}(n)$ for all $n \ge n_p$ and $i \in [\![1, k]\!]$ for which $e_p^{(i)} \ne e_p^{(i_p)}$. Let $n_\mathcal{P} := \max_{p \in \mathcal{P}} n_p$ (recall that $\mathcal{P}$ is a non-empty finite set), and for each $p \in \mathcal{P}$ and $i \in [\![1, k]\!]$ define $\Delta e_p^{(i)} := e_p^{(i)} - e_p^{(i_p)}$. Then set

$$\pi_n := \prod_{p \in \mathcal{P}} p^{e_p^{(i_p)}(n)} \quad \text{and} \quad \sigma_n := \sum_{i=1}^k \left( x_{i,0} \prod_{p \in \mathcal{P}} p^{\Delta e_p^{(i)}(n)} \right). \tag{7}$$

We have $|s_n| = \pi_n \cdot |\sigma_n|$, and we obtain from (5) that $\sigma_n \in \mathbf{Z} \setminus \{0\}$ for $n \ge n_\mathcal{P}$. Furthermore, having assumed $x_{i,j} > 0$ for all $(i, j) \in [\![1, k]\!] \times [\![1, \ell]\!]$ implies, together with (4) and (6), that

$$\lim_{n\to\infty} \prod_{p \in \mathcal{P}} p^{e_p^{(k)}(n) - e_p^{(i)}(n)} = \lim_{n\to\infty} \prod_{j=1}^\ell \left( \frac{x_{k,j}}{x_{i,j}} \right)^{n^j} = \infty, \quad \text{for each } i \in [\![1, k-1]\!]. \tag{8}$$

Consequently, we find that

$$|s_n| \sim |x_{k,0}| \cdot \prod_{j=1}^{\ell} x_{k,j}^{n^j} = |x_{k,0}| \cdot \prod_{p\in\mathcal{P}} p^{e_p^{(k)}(n)}, \quad \text{as } n \to \infty \tag{9}$$

and

$$|\sigma_n| = \frac{|s_n|}{\pi_n} \sim |x_{k,0}| \cdot \prod_{p\in\mathcal{P}} p^{\Delta e_p^{(k)}(n)}, \quad \text{as } n \to \infty. \tag{10}$$

We want to show that the sequence $(|\sigma_n|)_{n\ge 1}$ is eventually (strictly) increasing.

**Lemma 1.** *There exists $p \in \mathcal{P}$ such that $\Delta e_p^{(k)}(n) \to \infty$ as $n \to \infty$.*

*Proof.* Suppose the contrary is true. Then, for each $p \in \mathcal{P}$ we have $e_p^{(k)} = e_p^{(i_p)}$, since $\Delta e_p^{(k)}(n)$ is basically a polynomial with integral coefficients in the variable $n$ and $\Delta e_p^{(k)}(n) = e_p^{(k)}(n) - e_p^{(i_p)}(n) \ge 0$ for $n \ge n_{\mathcal{P}}$. Therefore, we get from (8) that

$$\prod_{p\in\mathcal{P}} p^{e_p^{(i_p)}(n)} \le \prod_{p\in\mathcal{P}} p^{e_p^{(i)}(n)} \le \prod_{p\in\mathcal{P}} p^{e_p^{(k)}(n)} = \prod_{p\in\mathcal{P}} p^{e_p^{(i_p)}(n)}$$

for all $n \ge n_{\mathcal{P}}$ and $i \in [\![1,k]\!]$. But this is impossible, as it implies that $e_p^{(i)} = e_p^{(i_p)}$ for all $p \in \mathcal{P}$ and $i \in [\![1,k]\!]$, and hence, in view of (6), $s_n = (x_{1,0} + \cdots + x_{k,0}) \cdot \prod_{p\in\mathcal{P}} p^{e_p^{(i_p)}(n)}$ for all $n$. ∎

Now, let $A := 2\mathfrak{z}^2$ (this is just a convenient value for $A$: We make no effort to try to optimize it, and the same is true for other constants later on). Since $\Delta e_p^{(k)}$ is eventually non-decreasing for every $p \in \mathbf{P}$ (recall that $\Delta e_p^{(k)}$ is a polynomial function and $\Delta e_p^{(k)}(n) \ge 0$ for all large $n$), we obtain from (5), (9), (10), and Lemma 1 that there exists $n_0 \ge \max(2, n_{\mathcal{P}})$ such that

$$\sigma_n^2 \le s_n^2 < A^{n^\ell} \quad \text{and} \quad 0 \ne |\sigma_n| < |\sigma_{n+1}|, \quad \text{for } n \ge n_0. \tag{11}$$

From here on, the proof of Case (i) splits, as we present two different approaches that can be used to finish it, the first of them relying on a theorem of Evertse from [5], and the second using only elementary methods (as anticipated in the introduction).

1st Approach: Let $\mathfrak{y} := \mathfrak{z} \cdot \prod_{i=1}^k |x_{i,0}|$ and $B := \max(n_0^\ell, (2^{35}k^2)^{2k^3\mathfrak{y}\ell\log A})$. We will need the following:

**Lemma 2.** *There is a sequence $(r_\kappa)_{\kappa\ge 0}$ of integers $\ge n_0$ such that $r_\kappa^\ell \le B^{\otimes(\kappa+1)}$ and $\omega(s_{r_\kappa}) \ge \kappa$ for every $\kappa \in \mathbf{N}$.*

*Proof.* Set $r_0 := n_0$, fix $\kappa \in \mathbf{N}^+$, and suppose we have already found an integer $r_{\kappa-1} \ge n_0$ such that $r_{\kappa-1}^\ell \le B^{\otimes\kappa}$ and $\omega(s_{r_{\kappa-1}}) \ge \kappa - 1$: Notice how these conditions are trivially satisfied for $\kappa = 1$, because $r_0^\ell = n_0^\ell \le B = B^{\otimes 1}$ and $\omega(x) \ge 0$ for all $x \in \mathbf{Z}$.

Accordingly, denote by $\mathcal{S}_\kappa$ the set of prime divisors of $\mathfrak{y}\cdot s_{r_{\kappa-1}}$, and for all $n \ge n_0$ and $i \in [\![1,k]\!]$ let $X_i(n) := s_n^{-1} \cdot \prod_{j=0}^{\ell} x_{i,j}^{n^j}$ (note that these quantities are well defined, since we have by (5) that $s_n \ne 0$ for $n \ge n_0$). A few remarks are in order.

Firstly, it is easy to check that, for every $n \ge n_0$, the $k$-tuple $\boldsymbol{X}_n := (X_1(n), \ldots, X_k(n)) \in \mathbf{Q}^k$ is a solution to the following equation (over the additive group of the rational field):

$$Y_1 + \cdots + Y_k = 1, \tag{12}$$

and we derive from (5) that it is actually a *non-degenerate* solution, where a solution $(Y_1, \ldots, Y_k)$ of (12) is called non-degenerate if $\sum_{i \in I} Y_i \neq 0$ for every non-empty $I \subseteq [\![1, k]\!]$.

Secondly, it is plain from our definitions that $\boldsymbol{X}_m = \boldsymbol{X}_n$, for some $m, n \geq n_0$, only if

$$\upsilon_p(X_i(m)) = \upsilon_p(X_i(n)), \quad \text{for all } p \in \mathbf{P} \text{ and } i \in [\![1, k]\!], \tag{13}$$

and we want to show that this, in turn, is possible only if $|\sigma_m| = |\sigma_n|$.

Indeed, let $p \in \mathbf{P}$ and $n \geq n_0$. By construction, the $p$-adic valuation of $\prod_{j=1}^{\ell} x_{i_p,j}^{n^j}$ is equal to $e_p^{(i_p)}(n)$, with $e_p^{(i_p)}(n)$ being zero if $p \notin \mathcal{P}$. Thus, we obtain from (7) that

$$\upsilon_p(X_{i_p}(n)) = \upsilon_p(x_{i_p,0}) + e_p^{(i_p)}(n) - \upsilon_p(s_n) = \upsilon_p(x_{i_p,0}) - \upsilon_p(\sigma_n).$$

It follows that, for $m, n \geq n_0$, (13) holds true only if $\upsilon_p(\sigma_m) = \upsilon_p(\sigma_n)$ for all $p \in \mathbf{P}$, which is equivalent to $|\sigma_m| = |\sigma_n|$. Accordingly, we conclude from (11) and the above that, for $m, n \geq n_0$ and $m \neq n$, $\boldsymbol{X}_m$ and $\boldsymbol{X}_n$ are *distinct* non-degenerate solutions of (12).

Thirdly, let $N_\kappa$ be the number of non-degenerate solutions $(Y_1, \ldots, Y_k)$ to (12) for which each $Y_i$ is an $\mathcal{S}_\kappa$-unit (i.e., lies in the subgroup of the multiplicative group of $\mathbf{Q}$ generated by $\mathcal{S}_\kappa$). We obtain from [5, Theorem 3] that $N_\kappa \leq (2^{35}k^2)^{k^3 g_\kappa}$, where

$$g_\kappa := |\mathcal{S}_\kappa| \leq \omega(\mathfrak{y}) + \omega(s_{r_{\kappa-1}}) \leq \mathfrak{y} + \log|s_{r_{\kappa-1}}| \overset{(11)}{\leq} \mathfrak{y} + r_{\kappa-1}^{\ell} \log A \leq \mathfrak{y} \cdot r_{\kappa-1}^{\ell} \log A.$$

Using that $r_{\kappa-1}^{\ell} \leq B^{\otimes \kappa}$ (by the inductive hypothesis), we thus conclude that

$$N_\kappa \leq (2^{35}k^2)^{B^{\otimes \kappa} k^3 \mathfrak{y} \log A} \leq B^{B^{\otimes \kappa}/(2\ell)}. \tag{14}$$

With this in hand, define $\wp := \prod_{p \,|\, s_{r_{\kappa-1}}} (p-1)$ and let $(t_h)_{h \geq 0}$ be the subsequence of $(s_n)_{n \geq 1}$ of general term $t_h := s_{h\wp + r_{\kappa-1}}$. We know from the above that there exists $h_\kappa \in [\![0, N_\kappa]\!]$ such that $t_{h_\kappa}$ is not an $\mathcal{S}_\kappa$-unit (note that $h\wp + r_{\kappa-1} \geq r_{\kappa-1} \geq n_0$ for all $h$), with the result that at least one prime divisor of $t_{h_\kappa}$ does not divide $s_{r_{\kappa-1}}$. On the other hand, a straightforward application of Fermat's little theorem shows that $p \mid t_{h_\kappa}$ for every $p \in \mathbf{P}$ such that $p \mid s_{r_{\kappa-1}}$. So, putting it all together, we find $\omega(s_{r_\kappa}) \geq 1 + \omega(s_{r_{\kappa-1}}) \geq \kappa$, where

$$\begin{aligned} r_\kappa := h_\kappa \wp + r_{\kappa-1} &\leq N_\kappa s_{r_{\kappa-1}} + r_{\kappa-1} \leq N_\kappa s_{r_{\kappa-1}} r_{\kappa-1} \overset{(11)}{\leq} A^{r_{\kappa-1}^{\ell}} N_\kappa r_{\kappa-1} \leq A^{2r_{\kappa-1}^{\ell}} N_\kappa \\ &\overset{(14)}{\leq} A^{2r_{\kappa-1}^{\ell}} B^{B^{\otimes \kappa}/(2\ell)} \leq A^{2B^{\otimes \kappa}} B^{B^{\otimes \kappa}/(2\ell)} \leq B^{B^{\otimes \kappa}/(2\ell)} B^{B^{\otimes \kappa}/(2\ell)} \end{aligned}$$

(recall from the above that $r_{\kappa-1}^{\ell} \leq B^{\otimes \kappa}$). This completes the induction step (and hence the proof of the lemma), since it implies $r_\kappa^{\ell} \leq B^{\frac{3}{4} B^{\otimes \kappa}} B^{\otimes \kappa} \leq B^{\otimes (\kappa+1)}$. ■

So to conclude, let $(r_\kappa)_{\kappa \geq 0}$ be the sequence of Lemma 2 and take $\theta := B^{\otimes 3}$. Then $\theta > 1$ and $\omega(s_{r_\kappa}) \geq \kappa > \text{slog}_\theta(r_\kappa)$ for all $\kappa \in \mathbf{N}^+$, because $r_\kappa \leq r_\kappa^{\ell} \leq B^{\otimes(\kappa+1)}$.

2ND APPROACH: Denote by $\mathcal{Q}_n$ the set of all prime divisors of $\sigma_n$ and let $\mathcal{Q}_n^\star := \mathcal{Q}_n \setminus \mathcal{P}$. It is clear that $\mathcal{Q}_n$ is finite for $n \geq n_0$ (recall that $\sigma_n \neq 0$ for $n \geq n_{\mathcal{P}}$). Thus, let

$$\lambda := \max_{p \in \mathcal{P}} \upsilon_p(\sigma_{n_0}) + \max_{p \in \mathcal{P}} \max_{1 \leq i \leq k} \Delta e_p^{(i)}(n_0),$$

and then

$$\alpha := k \cdot \max_{1 \le i \le k} |x_{i,0}| \cdot \prod_{p \in \mathcal{P}} p^{\lambda}, \quad \beta := \prod_{p \in \mathcal{P}} p^{\alpha-1}(p-1), \quad \text{and} \quad B := A^2\beta.$$

Lastly, suppose that, for a fixed $\kappa \in \mathbf{N}$, we have already found $r_0, \ldots, r_\kappa \in \mathbf{N}^+$ with $n_0 \le r_0 \le \cdots \le r_\kappa$, and define $\beta_\kappa := \beta \cdot \prod_{p \in \mathcal{Q}^\star_{r_\kappa}} p^{\upsilon_p(\sigma_{r_\kappa})}(p-1)$.

By taking $r_0 := n_0$ and $r_{\kappa+1} := \beta_\kappa + r_\kappa$, we obtain an increasing sequence $(r_\kappa)_{\kappa \ge 0}$ of integers $\ge n_0$ with the property that, however we choose a prime $p \in \mathcal{P}$ and an index $i \in [\![1, k]\!]$,

$$\Delta e_p^{(i)}(r_{\kappa+1}) \equiv \Delta e_p^{(i)}(r_\kappa) \bmod q^{\alpha-1}(q-1), \quad \text{for all } q \in \mathcal{P} \tag{15}$$

and

$$\Delta e_p^{(i)}(r_{\kappa+1}) \equiv \Delta e_p^{(i)}(r_\kappa) \bmod q^{\upsilon_q(\sigma_{r_\kappa})}(q-1), \quad \text{for all } q \in \mathcal{Q}^\star_{r_\kappa}, \tag{16}$$

where we use that $\Delta e_p^{(i)}$ is essentially a polynomial with integral coefficients, and $r_{\kappa+1} \equiv r_\kappa \bmod m$ whenever $m \mid \beta_\kappa$. In particular, (15) and a routine induction imply

$$\Delta e_p^{(i)}(r_\kappa) \equiv \Delta e_p^{(i)}(n_0) \bmod q^{\alpha-1}(q-1), \quad \text{for all } p, q \in \mathcal{P},\ i \in [\![1, k]\!], \text{ and } \kappa \in \mathbf{N}. \tag{17}$$

Also, since $r_\kappa \ge n_0$, there exists $B > A$ such that, for all $\kappa$,

$$r_{\kappa+1} \le r_\kappa + \beta \cdot \prod_{p \in \mathcal{Q}_{r_\kappa}} p^{\upsilon_p(\sigma_{r_\kappa})}(p-1) \le r_\kappa + \beta\sigma_{r_\kappa}^2 < r_\kappa + \beta A^{r_\kappa^\ell} \le \beta r_\kappa A^{r_\kappa^\ell} \overset{(11)}{<} B^{r_\kappa^\ell}. \tag{18}$$

Based on these premises, we prove a series of three lemmas. To ease notation, we denote by $I_p$, for each $p \in \mathcal{P}$, the set of all $i \in [\![1, k]\!]$ such that $e_p^{(i)} \ne e_p^{(i_p)}$, and we let $I_p^\star := [\![1, k]\!] \setminus I_p$.

**Lemma 3.** *$\mathcal{Q}_{r_\kappa} \subseteq \mathcal{Q}_{r_{\kappa+1}}$ for every $\kappa$.*

*Proof.* Pick $\kappa \in \mathbf{N}$ and $q \in \mathcal{Q}_{r_\kappa}$. If $i \in I_p$, then $\Delta e_p^{(i)}(n) = 0$ for all $n$, and hence $p^{\Delta e_p^{(i)}(n)} = 1$. If, on the other hand, $i \in I_p^\star$, then $\Delta e_p^{(i)}(n) > 0$ for $n \ge n_{\mathcal{P}}$, and we conclude from Fermat's little theorem that $p^{\Delta e_p^{(i)}(n)} \equiv 0 \bmod q$ if $q = p$, and $p^{\Delta e_p^{(i)}(n)} \equiv p^m \bmod q$ if $p \ne q$ and $\Delta e_p^{(i)}(n) \equiv m \bmod (q-1)$. So we get from (15), (16), and $r_{\kappa+1} > r_\kappa \ge n_0 > n_{\mathcal{P}}$ that

$$p^{\Delta e_p^{(i)}(r_{\kappa+1})} \equiv p^{\Delta e_p^{(i)}(r_\kappa)} \bmod q, \quad \text{for all } p \in \mathcal{P} \text{ and } i \in [\![1, k]\!],$$

which in turn implies

$$\sigma_{r_{\kappa+1}} \equiv \sum_{i=1}^k \left( x_{i,0} \prod_{p \in \mathcal{P}} p^{\Delta e_p^{(i)}(r_{\kappa+1})} \right) \equiv \sum_{i=1}^k \left( x_{i,0} \prod_{p \in \mathcal{P}} p^{\Delta e_p^{(i)}(r_\kappa)} \right) \equiv \sigma_{r_\kappa} \equiv 0 \bmod q.$$

This finishes the proof, since $\kappa \in \mathbf{N}$ and $q \in \mathcal{Q}_{r_\kappa}$ were arbitrary. ∎

**Lemma 4.** *Let $q \in \mathcal{P}$ and $\kappa \in \mathbf{N}$. Then $\upsilon_q(\sigma_{r_\kappa}) \le \alpha - 1$.*

*Proof.* The claim is straightforward if $\kappa = 0$, since $r_0 = n_0$ and $\upsilon_q(\sigma_{n_0}) \le \lambda < \alpha$. So assume for the rest of the proof that $\kappa \ge 1$. Then, we have from (7) that

$$\sigma_n = \sum_{i \in I_q} \left( x_{i,0} \prod_{p \in \mathcal{P}} p^{\Delta e_p^{(i)}(n)} \right) + \sum_{i \in I_q^\star} \left( x_{i,0} \prod_{q \ne p \in \mathcal{P}} p^{\Delta e_p^{(i)}(n)} \right), \quad \text{for all } n. \tag{19}$$

If $i \in I_q$, $n > n_0$ and $n \equiv n_0 \bmod \beta$, then $q^\alpha$ divides $\prod_{p\in\mathcal{P}} p^{\Delta e_p^{(i)}(n)}$, because $n \mid \Delta e_p^{(i)}(n)$ and $\Delta e_p^{(i)}(n) \neq 0$, hence $\alpha < \beta < \beta + n_0 \le n \le \Delta e_p^{(i)}(n)$.

On the other hand, it is seen by induction that $r_\kappa \equiv n_0 \bmod \beta$ (recall that $r_\kappa \equiv r_{\kappa-1} \bmod \beta$). Thus, we get from the above, equations (19) and (17), [1, Theorem 2.5(a)], and Euler's totient theorem that

$$\sigma_{r_\kappa} \equiv \sum_{i\in I_q^\star}\left(x_{i,0}\prod_{q\neq p\in\mathcal{P}} p^{\Delta e_p^{(i)}(r_\kappa)}\right) \equiv \sum_{i\in I_q^\star}\left(x_{i,0}\prod_{q\neq p\in\mathcal{P}} p^{\Delta e_p^{(i)}(n_0)}\right) \bmod q^\alpha. \tag{20}$$

But $\emptyset \neq I_q^\star \subseteq [\![1,k]\!]$, so it follows from (5) that

$$\begin{aligned} 0 &< \left|\frac{1}{\pi_{n_0}}\sum_{i\in I_q^\star}\prod_{j=0}^{\ell} x_{i,j}^{(n_0)^j}\right| = \left|\sum_{i\in I_q^\star}\left(x_{i,0}\prod_{q\neq p\in\mathcal{P}} p^{\Delta e_p^{(i)}(n_0)}\right)\right| \\ &\le \max_{1\le i\le k}|x_{i,0}|\cdot\sum_{i\in I_q^\star}\prod_{q\neq p\in\mathcal{P}} p^{\Delta e_p^{(i)}(n_0)} \le k\cdot\max_{1\le i\le k}|x_{i,0}|\cdot\prod_{p\in\mathcal{P}} p^\lambda = \alpha < q^\alpha, \end{aligned}$$

which, together with (20), yields $\upsilon_q(\sigma_{r_\kappa}) < \alpha$. ∎

**Lemma 5.** *Let $\kappa \in \mathbf{N}^+$ and $q \in \mathcal{Q}_{r_\kappa}$. Then $\upsilon_q(\sigma_{r_\kappa}) = \upsilon_q(\sigma_{r_{\kappa+1}})$.*

*Proof.* If $q \notin \mathcal{P}$, then we infer from (7) and (16), [1, Theorem 2.5(a)], and Euler's totient theorem that $\sigma_{r_{\kappa+1}} \equiv \sigma_{r_\kappa} \bmod q^{\upsilon_q(\sigma_{r_\kappa})+1}$, and we are done.

If, on the other hand, $q \in \mathcal{P}$, then we get from Lemma 4 that $\upsilon_q(\sigma_{r_1}) \le \alpha - 1$, which, along with (20), gives $\sigma_{r_\kappa} \equiv \sigma_{r_1} \bmod q^{\upsilon_q(\sigma_{r_1})+1}$, and consequently $\upsilon_q(\sigma_{r_\kappa}) = \upsilon_q(\sigma_{r_1})$. ∎

At this point, since $(|\sigma_n|)_{n\ge n_0}$ is an increasing sequence by (11) and $r_\kappa \ge n_0$ for all $\kappa \in \mathbf{N}^+$, we see from Lemmas 3-5 that $\emptyset \neq \mathcal{Q}_{r_\kappa} \subsetneq \mathcal{Q}_{r_{\kappa+1}}$, and hence $\omega(\sigma_{r_\kappa}) < \omega(\sigma_{r_{\kappa+1}})$. By induction, this implies $\omega(\sigma_{r_\kappa}) \ge \kappa$ for every $\kappa$.

On the other hand, if we let $\theta := \max(B^\ell\ell, r_1^\ell)$, then we get from (18) and another induction that $r_\kappa^\ell < \theta^{\otimes\kappa}$ for all $\kappa \in \mathbf{N}^+$, which, together with the above considerations, leads to $\omega(\sigma_{r_\kappa}) \ge \kappa > \operatorname{slog}_\theta(r_\kappa)$ and the desired conclusion.

**Case (ii):** *There do not exist $b_0, \ldots, b_\ell \in \mathbf{Q}$ such that $s_{2n-1} = \prod_{j=0}^{\ell} b_j^{(2n-1)^j}$ for all $n$.* Then, we are reduced to Case (i) by taking

$$y_{i,j} := \prod_{h=j}^{\ell} x_{i,h}^{(-1)^{h-j}\binom{h}{j}}, \quad \text{for } 1 \le i \le k \text{ and } 0 \le j \le \ell,$$

and by noting that for every $n \in \mathbf{N}^+$ we have $s_{2n-1} = t_{2n}$, where $(t_n)_{n\ge1}$ is the integer sequence of general term $\sum_{i=1}^{k}\prod_{j=0}^{\ell} y_{i,j}^{n^j}$ (we omit further details).

## 3. Proof of Corollary 3

Suppose for a contradiction that there are $c_1, \ldots, c_k \in \mathbf{Q}^+$ and $x_1, \ldots, x_k \in \mathbf{Q}\setminus\{0\}$ such that $|x_i| \neq |x_j|$ for some $i, j \in [\![1,k]\!]$ and $(\omega(u_n))_{n\ge1}$ is bounded, where $u_n := \sum_{i=1}^{k} c_i x_i^n$ for all $n$, and let $k$ the *minimal* positive integer for which this is pretended to be true.

Then $k \ge 2$, and we can assume that $|x_1| \le \cdots \le |x_k| \ne |x_1|$. Furthermore, we get from Theorem 1 that there must exist $c, x \in \mathbf{Q}^+$ such that $u_{2n} = cx^{2n}$. So now, we have two cases, each of which will lead to a contradiction (the rest is trivial and we may omit details):

**Case (i):** $x \le |x_k|$. We have $cy^{2n} = \sum_{i=1}^k c_i y_i^{2n}$ for all $n$, where $y_i := |x_i| \cdot |x_k|^{-1}$ for $1 \le i \le k$ and $y := x \cdot |x_k|^{-1}$. Let $h$ be the maximal index in $[\![2, k]\!]$ such that $y_{h-1} < y_k$, which exists because $y_1 < y_k$. Since $0 < y \le 1$ and $0 < y_i < 1$ for $1 \le i < h$, we find that

$$c \cdot \lim_{n \to \infty} y^{2n} = c_h + \cdots + c_k,$$

which can happen only if $y = 1$, as $c_h, \ldots, c_k > 0$. But then $c = c_1 + \cdots + c_k$, and consequently $\sum_{i=1}^{h-1} c_i y_i^{2n} = 0$ for all $n$, which is impossible, because $h \ge 2$ and $c_1, \ldots, c_{h-1} > 0$.

**Case (ii):** $x > |x_k|$. Then $c = \sum_{i=1}^k c_i z_i^{2n}$ for all $n$, where $z_i := |x_i| \cdot x^{-1}$ for $1 \le i \le k$. But this is still impossible, since $z_1, \ldots, z_k \in \,]0, 1[$, and hence $\sum_{i=1}^k c_i z_i^{2n} \to 0$ as $n \to \infty$.

## 4. Closing remarks

Let $\tau$ be an increasing function from $\mathbf{N}^+$ into itself. What can be said about the behavior of $\omega(s_{\tau(n)})$ as $n \to \infty$? And what about the asymptotic growth of the average of the function $\mathbf{R}^+ \to \mathbf{N} : x \mapsto \#\{n \le x : \omega(s_{\tau(n)}) \ge h\}$ for a fixed $h \in \mathbf{N}^+$?

In this paper, we have answered the first question in the case where $\tau$ is the identity or, more in general, a polynomial function (by the considerations made in the introduction). It could be interesting as a next step to look at the case where $\tau$ is a geometric progression, which however may be hard, when taking into account that it is a longstanding open problem to decide whether there are infinitely many composite Fermat numbers (that is, numbers of the form $2^{2^n} + 1$).

On the other hand, the basic question addressed in the present manuscript has the following algebraic generalization: Given a unique factorization domain $D$, let $\alpha_{i,j}$ be, for $1 \le i \le k$ and $0 \le j \le \ell$, some fixed elements in $D$, and for $x \in D$ let $\omega_D(x)$ denote the number of non-associate primes dividing $x$. What can be said about the sequence $(A_n)_{n \ge 1}$ of general term $\sum_{i=1}^k \prod_{j=0}^{\ell} \alpha_{i,j}^{n^j}$ if the sequence $(\omega_D(A_n))_{n \ge 1}$ is bounded? Does anything along the lines of Theorem 1 hold true?

## Acknowledgements

The authors are grateful to Carlo Sanna (Università di Torino, IT) for useful comments. Moreover, they are greatly indebted to an anonymous referee for outlining a proof of Lemma 2 in the case when the numbers $x_{i,j}$ are positive, and most notably for explaining how to use [5, Theorem 3] to bound, in the notations of the same proof, the number of non-degenerate $\mathcal{S}_\kappa$-unit solutions of equation (12), which is the key step in the first approach ("Approach 1") to the proof of Theorem 1.

## References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergrad. Texts Math., Springer-Verlag: New York, 1976.

[2] A. Baker and G. Wüstholz, *Logarithmic forms and Diophantine geometry*, New Mathematical Monographs **9**, Cambridge University Press: Cambridge, 2007.
[3] G. Everest, A. van der Poorten, I. Shparlinski, and T. Ward, *Recurrence Sequences*, Math. Surveys Monogr. **104**, Amer. Math. Soc.: Providence (RI), 2003.
[4] J.-H. Evertse, *On sums of S-units and linear recurrences*, Compositio Math. **53** (1984), No. 2, 225–244.
[5] ———, *The number of solutions of decomposable form equations*, Invent. Math. **122** (1995), No. 3, 559–601.
[6] F. Luca, *Prime divisors of binary holonomic sequences*, Adv. Appl. Math. **40** (2008), No. 2, 168–179.
[7] H. P. Schlickewei, *Linearformen mit algebraischen Koeffizienten*, Manuscripta Math. **18** (1976), 147–185.
[8] R. A. Smith and M. V. Subbarao, *The average number of divisors in an arithmetic progression*, Canad. Math. Bull. **24** (1981), No. 1, 37–41.
[9] T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge Tracts in Mathematics **87**, Cambridge Univ. Press: Cambridge, 1986.
[10] I. E. Shparlinski, *Prime divisors of recurrent sequences*, Isv. Vyssh. Uchebn. Zaved. Math. **215** (1980), 101–103.
[11] C. L. Stewart, *On divisors of terms of linear recurrence sequences*, J. reine angew. Math. **333** (1982), 12–31.
[12] ———, *On the greatest prime factor of terms of a linear recurrence sequence*, Rocky Mountain J. Math. **15** (1985), No. 2, 599–608.
[13] ———, "On Prime Factors of Terms of Linear Recurrence Sequences", pp. 341–359 in J. M. Borwein, I. Shparlinski, and W. Zudilin (eds.), *Number Theory and Related Fields. In Memory of Alf van der Poorten*, Springer Proc. Math. Stat. **43**, Springer: New York, 2013.
[14] A. J. van der Poorten and H. P. Schlickewei, *The growth conditions for recurrence sequences*, Macquarie Math. Reports 82-0041 (1982).
[15] K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. **3** (1892), 265–284.

Department of Decision Sciences, Università L. Bocconi, via Roentgen 1, 20136 Milano, Italy
*E-mail address*: leonetti.paolo@gmail.com
*URL*: https://sites.google.com/site/leonettipaolo/

Department of Mathematics, Texas A&M University at Qatar, PO Box 23874 Doha, Qatar
*Current address*: Institute for Mathematics and Scientific Computing, University of Graz, NAWI Graz | Heinrichstr. 36, 8010 Graz, Austria
*E-mail address*: salvatore.tringali@uni-graz.at
*URL*: http://imsc.uni-graz.at/tringali