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Wall crossing structure from quantum phenomena to Feynman Integrals

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Abstract: A growing body of evidence suggests that the complexity of Feynman integrals is best understood through geometry. Recent mathematical developments [arXiv:2402.07343] have illuminated the role of exponential integrals as periods of twisted de Rham cocycles over Betti cycles, providing a structured approach to tackle this problem in many situations. In this paper, we apply these concepts to show how families of physically relevant integrals, ranging from exponentials to logarithmic multivalued functions, can be recast as twisted periods of differential forms over homology cycles. In the case of holomorphic exponents, we provide explicit decompositions as thimble expansions and reveal a geometric wall-crossing structure behind the analytic continuation in parameters. We then show that the generalization to multivalued functions provides the right framework to describe Feynman integrals in the Baikov representation, where the multivaluedness is governed by the logarithm of the Baikov polynomial. In this context, the thimble decomposition aligns with the decomposition into Master Integrals. We highlight how the wall-crossing structure allows for a sharp count of independent Master Integrals (or periods), circumventing complications arising from Stokes phenomena. Additionally, we study the large-parameter expansions of these integrals, whose coefficients correspond to periods of standard (co-)homology associated with families of algebraic varieties, and which reveal the dominant basis elements in different sectors of the wall crossing structure. This unifies perturbative expansions and geometric representation theory under a single cohomological framework.

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1 Introduction

Computing integrals is one of the main activities of physicists: evaluate fluxes of electric or magnetic fields, determine expectation values for processes in quantum mechanics, calculate averages in statistical physics, or infinite dimensional integrals in defining quantum field theories, and so on and so forth. More broadly, many physical problems reduce to solving systems of nonlinear partial differential equations [2–6] (see [7, 8] for recent developments in reduction methods). In very few cases, this can be done exactly, but generically it must be done perturbatively with various techniques. This is how Feynman integrals, for example, find their origin. The ever-increasing precision with which modern technologies allow us to test theory against experiments requires ever-increasing precision in solving the equations that characterize the theory. In most cases, this means being able to compute more and more difficult integrals. Nevertheless, certain kinds of integrals are easier than others. This is what we learn from Stokes' theorem: when we consider fluxes of closed forms over closed cycles, we can deform the cycle and add exact forms to the integrands without changing the final result. Here is where homology and cohomology start playing a central role.

In the simplest cases, mathematically, these integrals correspond to periods of elements of a de Rham cohomology over closed cycles of a dual homology (closed manifolds up to boundaries). However, more general cases can be considered by suitably identifying the correct cohomology theory entering the game. An important example is the case of hypergeometric type integrals and their generalizations, like GKZ equation systems and Euler integrals, where it has been realized that they can be again understood in terms of co-homology after replacing the de Rham cohomology with its twisted version, and the dual cohomology with one realized in terms of open twisted cycles [9–28].

In [29], observing a similar underlying structure in Feynman integrals in QFT on Minkowski spacetime, it was proposed that the same strategies be applied to the computation of scattering amplitudes. The key advantage lies in the fact that both homology and cohomology rings are finitely generated and endowed with a non-degenerate internal product, the topological intersection product for homology and its dual in cohomology, defined between closed forms. In this way, a given Feynman integral can be identified as an element of a finite dimensional vector space endowed with a double structure suitable for projecting any vector to a given basis, given precisely by the intersection products of (co)homology. Moreover, when a Hodge structure is available, important quadratic relations, generalizing the Riemann quadratic relations, can be determined, see [30, 31]. The identification of Feynman integrals with periods of a suitable cohomology [32–40] thus allows to replace the integration by part identities (IBP) ([41, 42]) with intersection projection, that provide a more systematic procedure, both for determining a basis of master integrals (MI) and to decompose the vector space accordingly, as well as for finding differential equations and quadratic identities satisfied by the MI, see [43–58] for several successful applications to physics. The efficiency of this strategy depends on identifying the appropriate cohomology and efficient ways to calculate the intersection product [59–66]. Recently, [67], it has been shown that intersection theory plays a role in more general situations, beyond the realm of Feynman integrals, for integrals involving generalized orthogonal polynomials and computations of matrix elements in quantum mechanics, Green's functions in field theories, higher-order moments of probability distributions, suggesting a deep intertwinement between physics, geometry, and statistics.[1]

Driven by emerging perspectives that reveal integration theory as a unifying language in mathematical physics, in the present paper we started investigating a quite general approach that can be applied to several questions of physical interest, like Feynman integrals, Fourier integrals [69], scattering in curved spacetime [70, 71], and the examples investigated in [67]: the exponential integrals. Our analysis will be based on [1], where exponential integrals, and the related wall-crossing structures, are analyzed in light of a series of isomorphisms between the twisted de Rham and Betti cohomologies in their local and global versions. In the present paper, we will mainly concentrate on the case where the exponential integral is defined starting from a holomorphic function on a complex manifold X, $f : X \to \mathbb{C}$, so that

$$I_\Gamma(f, \gamma) = \int_\Gamma e^{-\gamma f} \mu,$$

[1] It is not trivially expected an arbitrary generalization of the applicability of these methods to any situation, given that "being a period" is a special situation, see [68] for an introduction to this idea.

where we assume f to have a finite number of isolated critical points, γ is a non-vanishing complex parameter, μ is a holomorphic volume form over X, and Γ is an open integration chain. This kind of integrals are met in several physical applications, where one evaluates them asymptotically for large values of γ, by using the saddle point approximations. The associated thimbles indeed represent selected basis of integration cycles [72], relative to some subset $D_0 \subset X$ of positive codimension, and allow to understand the cohomology structures underlying the integral. The twisted de Rham cohomology is determined by the exact form df, the twisting being given by the covariant differential $\nabla_f = d + df\wedge$. To the triple (X, D_0, f), one can construct four Local Systems, given by the local and global twisted de Rham and Betti cohomologies, all deeply related. In these terms, exponential integrals can be interpreted as periods of such cohomologies. In proving this, a key role is played by the Wall Crossing Structures associated to the integrals, which appear when the parameter γ meets Stokes lines in the complex plane. The proof of these facts, given in [1], relates on a generalization of the Riemann-Hilbert problem along the Stokes rays.

We will review these facts in section 2, adapting the notation and formalism to the application to physics we have in mind. In a sequence of recent papers, it has been shown that the Skyrme model admits exact analytic solutions describing nuclear matter in different pasta states [73–75]. In particular, for the gauged Skyrme model one finds solutions representing baryonic layers at finite baryon density in the presence of a constant magnetic field [76]. The grand-canonical partition function of such system is expressed in terms of a Pearcey integral. Its importance relies on the fact that it provides low energy non-perturbative effective description of chromodynamics and gives the occasion to replace cumbersome numerical analyses with analytical studies. The phase space of this system strongly depends on the Stokes lines of the partition function, which also determine critical curves in the $\mu_B - B_{ext}$ plane, μ_B being the (complexified) chemical potential of the external magnetic field B_{ext}. In section 3 we will interpret the Pearcey integral as an exponential integral and apply the general theory to it in order to analyze the Stokes phenomenon. Interestingly, the Stokes phenomenon also plays a significant role in analogue gravity and in the interpretation of the Hawking effect, [77–80].

In this work, we further consider extensions of the previous analysis to include exponential integrals involving multivalued functions, which serve as a tool for computing Feynman integrals. In section 4, we briefly review how the constructions related to the triple (X, D_0, f) can be generalized to the triple (X, D_0, α), where the 1-form α, representing the twisting of the covariant differential, is now a generic closed holomorphic form rather than necessarily an exact one, as developed in [1]. Our motivation is that for Feynman integrals the function f is replaced by the logarithm of a polynomial function, $f = \log \mathcal{B}$, which is thus not well defined globally, while its differential is. We will elaborate on a general strategy for applying such tools to Feynman integrals. Our strategy will provide an interpretation that is not restricted by specific underlying geometries or special assumptions on the spacetime dimensions. This will be illustrated through a concrete example in section 4.3.

A more extended analysis of the multivalued case in relation to Feynman integral will be presented in a companion paper [81].

Finally, we will conclude with a discussion of our results and future perspectives in section 5.

Section 2 can be skipped by readers already familiar with the theory in [1].

2 Exponential integrals for holomorphic functions

Exponential integrals are ubiquitous in physics, particularly in path integral computations across any quantum field theory, including conformal field theory correlators and non-perturbative analyses in string theory. In this section, we provide a concise overview of the mathematical techniques developed to handle these integrals, while referring the reader to [1] for more detailed information.

Let X be a smooth n-dimensional complex affine algebraic variety, $\mathcal{O}_X$ its structure sheaf and Ω^k_X the sheaf of differential k-forms on X. Given a holomorphic function $f \in \mathcal{O}(X) = \Gamma(X, \mathcal{O}_X)$, a Borel-Moore n-chain Γ (locally compact) and an algebraic volume form $\mu \in \Gamma(X, \Omega^n_X)$, one defines the exponential integral of f over Γ with respect to μ as:

$$I \equiv I_\Gamma(f) = \int_\Gamma e^{-f}\mu. \tag{2.1}$$

Since we are working with smooth algebraic varieties, we can assume the support of Γ to be an integer linear combination of closed oriented submanifolds. However, differently from ordinary homology, Γ may have a nonempty boundary $\partial\Gamma$. We will assume that the boundaries of the integration cycles are contained in a closed algebraic subset $D_0 \subset X$ of strictly positive codimension, $(\mathrm{Supp}(\partial\Gamma) \subset D_0)$. Therefore, if the integration chain Γ is such that the map

$$Re(f)|_{\mathrm{Supp}(\Gamma)} : \mathrm{Supp}(\Gamma) \to \mathbb{R} \tag{2.2}$$

is proper[2] and bounded from below, the exponential integral (2.1) is absolutely convergent.

Furthermore, we can even generalize the notion of exponential integral by rescaling the function $f \mapsto \gamma f$ and studying how the structure of the resulting integral

$$I(\gamma) = I_\Gamma(f, \gamma) = \int_\Gamma e^{-\gamma f}\mu. \tag{2.3}$$

depends on the complex parameter $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$.[3]

For generic γ, the integral $I(\gamma)$ can be expressed as a linear combination of exponential integrals over special integration cycles, called *thimbles*. These are real, non-compact cycles formed by the gradient flow lines of $Re(\gamma f)$ with respect to an auxiliary Hermitian metric on X. In general, these gradient flow lines, which originate from a critical point, do not cross any other critical point along their trajectory. However, as the argument $arg(\gamma)$ varies, there exist special values of $\gamma \in \mathbb{C}^*$ at which this condition fails, leading to a change in the number of independent gradient flow lines. When one of these special loci, known as *Stokes lines*, is crossed in $\mathbb{C}^*$, the linear combination of thimbles undergoes a discontinuous change (jump) described by a Stokes automorphism.

[2] The pre-image of any compact is compact.

[3] In order to emphasize the variable we will often use the notation $\mathbb{C}_\gamma$ to mean the copy of $\mathbb{C}$ where γ takes values.

The collection of Stokes automorphisms along the plane $\mathbb{C}^*_\gamma$ forms the wall crossing structure associated to the integrals (2.3), which coincides with the one arising from the holomorphic version of Morse theory, see [82] for an introduction to complex Morse theory. Exponential integrals can also be placed within the framework of exponential Hodge theory and interpreted as periods. In particular, they can be embedded into a generalized Riemann-Hilbert correspondence to study the relationship between de Rham and Betti cohomologies, both at the local and global levels. In the global setting, the isomorphism between these two cohomologies associated with the triple (X, D_0, f) is precisely realized through the exponential integral. In the next subsections, we will study in detail the four cohomologies associated with this triple, each of which defines a vector bundle over $\mathbb{C}^*_\gamma$, and we will discuss their mutual relations.

Let us define the *bifurcation set* $S^* \subset \mathbb{C}$ as the minimal finite set of points such that for any $t \in \mathbb{C} \setminus S^*$ there exists an open neighborhood U (in analytic topology[4]) and a homeomorphism $f^{-1}(U) \simeq U \times f^{-1}(t)$ which is compatible with the natural projections on both spaces to U and such that it induces a homeomorphism:

$$f^{-1}(U) \cap D_0 \simeq U \times \left(f^{-1}(t) \cap D_0\right). \tag{2.4}$$

Smoothness implies X is a complex manifold, i.e. it locally looks like $\mathbb{C}^n$. Since f is continuous, and $U \subset \mathbb{C}$ is open, $f^{-1}(U)$ can be identified with an open set of $\mathbb{C}^n$. If U does not contain bifurcation points, the above definition states that $f : \mathbb{C}^n \supset f^{-1}(U) \to U$, defines a local fibration on $\mathbb{C}$ (indeed a fibration on U), whose fiber is $f^{-1}(t) \cong Spec[\mathbb{C}[x_1, \ldots, x_n]/\langle f \rangle]$,[5] with $t \in U$, up to constant deformations (that are deformations depending trivially on t). Such fiber is smooth whenever the Milnor algebra[6] (Chiral ring)

$$\mathcal{M}_f = \frac{\mathcal{O}_{\mathbb{C}^n,\mathbf{z}}}{\langle \partial_{x_1} f, \ldots, \partial_{x_n} f \rangle} \tag{2.5}$$

is trivial, that is, if its dimension μ, called (local) *Milnor number*, vanishes [83]. Here $\mathcal{O}_{\mathbb{C}^n,\mathbf{z}}$ is the stalk at $\mathbf{z}$ of $\mathcal{O}(X)$, that is the ring of germs of power series converging in some neighborhood of $\mathbf{z}$, $f(\mathbf{z}) = t$. If $\mu = 0$ for any $t' \in \mathrm{U}(f(\mathbf{z}))$, Ehresmann's lemma [84] implies that f is a locally trivial fibration over U. On the other hand, if non zero Milnor numbers arise, the transition functions are constrained by elements of the Jacobian and the fibration cannot be trivial: the bifurcation set contains at least the set $S \equiv \{t_i \equiv f(\sigma_i)\}_i$ of critical values of f.

In the latter case, one can still define a locally trivial fibration M on $U_i^\odot \equiv \mathrm{U}(t_i)\backslash\{t_i\}$, called *Milnor fibration* [85–89], whose fiber $M_{\mathbf{z}_i}$ is a CW complex homotopy equivalent to a bouquet of $\mu_i \equiv \mu|_{t_i}$ copies of $(n-1)-$spheres. Each of such spheres, or equivalently each

[4]Algebraic geometry makes use of the Zariski topology. However, since we have invoked smoothness, we can always view X as a complex manifold and use the corresponding topology. This is called analytification of the topology

[5]With $\langle f \rangle$ we mean the ideal generated by f in $\mathbb{C}[x_1, \ldots, x_n]$, e.g. the elements of the form gf, $g \in \mathbb{C}[x_1, \ldots, x_n]$. Therefore, $R := \mathbb{C}[x_1, \ldots, x_n]/\langle f \rangle$ is a ring. On the other hand, $f = 0$ determines an affine subvariety Y of $\mathbb{C}^n$. If $x \in Y$ and $h \in R$, we have an evaluation map $ev_x : R \to \mathbb{C}$ which is a homomorphism. The map $ev : Y \to R$, $x \to ev_x$, gives a bijection between Y as a subset of $\mathbb{C}^n$, and the set of homomorphisms $R \to \mathbb{C}$. The latter is called $Spec[\mathbb{C}[x_1, \ldots, x_n]/\langle f \rangle]$.

[6]Also called local Jacobian ring.

element Δ_i of $H_{n-1}(M_{\mathbf{z}_i}, \mathbb{C})$,[7] is called (algebraic) *vanishing cycle*. Their denomination follows from the fact that they shrink to zero when approaching the critical point. The importance of the role they play here derives from Brieskorn and Malgrange [90, 91] proof of the isomorphism between the homology generated by the vanishing cycles and the hypercohomology of the De-Rham complex twisted by middle extended Gauss-Manin connection:

$$H_{n-1}(M_z, \mathbb{C}) \cong \mathbb{H}(\Omega_X^\bullet, \nabla_{GM}^{\text{mid}}), \tag{2.6}$$

which means that, in some sense, the homology of the full space is determined by the homology of the fiber.[8] The *Picard-Lefschetz theory* [92, 93] (see appendix A for a concise review) provides a concrete tool for determining and studying these vanishing cycles.

As we will see explicitly in section 3.1, the set of n-dimensional(real) manifolds $\mathcal{T}_i \simeq \Delta_i \times \mathbb{R}^+ \subset X$ corresponding to the traces of the vanishing cycles along the vanishing directions in the base space, called Picard-Lefschetz thimbles, provide a basis of thimbles for the global Betti cohomology associated with the triple (X, D_0, f).

Suppose now X can be compactified to a smooth projective variety $\overline{X}$ such that f extends to a regular map (t.i. everywhere defined):

$$\overline{f} \;:\; \overline{X} \longmapsto \mathbb{P}^1. \tag{2.7}$$

We can decompose $\overline{X} - X = D_h \sqcup D_v$, where the *vertical divisor* $D_v = \overline{f}^{-1}(\infty)$ is the locus at infinity where f diverges, and the *horizontal divisor* D_h is the locus at infinity where f has finite limit. In the following we will assume the set $D_0 \cup D_v \cup D_h$ is a normal crossing divisor and that no critical points lie at infinity nor at D_0. With this, we mean that the restriction of f to D_0 or to the infinity locus,[9] the restricted function has no critical points, which implies $S^* = S$. Finally, suppose that no degeneration of critical points occurs.

2.1 Twisted de Rham cohomology

In the previous paragraph we introduced the triple (X, D_0, f) and the exponential integral (2.1) where μ is a holomorphic top form on X. From a cohomological perspective μ is closed with respect to the differential

$$\nabla_f \;=\; d \;-\; df \;\wedge . \tag{2.8}$$

Since the connection is flat, this differential gives rise to a complex of sheaves (in Zariski topology):

$$\Omega_X^\bullet \;=\; \Omega_X^0 \xrightarrow{\nabla_f} \Omega_X^1 \xrightarrow{\nabla_f} \dots \xrightarrow{\nabla_f} \Omega_X^n. \tag{2.9}$$

[7] $H_k(S^{n-1}, \mathbb{C}) = \mathbb{C}$, for $k = 0, n-1$ and vanishes otherwise.

[8] One can get an idea as follows. In the above local fibration all fibers are isomorphic so have equal (co)homology. This determines a vector bundle over the complement of the bifurcation points, with fibers the (co)homology groups, and whose transition functions are nontrivial only around the bifurcation points, so they are locally constant. In this sense, one can think of (co)homology classes as functions of the base point t. The Gauss-Manin connection is essentially a flat connection telling how the (co)homology classes change along the basis, t. i. how to take their covariant derivative in t.

[9] In general, the infinity locus is not a submanifold but rather a stratifold. Thus, one has to check that the restrictions of f to each open stratum has no critical points.

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In order to incorporate the boundary divisor D_0, we restrict to the subcomplex $\Omega^\bullet_{X,D_0}$ of $\Omega^\bullet_X$ of forms with support on $X \setminus D_0$. This subcomplex is the basis for the following definition of global cohomology.

Definition: [GLOBAL TWISTED DE RHAM, [1], DEF.2.2.1]
The global twisted de Rham cohomology is the graded abelian group:

$$H^\bullet_{dR,\text{global}}\left((X,D_0),f\right) = \mathbb{H}^\bullet\left(X_{Zar}, (\Omega^\bullet_{X,D_0}), \nabla_f\right) \tag{2.10}$$

of equivalence classes of forms on $X \setminus D_0$ with respect to the differential ∇_f.

Notice, for instance, that any 1-form α closed with respect to the standard de Rham differential, yields a ∇_f-closed 1-form $e^f\alpha$:

$$\nabla_f(e^f\alpha) = d(e^f\alpha) - df \wedge e^f\alpha = e^f d\alpha + e^f df \wedge \alpha - df \wedge e^f\alpha = 0. \tag{2.11}$$

If we now fix $\gamma \in \mathbb{C}^*_\gamma$, and replace $f \mapsto \gamma f$, we obtain the graded $\mathbb{C}-$vector space

$$H^\bullet_{dR,\text{global},\gamma}\left((X,D_0),f\right) = H^\bullet_{dR,\text{global}}\left((X,D_0),\gamma f\right). \tag{2.12}$$

In addition to this global version of cohomology, one may also study the cohomology localized near each critical point of f.

Definition: [LOCAL TWISTED DE RHAM, [1], DEF.2.3.3]
Let $\Sigma = \{\sigma_i \in X \setminus D_0 | df(\sigma_i) = 0\}$ be the set of critical loci of f in $X \setminus D_0$. The local twisted de Rham cohomology associated to the triple (X, D_0, f) is the $\mathbb{C}[[1/\gamma]]$-module:[10]

$$H^\bullet_{dR,\text{local}}\left((X,D_0),f\right) = \bigoplus_{i\in\Sigma} \mathbb{H}^\bullet\left(U_{\text{form}}(\sigma_i), (\Omega^\bullet_{X,D_0}\left[[\gamma]\right], (1/\gamma)d - df\wedge)\right) \tag{2.13}$$

where $U_{\text{form}}(\sigma_i)$ is the formal neighborhood of the critical locus $\sigma_i \in X$. Each summand is called local de Rham cohomology associated with σ_i (or $t_i = f(\sigma_i) \in S$) and it is denoted with $H^\bullet_{dR,\text{loc},\sigma_i}(X, D_0, f)$.

With the above definitions in mind, the following proposition ([1], Prop. 2.3.4) summarizes the key structural properties of the global de Rham cohomology.

Proposition:
Assume that f is proper, and set $\tau = 1/\gamma$. Then

(i) The coherent sheaf $\mathcal{H}^\bullet_{dR,\text{global}}(X,f)$ on $\mathbb{C}$, defined as:

$$\mathcal{H}^\bullet_{dR,\text{global}}(X,f) = \mathbb{H}^\bullet_{Zar}\left(X \times \mathbb{C}_\tau, (pr^*_X(\Omega^\bullet_X, \tau d_X - df\wedge))\right) \tag{2.14}$$

gives rise to a graded vector bundle over $\mathbb{C}_\tau$.[11] *Its restriction to $\mathbb{C}^*_\tau$ carries a flat connection ∇_τ, encoding how cohomology varies with the parameter τ.*

[10]Notation: $\mathbb{C}[[1/\gamma]]$ is the ring of formal power series of $\frac{1}{\gamma}$ with coefficients in $\mathbb{C}$. A $\mathbb{C}[[1/\gamma]]-$module is an abelian group equipped with an induced action of $\mathbb{C}[[1/\gamma]]$.

[11]Notation: pr^*_X is the map induced from the projection of $X \times \mathbb{C}_t$ onto X.

(ii) *The connection* ∇_τ *has a regular singularity at* $\tau = \infty$ *(i.e.* $\gamma = 0$*) and a second order pole at* $\tau = 0$ *(i.e.* $\gamma = \infty$*). As we approach the point* $\tau = 0$*, the global connection* ∇_τ *splits into a direct sum of blocks, each of which is the tensor product of an exponential factor* $e^{t_i\gamma}$ *(rank 1 irregular D-module on* $\mathbb{C}$*) and a regular connection:*

$$\left(\mathcal{H}^\bullet_{dR,\text{global}}(X,f), \nabla_\tau\right) \simeq \bigoplus_{i\in S} e^{t_i\gamma} \otimes (E_i, \nabla_i)\,. \tag{2.15}$$

In physical parlance this is the statement that, as $\tau \mapsto 0$ *(i.e.* $\gamma \mapsto \infty$*), the integral localizes around each critical point* σ_i *giving an irregular contribution* $e^{t_i\gamma}$ *times a regular contribution solution of the system* (E_i, ∇_i)*.*

(iii) *The fiber of* $\mathcal{H}^\bullet_{dR,\text{global}}(X,f)$ *at* $\tau = 0$ *(i.e.* $\gamma = \infty$*) is isomorphic to the sum*

$$\bigoplus_{i\in S} \mathbb{H}^\bullet\left(U_{\text{form}}(\sigma_i), (\Omega^\bullet, -df\wedge)\right). \tag{2.16}$$

(iv) *Formally near* $\tau = 0$*, the global twisted de Rham cohomology can be reconstructed using the local pieces around each critical point via the following global-to-local isomorphism:*

$$\varphi_{dR}\ :\ H^\bullet_{dR,\text{global}}((X,D_0),f) \otimes_{\mathbb{C}[\gamma]} \mathbb{C}[[\gamma]] \ \simeq\ H^\bullet_{dR,\text{loc}}((X,D_0),f) \tag{2.17}$$

(v) *For any* $\gamma \in \mathbb{C}^*$ *there is a non-degenerate pairing*

$$H^\bullet_{dR,\text{global},-\gamma}(X,f) \otimes H^\bullet_{dR,\text{global},\gamma}(X,f) \quad \longmapsto \quad \mathbb{C}\left[-2\dim_{\mathbb{C}} X\right] \tag{2.18}$$

which extends to a non-degenerate pairing at $\gamma = \infty$ *(i.e.* $\tau = 0$*). This is the twisted Poincaré duality, shifting cohomological degree by* $-2\dim_{\mathbb{C}} X = -2n$*.*

We will show how to compute it concretely in section (3).

2.2 Betti (co-)homology

To complement the twisted de Rham picture, we now introduce the corresponding Betti (co)-homology groups, which capture the topology of chains on X relative to the level sets of f at infinity. We begin by fixing a real constant $c > 0$ and considering the singular relative homology

$$H_\bullet\left(X, D_0 \cup f^{-1}(Re(z) \geq c), \mathbb{Z}\right) \simeq H_\bullet\left(X, D_0 \cup f^{-1}(c), \mathbb{Z}\right). \tag{2.19}$$

Once $c > max_{\sigma\in\Sigma} Re(f(\sigma))$, the critical points do not lie on the boundary, and then the relative homology stabilizes (i.e. it is the same replacing c with any $c' > c$).

Definition: [Global Betti, [1], Def.2.4.1]
The global Betti homology of (X, D_0, f) *is*

$$H_\bullet^{\text{Betti,global}}((X,D_0),f,\mathbb{Z}) := H_\bullet\left((X,D_0), f^{-1}(\infty), \mathbb{Z}\right) \tag{2.20}$$

and, similarly, the global Betti cohomology is

$$H^\bullet_{\text{Betti,global}}((X,D_0),f,\mathbb{Z}) \equiv H^\bullet\left((X,D_0), f^{-1}(\infty), \mathbb{Z}\right), \tag{2.21}$$

where the infinity means selecting the stabilized (co)homology.

As in the case of de Rham cohomology, we consider the rescaling of the function $f \mapsto \gamma f$, which allows to define the global Betti cohomology at any point in the plane $\mathbb{C}^*_\gamma$.

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Definition: [[1], Def.2.4.2]
Let $\gamma \in \mathbb{C}^*_\gamma$. *For each fixed* γ *we define the graded abelian group:*

$$H^\bullet_{\text{Betti,global},\gamma}((X, D_0), f, \mathbb{Z}) \equiv H^\bullet\left((X, D_0), (\gamma f)^{-1}(\infty), \mathbb{Z}\right) \tag{2.22}$$

By extending these groups from $\mathbb{Z}$ to $\mathbb{Q}$, one obtains a definition for the Poincaré duality in this global Betti setting.

Proposition: [Poincaré duality, [1], Prop.2.4.3]
Let $X' = \overline{X} - D_v - \overline{D}_0$ *and* $D'_0 = D_h - (D_h \cap D_v)$. *We have the following isomorphism:*

$$H_\bullet^{\text{Betti,global}}((X, D_0), f) \simeq H^\bullet_{\text{Betti,global}}((X', D'_0), -f)\,[2\dim_{\mathbb{C}} X]\,. \tag{2.23}$$

The family of abelian groups (2.22) over the whole space $\mathbb{C}^*_\gamma$ defines a *Local System* (a locally constant sheaf) over $\mathbb{C}^*_\gamma$ denoted as $\mathcal{H}^\bullet_{\text{Betti,global}}((X, D_0), f)$.

Now, we want to relate these groups to local data. In order to do this, we look at the codomain $\mathbb{C}_t$ of the function f as the real plane $\mathbb{R}^2$ and choose an open region U, whose closure $B = \bar{U}$ is a submanifold of $\mathbb{R}^2$ isomorphic to a unit disc. Assuming that the boundary ∂B does not intersect the critical locus of f, we fix an arbitrary point $t_0 \in \partial B$. The idea is the following. For each $k \in \mathbb{Z}_{\geq 0}$ and $k < n$, we associate to the pair (B, t_0) the abelian group

$$V^k(B, t_0) := H^k\left(f^{-1}(B), (D_0 \cap f^{-1}(B)) \cup f^{-1}(t_0), \mathbb{Z}\right) \tag{2.24}$$

and we look at it as a vector space. We now assume that all finite, non-degenerate critical values of f lie in the interior of B. Since they are isolated points, we can find a finite number of subsets $B_i = \bar{U}_i \subset B$, each containing exactly one critical value and such that they have vanishing intersection in B and intersect ∂B precisely in the same marked point b. By retracting B to the bouquet of B_i, one gets the isomorphism

$$V(B, b) \simeq \bigoplus_j V(B_j, b). \tag{2.25}$$

This allows us to explore each component separately and study cohomologies with a single critical point. This leads one to introduce the following definitions. Let us assume $D_0 = \emptyset$ (X is projective) to lighten notation.

Definition: [local Betti cohomology, [1], Def.2.5.1]
For each critical value t_i, *a small positive* ϵ *and* $\gamma \in \mathbb{C}^*$, *we define the local Betti cohomology associated with the pair* (t_i, γ) *as the graded abelian group*

$$H^\bullet_{\text{Betti,local},t_i,\gamma}(X, f) = V\left(D(\gamma t_i, \epsilon), t_{\theta_\gamma}\right) = H^\bullet\left((\gamma f)^{-1}(D(\gamma t_i, \epsilon)), f^{-1}(t_{\theta_\gamma}), \mathbb{Z}\right) \tag{2.26}$$

where $D(\gamma t_i, \epsilon)$ *is a closed disc in* $\mathbb{C}$ *of radius* ϵ *centered in* γt_i *and* t_{θ_γ} *is the point on the boundary of the disc such that* $t_{\theta_\gamma} = \gamma t_i + \epsilon e^{i\theta_\gamma}$ *with* $\theta_\gamma = \pi - \arg(\gamma)$.

At fixed γ, *the direct sum of these cohomology groups for each* $t_i \in S$ *form the local Betti cohomology* $H^\bullet_{\text{Betti,local},\gamma}(X, f)$:

$$H^\bullet_{\text{Betti,local},\gamma}(X, f) = \bigoplus_{t_i \in S} H^\bullet_{\text{Betti,local},t_i,\gamma}(X, f). \tag{2.27}$$

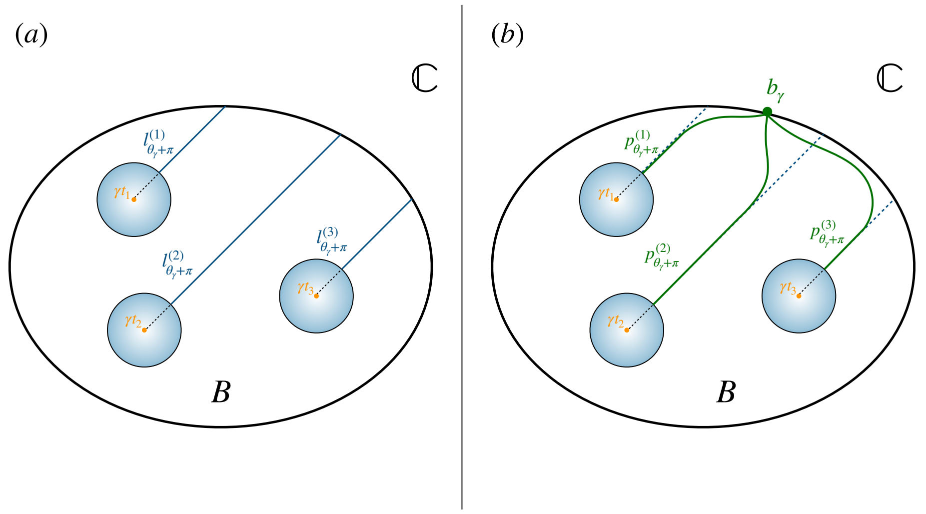

Figure 1. Pictorial representation of the homotopy deformation implementing the boundary retraction that induces the isomorphism (2.28).

Similar to the global case, the family of local Betti cohomologies over the space $\mathbb{C}^*_\gamma$ forms a Local System denoted as $\mathcal{H}^\bullet_{\text{Betti,local}}(X, f)$.

Now we will make use of the description of the local and global cohomology groups as vector spaces $V(B_j, t_j)$ in order to relate them to each other. First, let us construct a sufficiently large disc $B \subset \mathbb{C}$ containing all the critical values of S, and for each critical value in $t_i \in S$ let us construct its proper disc $D(\gamma t_i, \epsilon)$ with marked point $t^{(i)}_{\theta_\gamma}$ on its boundary. From each of these points $t^{(i)}_{\theta_\gamma}$ let us construct a ray $l^{(i)}_{\theta_\gamma+\pi}$ in the direction $\theta_\gamma + \pi$. The resulting configuration consists of a set of parallel lines originating from the small discs and terminating at the boundary of the large disc, as depicted in figure 1-(a). At this point, we can construct a homotopy of the large disc such that the deformed rays $l^{(i)}_{\theta_\gamma+\pi} \mapsto p^{(i)}_{\theta_\gamma+\pi}$ intersect at a unique point b_γ on the boundary of the large disc (figure 1-(b)).

For all the $\gamma \in \mathbb{C}^*$ that do not belong to the Stokes rays, defined below, the retraction of the complement of the big disc with respect to the union of the small discs and the paths $p^{(i)}_{\theta_\gamma+\pi}$ gives rise to the Betti local to global isomorphism:

$$\varphi_{\text{Betti}} \;:\; \mathcal{H}^\bullet_{\text{Betti,local}}(X, f) \xrightarrow{\sim} \mathcal{H}^\bullet_{\text{Betti,global}}(X, f). \tag{2.28}$$

We call φ_θ the restriction of this isomorphism along a specific ray θ in $\mathbb{C}^*_\gamma$. We can think of it as generated by the embedding of a neighborhood of the ray (intersected with B) in B.

Definition: [Stokes ray, [1] Def.2.5.3]
We call the ray $s_\theta = \{\gamma | \arg(\gamma) = \pi - \theta_{ij} = \theta\} = \mathbb{R}_{\geq 0} \cdot (t_i - t_j)^{-1} \subset \mathbb{C}_\gamma$ with $\theta_{ij} = \arg(t_j - t_i)$ a Stokes ray.
Rays with vertex at the origin that are not Stokes rays are called generic rays.

Notice that there can be more copies $t_a, t_b \in \mathbb{C}_t$ of critical points such that $\arg(t_a - t_b) = \arg(t_i - t_j)$. All these copies give the same Stokes ray. Whenever γ lies on the Stokes ray of slope $\pi - \arg(t_i - t_j)$ in the plane $\mathbb{C}_\gamma$, the corresponding line $l^{(i)}_{arg(t_i-t_j)}$ in the $\mathbb{C}_{\gamma t}$-plane,

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used to construct the Betti isomorphism φ_{Betti}, passes through both points γt_i and γt_j before reaching the boundary of the large disc. For all other points nothing special happens. Therefore, we see that for each Stokes ray s_θ (namely a Stokes ray with slope θ), we have an isomorphism T_θ among the graded abelian groups $H^\bullet_{\text{Betti,local},\gamma}(X, f)$ with $\arg(\gamma)$ sufficiently close to θ. Concretely, choose a sector in the γ-plane with boundary rays at angles $\theta_\pm = \theta \pm \epsilon$. These two rays lift to sectors in the $\mathbb{C}_{\gamma t}$-plane whose retraction paths avoid all but one critical value, so for each of those values the two deformations give homotopic maps and hence the same identification of local Betti groups.

However, when $\theta = \pi - \arg(t_i - t_j)$, the corresponding rays in $\mathbb{C}_{\gamma t}$ pass through both γt_i and γt_j. In that case the edges cannot be deformed into one another without crossing the line $l^{(i)}_{\theta_{ij}}$. The required isomorphism for the jump, given by $\varphi^{-1}_{\theta^-} \circ \varphi_{\theta^+}$, is implemented by the operator:

$$T_\theta = \mathbb{1} + \sum_{\substack{i \neq j \\ \arg(t_i - t_j) = -\theta}} T_{ij}, \tag{2.29}$$

where

$$T_{ij} \;:\; H^\bullet\left(D(t_i, \epsilon), t_{\theta_{ij}}; \mathbb{Z}\right) \quad\longmapsto\quad H^\bullet\left(D(t_j, \epsilon), t_{\theta_{ij}}; \mathbb{Z}\right). \tag{2.30}$$

With this isomorphism we can glue the Local System $\mathcal{H}_{\text{Betti,local}}(X, f)$ across the Stokes rays. Equipped with these Stokes automorphisms, the local system $\mathcal{H}_{\text{Betti,local}}(X, f)$ over the circle $S^1 = \{|\gamma| = const\}$ provides a concrete example of an analytic wall-crossing structure.

To this point we have introduced four cohomology theories, de Rham and Betti in their global and local versions, each pair related by its own isomorphism. In the statements that follow, we will establish the comparison isomorphisms between the de Rham and Betti frameworks.

Definition: [EXPONENTIAL PERIOD MAP]
The integration over cycles defines a non-degenerate pairing

$$\int \;:\; H^{\text{Betti,global}}_\bullet\left((X, D_0), f\right) \otimes H^\bullet_{dR,\text{global}}\left((X, D_0), f\right) \quad\longmapsto\quad \mathbb{C}, \tag{2.31}$$

called Exponential Period Map.

From this pairing, for each $\gamma \in \mathbb{C}^*_\gamma$, we can construct the following isomorphism, [1], Prop.2.7.1:

$$\varphi_\gamma \;:\; H^\bullet_{dR,\text{global},\gamma}(X, f) \simeq H^\bullet_{\text{Betti,global},\gamma}(X, f) \otimes \mathbb{C}. \tag{2.32}$$

Then we can refer to the integrals (2.3) as exponential periods of de Rham cocycles over Betti cycles.

Finally, by promoting the Local Systems over $\mathbb{C}^*_\gamma$ to vector bundles with connection $\nabla_{\tau=1/\gamma}$, we have the following local version of the isomorphism (2.32) ([1], Prop.2.7.2)

$$RH^{-1}_{\text{loc}}\left(\mathcal{H}^\bullet_{\text{Betti,local}}((X, D_0), f) \otimes \mathbb{C}\right) \simeq \mathcal{H}^\bullet_{dR,\text{local}}((X, D_0), f), \tag{2.33}$$

where RH^{-1}_{loc} is the Riemann-Hilbert inverse functor from the category of Local Systems of complex vector spaces to the category of regular singular connections of vector spaces over $\mathbb{C}[[t]]$.

2.3 WCS for exponential integrals

One of the main consequences of our initial generalization of the exponential integral (2.1) to the one-parameter family (2.3), achieved by rescaling the function $f \mapsto \gamma f$, is the emergence of Stokes phenomena for specific values of the parameter $\gamma \in \mathbb{C}^*$. For these special values γ^*, the number of independent lines $l^{(i)}_{\theta_{\gamma^*+\pi}}$ used to construct the Betti local to global isomorphism decreases, leading to discrete changes in the graded abelian groups $H^{\bullet}_{\text{Betti,local},\gamma}(X, f)$. These changes are controlled by wall crossing formulas. In this section we discuss the wall crossing structure for exponential integrals [1, 94], which provides a generalization of the $2d$ version used by Cecotti and Vafa in [95].

Let us fix a region $\mathcal{R} \subset \mathbb{C}_\gamma$, such that for any $\gamma \in \mathcal{R}$, the exponential integral (2.3) is an analytic function of γ, depending on Γ. Notice that if $\text{Supp}(\Gamma)$ is compact, then the region is unrestricted. Otherwise, it is necessary to ensure that γf is bounded from below. In general, if $\gamma_0 \in \mathcal{R}$, then $\mathbb{R}_{\geq 0}\gamma_0 \subset \mathcal{R}$.

If we do not fix the integration cycle but we keep the volume form fixed, we can interpret $I(\gamma)$ as a morphism of sheaves of abelian groups on $\mathbb{C}^*_\gamma$

$$\begin{aligned} \mathcal{H}^{\text{Betti,global}}_{\bullet}((X, D_0), f) &\longmapsto \mathcal{O}^{an}_{\mathbb{C}^*_\gamma} \\ \Gamma &\longmapsto \int_\Gamma e^{-\gamma f}\,\mu. \end{aligned} \tag{2.34}$$

If we choose γ_0 lying on a generic ray in $\mathbb{C}^*_\gamma$, then, for any γ in a small sector $V \subset \mathcal{R}$ containing the ray $\mathbb{R}_{\geq 0} \cdot \gamma_0$ (see figure 2-(a)), the canonical isomorphism between global and local Betti homologies induced by (2.28) is well defined and it gives rise to the following morphism among sheaves:

$$\bigoplus_{t_i \in S} \mathcal{H}^{\text{Betti,local},\gamma,t_i}\left(f^{-1}(V), f\right) \longmapsto \mathcal{O}^{an}_{\mathbb{C}^*_\gamma}(V). \tag{2.35}$$

Let us now choose a Stokes ray s_θ and consider a new small sector $V = V^+ \cup V^-$ in the plane $\mathbb{C}^*_\gamma$ containing the ray (see figure 2-(b)). We choose two bases $\left\{\Gamma^+_{(i)}\right\}$ and $\left\{\Gamma^-_{(i)}\right\}$ for the local Betti homology in the sectors V^+ and V^-, respectively, corresponding to the angles $\theta^+ = \arg(\gamma^+) = \theta + \epsilon$ and $\theta^- = \arg(\gamma^-) = \theta - \epsilon$. With these choices, we can define two vector valued analytic functions:

$$\overline{I}^+(\gamma) = \begin{pmatrix} \int_{\Gamma^+_{(1)}} e^{-\gamma f}\mu \\ \int_{\Gamma^+_{(2)}} e^{-\gamma f}\mu \\ \dots \\ \dots \\ \int_{\Gamma^+_{(k)}} e^{-\gamma f}\mu \end{pmatrix} \qquad \overline{I}^-(\gamma) = \begin{pmatrix} \int_{\Gamma^-_{(1)}} e^{-\gamma f}\mu \\ \int_{\Gamma^-_{(2)}} e^{-\gamma f}\mu \\ \dots \\ \dots \\ \int_{\Gamma^-_{(k)}} e^{-\gamma f}\mu \end{pmatrix} \tag{2.36}$$

related by the following wall crossing formulas

$$\int_{\Gamma^-_{(i)}} e^{-\gamma f}\mu = \int_{\varphi^*_{\theta_+}(\Gamma^+_{(i)})} e^{-\gamma f}\mu + \sum_{\substack{j \neq i \\ \arg(t_i - t_j) = \theta}} \int_{(\varphi^*_{\theta^+} \circ T_{ij})\Gamma^+_{(i)}} e^{-\gamma f}\mu, \tag{2.37}$$

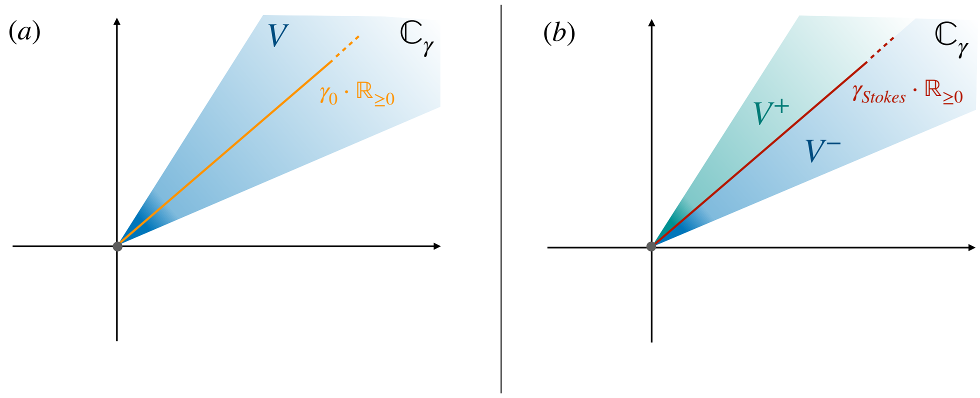

Figure 2. (a) Sector V in the complex plane $\mathbb{C}_\gamma$ containing the generic ray $\gamma_0 \cdot \mathbb{R}_{\geq 0}$. (b) Sectors V^+ and V^- in the complex plane $\mathbb{C}_\gamma$ separeted by the Stokes ray $\gamma_{Stokes} \cdot \mathbb{R}_{\geq 0}$.

where φ^* denotes the dual isomorphism to the one in (2.28). Roughly speaking, these formulas describe the analytic continuation of the function $\overline{I}^-(\gamma)$ from the sector V^- to the adjacent sector V^+ across the Stokes ray.

In the special case in which f is a Morse function with k different critical points, there is a special basis for the local and global Betti homologies for each direction $\theta = \pi - \arg(\gamma)$ which is the one of Lefschetz thimbles $th_{i,\theta+\pi}$. By definition, $th_{i,\theta+\pi}$ is the union of gradient lines of the function $Re(e^{-i\theta} f)$ emerging from the critical point σ_i, while $f(th_{i,\theta+\pi})$ is the line with direction $\theta + \pi$ emerging from the critical value $f(\sigma_i) = t_i$. Using this basis of thimbles, we can define the following collection of integrals for any generic direction $\gamma \in \mathbb{C}^*$, with $\theta = \pi - \arg(\gamma)$, such that γ does not lie on a Stokes ray:

$$I_i(\gamma) = \int_{th_{i,\theta+\pi}} e^{-\gamma f} \mu. \tag{2.38}$$

Let us suppose to have defined them along the direction $\theta_- = \theta - \epsilon$, where θ identifies now a Stokes ray. Then, when we move toward $\theta_+ = \theta + \epsilon$ through the Stokes line, the integrals undergo a discontinuous jump according to (2.37)

$$I_i \mapsto I_i + n_{ij} I_j, \tag{2.39}$$

where n_{ij} are integers counting the number of gradient trajectories of $Re(e^{i\theta_{ij}} f)$ joining the critical values σ_i and σ_j. Equivalently, n_{ij} is the intersection index of the opposite thimbles $th_{i,\theta_+ +\pi}$ and $th_{j,\theta_- -\pi}$ emerging from the critical points σ_i and σ_j.

As $\gamma \mapsto \infty$, the integrals (2.38) admit a power expansion:

$$I_i(\gamma) = e^{-\gamma t_i} \sum_{\lambda} c_{i,\lambda} \gamma^{-\lambda-1}, \tag{2.40}$$

for some $c_{i,\lambda} \neq 0$.

In order to analyze this series let us start isolating the exponential dependence at the critical point

$$I_i(\gamma) = e^{-\gamma t_i} I_i^{\text{mod}}(\gamma) \tag{2.41}$$

and define the new variable

$$s = f(\mathbf{z}) - t_i. \tag{2.42}$$

Since the function $\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\theta}\mathrm{f})$ remains constant along the cycle $th_{i,\theta+\pi}$, the variable s ranges over the real interval from zero to infinity. Let us denote by $\Delta_i(s)$ the $(n-1)$-dimensional closed hypersurfaces defined by the level equations $f(\mathbf{z}) = s = \mathit{const}$. These level sets are known as vanishing cycles of the homology group $H_{n-1}\left(f^{-1}(s), \mathbb{Z}\right)$ [96] (see appendix A for a concise introduction to the topic). When γ does not lie on a Stokes ray, the trace of these vanishing cycles along the variation of s in the range $[0; +\infty[$ span the thimble $th_{i,\theta+\pi}$

$$th_{i,\theta+\pi} = \bigcup_{s\geq 0} \Delta_i(s). \tag{2.43}$$

Using the Gelfand-Leray form $\frac{\mu}{ds}\Big|_{\Delta_i(s)}$, the exponential integral (2.41) can be rewritten as

$$I_i^{\mathrm{mod}}(\gamma) = \int_0^\infty ds e^{-\gamma s} \mathrm{vol}_{\Delta_i}(s), \tag{2.44}$$

where

$$\mathrm{vol}_{\Delta_i}(s) = \int_{\Delta_i(s)} \frac{\mu}{ds}\Big|_{\Delta_i(s)} \tag{2.45}$$

denotes the volume of the $(n-1)-$dimensional vanishing cycles $\Delta_i(s)$ in the family (2.43). Note that the modified integral $I_i^{\mathrm{mod}}(\gamma)$ can be interpreted as the Laplace transform of $\mathrm{vol}_{\Delta_i}(s)$. On the other hand, the function $\mathrm{vol}_{\Delta_i}(s)$ can be read as the pairing between the holomorphic cohomology class $\left[\frac{\mu}{ds}\right] \in H^{n-1}\left(f^{-1}(s), \mathbb{C}\right)$ and the homology class $[\Delta_i] \in H_{n-1}\left(f^{-1}(s), \mathbb{Z}\right)$. According to the resolution of singularities theorem (see for example [97]), this function admits an absolutely convergent power series expansion for $0 < s \ll \varepsilon$ of the form:

$$\mathrm{vol}_{\Delta_i}(s) = \sum_\lambda \sum_{0\leq k\leq k_{\max}} a_{\lambda,k} s^\lambda \log(s)^k. \tag{2.46}$$

The numbers λ correspond to the eigenvalues $e^{2\pi i\lambda}$ of the monodromy operator M_i acting on $H_{n-1}(f^{-1}(s), \mathbb{Z})$ when we turn around the singularity $s = 0$, while the integer $k_{\max} + 1$ determines the size of the largest Jordan block associated with that eigenvalue. Taking the total differential of the definition (2.42), we obtain

$$ds = \frac{\partial f}{\partial z^1} dz^1 + \frac{\partial f}{\partial z^2} dz^2 + \cdots + \frac{\partial f}{\partial z^n} dz^n. \tag{2.47}$$

This expression shows that the function $\mathrm{vol}_{\Delta_i}(s)$ develops potential singularities in the complex s-plane whenever all partial derivatives of f vanish simultaneously, that is, at the critical points of f. Consequently, a series expansion of $\mathrm{vol}_{\Delta_i}(s)$ in powers of s will have a radius of convergence determined by the distance to the nearest singularity on the same Riemann sheet.

Substituting the expansion (2.46) into the exponential integral (2.41), and using the following identity:

$$\int_0^\infty e^{-\gamma s} s^\lambda (\log s)^k ds = \frac{d^k}{d\lambda^k}\left[\gamma^{-(\lambda+1)} \Gamma(\lambda+1)\right] \tag{2.48}$$

we obtain

$$I_i(\gamma) = e^{-\gamma t_i} \sum_\lambda \sum_k a_{\lambda,k} \int_0^\infty dse^{-\gamma s} s^\lambda (\log s)^k = e^{-\gamma t_i} \sum_\lambda \sum_k a_{\lambda,k} \frac{d^k}{d\lambda^k} \left[\gamma^{-(\lambda+1)} \Gamma\left(\lambda+1\right)\right]. \tag{2.49}$$

Comparing this result with the power series in (2.40), we have

$$c_{i,\lambda} = \frac{1}{\gamma^{-(\lambda+1)}} \sum_k a_{\lambda,k} \frac{d^k}{d\lambda^k} \left[\gamma^{-(\lambda+1)} \Gamma\left(\lambda+1\right)\right]. \tag{2.50}$$

One of the main advantages of constructing the γ-expansions of the same integral evaluated in the thimble basis is that it enables a direct comparison among the various integrals $I_i(\gamma)$(the Master Integrals) associated with different homology classes of integration contours within the same sector of the $\mathbb{C}_\gamma$ plane. In some sectors, certain thimbles may dominate over the others.

3 Holomorphic Morse theory

In this section, we present a concrete application of the formalism developed in the previous paragraphs, accompanied by a discussion of its connection to holomorphic Morse theory. This connection not only provides deeper geometric insight into the structure of the theory but also clarifies the role of the abstract objects we introduced in the construction of Betti cohomology.

In order to build geometric intuition and develop familiarity with the setup, we now focus on the case $X \cong \mathbb{C}^n$ and consider exponential integrals of the form

$$I(f,\gamma) = \int_\Gamma e^{-\gamma f(\mathbf{z})} g(\mathbf{z}) d^n\mathbf{z}, \tag{3.1}$$

where $f(\mathbf{z})$ is a holomorphic function and $g(\mathbf{z})d^n\mathbf{z}$ is a holomorphic n-form on X. The aim of the procedure is the one to provide a basis for the integration contours, for any $\gamma \in \mathbb{C} \setminus \{0\}$, such that the integral (3.1) converges. As γ varies over its domain, the admissible integration contours Γ must be deformed accordingly to ensure convergence.

Let us define the set D_N in X as

$$D_N = \{\mathbf{z} \in \mathbb{C}^n | Re(\gamma f(\mathbf{z})) \geq N\}, \tag{3.2}$$

for $N \in \mathbb{R}$, with $|N| \gg 1$. This subset of X consists in general of different disconnected components (an illustrative example is given by the blue regions in figure 3). Any reasonable cycle Γ for (3.1) should connect two distinct regions of this subset, namely it should be a non-compact n-cycle of X with boundaries in D_N, i.e. an element of the relative homology $H_n(X, D_N, \mathbb{Z})$. The condition on the boundaries is just part of the requirements that our integration contours have to satisfy. To ensure that the integrals are well-behaved, we must also impose conditions on the portions of the cycles extending into the complementary region $X \setminus D_N$. In particular, the cycles must avoid regions of X where $Re(\gamma f(\mathbf{z})) \mapsto -\infty$, as such behavior would lead to divergence. Furthermore, to prevent oscillations, we must impose the condition that $\mathrm{Im}(\gamma \mathrm{f}(\mathbf{z}))$ remains constant along Γ, ensuring that we can factor out the phase e^{ic} and reduce the problem to a real-valued integral.

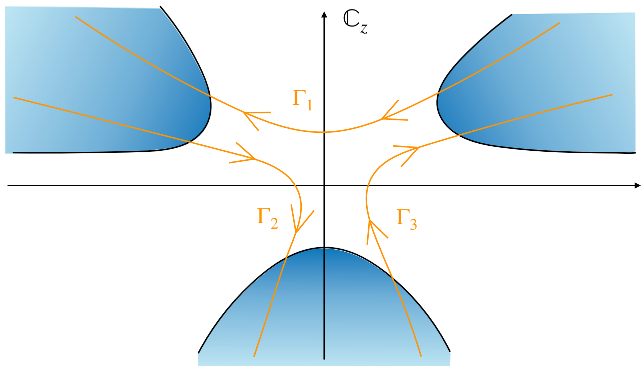

Figure 3. The blue areas in the complex plane represent the regions D_N for $f(\mathbf{z} = Ai(\mathbf{z}))$ and large N. Despite $X = \mathbb{C}$ is contractible, D_N can be the union of different disjoint pieces. The 1-cycles $\Gamma_i \in H_1(X, D_N, \mathbb{Z})$ must connect distinct components of D_N.

The techniques described in section 2 provide a systematic method for analyzing the cycles in this relative homology, constructing a basis for them, and defining a well-behaved intersection pairing.

3.1 Relative homology

Let $H_k(X, D_N, \mathbb{Z})$ be the k-homology group of X, on $\mathbb{Z}$, relative to D_N. The elements of this group, called relative cycles, are equivalence classes of k-chains in X whose boundaries lie in D_N, modulo those chains that are homologous to chains entirely contained in D_N. Notice that, in the limit $N \mapsto +\infty$, this homology group corresponds, up to Poincaré duality as given in (2.23), to the Betti homology group defined in (2.22), with $D_0 = \emptyset$. By applying the constructions outlined in section 2 we can determine the dimension of this relative homology group and construct an explicit basis for it. In doing so, we recover the same geometric objects that arise in Morse theory, which analyzes the topology of X by studying the properties of the differential functions defined on it.

In the present case, the function we will use to carry out the analysis is the height function

$$h = Re(\gamma f(\mathbf{z})). \tag{3.3}$$

The set Σ of critical points of this function h coincides with the one of f because of the Cauchy-Riemann equations. A critical point is said to be non-degenerate if the Hessian matrix associated to h in that point is invertible. If all critical points are non-degenerate, the height function is a well-defined Morse function. The number of negative eigenvalues of the Hessian equips critical points of an index, called Morse index. For non-degenerate holomorphic functions on complex manifolds of complex dimension n, the Morse indices are all equal to n. Consequently, the Betti inequalities, which provide lower bounds on the dimensions of the homology groups, are saturated

$$\mathrm{rank}[H_k(X, D_N, \mathbb{R})] = \begin{cases} 0, & k < n, \\ \#\Sigma, & k = n. \end{cases} \tag{3.4}$$

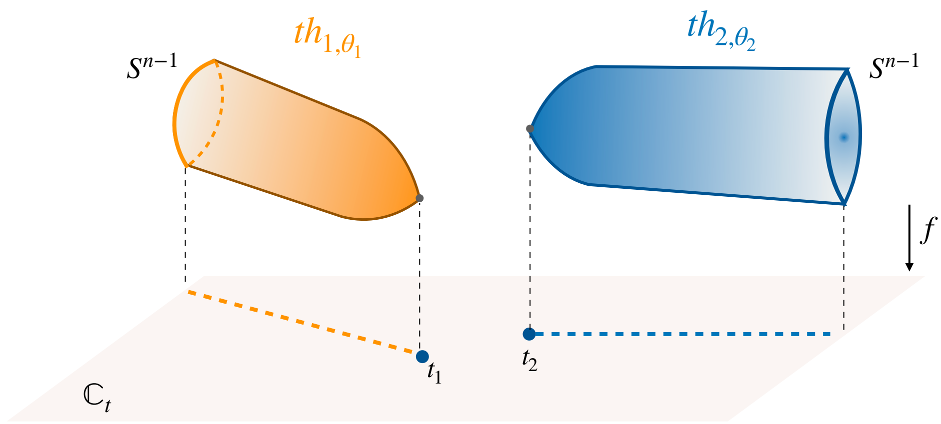

Figure 4. Pictorial representation of two thimbles: th_{1,θ_1} and th_{2,θ_2} constructed as continuations of the vanishing cycles Δ_i and Δ_j, diffeomorphic to S^{n-1}, along the paths $t_1 + e^{i\theta_1}\mathbb{R}_{\geq 0}$ and $t_2 + e^{i\theta_2}\mathbb{R}_{\geq 0}$ in $\mathbb{C}_t$.

This provides a direct way to compute the dimension of $H_n(X, D_N, \mathbb{Z})$. Let us now determine a basis for this group. If the function h is perfect,[12] Morse theory provides a way to construct a relative n-cycle Γ_i for each critical point in Σ. Let us make the simplifying assumption that all the critical points $\sigma_i \in \Sigma$ are isolated points in X, and the corresponding distinct critical values form the set:

$$S = \{t_i \in \mathbb{C} |\, f(\sigma_i) = t_i\}\,. \tag{3.5}$$

For each critical point $\sigma_i \in X$, there is a unique vanishing cycle $\Delta_i \subset f^{-1}(t_i)$ diffeomorphic to S^{n-1}. An explicit method to construct these cycles is provided in the appendix A. Moreover, for each critical point σ_i and a generic direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we can construct the Lefschetz thimble $th_{i,\theta} \sim \mathbb{R}^n \subset X$ as the continuation of the vanishing cycle along the path $t_i + e^{i\theta}\mathbb{R}_{\geq 0} \subset \mathbb{C}_t$ (see figure 4 for a conceptual visualization), ill-defined only if $\theta = \arg(t_j - t_i)$ for some $t_j \neq t_i$.

Among these thimbles, we aim to select a basis for the relative homology. This is achieved by considering the continuations of vanishing cycles along special paths of the form $t_i + e^{i\theta_i}\mathbb{R}_{\geq 0}$ in $\mathbb{C}_t$ which start from the critical points t_i and reach $t = \infty$ while maintaining a constant phase $\theta_i = \mathrm{Im}(\gamma \mathrm{f}(\mathrm{t_i}))$. These paths are solutions of the gradient flow equations:

$$\frac{du^i}{dt} = +g^{ij}\frac{\partial h}{\partial u^j}, \tag{3.6}$$

where u^i are real coordinates on X and g_{ij} is a Riemannian metric on X. These paths define the steepest ascent Lefschetz thimbles Γ_i^+, which have the key properties that the function h increases monotonically along them, and, if $h = Re(\gamma f)$ for a holomorphic function f, then the imaginary part $\mathrm{Im}(\gamma \mathrm{f})$ remains constant along the thimble. If Γ_i^+ contains exactly one critical point, it corresponds to a *good* Lefschetz thimble. Otherwise, it is referred to as a *Stokes line*.[13] Assuming no Stokes lines are present in our set of thimbles, the number of thimbles

[12]The differences between the indices of distinct critical points of h are never equal to ± 1.

[13]Note that we use the term "*Stokes rays*" to refer to the semi-infinite lines in the complex plane $\mathbb{C}_\gamma$ as established in the definition from section 2.2. Instead, we use the term "*Stokes lines*" to denote Lefschetz

exactly matches the rank of the relative homology group. To prove that they indeed generate the homology group, namely that they are independent cycles, we need to establish a method for uniquely decomposing any element $\Gamma \in H_n(X, D_N, \mathbb{Z})$ as a linear combination of the form

$$\Gamma = \sum_{\sigma\in\Sigma} n_\sigma \Gamma_\sigma^+. \tag{3.7}$$

Such a decomposition exists, and the coefficients n_σ that appear on it are integer numbers representing the intersection between the cycle Γ and the basis of steepest descent Lefschetz thimbles Γ_i^- which belong to the dual homology group $H_n(X, D_{-N}, \mathbb{Z})$. Here, $D_{-N} = \{\mathbf{z} \in X | Re(\gamma f(z)) \leq -N\}$ with $N \in \mathbb{R}$ taken to be sufficiently large. The new thimbles Γ_i^- are solutions of the gradient flow equations with opposite sign:

$$\frac{du^i}{dt} = -g^{ij}\frac{\partial h}{\partial u^j}. \tag{3.8}$$

They represent the downward-flowing cycles associated with each critical point $\sigma_i \in \Sigma$. These cycles retain the property that $\mathrm{Im}(\gamma \mathrm{f}(\mathrm{z}))$ remains constant along them, while the function h decreases monotonically. In the absence of Stokes rays, the intersection pairing is given by:

$$\langle \Gamma_i^+, \Gamma_j^- \rangle = \delta_{ij}. \tag{3.9}$$

It is straightforward to evaluate this formula for perfect Morse functions with no flows between distinct critical points. Indeed, if there are no flows between two distinct points $\sigma_i, \sigma_j \in \Sigma$, the corresponding Lefschetz thimbles do not intersect. This is because they are associated with different constant values of the phase $\mathrm{Im}(\gamma \mathrm{f})$, which remains constant along each thimble. Conversely, the thimbles $\Gamma_\sigma^\pm$ follow paths along which the function h is monotonically increasing or decreasing. As a result, they intersect exactly once, at the critical point σ itself.

Therefore, a generic integration contour $\Gamma \in H_n(X, D_N, \mathbb{Z})$ can be decomposed in terms of the paths Γ_i^+ for generic $\gamma \in \mathbb{C}^*$, away from a Stokes ray, as in (3.7), with coefficients uniquely determined by

$$n_\sigma = \langle \Gamma, \Gamma_\sigma^- \rangle. \tag{3.10}$$

They count, with appropriate orientation, the number of downward flows from each critical point σ to Γ.

Let us consider the example where γ is purely imaginary, and the function f is given by the quotient $P_1(\mathbf{z})/P_2(\mathbf{z})$, where P_1 and P_2 are polynomials with real coefficients. We take the integration cycle $\Gamma = \Gamma_\mathbb{R}$ to be the product of n real lines in $\mathbb{C}^n$. Let us partition the set of critical points Σ into three subsets:

$$\Sigma = \Sigma_\mathbb{R} + \Sigma_{\leq 0} + \Sigma_{>0}, \tag{3.11}$$

where $\Sigma_\mathbb{R}$ denotes the set of critical points lying on the real axis, $\Sigma_{\leq 0}$ consists of critical points off the real axis for which the associated value of h satisfies $h \leq 0$, and $\Sigma_{>0}$ includes those off the real axis for which $h > 0$.

thimbles in $\mathbb{C}^n$ that contain more than one critical point. Each time $\gamma \in \mathbb{C}_\gamma$ lies on a Stokes ray, Stokes lines appear in $\mathbb{C}^n$ as a manifestation of the associated Stokes phenomenon.

For critical points lying on the real line, since $\gamma \in Im$, the function $h = Re(\gamma f)$, vanishes identically. In particular, we have $h_\sigma = 0$ for all $\sigma \in \Sigma_\mathbb{R}$. Because h strictly decreases along downward gradient flows, there can be no such flows starting at $\sigma \in \Sigma_\mathbb{R}$ that remain on the real line. Consequently, the only intersection between the downward Lefschetz thimble Γ_σ^- and the real cycle $\Gamma_\mathbb{R}$ is the point σ itself:

$$\sigma \in \Sigma_\mathbb{R} \qquad \Longrightarrow \qquad n_\sigma = \langle \Gamma_\mathbb{R}, \Gamma_\sigma^- \rangle = 1. \tag{3.12}$$

If $\sigma \in \Sigma_{\leq 0}$, no downward flows originating from σ intersect $\Gamma_\mathbb{R}$. This follows from the fact that h is strictly decreasing along downward flows, and by definition, $h \leq 0$ for points in $\Sigma_\leq$. Thus, we have:

$$\sigma \in \Sigma_{\leq 0} \qquad \Longrightarrow \qquad n_\sigma = \langle \Gamma_\mathbb{R}, \Gamma_\sigma^- \rangle = 0. \tag{3.13}$$

Finally, if $\sigma \in \Sigma_{>0}$, it is in principle possible for downward flows originating from σ to intersect the real section $\Gamma_\mathbb{R}$. The precise number of such intersections depends on the specific geometry of the function and must be determined case by case. Altogether, we obtain the decomposition:

$$\Gamma_\mathbb{R} = \sum_{\sigma \in \Sigma_\mathbb{R}} \Gamma_\sigma^+ + \sum_{\sigma \in \Sigma_{>0}} n_\sigma \Gamma_\sigma^+. \tag{3.14}$$

3.2 Stokes rays

In the previous paragraph, we described the relative homology $H_n(X, D_N, \mathbb{Z})$, constructed a basis of thimbles for it, and defined an intersection pairing with the dual homology $H_n(X, D_{-N}, \mathbb{Z})$ to express a generic cycle $\mathcal{C} \in H_n(X, D_N, \mathbb{Z})$ in terms of this basis. However, as we explained in section 2.2, the presence of Stokes rays affects the well-definedness of certain Lefschetz thimbles, making the previous construction insufficient. In this section, we explain why some Lefschetz thimbles become ill-defined in the presence of Stokes phenomena and how the framework introduced earlier can be adapted to restore consistency.

The key perspective we adopt is to construct a description of the homology group $H_n(X, D_N, \mathbb{Z})$ that ensures a well-defined pairing (3.9) for any value of $\gamma \in \mathbb{C}_*$. This is achieved by first establishing the structure for a specific γ where no Stokes rays appear, as we did in the previous section, and then extending it across the entire $\mathbb{C}_*$.

Stokes lines are solutions of the gradient flow equation (3.6) that cross at least two critical points of the function $\gamma f(\mathbf{z})$. Since the imaginary part of this function is preserved along the flows, we have

$$\mathrm{Im}(\gamma \mathrm{f}(\mathbf{z}))\,|_{\sigma_\mathrm{i}} = \mathrm{Im}(\gamma \mathrm{f}(\mathbf{z}))\,|_{\sigma_\mathrm{j}} \tag{3.15}$$

for any point in the thimbles Γ_i^+ and Γ_j^+. Moreover, since the number of critical points is finite, there can only be a finite number of Stokes lines. By assumption, $\gamma f(\mathbf{z})$ takes distinct values at different critical points, meaning that Stokes lines appear only for specific values of γ. For $\sigma_i \neq \sigma_j \in \Sigma$, the loci

$$l = \{\gamma \in \mathbb{C}_* \mid \mathrm{Im}(\gamma \mathrm{f}(\mathbf{z}))\,|_{\sigma_\mathrm{i}} = \mathrm{Im}(\gamma \mathrm{f}(\mathbf{z}))\,|_{\sigma_\mathrm{j}}\} \tag{3.16}$$

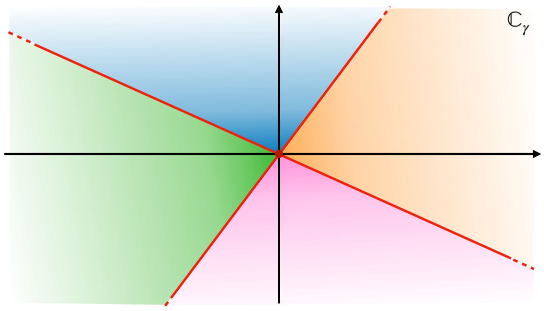

Figure 5. The $\mathbb{C}_\gamma$ plane: the red lines indicate the Stokes lines, across which discontinuities arise in the definition of the Lefschetz thimble structure for the corresponding exponential integrals. The differently colored regions represent the distinct petals of the fan.

in the complex plane $\mathbb{C}_\gamma$, define regions where the Lefschetz thimble structure undergoes discontinuities: they are the Stokes rays discussed in section 2.2. These rays always pass through the origin; however, since $\{0\} \notin \mathbb{C}_\gamma$, they remain disconnected and form straight lines radiating outward from the center. As a result, the complex γ-plane is divided into a fan-like structure composed of distinct sectors — referred to as petals in figure 5. At this point, the procedure is to fix γ within a specific petal of the fan, say the zeroth region (0), away from any Stokes rays, and define the Lefschetz thimble structure for the corresponding integral. We can then vary γ along the complex plane. As we cross a Stokes ray s_θ, associated with a Stokes line between the critical points σ_i and σ_j, for which $h_{\sigma_i} < h_{\sigma_j}$, the corresponding thimbles Γ_i^+ and Γ_j^+ undergo a discontinuous jump to the adjacent region (1) of the form:

$$\begin{pmatrix} \Gamma_i^{+(1)} \\ \Gamma_j^{+(1)} \end{pmatrix} = \begin{pmatrix} 1 & \Delta_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_i^{+(0)} \\ \Gamma_j^{+(0)} \end{pmatrix}, \qquad \text{for} \quad h_{\sigma_i} < h_{\sigma_j} \tag{3.17}$$

where the integers Δ_{ij} receive a contribution ± 1 for each upflow line from σ_i to σ_j, with the sign depending on cycles orientation and on the direction from which γ crosses the Stokes line. This is nothing but the intersection number of the corresponding vanishing cycles expressed by the Picard-Lefschetz formula (A.47), up to a sign depending on the relative orientation of the cycles, when we cross the cut line starting from $t_j = f(\sigma_j)$ in the plane $\mathbb{C}_t$:

$$\Delta_{ij} = (\pm 1)\Delta_i \circ \Delta_j. \tag{3.18}$$

This means that the new thimble $\Gamma_i^{+(1)}$ in the region (1) is associated to the new vanishing cycle

$$\Delta_i^{(1)} = \Delta_i^{(0)} \pm (\Delta_i \circ \Delta_j)\, \Delta_j^{(0)}. \tag{3.19}$$

In order for the decomposition (3.7) to be continuous, the coefficients n_{σ_i} and n_{σ_j} transform across the ray by

$$\begin{pmatrix} n_{\sigma_i} \\ n_{\sigma_j} \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ -\Delta_{ij} & 1 \end{pmatrix} \begin{pmatrix} n_{\sigma_i} \\ n_{\sigma_j} \end{pmatrix}. \tag{3.20}$$

To understand the reason for these jumps and the meaning of the integer coefficients Δ_{ij} appearing in the jump matrix in (3.17) let us consider a simple one-dimensional example.

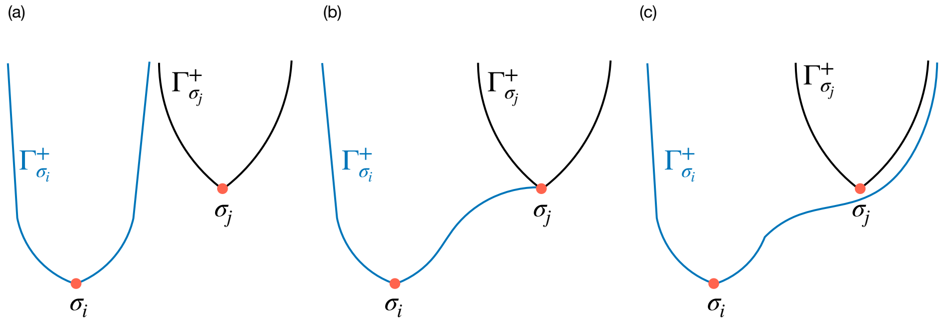

Figure 6. Jump of the thimble Γ_i^+ across a Stokes ray. From left to right, γ crosses a Stokes ray: (b), (c).

Suppose that for a suitable γ, away from any Stokes line, we have two critical points σ_i and σ_j with distinct values of $h_{\sigma_i} < h_{\sigma_j}$ and distinct imaginary parts $\text{Im}(\gamma \text{f}(\mathbf{z}))$, for which we can define two distinct thimbles without any intersection (see figure 6(a)). As we move γ towards a Stokes line, the thimble Γ_i^+ is continuously deformed until it crosses the thimble Γ_j^+ at the critical point σ_j (see figure 6(b)). At this point, the first thimble is no longer well defined. As we continue moving γ across the Stokes line, the support of the thimble Γ_i^+ continues to be deformed on the other side of the thimble Γ_j^+, as shown in figure 6(c). The comparison between the representations (a) and (c) in figure 6 illustrates the jump.

The number of upward flows from σ_i that intersect the point σ_j along a Stokes ray, counted with appropriate sign based on the orientation, gives the number Δ_{ij} in the matrix (3.17).

Remark: an interesting relation between the total monodromy acting on the thimbles after a transformation $\gamma \mapsto e^{2\pi i}\gamma$ and the transformation of a basis of $(n-1)-$forms dual to the vanishing cycles has been pointed out in [95]. Let us start from a regular point γ and let us transport it along a circle in a clockwise direction with $|\gamma|$ fixed. Each time that γ crosses a Stokes ray l_{ij}, corresponding to the crossing of the Stokes line connecting the critical points σ_i and σ_j, we have a change on the thimble basis given by (3.17). Let us indicate the matrix giving the jump as

$$M_{ij} = \mathbb{1} + A_{ij}, \tag{3.21}$$

where the only non-zero entry in the matrix A_{ij} is $(ij) = \Delta_{ij}$ counting the intersection number among the vanishing cycles Δ_i and Δ_j. After a tour of π around the origin, the total change on the basis of thimbles is

$$S = \prod_{l_{ij}} M_{ij}. \tag{3.22}$$

In the second half sector, beyond π, each time λ crosses a Stokes line we have a jump given by:

$$M_{ji}^{-t} = \mathbb{1} - A_{ji}. \tag{3.23}$$

The total jump along this second half circle is represented by the matrix:

$$S^{-t} = \prod_{l_{ij}} M_{ji}^{-t}. \tag{3.24}$$

The full monodromy is defined via

$$H = S^{-t}S. \tag{3.25}$$

This matrix is invariant under deformations of the function $\gamma f(z)$ and it is quasi-unipotent.[14] Then, its eigenvalues are always roots of the unity

$$\text{Eigenvalues of H} \qquad \longrightarrow \qquad \left\{e^{2\pi i q_k}, \qquad q_k \in \mathbb{Q}\right\}.$$

3.3 Twisted de Rham cohomology

Let us now move to the cohomological side of the exponential pairing, explicitly showing its construction in the one variable case $X = \mathbb{C}$, considering as holomorphic function the polynomial $f = \mathcal{P}_\ell \in \mathbb{C}[z]$ of degree ℓ. Adding to $\mathbb{C}$ the divisor $D_v = p = \{\infty\}$, we end up with the good normalization $\overline{X} = \mathbb{P}^1$, on which $\mathcal{P}_\ell$ naturally extends to $\overline{\mathcal{P}}_\ell : \mathbb{P}^1 \to \mathbb{P}^1$. Denoting the twisted differential as $\nabla \equiv (\gamma^{-1}d + d\mathcal{P}_\ell\wedge)$, let consider the complex of sheaves:

$$(\Omega^\bullet_{\mathbb{P}^1,p}, \nabla) : 0 \to \mathcal{O}_{\mathbb{P}^1}(*p) \xrightarrow{\nabla} \Omega^1_{\mathbb{P}^1}(*p) \to 0. \tag{3.26}$$

On $\mathbb{P}^1$, meromorphic functions with only allowed pole at infinity are in fact polynomials on $\mathbb{C}$, thus we have:

$$\begin{aligned} &\mathcal{O}_{\mathbb{P}^1}(*p) \cong \mathbb{C}[z], \\ &\Omega^1_{\mathbb{P}^1}(*p) \cong \{P(z)dz | P(z) \in \mathbb{C}[z]\}. \end{aligned} \tag{3.27}$$

By definition (2.10), the global twisted de Rham cohomology is

$$H^\bullet_{dR,t}(\mathbb{C}, d\mathcal{P}_n) = \mathbb{H}(\mathbb{P}^1, (\Omega^\bullet_{\mathbb{P}^1,p}, \nabla)) = H^\bullet(R\Gamma(\mathbb{P}^1, (\Omega^\bullet_{\mathbb{P}^1,p}, \nabla))) \tag{3.28}$$

We can compute the hypercohomology by means of the Grothendieck spectral sequence with second page

$$E_2^{p,q} = R^p\Gamma(\mathbb{P}^1, H^q(\Omega^\bullet_{\mathbb{P}^1,p}, \nabla)). \tag{3.29}$$

Let us firstly determine the cohomology of $(\Omega^\bullet_{\mathbb{P}^1,p}, \nabla)$:

$$H^\bullet(\Omega^\bullet_{\mathbb{P}^1,p}, \nabla) = \ker\nabla \oplus \operatorname{coker}\nabla. \tag{3.30}$$

Thus, the computation reduces to the calculation of the kernel and the cokernel of the twisted differential. For $g(z) \in \mathbb{C}[z]$, the stalk of $\ker\nabla$ is

$$\ker\nabla = \{g(z) \in \mathbb{C}[z] \mid \gamma^{-1}g'(z) + \mathcal{P}'_\ell(z)g(z) = 0\}, \tag{3.31}$$

for $\gamma \in \mathbb{C}^*$, the constrain on $g(z)$ becomes:

$$g(z) = e^{-\gamma\mathcal{P}_\ell(z)}, \tag{3.32}$$

[14] This means that some power of it is unipotent: the sum of the identity plus a nilpotent matrix.

that cannot be a polynomial unless $\ell = 0$. Therefore:

$$\ker \nabla = \emptyset. \tag{3.33}$$

The cokernel of ∇ measures the failure of ∇ to be surjective: away from the critical points of $\mathcal{P}_\ell$, ∇ is locally surjective and its cokernel vanishes. On the other hand, near each critical point the equation $\nabla g = \eta$ for $\eta \in \Omega^1_{\mathbb{P}^1}(*p)$ may not have a polynomial solution for g. This generates a one dimensional obstruction to surjectivity, thus the stalk $\mathrm{coker}(\nabla)_\sigma \cong \mathbb{C}$. The cokernel of ∇ is a direct sum of skyscraper sheaves with support on critical points

$$\mathrm{coker}\nabla = \bigoplus_{\sigma\in\Sigma} \mathbb{C}. \tag{3.34}$$

An alternatively and, for our purposes, more interesting way to determine it consists to notice that the cokernel of ∇ is isomorphic to the Jacobian ring associated to $\mathcal{P}_\ell$

$$\mathrm{coker}\nabla \cong \frac{\mathbb{C}[z]}{\mathrm{Im}(\nabla)} \cong \frac{\mathbb{C}[z]}{(\mathcal{P}'_\ell(z))} = J_{\mathcal{P}_\ell}. \tag{3.35}$$

One can indeed prove the image of ∇ is isomorphic to the ideal generate by $\mathcal{P}'_\ell$. The Jacobian ring, as a $\mathbb{C}$ vector space, is:

$$J_f = \mathrm{span}\{1, z, \ldots, z^{\ell-2}\} \cong \mathbb{C}^\mu, \tag{3.36}$$

with $\mu = \ell - 1$ the total Milnor number.

Because of the vanishing of the higher cohomology groups of $(\Omega^\bullet_{P^1}(*p), \nabla)$, the terms $E_2^{p,q}$ vanish for $q > 1$ and because of the acyclicality of skyscraper sheaves $E_2^{p,q} = 0$ for $p > 0$. Thus, the only possibly non zero "turning page" differential could be $d_2 : E_2^{0,1} \to E_2^{2,0}$, but $E_2^{2,0}$ is also zero. Hence, the spectral sequence degenerates at page E_2. Therefore,

$$\begin{aligned} H^n(R\Gamma(\mathbb{P}^1, (\Omega^\bullet_{\mathbb{P}^1,p}, \nabla))) &= \bigoplus_{p+q=n} E_\infty^{p,q} = \bigoplus_{p+q=n} E_2^{p,q} \\ &= \bigoplus_{p+q=n} R^p\Gamma(\mathbb{P}^1, H^q(\Omega^\bullet_{\mathbb{P}^1,p}, \nabla)). \end{aligned} \tag{3.37}$$

That is:

$$\begin{aligned} \mathbb{H}^0(\mathbb{P}^1, (\Omega^\bullet_{\mathbb{P}^1,p}, \nabla)) &\cong H^0(\mathbb{P}^1, \ker\nabla) \cong 0, \\ \mathbb{H}^1(\mathbb{P}^1, (\Omega^\bullet_{\mathbb{P}^1,p}, \nabla)) &\cong H^1(\mathbb{P}^1, \ker\nabla) \oplus H^0(\mathbb{P}^1, \mathrm{coker}\nabla) \\ &\cong H^0(\mathbb{P}^1, \mathbb{C}^{\ell-1}) \cong \mathbb{C}^{\ell-1} \otimes H^0(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{C}^{\ell-1}, \end{aligned} \tag{3.38}$$

where the first isomorphism in the last line is given by the universal coefficient theorem. Finally, the global de Rham cohomology is

$$\begin{aligned} H^0_{dR,t}(\mathbb{C}, d\mathcal{P}_\ell) &\cong 0, \\ H^1_{dR,t}(\mathbb{C}, d\mathcal{P}_\ell) &\cong \mathbb{C}^{\ell-1}. \end{aligned} \tag{3.39}$$

As we can see, the global twisted de Rham cohomology is independent on the possible coalescence of critical points. The point is that the global Jacobian ring has dimension equal to the total Milnor number (3.36), which takes into account any possible multiplicity m_i. So, although the support of coker(∇) and its local structure changes

$$\mathrm{coker}(\nabla) = \bigoplus_{\Sigma} \mathbb{C}_{\sigma_i}^{m_i}, \tag{3.40}$$

its global sections remain always $\Gamma(\mathrm{coker}(\nabla)) \cong \mathbb{C}^{\ell-1}$. The sheaf coker$\nabla$ is not sensitive to the coalescence of critical points, due to its naively construction as a direct sum of skyscraper shaves, which loses information about the local structure. In order to recover such information, we need to turn it into a perverse sheaf and to consider a suitable extension of the twisted de Rham complex, i.e. of ∇.

Given a connection defined on a open dense subset $U \subset X$ (smooth locus), its extension along a divisor $D = X \backslash U$ come with a prescription about its behaviour near D. The ones we are possibly interested in are the so called Middle and Logarithmic extensions: the first one, arising in the context of perverse sheaves, avoids the addition of unnecessary singularities while preserving key invariants; on the contrary, the second one allows for the connection to have logarithmic singularities near the divisors. Although ∇ is defined on the whole complex plane $\mathbb{C}$, it does not define a local system $\mathcal{L}$ on it, because locally constancy fails on Σ, due to the obstructions arising in solving $\nabla s = 0$: flat sections do not freely generate the cohomology. In fact, ∇ defines a local system on $\mathbb{C}\backslash(D \cup \Sigma)$. Such obstructions arise as a consequence of the non trivial monodromy around critical points (and around branch points for a multivalued function). Indeed, near a critical point σ_i, the expansion $\mathcal{P}_\ell(z) \sim \mathcal{P}_\ell(\sigma_i) + c(z-\sigma_i)^{m_i+1}$ shows that the monodromy has a Jordan block of size m_i, and it becomes unipotent in full degenerate case. Thus, the number of critical points influences the rank of coker∇ by reducing it by the size of the Jordan blocks of the monodromy matrices. Explicitly, if $\mathcal{P}_\ell$ has $\ell-1$ distinct critical points, the monodromy acts on flat sections via distinct eigenvalues, thus $\mathcal{L}$ ha no invariant subspaces and coker$\nabla \cong \mathbb{C}^{\ell-1}$: each critical point contributes with independent obstructions. Instead, if $\mathcal{P}_\ell$ has only one critical point with multiplicity $\ell-1$, the monodromy matrix becomes unipotent ($\ell-1$ equal eigenvalues), introducing $\ell-2$ relations among obstructions and thus coker$\nabla = \mathbb{C}$. The solutions to $\nabla s = 0$ is $s(z) = e^{-\gamma \mathcal{P}_\ell(z)}(\sum_{i=0}^{\ell-1}(c_i \log^i(z-\sigma))$. Thus:

$$(\mathrm{coker}\nabla)_{\mathrm{mid}} = \bigoplus_{\sigma\in\Sigma} \mathbb{C} = \mathbb{C}^{n_c}, \tag{3.41}$$

with n_c, the number of distinct critical points. Taking into account the monodromy in this way, equivalently, means to restrict on sections with moderate growth, that is to consider the middle (mid) extension $(\Omega\bullet_{\mathbb{C},D})_{\mathrm{mid}}$. Notice that no prescription along the divisor D is added. In particular, the mid extension is independent on D.

We want just to add a comment on perversity, without dwelling too much on the subject;[15] in this context, we could forget about it, since its consideration is necessary only for categorical reasons, but totally irrelevant for the purposes of the present calculations. Consider, for instance, a skyscraper sheaf δ. It fails to be perverse because it does not satisfy co-support conditions, however we can easily make it into a perverse sheaf by just shifting it by $[-1]$, meaning now $H^{i-1}(\delta[-1]) = H^i(\delta)$. The "perversification" of coker∇ then just imply a unit shift to the left of the spectral sequence, leaving, in fact, hypercohomology unchanged.

[15]Readers interested in a deeper understanding of perversity are referred to the lecture notes [98].

Finally, supposing only one critical point has multiplicity m

$$\begin{aligned} H^0_{dR,t,\text{local}}(\mathbb{C}, d\mathcal{P}_\ell)_{(m)} &\cong 0, \\ H^1_{dR,t,\text{local}}(\mathbb{C}, d\mathcal{P}_\ell)_{(m)} &\cong \mathbb{C}^{\ell-m}, \end{aligned} \tag{3.42}$$

We will need this refinement in the next section when considering the case of degenerate points.

3.4 Pearcey's integral

As a first application of the Lefschetz thimble decomposition discussed above, we examine a Pearcey's integral [99], appearing in [76] as the grand-canonical partition function of the gauged Skyrme model, describing baryonic layers living at finite baryon density within a constant magnetic field. We want to study the integral

$$\mathcal{P}(a) = \int_{-\infty}^{+\infty} dz\, e^{-a\left(z^4+bz^2+cz+d\right)}, \tag{3.43}$$

for generic values of the real parameters a, b, c, d. This generalizes the case studied in [67].

Following the prescription of the previous section, let proceed extending the polynomial argument of the exponential to a holomorphic function over $\mathbb{C}_z$, by complexifying both the variable z and the parameters. In particular, the real parameter a is promoted to the complex parameter γ, over which we will build the wall crossing structure. We then analyze the integral over a generic contour Γ in $\mathbb{C}_z$

$$\mathcal{P}(\gamma) = \int_\Gamma dz\, e^{-\gamma\left(z^4+bz^2+cz+d\right)}, \tag{3.44}$$

and seek a basis of integration cycles along which the integral remains convergent. Once a basis and a intersection product (in homology) are identified, the real integration contour can be decomposed, with integral coefficients, in terms of such basis. As a result, the integral in (3.43) becomes a linear combination of integrals evaluated over the basis. For large values of the parameters, these basis integrals admit an asymptotic expansion, which is then transferred to the initial integral. The expectation is that for different values of the parameters, both the basis for the integration contours and the decomposition of the real line in terms of them will be modified.

The set of critical points of the holomorphic function

$$f(z) \equiv z^4 + bz^2 + cz + d, \tag{3.45}$$

i.e. the solutions of the cubic equation $f'(z) = 4z^3 + 2bz + c = 0$, can be compactly written as

$$\Sigma = \left\{ -\frac{b}{\sqrt[3]{3(-9c+\sqrt{3\Delta})}} + \frac{\sqrt[3]{-9c+\sqrt{3\Delta}}}{2\sqrt[3]{9}},\, \frac{(1\pm i\sqrt{3})b}{2\sqrt[3]{3(-9c+\sqrt{3\Delta})}} - \frac{(1\pm i\sqrt{3})\sqrt[3]{-9c+\sqrt{3\Delta}}}{4\sqrt[3]{9}} \right\}, \tag{3.46}$$

where according to the sign of the discriminant Δ, three different situation arise:

$$\Delta \equiv 8b^3 + 27c^2 \qquad \begin{cases} > 0 & 1 \text{ real and 2 complex conjugate solutions,} \\ < 0 & 3 \text{ real different solutions,} \\ = 0 & 3 \text{ real solutions with at least a multiple root.} \end{cases} \tag{3.47}$$

We will analyze these three cases separately, since each one defines a different connected region in the parameter space $U = \{(b,c) \in \mathbb{C}^2\}$, and on each region we can define a distinct local system of Betti homologies $H^{\bullet}_{\text{Betti,glob},\gamma}(\mathbb{C}, f_{(a,b)}, \mathbb{Z})$ equipped with its own wall-crossing structure.

Positive discriminant. Let us firstly consider the case where $\Delta > 0$.

For concreteness, we fix the parameters to $(b,c,d) = (3/2, 7, -1)$ and carry out the explicit computations for this choice. The critical points are computed to be

$$\sigma_i \in \Sigma = \left\{-1, \frac{1}{2}(1+i\sqrt{6}), \frac{1}{2}(1-i\sqrt{6})\right\}, \tag{3.48}$$

where $f(z)$ takes (respectively) the critical values

$$t_i \equiv f(\sigma_i) \in \mathcal{S} = \left\{-\frac{11}{2}, \left(\frac{11}{16}+3i\sqrt{6}\right), \left(\frac{11}{16}-3i\sqrt{6}\right)\right\}. \tag{3.49}$$

The non-degeneracy of the Hessian at each critical point, together with the fact that all Morse indices equal one guarantee the saturation of (3.4), thus $\dim H_1(X, D_N, \mathbb{Z}) = 3$. Here $D_N \subset \mathbb{C}$ denotes the union of the four connected regions in the complex $z = (u,v)$ (shaded blue in figure 7) where the Morse function $h(u,v) = Re(\gamma f(u,v)) > N$.

Using (3.15), we find that the Stokes' rays are the three lines

$$\begin{aligned}
&l_0 : Re(\gamma) = -\frac{11}{16}\sqrt{\frac{3}{2}}\mathrm{Im}(\gamma), && \text{where} \quad \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_1} = \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_2},\\
&l_1 : Re(\gamma) = 0, && \text{where} \quad \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_2} = \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_3},\\
&l_2 : Re(\gamma) = \frac{11}{16}\sqrt{\frac{3}{2}}\mathrm{Im}(\gamma), && \text{where} \quad \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_1} = \mathrm{Im}(\gamma \mathrm{f(z)})|_{\sigma_3},
\end{aligned} \tag{3.50}$$

resulting in a splitting of the $\mathbb{C}_\gamma$ plane in three regions with different thimbles structures. Let us fix $\gamma = 1$, lying in the first petal of the fan (orange region labeled with (0) on the right side of figure 7). We identify the three thimbles $\Gamma_i \equiv \Gamma_{\sigma_i}$ as the paths passing through a critical point and keeping constant the imaginary part of the Morse function (figure 7):

$$\begin{aligned}
H_1(X, D_N, \mathbb{Z}) &= \mathrm{span}\{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+\} \cong \mathbb{Z}^3,\\
H_1(X, D_N, \mathbb{Z})^\vee &= \mathrm{span}\{\Gamma_1^-, \Gamma_2^-, \Gamma_3^-\} \cong \mathbb{Z}^3.
\end{aligned} \tag{3.51}$$

Let us set $f(z) = t$ and look at the preimage

$$f^{-1}(t) = \begin{pmatrix} z_1(t)\\ z_2(t)\\ z_3(t)\\ z_4(t)\end{pmatrix}. \tag{3.52}$$

When approaching a critical point σ_i, the four point fiber degenerates to a three point set, identifying a vanishing cycle Δ_i. We have

$$\Delta_1 = \{z_3\} - \{z_4\}, \qquad \Delta_2 = \{z_1\} - \{z_4\} \quad \text{and} \quad \Delta_3 = \{z_1\} - \{z_4\}. \tag{3.53}$$

The monodromy matrices acting on this base of vanishing cycles are computed to be (see appendix A.3 for details):

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = M_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.54}$$

By using (A.47) and (3.18) we compute the intersection numbers

$$\Delta_{12} = 1, \qquad \Delta_{13} = 1 \quad \text{and} \quad \Delta_{23} = -2. \tag{3.55}$$

Note that the intersection numbers Δ_{ij} are defined in (3.18) up to a sign depending on the orientation of the cycles. When crossing a Stokes' line, the base of thimbles undergoes a change of the type (3.17). Let $\Gamma_i^{+(k)}$ be the vector of thimbles in the (k)-th sector of the fan. With the clockwise ordering showed in figure 7 and $T^{\theta_{(k)}}$[16] the jump matrix associated to the Stokes line $l_{(k)} \to s_{\theta_k}$ connecting the (k)-th and the $(k+1)$-th sectors of the fan,

$$\Gamma_i^{+(k+1)} = T_{ij}^{\theta_{(k)}} \Gamma_j^{+(k)}, \tag{3.56}$$

we have

$$\begin{aligned} &Re(\gamma f(z))|_{\sigma_1} < Re(\gamma f(z))_{\sigma_2} && \text{along } l_0, \\ &Re(\gamma f(z))|_{\sigma_2} > Re(\gamma f(z))_{\sigma_3} && \text{along } l_1, \\ &Re(\gamma f(z))|_{\sigma_1} > Re(\gamma f(z))_{\sigma_3} && \text{along } l_2, \end{aligned} \tag{3.57}$$

so that

$$T^{\theta_{(0)}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T^{\theta_{(1)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \qquad T^{\theta_{(2)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{3.58}$$

These matrices define the wall crossing structure in $\mathbb{C}_\gamma \times U_{b,c}^{\Delta>0}$, in which the walls are defined on subregions of this space where exactly two critical values are aligned

$$W_{ij} = \left\{ (\gamma, u) \in \mathbb{C}^* \times U_{b,c}^{\Delta>0} | \mathrm{Im}\left(\gamma(\mathrm{t_i(u)} - \mathrm{t_j(u)})\right) = 0 \right\}. \tag{3.59}$$

They correspond to walls of the second type in the sense of [100].

After a complete round of 2π, we get the full monodromy matrix

$$S = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix}, \tag{3.60}$$

with eigenvalues $-1, -1, 1$.

Let us now consider the case of non positive discriminant. Then, singular points lie on the real axes of $\mathbb{C}_z$, this meaning that a Stokes' line appears along $\mathrm{Im}(\gamma) = 0$, splitting the $\mathbb{C}_\gamma$ plane into two regions, corresponding to the upper and lower half planes. It is worthwhile to emphasize that the a priori naive choice of a real γ, in this case, would give rise to a wrong description, since we would have set precisely on the Stokes' line. In order to proceed, let us thus set $\gamma = i$.

[16] Matrix representation of the operator dual to T_θ in (2.29).

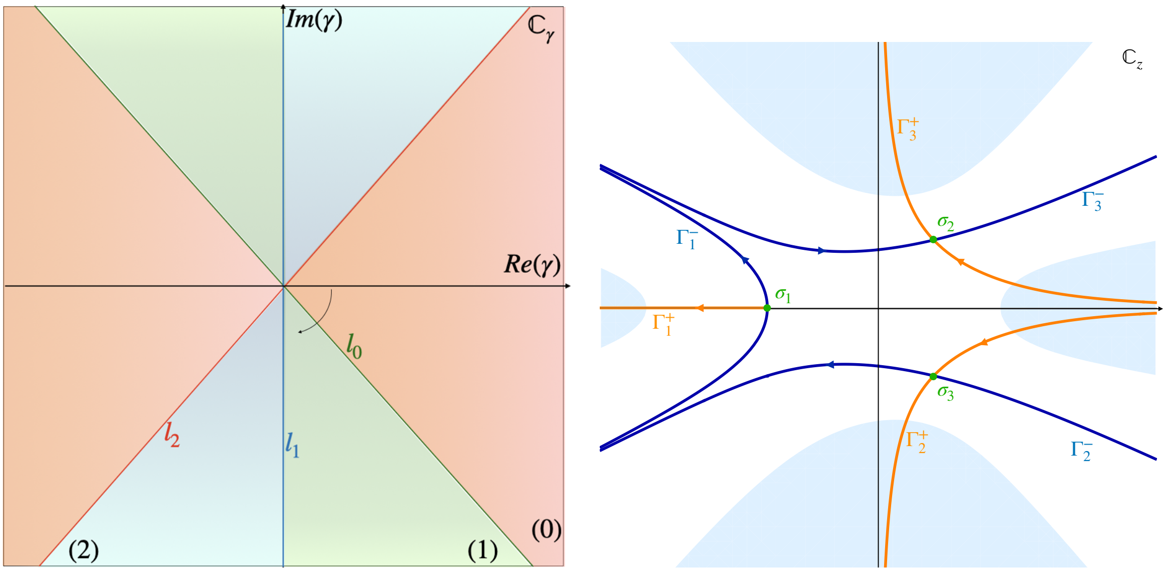

Figure 7. Positive discriminant case $(\gamma, b, c, d) = (1, 3/2, 7, -1)$. (Left) Stokes' lines on the $\mathbb{C}_\gamma$ plane. (Right) Ascendant paths Γ_i^+ spanning $H_1(\mathbb{C}_z, D_N, \mathbb{Z})$ (in orange), descendant paths Γ_i^- spanning $H_1(\mathbb{C}_z, X_{-N}, \mathbb{Z})$ (in blue).

Negative discriminant. Firstly, let us consider the case $\Delta < 0$, characterized by three different real critical points. The local Betti homology generated by local thimbles, shown in figure 8(right), is pretty much the same as in the positive discriminant case, being it of course unaffected by the reality of critical points. However, as early emphasized, the relevant peculiarity appears in the thimbles structure, due to a Stokes' line l_0 on the real axis of the $\mathbb{C}_\gamma$ plane, (see figure 8(left)). Setting $(b, c, d) = (-1, 1/2, -1)$, we get

$$\Sigma = \left\{ \frac{1}{2}, \frac{1}{4}(1 \pm \sqrt{5}) \right\} \quad \text{and} \quad \mathcal{S} = \left\{ -\frac{15}{16}, \frac{1}{32}(-41 \pm 5\sqrt{5}) \right\}. \tag{3.61}$$

Proceeding as above, we identify three vanishing cycles

$$\Delta_1 = \{z_1\} - \{z_2\}, \quad \Delta_2 = \{z_3\} - \{z_4\}, \quad \Delta_3 = \{z_1\} - \{z_4\} \tag{3.62}$$

and the corresponding monodromy matrices

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \; M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.63}$$

In order to determine the jump matrices, we have to compute the intersection numbers among thimbles. However, the Morse function vanishes in all critical points. In order to avoid it, we slightly move away from the imaginary axis setting $\gamma = i + \delta$. We get

$$Re(\gamma f(z))|_{\sigma_3} > Re(\gamma f(z))|_{\sigma_1} > Re(\gamma f(z))_{\sigma_2}, \qquad \text{along } l_0. \tag{3.64}$$

Note that in this case the ray l_0 corresponds to a Stokes line intersecting three distinct critical values. The natural generalization of the jump matrix (3.17) in this case accounts

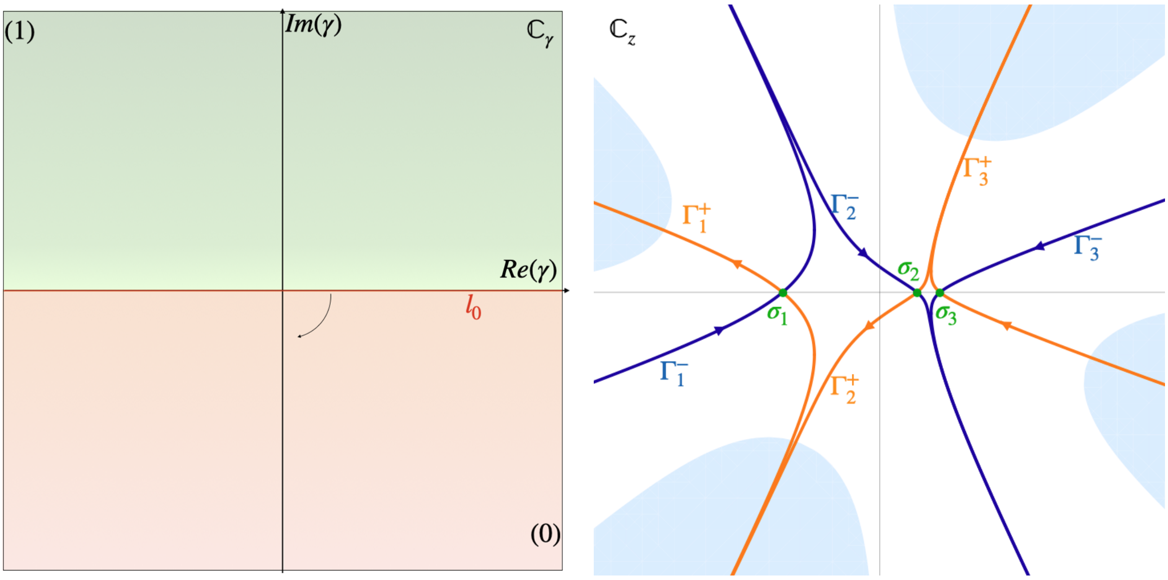

Figure 8. Negative discriminant case $(\gamma, b, c, d) = (i, -1, 1/2, -1)$. (Left) Stokes' lines on the $\mathbb{C}_\gamma$ plane. (Right) Ascendant paths Γ_i^+ spanning $H_1(\mathbb{C}_z, D_N, \mathbb{Z})$ (in orange), descendant paths Γ_i^- spanning $H_1(\mathbb{C}_z, X_{-N}, \mathbb{Z})$ (in blue).

for the double jump of $\Gamma_2^{+(0)}$: the first caused by crossing the branch cut emanating from t_1 in the $\mathbb{C}_t$ plane, and the second by crossing the cut associated with $t_3 \in \mathbb{C}$. Then, when we cross the line l_0 in the clockwise direction, the corresponding jump matrix is given by the following upper triangular matrix:

$$\begin{pmatrix} \Gamma_2^{+(1)} \\ \Gamma_1^{+(1)} \\ \Gamma_3^{+(1)} \end{pmatrix} = \begin{pmatrix} 1 & \Delta_{21} & \Delta_{23} - \Delta_{21}\Delta_{13} \\ 0 & 1 & \Delta_{13} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_2^{+(0)} \\ \Gamma_1^{+(0)} \\ \Gamma_3^{+(0)} \end{pmatrix}, \tag{3.65}$$

where intersection numbers among vanishing cycles Δ_{ij} are computed with (A.47). Reordering the base of thimbles, we get

$$T^{(0)} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.66}$$

and

$$S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}, \tag{3.67}$$

with eigenvalues $i, -i, 1$. The matrix (3.66) define the wall crossing structure in $\mathbb{C}_\gamma^* \times U_{(b,c)}^{\Delta<0}$, in which the walls are defined by subregions of this space where three critical values are aligned. They correspond to walls of the first type in the sense of [100].

Vanishing discriminant. The last case we want to discuss is $\Delta = 0$, where two or all three critical points coalesce. Comparing figures 8(right) and 9(left) we observe that as σ_2

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and σ_3 coalesce, the upward branches of Γ_2^+ and Γ_3^+, as well as the downward branches of Γ_2^- and Γ_3^-, begin to overlap with opposite orientations. As a result, the combination of these four paths yields only two independent thimbles:

$$\Gamma_2^+ + \Gamma_3^+ = \Gamma_{23}^+ \quad \text{and} \quad \Gamma_2^- + \Gamma_3^- = \Gamma_{23}^-, \tag{3.68}$$

This is a consequence of the fact the vanishing cycles Δ_2 and Δ_3, appearing in (3.53), identify the same homology class when the associated critical points coalesce.

Similarly, when the triple degeneration occurs,[17] shown in figure 9(right), four branches overlap, leading to a further reduction in the number of independent thimbles. We are left with:

$$\Gamma_1^+ + \Gamma_{23}^+ = \Gamma_{123}^+ \quad \text{and} \quad \Gamma_1^- + \Gamma_{23}^- = \Gamma_{123}^-. \tag{3.69}$$

Therefore, the Betti homology turn out to be

$$\begin{aligned} H_1(X, D_N, \mathbb{Z})_{(2)} &= \text{span}\{\Gamma_1^+, \Gamma_{23}^+\} \qquad \text{and} \quad H_1(X, D_N, \mathbb{Z})_{(2)}^\vee = \text{span}\{\Gamma_1^-, \Gamma_{23}^-\}, \\ H_1(X, D_N, \mathbb{Z})_{(3)} &= \text{span}\{\Gamma_{123}^+\}, \qquad \text{and} \quad H_1(X, D_N, \mathbb{Z})_{(3)}^\vee = \text{span}\{\Gamma_{123}^-\}, \end{aligned} \tag{3.70}$$

where bracket subscripts denote the multiplicity of coalescent critical points. In the cohomology side, we can compute the relative cohomology using (3.42). We obtain

$$\begin{aligned} H^1(X, D_N, \mathbb{C})_{(2)} &= \text{span}\{1, z\} \cong \mathbb{C}^2, \\ H^1(X, D_N, \mathbb{C})_{(3)} &= \text{span}\{1\} \cong \mathbb{C}. \end{aligned} \tag{3.71}$$

Notice that the universal coefficient theorem for cohomology explicitly shows the duality[18]

$$H^1(X, D_N, \mathbb{C})_{(i)} \cong \text{Hom}(H_1(X, D_N, \mathbb{Z}), \mathbb{C})_{(i)}. \tag{3.72}$$

4 Exponential integrals for closed forms and Feynman integrals

In the previous section, we associated to each triple (X, D_0, f) four Local Systems over $\mathbb{C}_\gamma^*$ corresponding to the global and local de Rham and Betti cohomologies. Moreover, we showed that, using this language, exponential integrals can be naturally interpreted as periods pairing these two types of (co)homologies. Here, still following [1], we extend the discussion to the more general setting of a triple (X, D_0, α) where X is a complex smooth algebraic variety, $D_0 \subset X$ is a normal crossing divisor, and α is a closed algebraic 1-form. This generalized setup allows us to consider exponential integrals defined by multivalued functions, which represent ubiquitous objects in physics. Understanding such integrals and developing general methods to evaluate them remains a challenging and often open problem across various domains of theoretical physics. The aim of this section is to outline a general framework in which exponential integrals of this type can be systematically studied. Our primary motivation comes from the study of Feynman integrals. In particular, we introduce here the study of Feynman integrals in the Baikov representation [101] using the language of

[17] For $b = c = 0$.

[18] $\text{Ext}(H_0(X, D_N, \mathbb{Z})_{(i)}, \mathbb{C}) = 0$.

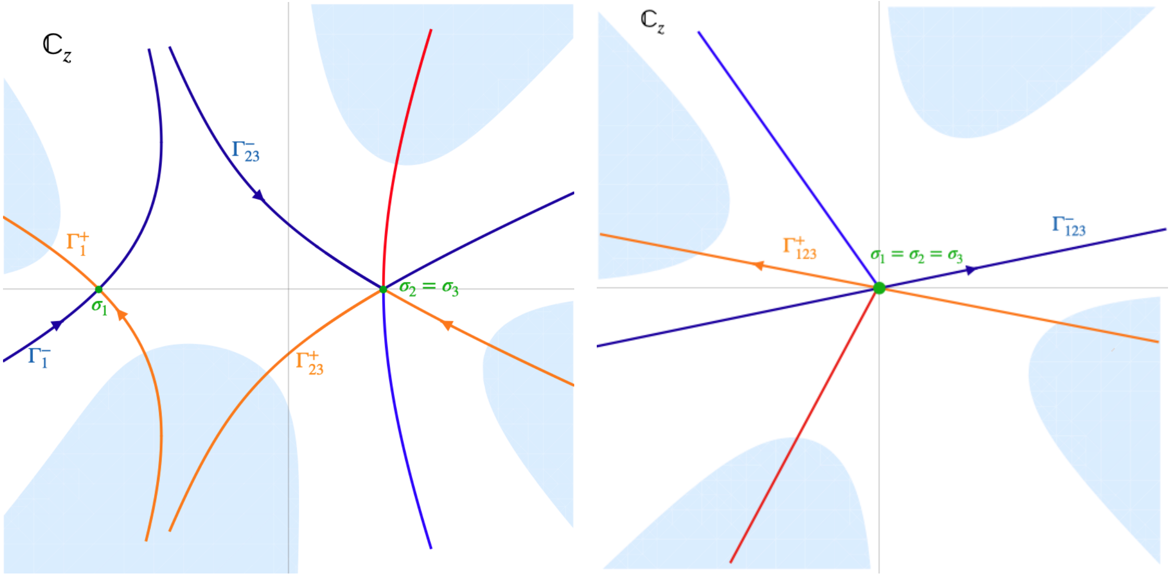

Figure 9. Vanishing discriminant case.
(Left) Double Degeneration, $(\gamma, b, c, d) = (i, -6, 8, -1)$. Ascendant paths Γ_i^+ spanning $H_1(\mathbb{C}_z, D_N, \mathbb{Z})$ (in orange), descendant paths Γ_i^- spanning $H_1(\mathbb{C}_z, X_{-N}, \mathbb{Z})$ (in blue).
(Right) Triple Degeneration, $(\gamma, b, c, d) = (i, 0, 0, -1)$. Ascendant paths Γ_i^+ spanning $H_1(\mathbb{C}_z, D_N, \mathbb{Z})$ (in orange), descendant paths Γ_i^- spanning $H_1(\mathbb{C}_z, X_{-N}, \mathbb{Z})$ (in blue).
(Both) Coalescing ascendant (red) and descendant (light blue) paths.

exponential integrals (see also [102] for a review. We remark that in [29] was realized for first time that Baikov representation allows to apply intersection theory methods to Feynman Integrals). A more detailed analysis will be provided in [81].

Any D dimensional, L-loops Feynman integral with E external legs, can be rewritten, using denominators as integration variables, in the so called Baikov representation

$$I(\{\nu_i\}) = \int_\Gamma \mathcal{B}^{-\gamma} \prod_{i=1}^{n} \frac{dz_i}{z^{\nu_i}}, \tag{4.1}$$

where $\mathcal{B}$, known as the Baikov polynomial, is a polynomial on the integration variables which coefficients depends on the masses of the internal particles and the external momenta, and $\gamma = -(D - E - L - 1)/2$. If $\gamma \in \mathbb{Q}$, the integral is multivalued but its underlying geometry[19] is a computable Riemann foliation obtained by gluing a finite number of sheets. In such cases, the integral (4.1) can still be interpreted as a standard pairing between the de Rham cohomology and the singular homology associated to this geometry. However, in the context of dimensional regularization, $\gamma \notin \mathbb{Q}$. As a result, we lose our geometric intuition behind the integral, and the previous interpretation of (4.1) as a period no longer applies. Reformulating the integrand as an exponential

$$I(\{\nu_i\}) = \int_\Gamma e^{-\gamma \log \mathcal{B}} \prod_{i=1}^{n} \frac{dz_i}{z^{\nu_i}} \tag{4.2}$$

[19]Here, by geometry we mean the geometric space where the integrand becomes single-valued.

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solves the issue on γ since it allows to interpret the integral as an exponential period for any $\gamma \in \mathbb{C}^*$ at the cost of dealing with a non-holomorphic function in the argument of the exponential.

Once the Feynman integral is expressed in exponential form, it can be interpreted as an exponential pairing between the de Rham cohomology and the Betti homology. For the latter, we have described above a geometric construction of a basis in terms of thimbles $th_{i,\gamma}$, and we have defined on it an internal product that can be computed purely from topological data. As we will show in this section, this construction remains valid even when the function f is multivalued. This allows us to express the integration contour Γ in (4.2) as a linear combination of this thimble basis:

$$\Gamma = \sum_{i=1}^{\dim H_n^{\text{Betti}}} \langle \Gamma, th_{i,\theta+\pi} \rangle \; th_{i,\theta}. \tag{4.3}$$

Using this linear combination for the contour, the integral (4.2) admits the following decomposition with respect to the coefficients defined by the internal product in homology:

$$I = \sum_{i=1}^{\dim H_n^{\text{Betti}}} \langle \Gamma, th_{i,\theta+\pi} \rangle \int_{th_{i,\theta}} e^{-\gamma \log \mathcal{B}} \prod_{j=1}^{n} \frac{dz_j}{z_j^{\nu_j}}. \tag{4.4}$$

From this perspective, the integrals evaluated over the thimbles that appear in the sum play the role of Master Integrals.

4.1 Twisted de Rham cohomology

The first important consequence of dealing with a multivalued function in the exponential is that the twisted de Rham cohomology side of the pairing is defined with respect to the differential

$$\nabla_\alpha = d - \alpha \wedge, \tag{4.5}$$

where α is a closed 1-form on the complex algebraic variety X that is not necessarily exact. This means that, in general, α cannot be written globally as $\alpha = df$ for some function f, but additional contributions may enter into its definition. To identify the various potential contributions, one needs to choose a suitable compactification $\overline{X}$ of X. However, the final result should ultimately be independent of the specific choice.

This "good" compactification for X must satisfy the properties stated in Proposition 3.1.1 of [1]. In particular it must be constructed using a set of normal crossing divisors D_h, D_v and $D_{\log}$

$$\overline{X} - X = D_h \cup D_v \cup D_{\log}, \tag{4.6}$$

such that

(i) Among all the normal crossing divisors $\overline{D}_0, D_h, D_v$ and $D_{\log}$ only D_v and $D_{\log}$ can have common irreducible components;

(ii) For any point $x \in \overline{X}$ there exists a small analytic neighborhood U and a closed meromorphic 1-form α locally given by the expression

$$\alpha = \alpha_{reg} + \alpha_{log} + \alpha_\infty . \tag{4.7}$$

The first contribution represent a regular form on U. The second contribution, $\alpha_{\log}$, can be expressed in local coordinates near a divisor D_{log} as

$$\alpha_{log} = \sum_i c_i d \log z_i , \tag{4.8}$$

where z_i are local coordinates in which D_{log} is given by $\prod_i z_i = 0$, and $c_i \in \mathbb{C} \setminus \{0\}$ represent the periods of α around the loci $z_i = 0$, computed as

$$c_i = \frac{1}{2\pi i} \oint_{S_i^1} \alpha , \tag{4.9}$$

the circle S_i^1 encircling D_{log} in a smooth point. The final contribution admits the form $\alpha_\infty = dF$ in local coordinates near D_v, where F is an analytic function of the form

$$F = \frac{c}{\prod_j z_j^{k_j}} \left(1 + o(1)\right) , \quad k_j \geq 1 \tag{4.10}$$

and D_v is locally defined by $\prod_j z_j = 0$ in those coordinates.

Once we have such a compactification, we can construct the sheaf $\Omega^\bullet_{\overline{X},D_{\log}}$ of differential forms on $\overline{X}$ with possible logarithmic poles along the divisors $D_{\log}$ and use this sheaf to construct the global de Rham cohomology at any $\gamma \in \mathbb{C}^*$ using the following definition.

Definition: [GLOBAL TWISTED DE RHAM, [1], DEF. 3.2.1]
Let $\gamma \in \mathbb{C}^*_\gamma$. The (twisted) de Rham cohomology is the graded abelian group constructed by the hypercohomology

$$H^\bullet_{dR,glob,\gamma}\left(X, D_0, \alpha\right) \;=\; \mathbb{H}^\bullet \left(\overline{X}, (\Omega^\bullet_{\overline{X},D}, \gamma^{-1} d + \alpha\wedge)\right) . \tag{4.11}$$

Also in the case of closed forms, one can define a local version of the cohomology and establish a global-to-local isomorphism, analogous to equation (2.17) for exact differentials. To provide this definition we need to introduce the subsheaf $\Omega^\bullet_{\overline{X},\alpha}$ of $\Omega^\bullet_{\overline{X}}(\log(\overline{D}_0 + D_h + D_v + D_{\log}))(-\overline{D}_0)$ consisting of forms $\eta \in \Omega^\bullet_{\overline{X}}(\log(\overline{D}_0 + D_h + D_v + D_{\log}))(-\overline{D})$ such that $\alpha \wedge \eta \in \Omega^\bullet_{\overline{X}}(\log(\overline{D}_0 + D_h + D_v + D_{\log}))(-\overline{D}_0)$, which means that the 1-form α can have poles of order one along the divisors $D_v \cup D_h \cup D_{\log}$.

Definition: [LOCAL TWISTED DE RHAM, [1], DEF. 3.2.1]
Let us indicate with $\mathcal{Z}(\alpha)$ the set of zeros of α on X and the zeros of the restriction of α on $D_0 \cup D_h$. We define the local (twisted) de Rham cohomology as

$$H^\bullet_{dR,loc,\gamma}\left(X, D_0, \alpha\right) \;=\; \bigoplus_{i\in I} \mathbb{H}^\bullet \left(U_{form}(z_i), (\Omega^\bullet_{U_{form}(z_i),\alpha}\left[\left[\gamma^{-1}\right]\right], \gamma^{-1} d + \alpha\wedge)\right) , \tag{4.12}$$

where $U_{\text{form}}(z_i)$ is the formal neighborhood of the component z_i of $\mathcal{Z}(\alpha)$.

The following isomorphism holds:

$$H^\bullet_{dR,glob,\gamma}\left(X, \alpha\right) \;\simeq\; H^\bullet_{dR,loc,\gamma}\left(X, \alpha\right) . \tag{4.13}$$

4.2 Betti cohomology

The direct construction of the global Betti cohomology is extremely technical, we just briefly recall it here for fixing notations, referring to [1] for details. Let $\widetilde{X}$ be the real oriented blow-up of $\overline{X}$ along $D = D_h \cup D_v \cup D_{log}$, that is the manifold with boundary, and possibly corners, obtained from $\overline{X}$ replacing D with the S^1- bundle of its normal bundle. Let $\pi : \widetilde{X} \to \overline{X}$ and $\Pi : X \to \widetilde{X}$ be the natural projection and embedding, respectively. Let $\mathcal{L}_{\alpha,\gamma}$ be the local system on X of flat sections of the trivial vector bundle on X with respect to $\nabla = (d+\gamma\alpha)$, i.e.

$$\mathcal{L}_{\alpha,\gamma}(U) = \ker \nabla|_U, \qquad U \subset X. \tag{4.14}$$

Let $D_k^{\mathbb{R}} = \{y \in \partial\widetilde{X} | \pi(y) \in D_k\}$, $k = h, v, log$, be the portion of the boundary of $\widetilde{X}$ whose points are projected onto D_k, and let us split each of them according to the growth behavior of F,

$$\begin{aligned} D_v^{\mathbb{R}\pm} &= \{\tilde{z} \in D_v^{\mathbb{R}} | \pm Re(\gamma F(\pi(\tilde{z})) \geq 0)\}, \\ D_{log,i}^{\mathbb{R}\pm} &= \{\tilde{z} \in D_{log}^{\mathbb{R}} | \mp Re(\gamma c_i) > 0)\}, \end{aligned} \tag{4.15}$$

where c_i is the residue of α in the $i-th$ irreducible component of $D_{log}^{\mathbb{R}}$. Next, let us define $\widetilde{X}^-$ as the subset of $\widetilde{X}$ obtained by removing the normal directions where F has a "bad" behavior:

$$\widetilde{X}^- = \widetilde{X} \backslash \left((D_{log}^{\mathbb{R}} - D_{log}^{\mathbb{R}+}) \cup (D_v^{\mathbb{R}} - D_v^{\mathbb{R}-}) \cup D_h \right), \tag{4.16}$$

with $D_{log}^{\mathbb{R}\pm} = \overline{\cup_i D_{log,i}^{\mathbb{R}\pm}}$. Let $i : \widetilde{X}^- \to \widetilde{X}$ its open inclusion.

Definition: [Global Betti cohomology, [1], Def.3.4.3]
The global Betti cohomology is defined as

$$\begin{aligned} H^\bullet_{\text{Betti,glob},\gamma}(X,\alpha) &\equiv H^\bullet\left(\widetilde{X}, \Pi_*(\mathcal{L}_{\alpha,\gamma}) \otimes i_!(\underline{\mathbb{Z}}_{\widetilde{X}^-})\right) \\ &\cong H^\bullet\left(\widetilde{X}, i_!(\underline{\mathbb{Z}}_{\widetilde{X}^-} \otimes i^*\Pi_*(\mathcal{L}_{\alpha,\gamma})\right) \cong H^\bullet\left(\widetilde{X}, i_! i^*\Pi_*(\mathcal{L}_{\alpha,\gamma})\right) \\ &\cong H^\bullet\left(\widetilde{X}, D_v^{\mathbb{R}+} \cup D_{log}^{\mathbb{R}-}, \Pi_*(\mathcal{L}_{\alpha,\gamma})\right). \end{aligned} \tag{4.17}$$

Let us describe the construction of the *local* Betti cohomology in a way similar to that presented in section 2.2 for the case of holomorphic functions. Fixing a Riemannian metric on X, for each $z_i \in \mathcal{Z}(\alpha)$, we can always choose a sufficiently small $\varepsilon-$neighborhood $U_{\varepsilon,i}(z_i) \subset X$ and a holomorphic function f_i defined on it such that, locally in this neighborhood,

$$\alpha = df_i. \tag{4.18}$$

For each $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, with $\theta = \arg(\gamma)$, we can define the graded $\mathbb{Z}-$module $H^\bullet_{\text{Betti,local},z_i,\gamma}$, analogous of (2.26), via the relative cohomology with respect to preimage of the point $t_i + \varepsilon e^{i\theta}$ in the boundary of $\overline{U}_{\varepsilon,i}(z_i)$:

$$H^\bullet_{\text{Betti,local},z_i,\gamma}(X,\alpha) = H^\bullet\left(U_{\varepsilon,i}(z_i), \overline{U}_{\varepsilon,i}(z_i) \cap f_i^{-1}\left(t_i + \varepsilon e^{i\theta}\right)\right), \tag{4.19}$$

with $t_i = f_i(z_i)$.

Definition: [LOCAL BETTI COHOMOLOGY, [1], DEF.3.4.6]
For fixed γ, the direct sum

$$H^\bullet_{\text{Betti,local},\gamma}(X,\alpha) = \bigoplus_{z_i\in\mathcal{Z}(\alpha)} H^\bullet_{\text{Betti,local},z_i,\gamma}(X,\alpha) \tag{4.20}$$

is called the local Betti cohomology.

Since the divisor D_h is empty, as we will see below, the only zeros of α contributing to $\mathcal{Z}(\alpha)$ arise from the domain $X = \mathbb{C}^n \setminus \{\mathcal{B}=0\}$. If the roots of $d\mathcal{B}$ do not lie on the hypersurface $\mathcal{B}=0$, the set $\mathcal{Z}(\alpha)$ coincides with the set Σ of critical points of the polynomial $\mathcal{B}$ within X. Since these are the only points contributing to the construction of the local Betti (co)homology, we can proceed in these cases analogously to the approach described for holomorphic functions.

Let us now discuss the global-to-local isomorphism for these Betti cohomologies. First, let us fix a new definition of Stokes rays in terms of zeros of the 1-form α rather than in terms of critical points as done in the case of holomorphic functions.

Definition: [STOKES RAY, [1], DEF. 3.9.1]
We call the ray $s_\theta = \{\gamma\,|\,\arg\gamma = \pi-\theta_{ij}=\theta\} = \mathbb{R}_{\geq 0}\cdot e^{i\theta}\subset\mathbb{C}_\gamma$ with $\theta = \arg\left(\int_{\Gamma_{ij}}\alpha\right)$, where Γ_{ij} is the homotopy class of paths in X joining the two points z_i and z_j in $\mathcal{Z}(\alpha)$, a Stokes ray. Rays with vertex at the origin that are not Stokes rays are called generic rays.

If γ does not lie on a Stokes ray, given the local system $\mathcal{L}_{\alpha,\gamma}$ associated with the holomorphic 1-form $\gamma\alpha$, we always have a well defined isomorphism

$$\varphi_{\arg\gamma}\,:\,H^\bullet_{\text{Betti,glob},\gamma}(X,\alpha)\simeq H^\bullet_{\text{Betti,local},\gamma}(X,\alpha). \tag{4.21}$$

Close to a Stokes ray s_θ, there exist two isomorphisms, $\varphi_{\theta+}$ and $\varphi_{\theta-}$, corresponding to angles immediately adjacent to the ray. The discrepancy between these isomorphisms is captured by the Stokes automorphism

$$\varphi^{-1}_{\theta-}\circ\varphi_{\theta+}\,:\,H^\bullet_{\text{Betti,local},\gamma}(X,\alpha)\mapsto H^\bullet_{\text{Betti,local},\gamma}(X,\alpha). \tag{4.22}$$

Just as for holomorphic functions, we can also associate a wall-crossing structure to the pair (X,α) by using the maps (4.22) as γ varies along S^1_θ in $\mathbb{C}^*_\gamma$. For a continuous family of pairs (X,α), as far as $\pi_0(\mathcal{Z}(\alpha))$ is locally constant, the corresponding wall crossing structures form a continuous family of WCS.

Now, let us assume that the zeros of α are isolated and simple. Then, for each $z_i\in\mathcal{Z}(\alpha)$, and its associated holomorphic function γf_i, we can construct the thimble th_{i,θ_γ} emanating from $z_i\in X$ by tracing the vanishing cycles $\Delta_i(s)$ on the level sets $f_i - f_i(z_i) = s$ for $s\in[0;+\infty)$ along the direction θ_γ. We say that th_{i,θ_γ} is compatible with α in the direction θ_γ if, for any point $x\in\Delta_i(s)$, as $s\mapsto+\infty$, we have $\gamma f_i(x)\mapsto+\infty$. In this case, we can define the pairing between the de Rham cohomology and the Betti homology class represented by th_{i,θ_γ} via the exponential integral

$$I_i(\gamma) = \int_{th_{i,\theta_\gamma}} e^{-\gamma f_i}\,\mu, \tag{4.23}$$

which is well defined only when the integral is convergent. Following a construction analogous to the one described in section 2, the integral (4.23) can rewritten as

$$I_i(\gamma) = e^{-\gamma f_i(z_i)} \int_0^{+\infty} ds \; e^{-\gamma s} \mathrm{vol}_{\Delta_i}(s). \tag{4.24}$$

where $\mathrm{vol}_{\Delta_i}(s)$ denotes the volume of the vanishing cycle $\Delta_i(s)$ on the level set $f_i - f_i(z_i) = s$. Since $\mathrm{vol}_{\Delta_i}(s)$ typically increases as $s \mapsto +\infty$, the converge of the integral (4.23) is ensured if and only if this volume growth is at most exponential. However, this condition may fail when $\dim_{\mathbb{C}} X \geq 3$.

We finally get to the point. Feynman integrals in the Baikov representation (4.1), involve multivalued logarithmic functions of the form

$$f(z_1, z_2, \ldots, z_n) = \log \mathcal{B}\left(z_1, z_2, \ldots, z_n\right). \tag{4.25}$$

The domain of definition of f is the complex manifold $X = \mathbb{C}^n \setminus \{\mathcal{B} = 0\}$, which excludes the zero locus of $\mathcal{B}$. A natural compactification of this space, satisfying the conditions outlined in Proposition 3.1.1 of [1], is the complex projective space $\overline{X} = \mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$, obtained introducing the additional divisors corresponding to the hyperplane at infinity and the zeros of the homogenized Baikov polynomial:

$$D_{\overline{\mathcal{B}}} \equiv \{[z_1, \ldots, z_n, \zeta] \in \mathbb{P}^n \mid \overline{\mathcal{B}}(z_1, \ldots, z_n, \zeta) = 0\}, \tag{4.26}$$

where $\overline{\mathcal{B}}$ is the extension of $\mathcal{B}$ to the compactification, with values in $\mathbb{P}^1$. Notice that f does not extend, but this is totally irrelevant. The two added divisors intersect non trivially in $D_{\overline{\mathcal{B}}} \cap \mathbb{P}^{n-1} = \{[z_1, \ldots, z_n, 0] \in \mathbb{P}^n \mid \overline{\mathcal{B}}(z_1, \ldots, z_n, 0) = 0\}$, i.e. the points at infinity of the compactification in $\mathbb{P}^n$ of the variety defined by the Baikov polynomial. Let us call the hyperplane at infinity

$$D_\infty \equiv \{[z_1, \ldots, z_n, 0] \in \mathbb{P}^n \}, \tag{4.27}$$

and finally write

$$\overline{X} - X = D_{\overline{\mathcal{B}}} \cup D_\infty. \tag{4.28}$$

Consider now the globally-defined 1-form

$$\alpha = d \log \overline{\mathcal{B}} = \frac{d\overline{\mathcal{B}}}{\overline{\mathcal{B}}}, \tag{4.29}$$

which clearly shows logarithmic poles along $D_{\overline{\mathcal{B}}}$ with residue $c_{\overline{\mathcal{B}}} = +1$, thus $D_{\overline{\mathcal{B}}} \subset D_{log}$. In order to study the behavior at infinity, let choose the coordinate $\eta^{-1} = \overline{\mathcal{B}}$: as η approaches zero, $\alpha = -d\eta/\eta$ exhibits again a logarithmic singularity, but with opposite residue: $c_\infty = -1$. We then finally conclude:

$$D_{log} = D_{\overline{\mathcal{B}}} \cup D_\infty \qquad \text{and} \qquad D_h = D_v = \emptyset. \tag{4.30}$$

The S^1-bundle with Euler class 1 over $D_\infty \cong S^n$ is given by the Hopf fibration: $D_\infty^{\mathbb{R}} \cong S^{2n-1}$. On the other hand $D_{\overline{\mathcal{B}}}^{\mathbb{R}}$ has Euler class $e = dc_1$, with d the degree of $\mathcal{B}$ and $c_1 = c_1(\mathcal{O}_{D_{\overline{\mathcal{B}}}})$ the first Chern class of the normal bundle of $D_{\overline{\mathcal{B}}}$ in $\mathbb{CP}^n$. Applying definitions (4.15) we get

$$D_{log}^{\mathbb{R}+} = \begin{cases} D_\infty^{\mathbb{R}}, & \text{if} \quad Re(\gamma) > 0, \\ D_{\overline{\mathcal{B}}}^{\mathbb{R}}, & \text{if} \quad Re(\gamma) < 0, \end{cases} \qquad \text{and} \qquad D_{log}^{\mathbb{R}-} = \begin{cases} D_\infty^{\mathbb{R}}, & \text{if} \quad Re(\gamma) < 0, \\ D_{\overline{\mathcal{B}}}^{\mathbb{R}}, & \text{if} \quad Re(\gamma) > 0, \end{cases} \tag{4.31}$$

and the global Betti cohomology (4.17) becomes

$$H^{\bullet}_{\text{Betti,glob},\gamma}(X,\alpha) \cong \begin{cases} H^{\bullet}\left(\widetilde{X}, D^{\mathbb{R}}_{\infty}, \Pi_*(\mathcal{L}_{\alpha,\gamma})\right), & \text{if} \quad Re(\gamma) > 0, \\ H^{\bullet}\left(\widetilde{X}, D^{\mathbb{R}}_{\overline{\mathcal{B}}}, \Pi_*(\mathcal{L}_{\alpha,\gamma})\right), & \text{if} \quad Re(\gamma) < 0. \end{cases} \tag{4.32}$$

For better readability, we now simplify the notation by setting $\mathcal{L} \equiv \mathcal{L}_{\alpha,\gamma}$. The key of the whole discussion lies in the behavior of the direct image $\Pi_*(\mathcal{L})$ that, by definition, is the extension of $\mathcal{L}$ to $\widetilde{X}$ by zero, i.e. the sheaf associating to $U \subset \widetilde{X}$ the group of sections of $\mathcal{L}$ on $U \cap X$. Note that no nonzero sections are entirely supported on the boundary, and since X is a deformation retract of $\widetilde{X}$, we get

$$H^k(\widetilde{X}, \Pi_*\mathcal{L}) \cong H^k(X, \mathcal{L}). \tag{4.33}$$

Therefore, the long exact sequence for the pair $(\widetilde{X}, D^{\mathbb{R}}_k)$ reduces to

$$\cdots \to H^k(\widetilde{X}, D^{\mathbb{R}}_k, \Pi_*\mathcal{L}) \to H^k(X, \mathcal{L}) \to H^k(D^{\mathbb{R}}_k, \Pi_*\mathcal{L}) \to H^{k+1}(\widetilde{X}, D^{\mathbb{R}}_k, \Pi_*\mathcal{L}) \to \cdots . \tag{4.34}$$

In order to compute (4.32), we have thus to determine $H^k(D^{\mathbb{R}}_k, \Pi_*\mathcal{L})$ and $H^k(X, \mathcal{L})$. Now, let $M_k \in \mathrm{GL}(r, \mathbb{C})$ be the monodromy matrix around D_k, with r the rank of $\mathcal{L}$. In order for a global section on X to be flatly extendable to the boundary $D^{\mathbb{R}}_k$, it must belong to $\ker(M_k - I)$. Indeed, subspaces that are invariant under the action of monodromies determine directions towards the boundary, along which global sections can be analytically continued. If M_k is semisimple with no eigenvalues 1 and no product relations (like, e.g., $\det(M_1 M_2) = 1$) occur, no nonzero section of $\mathcal{L}$ extends along the boundary $\partial\widetilde{X}$. In the opposite situation, when all M_k are the identity matrix, all global sections flatly extend along the boundary. In the general case, whenever M_k has some unitary eigeinvalue and/or relations among monodromies appear, some global sections extend, whereas others do not. Intuitively, the cohomology relative to a portion of the boundary kills all global sections that can be flatly extended to that boundary. Thus it depends on the possibility to extend global sections, encoded in $H^k(X, \mathcal{L})$, and on possible obstructions due to the topology of the boundary, encoded in $H^k(D^{\mathbb{R}}_k, \Pi_*\mathcal{L})$. Moreover, unless the local system is trivial (e.g. a constant sheaf), the isomorphism $H^k(-, \mathcal{L}) \cong H^k(-, \mathbb{C}) \otimes \mathcal{L}$ fails to be true, due to the torsion of $\mathcal{L}$ on $\mathbb{C}$,[20] and the cohomologies with coefficients in $\mathcal{L}$, significantly depends on the behaviour of the local system, encoded in the monodromy matrices defining it. This contribution must be computed case by case, however, we can compute, in fully generality, the contribution coming from the cohomologies with constant coefficients. We assume $D_{\overline{\mathcal{B}}}$ is smooth. In particular, $D_{\mathcal{B}} \equiv \{z \in \mathbb{C}^n | \mathcal{B}(z) = 0\}$ is a smooth affine variety. Therefore, we can proceed as follows. First, the Alexander duality theorem ensures that (here smoothness is irrelevant)

$$H^k(\mathbb{C}^n \backslash D_{\mathcal{B}}, \mathbb{C}) \simeq \tilde{H}_{2n-k-1}(D_{\mathcal{B}}, \mathbb{C}), \tag{4.35}$$

where $\tilde{H}_*$ is the reduced homology. Next, since $D_{\mathcal{B}}$ is smooth and has complex codimension 1, despite being noncompact, we can use the Poincaré duality to get (for $k \geq 1$)

$$H^k(\mathbb{C}^n \backslash D_{\mathcal{B}}, \mathbb{C}) \simeq \tilde{H}^{k-1}(D_{\mathcal{B}}, \mathbb{C}). \tag{4.36}$$

[20] Measured by the $\mathrm{Ext}(\mathbb{C}, \mathcal{L})$.

Finally, we have just to remember that for $q \geq 1$ one has $\tilde{H}^q = H^q$, while $\tilde{H}^0 = H^0/\mathbb{C}$ to get

$$H^k(X,\mathbb{C}) \cong \begin{cases} \mathbb{C}, & k=0 \\ 0, & k=1 \\ H^{k-1}(D_{\mathcal{B}},\mathbb{C}) \cong \mathbb{C}^{m_k}, & k=2,\dots,n \end{cases}, \tag{4.37}$$

with $m_k \equiv \dim H^{k-1}(D_{\mathcal{B}},\mathbb{C})$. The cohomologies of the boundaries components are computed via the Serre spectral sequence applied to the circle bundles $(S^1 \to D_j^{\mathbb{R}} \to D_j)$. The second page $E_2^{p,q} = H^p(D_j, H^q(S^1))$ is $E_2^{p,q} = H^p(D_j,\mathbb{C})$ for $q = 0,1$ and it vanishes otherwise. Both spectral sequences degenerate at page E_2. The case of $D_{\overline{\mathcal{B}}}$ clearly depends on $\mathcal{B}$ and must be computed case by case.[21] In particular, in the Serre spectral sequence the obstruction to lift the classes of S^1 to that of the total space of the bundle is $d_2^{p,1} : E_2^{p,1} \to E_2^{p+2,0}$. It acts as the cup product by the Chern class of the U(1) bundle.

For D_∞ we have that $H^{2j}(\mathbb{P}^{n-1},\mathbb{C}) = \mathbb{C}$ for $j = 0,1,\dots,n-1$, while are zero otherwise. Note that if we call H the hyperplane at infinity of $\mathbb{P}^{n-1}$, then, the generator of H^{2j} is H^j. On the other hand, H is exactly the first Chern class of the U(1) bundle. Thus, $d_2^{p,1} = \cdot \cup H$ is the cup product by H. It maps $H^{2j}(\mathbb{P}^{n-1},\mathbb{C}) \to H^{2j+2}(\mathbb{P}^{n-1},\mathbb{C})$ injectively for $j = 0,1,\dots,n-2$ and $H^{2n-2}(\mathbb{P}^{n-1},\mathbb{C})$ to 0. Therefore, the only surviving terms are $E_2^{0,0} \equiv \mathbb{C} \equiv H^0(D_\infty^{\mathbb{R}},\mathbb{C})$ and $E_2^{2n-2,1} \equiv \mathbb{C} \equiv H^{2n-1}(D_\infty^{\mathbb{R}},\mathbb{C})$. So

$$H^k(D_\infty^{\mathbb{R}},\mathbb{C}) \cong \begin{cases} \mathbb{C} & \text{for } k = 0, 2n-1 \\ 0 & \text{otherwise,} \end{cases} \tag{4.38}$$

and we finally obtain:

$$H^\bullet(\widetilde{X}, D_\infty^{\mathbb{R}},\mathbb{C}) \cong \mathbb{C}^{m_2} \oplus \ldots \oplus \mathbb{C}^{m_n}, \tag{4.39}$$

concentrated in degrees $2 \leq k \leq n$.

If $p = D_\infty \cap D_{\overline{\mathcal{B}}}$ is a critical point for $\overline{\mathcal{B}}$, $\mathcal{L}$ contains the monodromy M_∞ around p, and the twisted cohomology could be affected by its action. However, even in that case, because $\pi_1(S^{2n-1}) = 0$, any action is impossible. Therefore:

$$H^k(D_\infty^{\mathbb{R}},\mathcal{L}) \cong H^k(S^{2n-1},\mathbb{C}) \otimes \mathcal{L}. \tag{4.40}$$

In the following subsection we will use these results to explicitly compute (4.32) in a concrete example.

4.3 Elliptic fibers

We will now apply the generalized framework for multivalued functions described above, to a concrete example closely related to the family of l-loops banana integrals (figure 10) [34, 36, 103–105]. The latter, in Symanzik representation and $D = 2$, involves the integration of a homogeneous $(l+1)-$degree polynomial in $\mathbb{P}^l$, whose zeroes locus identifies a $(l-1)-$fold Calabi-Yau manifold. In particular, the two loops banana integral (sunrise diagram) is then associated to a family of elliptic curves ("1-fold Calabi-Yau") [32, 106, 107]. In order to

[21] Notice $D_{D_{\mathcal{B}}}^{\mathbb{R}}$ is twisted by $\mathcal{O}(degree(\mathcal{B}))$.

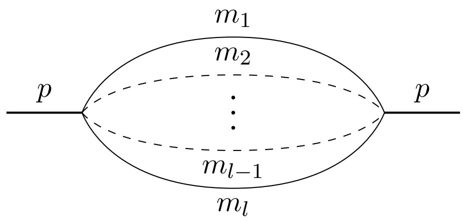

Figure 10. l loops banana diagrams.

make the details of the formalism above as clear as possible, we choose to apply it to the simplest case of the well-known Legendre family of elliptic curves, thus avoiding technical complications (non didactical examples will be provided in [81]).

Consider then the family of integrals

$$\mathcal{I} = \int_\Gamma \frac{dx \wedge dy}{[y^2 + x(x-1)(x-\lambda)]^\gamma} = \int_\Gamma e^{-\gamma \log\left[y^2+x(x-1)(x-\lambda)\right]} dx \wedge dy = \int_\Gamma e^{-\gamma \log \mathcal{B}(x,y;\lambda)} dx \wedge dy, \tag{4.41}$$

associated to the Legendre family of elliptic curves $\mathcal{E}_\lambda \equiv \{(x, y) \in \mathbb{C}^2 | \mathcal{B}(x, y, \lambda) = 0\}$, where $\lambda \in \mathcal{M}_{cs} = \mathbb{C} \setminus \{0, 1, \infty\}$ is a complex structure parameter. In the physical interpretation, λ can be thought of as parameterizing the masses of internal particles or the external momenta.

In this example we have $X = \mathbb{C}^2 \backslash \mathcal{E}_\lambda$, $\gamma \in \mathbb{C}^*_\gamma$ and the integration contour $\Gamma \in H_2\left(\mathbb{C}^2, D_0\right)$ is a singular Borel-Moore 2-chain with boundaries on the divisor D_0 defined as

$$D_0 = \lim_{N\mapsto\infty} X_N = \lim_{N\mapsto\infty} \{(x, y) \in X \mid Re\left(\gamma \log \mathcal{B}(x, y; \lambda)\right) \geq N\}\,. \tag{4.42}$$

The natural choice for the compactification is

$$\overline{X} \;=\; \mathbb{P}^2 \;=\; \mathbb{C}^2 \setminus \{\mathcal{E}_\lambda = 0\} \,\cup\, \left\{\overline{\mathcal{E}}_\lambda = 0\right\} \,\cup\, \mathbb{P}^1, \tag{4.43}$$

where $\mathcal{B}(x, y; \lambda)$ extends to

$$\overline{\mathcal{B}}(x, y, \eta; \lambda) = y^2\eta - x(x - \eta)(x - \eta\lambda), \tag{4.44}$$

with

$$d\,\log \overline{\mathcal{B}} = \frac{2\eta y dy + [y^2 + x^2 + x\lambda(x - 2\eta)]d\eta + [-3x^2 - \eta^2\lambda + 2x\eta(1 + \lambda)]dx}{y^2\eta - x(x - \eta)(x - \eta\lambda)}. \tag{4.45}$$

By analyzing the behavior of $\overline{\mathcal{B}}$ on $\mathbb{P}^2$ we can identify the types of divisors introduced in our compactification. In particular, we find

$$D_h = D_v = \emptyset \quad \text{and} \quad D_{log} = D_{\overline{\mathcal{B}}} \cup D_\infty, \tag{4.46}$$

with

$$\begin{aligned} D_{\overline{\mathcal{B}}} &= \overline{\mathcal{E}}_\lambda = \{[x : y : \eta] \in \mathbb{P}^2 | \overline{\mathcal{B}} = 0\}, \\ D_\infty &= \mathbb{P}^1 = \{[x : y : 0] \in \mathbb{P}^2\}, \end{aligned} \tag{4.47}$$

intersecting at $D_{\overline{\mathcal{B}}} \cap D_\infty \;=\; [0 \;:\; 1 \;:\; 0]$.

Since the divisors D_v and D_h are empty, the 1-form α receives a singular contributions only from the divisor $D_{\log}$. Therefore, we can write

$$\alpha = \alpha_{\log} + \alpha_{reg}. \tag{4.48}$$

Around $\mathcal{E}_\lambda \subset D_{log}$, the closed 1-form α is the holomorphic form

$$\alpha_{\log} = \frac{d\mathcal{B}}{\mathcal{B}}, \tag{4.49}$$

whose zeros are

$$\begin{aligned}\mathcal{Z}(\alpha) &= \{z_i = (x_i, y_i) \in X | d\mathcal{B}(x, y; \lambda)/\mathcal{B}(x, y, \lambda) = 0\} = \\ &= \left\{ \left(\frac{1}{3}\left[1 + \lambda - \sqrt{1 + \lambda(\lambda - 1)}\right], 0 \right), \left(\frac{1}{3}\left[1 + \lambda + \sqrt{1 + \lambda(\lambda - 1)}\right], 0 \right) \right\},\end{aligned} \tag{4.50}$$

with corresponding critical values

$$S = \{\log t_i \in \mathbb{C} | \log \mathcal{B}\,(z_i; \lambda)\}\,, \tag{4.51}$$

with

$$\begin{cases} t_1 = \frac{1}{27}\left(\sqrt{\lambda^2 - \lambda + 1} - \lambda - 1\right)\left(\lambda^2 - 4\lambda - (\lambda + 1)\sqrt{\lambda^2 - \lambda + 1} + 1\right) \\ t_2 = -\frac{1}{27}\left(\sqrt{\lambda^2 - \lambda + 1} + \lambda + 1\right)\left(\lambda^2 - 4\lambda + (\lambda + 1)\sqrt{\lambda^2 - \lambda + 1} + 1\right). \end{cases} \tag{4.52}$$

The map

$$\mathcal{B}: \quad X \quad \longmapsto \quad \mathbb{C}_t^*, \tag{4.53}$$

defines a non-trivial Lefschetz fibration over $\mathbb{C}_t^* = \mathbb{C}\backslash\{t_1, t_2\}$ for each $\lambda \in \mathcal{M}_{cs} = \mathbb{C}_\lambda\backslash\{0, 1, \infty\}$.

At fixed λ, the generic fiber $\mathcal{B}^{-1}(t)$ is the elliptic curve:

$$\mathcal{F}_t : \quad y^2 + x(x - 1)(x - \lambda) = t. \tag{4.54}$$

The badness of the fibration at the critical values in $\{t_1, t_2\}$ is measured in terms of the local monodromies acting on the homology group $H_1(\mathcal{F}_t)$ through the matrices:

$$M_i \; : \; H_1(\mathcal{F}_t) \quad \longmapsto \quad H_1(\mathcal{F}_t), \quad i = 1, 2. \tag{4.55}$$

To determine a basis for this homology and the therein representation of the monodromies we follow the description revisited in appendix A. We fix a non-critical point t_0 in $\mathbb{C}_t$ and construct two paths

$$u_i \; : \; [0; 1] \quad \longmapsto \quad \mathbb{C}_t, \tag{4.56}$$

each connecting the non-critical value $t_0 = u_i(0)$ to a critical value $t_i = u_i(1)$ without crossing any other critical point. For each path $u_i(t)$ we can define a family of 1-dimensional spheres in the level manifolds $\mathcal{F}_{u_i}$:

$$S_i(s) = \sqrt{u_i(s) - t_i} S^1, \tag{4.57}$$

that shrink to zero radius as we approach the critical point t_i. The homology classes $\Delta_i \in H_1(\mathcal{F}_{t_0})$ represented by these spheres are the Picard-Lefschetz vanishing cycles along the paths u_i and they form a basis for the homology $H_1(\mathcal{F}_{t_0})$.

To provide a clear visualization of the construction, we fix the parameter $\lambda = 3$ and carry out the explicit computations for this case. As long as $\pi_0(\mathcal{Z}(\alpha))$ remains invariant under variations of $\lambda \in \mathcal{M}_{cs}$ the wall crossing structures associated with the pairs $(X_\lambda, \alpha_\lambda)$ remain continuously connected to that of (X_3, α_3). The sets $\mathcal{Z}(\alpha)$ and S are:

$$\begin{aligned} \mathcal{Z}(\alpha) &= \left\{ z_1 = \left(\frac{1}{3}\left(\sqrt{7}+4\right), 0\right), z_2 = \left(\frac{1}{3}\left(-\sqrt{7}+4\right), 0\right)\right\}, \\ S &= \left\{ t_1 = -\frac{2}{27}\left(7\sqrt{7}+10\right), t_2 = \frac{2}{27}\left(7\sqrt{7}-10\right)\right\}. \end{aligned} \tag{4.58}$$

The level manifold $\mathcal{F}_{t_0}$ at the regular point t_0 is the graph of the two-valued function

$$y = \pm\sqrt{t_0 - (x-3)(x-1)x}, \tag{4.59}$$

namely, the double-covering of the x plane, branched at the points:

$$\begin{aligned} x_1 &= \frac{1}{6}\left(-2^{2/3}\sqrt[3]{3\sqrt{3}\sqrt{27t^2+40t-36}-27t-20} - \frac{14\sqrt[3]{2}}{\sqrt[3]{3\sqrt{3}\sqrt{27t^2+40t-36}-27t-20}} + 8\right), \\ x_2 &= \frac{1}{12}\left(2^{2/3}\left(1-i\sqrt{3}\right)\sqrt[3]{3\sqrt{3}\sqrt{27t^2+40t-36}-27t-20} + \frac{14\sqrt[3]{2}\left(1+i\sqrt{3}\right)}{\sqrt[3]{3\sqrt{3}\sqrt{27t^2+40t-36}-27t-20}} + 16\right), \\ x_3 &= \frac{1}{12}\left(2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{3\sqrt{3}\sqrt{27t^2+40t-36}-27t-20} + \frac{14\sqrt[3]{2}\left(1-i\sqrt{3}\right)}{\sqrt[3]{3\sqrt{3}\sqrt{27t^2+40t-36}-27t-20}} + 16\right). \end{aligned}$$

Let us choose the first cut from x_1 to x_3 and the second cut from x_2 to infinity.

As we move the value of t from t_0 to one of the critical values, the level manifold $\mathcal{F}_t$ is deformed and it becomes singular. In particular, when we approach t_1 we have that the branch point x_2 moves until overlaps with x_3, while when we approach t_2 the point x_1 moves towards the point x_3. From this construction we can draw the vanishing cycles Δ_1 and Δ_2 in $\mathcal{F}_{t_0}$ associated to the paths u_1 and u_2, respectively. The cycle Δ_1 encircles the points x_2 and x_3, while Δ_2 encircles x_1 and x_3.

Tracing the change in the positions of the three points x_j as we follow the counterclockwise-oriented closed loop $\tau_i \in \pi_1(\mathbb{C}_t \setminus \{t_1, t_2\}, t_0)$ encircling the critical point t_i we can deduce the corresponding monodromy action on the ordered basis $\{\Delta_1, \Delta_2\}$ of vanishing cycles. In particular, we obtain

$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{4.60}$$

Using the Picar-Lefschetz formula (A.58), we can derive the intersection form on $H_1(\mathcal{F}_t, \mathbb{Z})$ from these monodromies, expressed with respect to the chosen basis of vanishing cycles:

$$\Delta_i \circ \Delta_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.61}$$

These local monodromies characterize the type of singularity occurring at the critical points t_1 and t_2. In this particular case, where the fiber is an elliptic curve, we can refer to the singularity in t_1 as a MUM singularity, while the singularity in t_2 as a conifold point.

Betti cohomology. The divisors (4.47) intersect with normal crossing, thus the real oriented blow-up of $\mathbb{P}^2$ along D_{log} is given by union of blow-ups along the two divisors. The normal bundles of $D_\infty \cong S^2$ and $D_{\overline{\mathcal{B}}} \cong T^2$ in $\mathbb{P}^2$ are respectively the complex line bundles $\mathcal{O}(1)$ and $\mathcal{O}(3)$. The corresponding circle bundles respectively are the Hopf fibration $S^3 \to S^2$ and the 3-dimensional Heisenberg nilmanifold $Nil^3 \cong H_3(\mathbb{R})/H_3(\mathbb{Z})$, so $\widetilde{X}$ has boundary

$$\partial\widetilde{X} = S^3 \cup Nil^3, \tag{4.62}$$

where $\partial D^{\mathbb{R}}_\infty$ and $\partial D^{\mathbb{R}}_{\overline{\mathcal{B}}}$ are glued along the corner $S^1 \times S^1$, preimage of the intersection point.[22] We want to compute

$$\begin{aligned} &H^\bullet_{\text{Betti,glob},\gamma}(X,\alpha)(\widetilde{X}, S^3, \Pi_*(\mathcal{L}_{\alpha,\gamma})), \quad Re(\gamma) > 0,\\ &H^\bullet_{\text{Betti,glob},\gamma}(X,\alpha)(\widetilde{X}, D^R_{\overline{\mathcal{B}}}, \Pi_*(\mathcal{L}_{\alpha,\gamma})), \quad Re(\gamma) < 0. \end{aligned} \tag{4.63}$$

using the results discussed in section 4.2 and the monodromy matrices obtained in (4.60). The contributions coming from $D^{\mathbb{R}}_\infty \simeq S^3$, can be easily computed by the straight application of (4.40), yielding:

$$H^\bullet(D^{\mathbb{R}}_\infty, \mathcal{L}) \simeq H^\bullet(S^3, \mathbb{C}) \otimes \mathcal{L} \simeq \mathbb{C}^2 \oplus 0 \oplus 0 \oplus C^2. \tag{4.64}$$

The situation is much more involved in the case of $D^{\mathbb{R}}_{\overline{\mathcal{B}}}$, defined by the fibration

$$S^1 \hookrightarrow D^{\mathbb{R}}_{\overline{\mathcal{B}}} \to D_{\overline{\mathcal{B}}}, \tag{4.65}$$

which is nontrivial (since is generated by a nontrivial normal bundle of degree 9). The monodromies are nontrivial around the elliptic curve, so we can choose to assign them to a basis of generators of its homotopy π_1, say $\rho(a) = M_1$, $\rho(b) = M_2$. The Heisenberg structure gives a central extension such that the commutator (in the group theoretical sense) $[a,b] = \iota$ generates the homotopy of the fibre. This means that the representation must respect this relation and we must have

$$\rho(\iota) = M_1 M_2 M_1^{-1} M_2^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1. \end{pmatrix} \tag{4.66}$$

To compute the twisted cohomology we now use the isomorphism

$$H^\bullet(D^{\mathbb{R}}_{\overline{\mathcal{B}}}, \mathcal{L}) = H^\bullet(\pi_1(D^{\mathbb{R}}_{\overline{\mathcal{B}}}), V_\rho), \tag{4.67}$$

where V_ρ is the representation space of ρ seen as left ρ-module. The Nilmanifold can be represented by a CW-complex obtained by gluing a 3-cell to three 2-cells, next to three 1-cells and finally to a 0-cell. So we have that the j-chains C^j satisfy

$$C^0(D^{\mathbb{R}}_{\overline{\mathcal{B}}}, \mathcal{L}) \simeq V_\rho, \qquad C^1(D^{\mathbb{R}}_{\overline{\mathcal{B}}}, \mathcal{L}) \simeq V_\rho^3, \qquad C^2(D^{\mathbb{R}}_{\overline{\mathcal{B}}}, \mathcal{L}) \simeq V_\rho^3, \qquad C^3(D^{\mathbb{R}}_{\overline{\mathcal{B}}}, \mathcal{L}) \simeq V_\rho, \tag{4.68}$$

with $V_\rho \simeq \mathbb{C}^2$. Also, we can use that $H^j(\pi_1(D^{\mathbb{R}}_{\overline{\mathcal{B}}}), V_\rho) \simeq H^{3-j}(\pi_1(D^{\mathbb{R}}_{\overline{\mathcal{B}}}), V_{\rho^*})$, where ρ^* is the dual representation. These representations are irreducible so one finds that

$$H^j(\pi_1(D^{\mathbb{R}}_{\overline{\mathcal{B}}}), V_{\rho^*}) \simeq H^j(\pi_1(D^{\mathbb{R}}_{\overline{\mathcal{B}}}), V_\rho), \qquad j = 0, 1, \tag{4.69}$$

[22] Notice that it has multiplicity 3.

so we can reduce to the computations for $j = 0, 1$. The differential in the group cohomology is the standard one. We have to consider explicitly the differentials. If

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in C^0 \equiv \mathbb{C}^2 \tag{4.70}$$

is a 0-chain, one has that $d^0 f$ has to be a 1-chain so, for $g = a, b, \iota$, is defined by

$$d^0 f(g) = \rho(g)v - v. \tag{4.71}$$

This means that the elements of H^0 are the invariant vectors. So

$$0 = d^0 f(a) = \begin{pmatrix} 0 \\ f_1 \end{pmatrix}, \qquad 0 = d^0 f(b) = \begin{pmatrix} -f_2 \\ 0 \end{pmatrix}, \qquad 0 = d^0 f(\iota) = \begin{pmatrix} f_1 + f_2 \\ f_1 \end{pmatrix}, \tag{4.72}$$

which gives $H^0(\pi_1(D^{\mathbb{R}}_{\mathcal{B}}), V_\rho) = 0$.

Similarly, for $f \in C^1 \simeq \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$, one has (2-chains acts on pairs (g, g') of elements of π_1)

$$d^1 f((g, g')) = \rho(g)f(g') - f(gg') + f(g). \tag{4.73}$$

The explicit calculation of the kernel of d^1 is tedious but direct, and we leave the details to the readers. After quotienting by the image of d^0 one gets:

$$H^\bullet(D^{\mathbb{R}}_{\mathcal{B}}, \mathcal{L}) \cong 0 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus 0. \tag{4.74}$$

The last piece we have to compute is $H^\bullet(X, \mathcal{L})$. Here, we can use again the Alexander duality:

$$H^k(X, \mathcal{L}) \simeq H_c^{4-k-1}(\mathcal{E}_\lambda, \mathcal{L}^*)^*. \tag{4.75}$$

Next we use Poincaré duality to get (in finite dimensions biduals cancel out naturally)

$$H^k(X, \mathcal{L}) \simeq H^{k-1}(\mathcal{E}_\lambda, \mathcal{L}). \tag{4.76}$$

Therefore, we have to compute $H^k(\mathcal{E}_\lambda, \mathcal{L})$. $\mathcal{E}_\lambda$ is an affine elliptic curve so $H^2(\mathcal{E}_\lambda, \mathcal{L}) = 0$. $H^0(\mathcal{E}_\lambda, \mathcal{L})$ is determined by the vectors of V_ρ invariant under the action of $M_1 - I$ and $M_2 - I$ which, like before, give $H^0(\mathcal{E}_\lambda, \mathcal{L}) = 0$. The cellular decomposition of $\mathcal{E}_\lambda$ consists in a 2-cell, an two 1-cells (the 0-cell is missing. In cohomology this corresponds to $H^2 = 0$). Thus the chains are $C^0 \equiv V_\rho$, $C^1 = V_\rho^2$, $C^2 = 0$. It follows that $\ker d^1 = V_\rho^2 \simeq \mathbb{C}^4$, while $\operatorname{Im} d^0 \simeq \mathbb{C}^2$. We conclude that

$$H^\bullet(X, \mathcal{L}) = 0 \oplus 0 \oplus \mathbb{C}^2 \oplus 0 \oplus 0. \tag{4.77}$$

Finally, we can use (4.64), (4.74) and (4.77) in the long exact sequence (4.34), getting:

$$\begin{aligned} H^\bullet_{\text{Betti,glob},\gamma}(X, \alpha)(\widetilde{X}, S^3, \Pi_*(\mathcal{L}_{\alpha,\gamma})) &\cong 0 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus 0 \oplus \mathbb{C}^2, \\ H^\bullet_{\text{Betti,glob},\gamma}(X, \alpha)(\widetilde{X}, D^{\mathbb{R}}_{\mathcal{B}}, \Pi_*(\mathcal{L}_{\alpha,\gamma})) &\cong 0 \oplus 0 \oplus \mathbb{C}^2 \oplus \mathbb{C}^4 \oplus 0. \end{aligned} \tag{4.78}$$

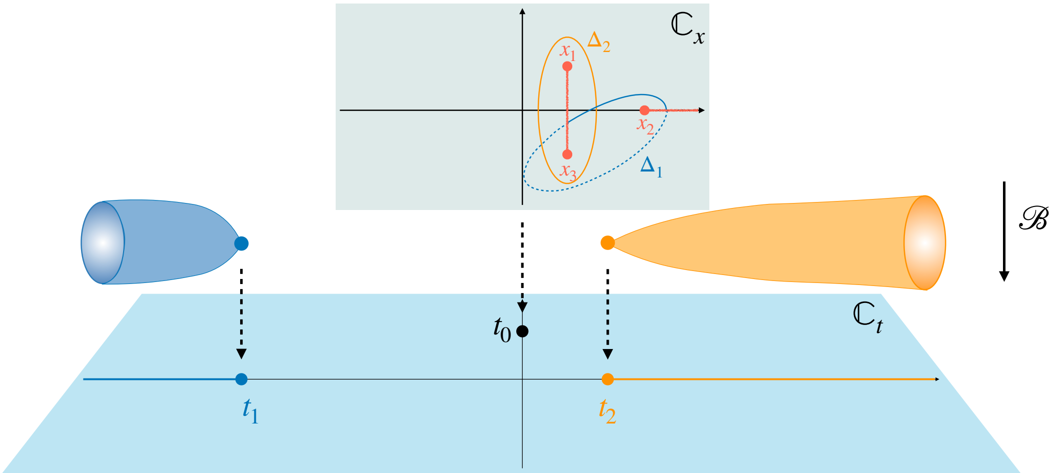

Figure 11. Representation of the construction of the thimbles in the double fibration $log : (P : X \mapsto \mathbb{C}_t) \mapsto \mathbb{C}_T$.

Thimbles construction. At this stage, we have all the necessary ingredients to construct the thimbles associated with the vanishing cycles Δ_1 and Δ_2, which form a basis for the local Betti homology groups $H_2^{\text{Betti,local},z_i,\gamma}(X,\alpha)$. As in the case of holomorphic functions, we begin by studying the homology for a fixed $\gamma \in \mathbb{C}^*_\gamma$, and then we analyze its analytic continuation, equipped with a wall-crossing structure.

Let us fix $\gamma = 1$. The step-ascend thimble associated with the vanishing cycle Δ_i is defined as the trace over a path in $\mathbb{C}^*_t$, starting from t_i, along which the imaginary part $\mathrm{Im(T)} = \mathrm{Im}(\gamma \log \mathrm{t})$ remains constant to the value $\mathrm{Im}(\gamma \log \mathrm{t_i}) = \mathrm{Im}(\log \mathrm{t_i}) = \arg(\mathrm{t_i})$, while the real part $Re(\gamma \log t) = Re(\log t)$ increases monotonically from $\log |t_i|$ to $+\infty$. A graphical illustration of the results is provided in figure 11. The Stokes rays in the plane $\mathbb{C}_\gamma$ are

$$
\begin{aligned}
s_{\theta_1} &= \left\{ \gamma \,|\, \arg\gamma = \arctan \frac{\pi}{\log|t_1| - \log|t_2|} \right\}, \\
s_{\theta_2} &= \left\{ \gamma \,|\, \arg\gamma = \arctan \frac{\pi}{\log|t_1| - \log|t_2|} + \pi \right\},
\end{aligned} \tag{4.79}
$$

that never stay along the real axis.

Now, if γ does not belong to a Stokes ray we can consider the collection of integrals evaluated along the thimbles:

$$
I_i(\gamma) = \int_{th_{i,\theta_\gamma}} e^{-\gamma \log \mathcal{B}(x,y)} dx \wedge dy, \tag{4.80}
$$

where $\log \mathcal{B}$ is exactly the function f such that $df = \alpha_{\log}$.

Using the parameterization $th_{i,\theta_\gamma} = \Delta_i(s) \times \mathbb{R}_{s\geq 0}$, we have

$$
I_i(\gamma) = e^{-\gamma \log t_i} \int_0^{+\infty} e^{-\gamma s} \mathrm{vol}_{\Delta_i}(s) ds, \tag{4.81}
$$

where $\mathrm{vol}_{\Delta_i}(s)$ is the volume of the vanishing cycle in the fiber $\mathcal{B}^{-1}(s)$ defined with respect to the Gelfand-Leray form $\frac{dx \wedge dy}{d\mathcal{B}/\mathcal{B}}$, namely:

$$
\mathrm{vol}_{\Delta_i}(s) = \int_{\Delta_i(s)} \frac{dx \wedge dy}{d\mathcal{B}/\mathcal{B}} = \int_{\Delta_i(s)} (t_i + s) \frac{dx}{2y(s)}. \tag{4.82}
$$

In the integrand we can recognize the holomorphic form $\omega^{1,0} = dx/y$, so the resulting integrals are precisely the periods of this form with respect to the basis of vanishing cycles for the family of varieties $\mathcal{F}_t$:

$$\mathbf{\Pi}(t) = \begin{pmatrix} \int_{\Delta_1} \frac{dx}{y} \\ \int_{\Delta_2} \frac{dx}{y} \end{pmatrix} . \tag{4.83}$$

Since we know the monodromies (4.60) around the critical values t_i, we can compute the expansions for $0 \leq s \leq \epsilon$ using the Nilpotent Orbit Theorem [108]:

$$\mathbf{\Pi}(\tilde{s}) = e^{\tilde{s}N_i} \left(\mathbf{a}_0 + \mathbf{a}_1 e^{2\pi i \tilde{s}} + \dots \right) , \tag{4.84}$$

where N_i is the Nilpotent matrix encoding the unipotent part of the monodromy M_i:

$$N_i = \log \left(M_i^{(u)} \right) . \tag{4.85}$$

The variable $\tilde{s}$ is related to the coordinate s by the following transformation

$$\tilde{s} = \frac{1}{2\pi i} \log s, \tag{4.86}$$

and we have

$$\mathbf{a}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;, \quad \text{or} \quad \mathbf{a}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.87}$$

for th_{1,θ_γ} and th_{2,θ_γ} respectively.

This explicit construction shows the existence of two Lefschetz thimbles that generate the local Betti homology, which is isomorphic to the global version (4.78), via the isomorphism (4.21).

In order to compare the two Master Integrals $I_i(\gamma)$ for $\gamma \mapsto \infty$ within the same sector of the $\mathbb{C}_\gamma$ plane, we associate to each of them the corresponding power series expansion in γ, as in (2.40). Following the discussion in section 2.3, the coefficients $c_{i,\lambda}$ governing the series can be computed from the closed formula (2.50), which is entirely determined by the local monodromies (4.60) around the critical points. In the present case, both monodromy matrices have a single Jordan block of dimension 2×2 associated with the eigenvalue $+1$. Consequently, for both cases we obtain the following non-trivial coefficient:

$$c_{i,0} = \frac{1}{\gamma^{-1}} \left[a_{0,0}\gamma^{-1} + a_{0,1} \left(\gamma^{-\lambda-1}\Gamma(\lambda+1)\psi^{(0)}(\lambda+1) - \gamma^{-\lambda-1}\log(\gamma)\Gamma(\lambda+1) \right)_{\lambda=0} \right] . \tag{4.88}$$

This shows that the two Master Integrals associated with the different homology classes represented by thimbles are of the same order in the same sector of the $\mathbb{C}_\gamma$ plane.

5 Conclusions

The analysis of physical systems across various domains, from quantum mechanics to statistical physics and from quantum field theory to string theory, often necessitates the computation of increasingly complex integrals. Developing systematic methods to address their computation remains a central challenge in theoretical physics. One of the most powerful techniques for simplifying certain classes of integrals, those expressible as periods of de Rham cocycles over closed cycles on smooth manifolds, is provided by Stokes' theorem. By fixing appropriate bases in the relevant cohomology and homology spaces, the integration of basis cocycles over basis cycles yields a set of simpler integrals encoded in the period matrix. Stokes' theorem then allows any integral within the family to be reduced to a linear combination of these fundamental integrals, whose coefficients are interpretable as intersection numbers in either cohomology or homology.

A natural and compelling extension of this framework would be to encompass broader classes of integrals, particularly those encountered in physics. However, a significant obstacle arises in cases involving multivalued or otherwise intricate integrals, where geometric intuition is lost, and the appropriate cohomology/homology needed to define the pairing required to interpret the integral as a period is no longer evident.

In this work, we propose a systematic approach to identifying the appropriate (co)-homological structures to apply in a large classes of physical integrals. Leveraging recent mathematical developments, we employ twisted de Rham cohomology and Betti homology over complex manifolds to rigorously treat exponential-type integrals as periods. This framework accommodates a wide range of physically relevant integrals, including quantum mechanical partition functions, conformal correlators, and, importantly, Feynman integrals. In the latter case this interpretation becomes viable through a generalization of established techniques for exponential integrals involving holomorphic functions in the exponent, extended to accommodate multivalued functions. Indeed, Feynman integrals expressed in the Baikov representation, as in any parametric representation as well, naturally admit such a reformulation, where the role of the multivalued function is played by the logarithm of a polynomial, the Baikov polynomial.

A key ingredient in this framework is the study of the complex analytic continuation of a real parameter γ, which appears as a prefactor in the exponent. This continuation induces a wall crossing structure, known in physics as the Cecotti-Vafa wall crossing structure [95], on the complex γ-plane, $\mathbb{C}^*_\gamma$, where four distinct local systems can be defined: local and global versions of twisted de Rham and Betti (co)homologies. The complex plane $\mathbb{C}_\gamma$ is partitioned into sectors by Stokes rays, and within each sector, one can define a canonical basis for each of the four (co)homologies. As γ crosses a Stokes ray in this fan, these bases undergo discontinuous transformations encoded by Stokes automorphisms.

After a concise review of the mathematical tools required for this analysis, along with a reformulation of the formalism suited to the physical contexts of interest, we present our main ideas for applying these techniques in physics and outline the objectives we aim to achieve. The main results of the present work are the following:

- We perform an explicit analysis of the wall-crossing structure and associated Stokes phenomena for the Lefschetz thimble decomposition of a class of exponential Pearcey integrals arising in the grand-canonical partition function of gauged Skyrme models, which describe nuclear matter in various pasta phases.

- We introduce, using a language accessible to physicists, the recent mathematical framework developed by Kontsevich and Soibelman [1] for studying wall-crossing structures in exponential integrals involving multivalued functions. We argue that this is the appropriate framework for interpreting Feynman integrals in the Baikov representation as periods.

- We present a concrete example of this generalization applied to the Legendre family of elliptic curves. The associated thimble decomposition yields a basis of simplified integrals, expressible in terms of standard elliptic integrals of the first and second kind.

- We propose that the decomposition of Feynman integrals into simpler components via this formalism matches the standard notion of Master Integral decomposition. This correspondence offers a more geometric and algebraic perspective on the structure of these integrals. Moreover, we argue that the analytic continuation in the parameter γ and the study of the associated wall crossing structure provide a principled method for enumerating Master Integrals, thereby avoiding ambiguities linked to Stokes phenomena.

- We analyze the large-parameter asymptotic expansion of exponential integrals expressed over a basis of Lefschetz thimbles, where the expansion coefficients correspond to periods of standard (co)homology classes associated with families of algebraic varieties. The existence of a well-defined pairing between the global de Rham and Betti (co)homology imposes a constraint on these periods: the volume growth of these standard cycles must not exceed an exponential rate.

We believe that this initial analysis already provides a solid background to motivate a more systematic investigation into the applications of these methods in physics. In particular, we propose the following directions for future work:

- A detailed study of the implementation of the method to perform Master Integral decompositions in families of Feynman integrals with free external kinematic parameters (such as masses of the internal particles and external momenta). A thorough analysis, including several concrete examples, will be presented in the companion paper [81].

- The implementation of the computation for integrals with a larger number of variables, for instance, those associated with families of higher-dimensional Calabi-Yau manifolds, such as the banana integrals [81].

- An extension of the framework to the analysis of string amplitudes, where the contribution of external states, implemented by the insertion of vertex operators, is controlled by Koba-Nielsen factors.

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A Picard-Lefschetz theory

The *Picard-Lefschetz theory* [93] is the complex analogous of the Morse theory that studies the topology of level sets of complex analytic functions.

Let us begin by considering the following simple example of a two variable function:

$$f(z,w) = z^2 + w^2. \tag{A.1}$$

The function $f(z,w)$ has a unique critical point:

$$\begin{cases} \partial_z f(z,w) = 0 \\ \partial_w f(z,w) = 0 \end{cases} \qquad \longmapsto \qquad (z,w) = (0,0). \tag{A.2}$$

We refer to the value of the function $f(z,w)$ at a critical point as a *critical value*, in the present case $f(0,0) = 0$. The *critical set* is the set of points in $\mathbb{C}^2$ where the function f takes the critical value:

$$V_0 = \Big\{(z,w)|z^2 + w^2 = 0\Big\}. \tag{A.3}$$

For any other value $f(z,w) = t$, different from the critical one, we call *level sets* the loci

$$V_t = \Big\{(z,w)|z^2 + w^2 = t\Big\}. \tag{A.4}$$

In order to figure out the topology of these level sets we can consider the Riemann surfaces associated with the function (A.1)

$$w = \sqrt{(t - z^2)}. \tag{A.5}$$

These surfaces can be obtained gluing together two copies of the complex plane z with a cut along the segment $(-\sqrt{t}, \sqrt{t})$, as showed in figure 12, resulting in a surface topologically equivalent to a cylinder. When $t = 0$, the corresponding critical level set consists of two lines intersecting at the point 0.

Consider now the fibration $f : \mathbb{C}^2 \mapsto \mathbb{C}_t \setminus \{0\}$ over the space $\mathbb{C}_t^* = \mathbb{C} \setminus \{0\}$, which fibers are Riemann surfaces representing the non-critical level sets V_t. Note that we removed from the base the point $t = 0$, which correspond to the singular fiber V_0. Let us now consider the circular path around $t = 0$

$$t(\tau) = e^{2\pi i \tau}\alpha \qquad 0 \leq \tau \leq 1, \quad \alpha > 0. \tag{A.6}$$

We aim to study how the fiber V_t varies along this path by tracking the motion of the branch points $z = \pm\sqrt{t(\tau)}$ in the complex planes $\mathbb{C}_z^{\pm}$ as the parameter τ evolves. We observe that these branch points rotate counterclockwise around $z = 0$ undergoing a half-turn (rotation by π) at $\tau = 1$ (see figure 13). Considering the fibration $V \longrightarrow I$, with $I \equiv [0,1]$, defined by $V_{t(\tau)} \mapsto \tau$, we can associate to the closed curve $t(\tau)$ in (A.6) a continuous map

$$\Gamma : V \longrightarrow V_{t(0)}, \tag{A.7}$$

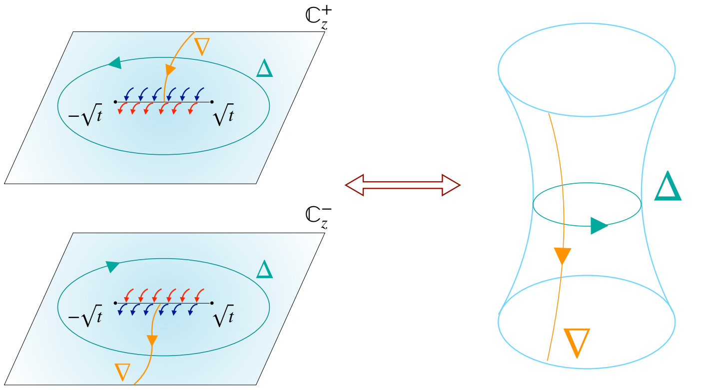

Figure 12. Gluing of the two Riemann sheets along the two edges of the cut from $-\sqrt{t}$ to $\sqrt{t}$. The resulting surface is topologically equivalent to a cylinder.

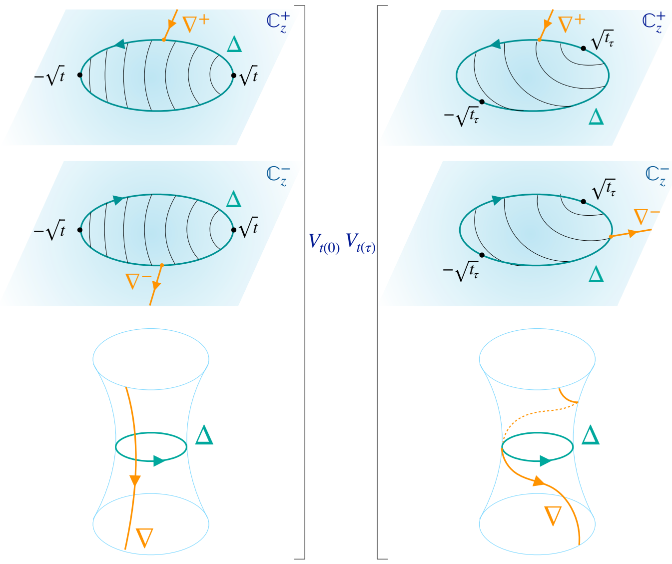

Figure 13. Action of Γ_τ on the vanishing and covanishing cycles.

such that for any τ, the map $\Gamma_\tau : V_{t(\tau)} \longrightarrow V_{t(0)}$ defined by $\Gamma_\tau(\cdot) = \Gamma(\tau, \cdot)$ is a diffeomorphism, and $\Gamma_0 = id$. Since $t(1) = t(0)$, the corresponding map $h = \Gamma_1 : V_{t(0)} \longrightarrow V_{t(1)}$ is called *monodromy map.*

We are interested to know how this map acts on the first homology group of the fiber V_t, which is generated by the 1-cycle Δ represented in figure 12. This 1-cycle Δ is called *Picard-Lefschetz vanishing cycle*, due to the fact it shrinks to a point when $t \to 0$. Its transversely intersecting cycle ∇ is called *covanishing cycle* and it generates the first homology group $H_1^{BM}(V_t, \mathbb{Z}) \simeq \mathbb{Z}$. Here, $H_1^{BM}(V_t)$ is the first Borel-Moore homology of the non-compact space V_t. This homology admits chains that may be infinite in extent but are restricted to be finite in any compact region. A vertical line in the cilinder V_t is locally finite because, in any compact sub-interval of the vertical direction the line is finite. Even though the line can extend indefinitely along the cylinder, within any small, bounded region it is just a finite segment. Since $H_1^{BM}(V_t,\mathbb{Z}) \simeq H_1(C,\partial C;\mathbb{Z})$, where C is the compact cylinder, it is easy to prove that $H_1^{BM}(V_t,\mathbb{Z}) \simeq \mathbb{Z}$ with generator ∇.

In figure 13 we show how the diffeomorphism Γ_τ acts on these two cycles. In the $\mathbb{C}_z^-$ foil, after the action of Γ_τ, ∇^- is rigidly transported along the counterclockwise direction of the rotation. As depicted in figure 14 we can deform homotopically the support of ∇_τ^- to the support of ∇_0^- through a connected path γ in $\mathbb{C}_z^-$ such that:

$$\nabla_\tau = \nabla_\tau^+ + \nabla_\tau^- \approx \nabla_0^+ + \tilde{\nabla}^- = \nabla_0^+ + \gamma + \nabla_0^- . \tag{A.8}$$

In terms of the homology we have:

$$[\nabla_\tau] \sim [\nabla_0] + [\gamma] \sim [\nabla_0] , \tag{A.9}$$

if γ in contractible. If $\tau = 1$ the path γ closes around the hole and

$$[\nabla_1] = \nabla_0 + \Delta. \tag{A.10}$$

In particular, one gets that, up to homotopies, the monodromy map h acts as the identity outside a compact set around $z = 0$ and non-trivially inside this set. More precisely, it maps the vanishing cycle Δ into itself and it acts as the (homotopically equivalent) identity in the part of ∇ extending outside the compact set and with the following transformation inside the set:

$$h\nabla = \nabla - \Delta. \tag{A.11}$$

It is worth to mention that while the map h defined on the fibration $V \to I$ depends on the choice of the diffeomorphisms Γ, when induced to the (co)homology it becomes independent on such a choice. This action allows us to define a function, called *variation map*, mapping a cycle with closed support to a cycle with compact support:

$$\begin{aligned} \mathrm{Var} \quad : \quad H_1^{BM}(V_t,\mathbb{Z}) \quad &\longmapsto \quad H_1(V_t,\mathbb{Z}) , \\ \nabla \quad &\longmapsto \quad (\Delta \circ \nabla)\Delta, \end{aligned} \tag{A.12}$$

where $(\Delta \circ \nabla)$ denotes the intersection pairing between the cycles Δ and ∇, defined as the number of topological intersections counted with a sign depending by the relative orientation of the two cycles.

The main objects we defined so far are the *vanishing cycles*, the *monodromy* and the *variation map* for a two-variables function. Defining these objects for arbitrary functions of several variables is a challenging problem that remains unsolved in general. Picard-Lefschetz theory offers a powerful method to address this problem by using *deformation theory* techniques.

A.1 Monodromy, variation operators and vanishing cycles

In this paragraph, we extend the above discussion to the case of holomorphic functions in several complex variables and provide formal definitions of the concepts introduced in the previous example.

Let

$$f : \mathcal{M}^n \quad \longmapsto \quad \mathbb{C} \tag{A.13}$$

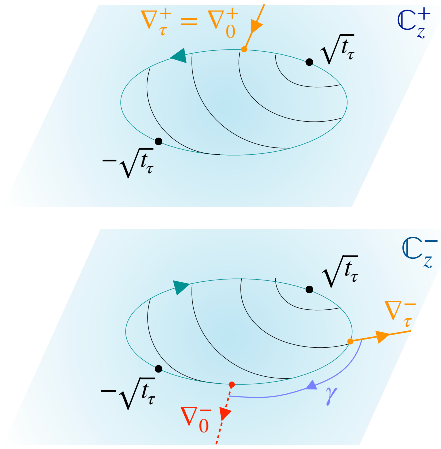

Figure 14. Homotopically deformation of ∇_τ^- over ∇_0^-.

be a holomorphic function on a n-dimensional complex manifold $\mathcal{M}^n$. Let U be a contractible compact region in the target space with smooth boundary ∂U, and let us assume f has a finite number of critical points $\Sigma = \{\sigma_i\}_{i=1}^{\mu}$ with critical values $t_i = f(\sigma_i)$ on $\overline{U}$. Let us indicate with F_t the level set of the function f at $t \in U$:

$$F_t = \{\mathbf{z} \in \mathbb{C}^n | f(\mathbf{z}) = t\}\,. \tag{A.14}$$

If $t \in U$ is not a critical value, then F_t is a $(n-1)-$dimensional complex manifold with smooth boundary. Let t_0 be a non-critical value in the boundary ∂U and let us construct for each class of loops $[\gamma] \in \pi_1(U \setminus \{t_i\}, t_0)$ a continuous family of mappings $\Gamma_\tau : F_{t(\tau)} \mapsto F_{t_0}$, for which $\Gamma_0 = id$. Then, $h_\gamma = \Gamma_1$, transforming the non-singular level F_{t_0} into itself, defines the *monodromy map* along the loop γ. Note that the map h_γ depends on the specific path γ we are considering.

Definition [Monodromy operator]:
We call monodromy operator of the loop γ the action $h_{\gamma *} = h_{[\gamma]}$ of the transformation h_γ on the homology of the non-singular level set $H_*(F_{t_0})$.

The transformation h_γ also induces an automorphism $h_{[\gamma]}^{(r)}$ in the relative homology group $H_\bullet(F_{t_0}, \partial F_{t_0})$ of the non-singular level set F_{t_0} modulo its boundary. This homology is isomorphic to the homology of cycles with closed support:

$$H_\bullet(F_{t_0}, \partial F_{t_0}) \simeq H_\bullet^{BM}(F_{t_0} \setminus \partial F_{t_0})\,. \tag{A.15}$$

Since the action h_γ is trivial on the boundary ∂F_t, then the difference between $h_{[\gamma]}^{(r)}\delta$ and $\delta \in H_\bullet(F_{t_0}, \partial F_{t_0})$ is a cycle in $H_\bullet(F_{t_0})$.

Definition [Variation]:
The homomorphism

$$\mathrm{Var}_\gamma \quad : \quad H_\bullet(F_{t_0}, \partial F_{t_0}) \quad \longmapsto \quad H_\bullet(F_{t_0}) \tag{A.16}$$

is called the variation operator over the loop γ.

Using the natural homomorphism

$$i_* \quad : \quad H_\bullet (F_{t_0}) \quad \longmapsto \quad H_\bullet (F_{t_0}, \partial F_{t_0}) \tag{A.17}$$

induced by the inclusion $F_{t_0} \subset (F_{t_0}, \partial F_{t_0})$, we can write the following relations connecting the automorphisms $h_{[\gamma]}$ and $h^{(r)}_{[\gamma]}$:

$$\begin{aligned} h_{[\gamma]} &= id + \mathrm{Var}_\gamma \cdot i_* \\ h^{(r)}_{[\gamma]} &= id + i_* \cdot \mathrm{Var}_\gamma . \end{aligned} \tag{A.18}$$

If the class $[\gamma] \in \pi_1 (U \setminus \{z_i\}, z_0)$ is given by $[\gamma] = [\gamma_1] \cdot [\gamma_2]$,[23] then

$$\begin{aligned} &\mathrm{Var}_\gamma = \mathrm{Var}_{\gamma_1} + \mathrm{Var}_{\gamma_2} + \mathrm{Var}_{\gamma_2} \cdot i_* \cdot \mathrm{Var}_{\gamma_1} \\ &h_{[\gamma]} = h_{[\gamma_2]} \cdot h_{[\gamma_1]} \\ &h^{(r)}_{[\gamma]} = h^{(r)}_{[\gamma_2]} \cdot h^{(r)}_{[\gamma_1]} . \end{aligned} \tag{A.19}$$

Let us suppose all the critical points are non-degenerate and the corresponding critical values are different:[24] such a function is said to be *Morse*.

Definition [Monodromy group]:
The map

$$\pi_1 (U \setminus \{z_i\}, z_0) \quad \longrightarrow \quad Aut\, (H_\bullet(F_{t_0})) , \tag{A.20}$$

$$[\gamma] \quad \longmapsto h_{\gamma *} \tag{A.21}$$

is called monodromy representation of $\pi_1 (U \setminus \{t_i\}, t_0)$*. The imagine of this map defines what we call the Monodromy group of the Morse function* f*.*

Now, we construct a path $u : [0, 1] \mapsto U$ joining the non-critical value $t_0 = u(0) \in \partial U$ to some critical value $t_i = u(1) \in U$ without crossing any other critical value. The Morse lemma tells us that, given a holomorphic Morse function f, it always exists a local set of coordinates in a neighbourhood of the non-degenerate critical point p_i such that the function takes the form

$$f(z_1, \ldots, z_n) = t_i + \sum_{j=1}^{n} z_j^2 . \tag{A.22}$$

Then, for each path u, we can define a family of $(n-1)-$dimensional spheres in the level manifolds $F_{u(\tau)}$. For each point of the path $u(\tau)$ the level set $F_{u(\tau)}$ is a hyperboloid equivalent to a trivial fibration with base a $(n-1)-$dimensional sphere of radius$\sqrt{|u(\tau) - t_i|}$:

$$S(\tau) = \sqrt{u(\tau) - t_i} S^{n-1} . \tag{A.23}$$

In particular, we have that the sphere $S(1)$ reduces to the critical point p_i.

[23] Note that in the composition of homology classes, we follow the convention of right multiplication.
[24] This second requirement is not strictly necessary to define a Morse function.

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Definition [Vanishing cycle of Picard-Lefschetz]:
The homology class $\Delta \in H_{n-1}(F_{t_0})$ represented by the $(n-1)-$dimensional sphere $S(0)$ in F_{t_0} is called vanishing cycle of Picard-Lefschetz along the path u.

Note that the homotopy class of $u \in U$ uniquely defines the homology class of the vanishing cycle Δ modulo orientation.

Definition [Distinguished Basis]:
The set of cycles $\Delta_1, \ldots, \Delta_\mu \in H_{n-1}(F_{t_0})$, with t_0 non-singular, is called distinguished if:

(i) The cycles Δ_i are vanishing along non-self-intersecting paths u_i reaching the critical values t_i;

(ii) The unique common point of u_i and u_j for $i \neq j$ is $u_i(0) = u_j(0) = t_0$;

(iii) The paths $u_i, \ldots, u_\mu$ are numbered in the order in which they enter to the point t_0 counting clockwise starting from the boundary ∂U of U.

Example 1: $f(z) = z^3 - 3\lambda z$.
Let us consider the Morse function $f(x) = z^3 - 3\lambda z$, with $\lambda \in \mathbb{R}_+$, and let us construct a distinguished basis of vanishing cycles. This function is a deformation of the function $f(z) = z^3$ and it has two critical points in the real line

$$\mathcal{C}: \quad \bar{z}_1 = \sqrt{\lambda}, \qquad \bar{z}_2 = -\sqrt{\lambda}, \tag{A.24}$$

with corresponding critical values are

$$t_1 = -2\lambda\sqrt{\lambda}, \qquad t_2 = 2\lambda\sqrt{\lambda}. \tag{A.25}$$

Let us choose as non-critical reference point $t_0 = 0$ and let us construct the paths u_1 and u_2 connecting the critical values with t_0. The level manifold at $t_0 = 0$ consists of three points:

$$F_{t_0}: \quad z^3 - 3\lambda z = 0 \quad \to z_1 = -\sqrt{3\lambda}, \quad z_2 = 0, \quad z_3 = \sqrt{3\lambda}. \tag{A.26}$$

In this example the level manifold for a generic regular point t is given by the condition $f(z) = t$ which admits three point solutions. The vanishing cycles, when we approach the critical values t_1 and t_2, are the differences

$$\Delta_1 = \{z_3\} - \{z_2\}, \qquad \Delta_2 = \{z_2\} - \{z_1\} \tag{A.27}$$

between the zeroth homology classes represented by the points.

Choosing the set U as depicted in figure 15 the cycles Δ_1 and Δ_2 form a distinguished basis for $H_1(F_{t_0})$.

Example 2: $f(x,y) = x^3 - 3\lambda x + y^2$.
In this second example we consider the function of two variables $f(x,y) = x^3 - 3\lambda x + y^2$, which is a deformation through the small real parameter λ of the function $f(x,y) = x^3 + y^2$. The set of critical points in $\mathbb{C}^2$ with their corresponding critical values is

$$\mathcal{C}: \quad \begin{cases} P_1 : (x,y) = \left(\sqrt{\lambda}, 0\right) & \to \quad t_1 = -2\lambda\sqrt{\lambda} \\ P_2 : (x,y) = \left(-\sqrt{\lambda}, 0\right) & \to \quad t_2 = 2\lambda\sqrt{\lambda}. \end{cases} \tag{A.28}$$

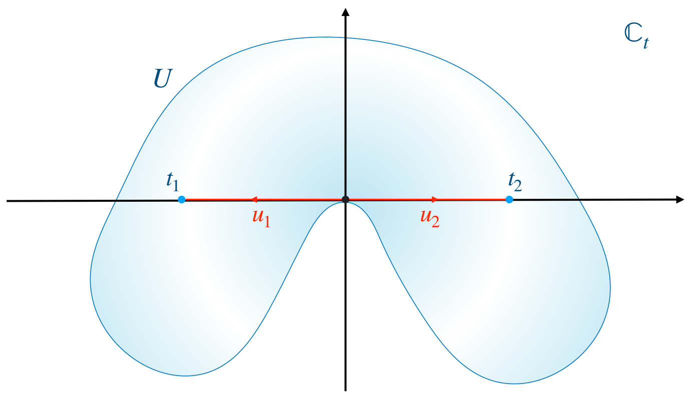

Figure 15. Choice of the set U and the paths u_1 and u_2 in the codomain of the function $f(z) = z^3 - 3\lambda z$.

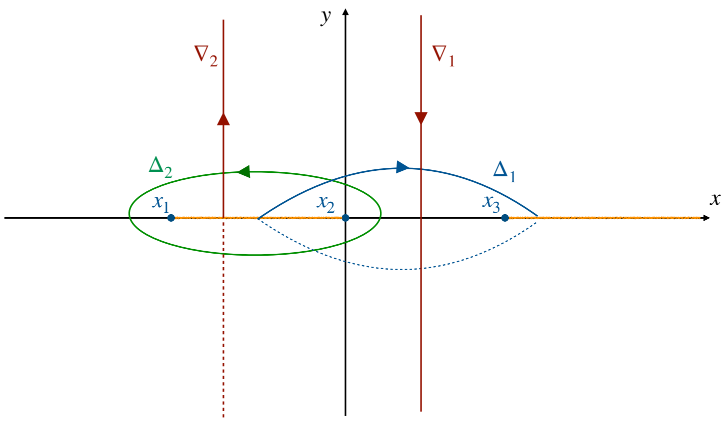

Figure 16. Vanishing and Covanishing cycles.

As in the previous example we can consider the paths u_1 and u_2 joining the two critical values with the non-critical value $t_0 = 0$. The level manifold in this regular point is the graph of the two-valued function $y = \pm\sqrt{x^3 - 3\lambda x}$, namely the double-covering of the x complex plane branched between the points $x_1 = -\sqrt{3\lambda}$ and $x_2 = 0$ and $x_3 = \sqrt{3\lambda}$ and infinity.

As we move the value of t from 0 to one of the two critical values, the level manifold $f(x, y) = t$ is deformed and becomes singular at $t = t_1$ and $t = t_2$. In particular, when we approach t_1 we have that the branch point x_2 moves until it overlaps x_3, while, when we approach t_2 the point x_2 moves towards the point x_1. From this construction, we can draw the vanishing cycles corresponding to the paths u_1 and u_2: we obtain Δ_1 encircling the points x_2 and x_3, and Δ_2 encircling the points x_1 and x_2 (see figure 16).

Definition [Simple loops]:
A simple loop is an element τ_i of $\pi_1(U \setminus \{t_i\}, t_0)$ represented by the loop going along the path u_i from t_0 to t_i, then encircling t_i with an anticlockwise path and returning along u_i to t_0.

The region $(U \setminus \{t_i\}_{i=1}^{\mu}, t_0)$ is homotopically equivalent to a bouquet of μ circles. Then, the fundamental group $\pi_1(U \setminus \{t_i\}, t_0)$ is a free group with μ generators $\tau_1, \tau_2, \ldots, \tau_\mu$.

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Definition [Weakly distinguished]:
The set of vanishing cycles $\Delta_1, \ldots, \Delta_\mu$ *defined by the paths* $u_1, \ldots, u_\mu$ *is called weakly distinguished if* $\pi_1 (U \setminus \{t_i\}, t_0)$ *is the free group generated by the simple loops* $\tau_1, \ldots, \tau_\mu$ *associated to the paths* $u_1, \ldots, u_\mu$.

We have that if the paths $\{u_i | i = 1, \ldots, \mu\}$ define a weakly distinguished set of vanishing cycles Δ_i in the $(n-1)-$homology group of the non-singular level manifold, then, the monodromy group of the function f is generated by the monodromy operators $h_{\tau_i *} = h_{[\tau_i]}$. Hence, the monodromy group of f is always a group generated by μ generators.

Definition [Picard-Lefschetz operator]:
The monodromy operator

$$h_i = h_{\tau_i *} : H_\bullet (F_{t_0}) \longmapsto H_\bullet (F_{t_0}) \tag{A.29}$$

of the simple loop τ_i *is called the* i^{th} *Picard-Lefschetz operator.*

In the Example 1, we can take trace of the change of the position of the three points z_i when we move t along the paths τ_1 and τ_2. We observe that along the path τ_1 the point z_2 approaches the point z_3, then they make a half-turn around a common centre and move again away one from the other. The point z_1 stays fixed. Then, we deduce the following monodromy action on the vanishing cycles:

$$h_1 \Delta_1 = -\Delta_1, \qquad h_1 \Delta_2 = \Delta_1 + \Delta_2. \tag{A.30}$$

In the same way we can deduce

$$h_2 \Delta_1 = \Delta_1 + \Delta_2, \qquad h_2 \Delta_2 = -\Delta_2. \tag{A.31}$$

The cycles Δ_i are in the homology group $H_{n-1} (F_t)$ of the non singular level manifold F_t. Moreover, we are interested also in the homology group $H_{n-1} (F_t, \partial F_t)$, which is dual to the group $H_{n-1} (F_t)$. In the present case it is generated by two cycles ∇_i such that

$$(\nabla_i \circ \Delta_j) = \delta_{ij}. \tag{A.32}$$

We can choose

$$\nabla_1 = \{z_3\}, \qquad \nabla_2 = -\{z_1\}, \tag{A.33}$$

for which we have the following variations:

$$\begin{aligned} &\mathrm{Var}_{\tau_1} \nabla_1 = \{z_2\} - \{z_3\} = -\Delta_1, && \mathrm{Var}_{\tau_1} \nabla_2 = 0, \\ &\mathrm{Var}_{\tau_2} \nabla_1 = 0, && \mathrm{Var}_{\tau_2} \nabla_2 = -\{z_2\} + \{z_1\} = -\Delta_2. \end{aligned} \tag{A.34}$$

We can now consider the loop $\tau = \tau_2 \tau_1$ that turns around the point $t_0 = 0$ encircling the two critical values t_1 and t_2 in a positive counterclockwise direction. The monodromy transformation associated to this loop permutes the points $z_3 \mapsto z_2 \mapsto z_1 \mapsto z_3$, then,

$$\begin{aligned} h_\tau \Delta_1 &= \{z_2\} - \{z_1\} = \Delta_2, \\ h_\tau \Delta_2 &= \{z_1\} - \{z_3\} = -\Delta_1 - \Delta_2, \end{aligned} \tag{A.35}$$

and

$$\begin{aligned} \mathrm{var}_\tau \nabla_1 &= \{z_2\} - \{z_3\} = -\Delta_1, \\ \mathrm{var}_\tau \nabla_2 &= -\{z_3\} + \{z_1\} = -\Delta_1 - \Delta_2. \end{aligned} \tag{A.36}$$

The monodromy group of the Morse function f is generated by the Picard-Lefschetz operators h_{τ_1} and h_{τ_2}. All the elements of this group preserve the intersection product of the group $H_0(F_t)$, for t non-critical, generated by the vanishing cycles Δ_i. The monodromy group is the group S_3 of permutations of three elements.

Now, let us construct the monodromy group for the Example 2. Drawing the analogous of the figure 14, we can deduce the action of the Picard-Lefschetz operator on the vanishing cycles to be

$$\begin{aligned} h_1\Delta_1 &= \Delta_1, & h_1\Delta_2 &= \Delta_1 + \Delta_2, \\ h_2\Delta_1 &= \Delta_1 - \Delta_2, & h_2\Delta_2 &= \Delta_2, \end{aligned} \tag{A.37}$$

and the following variation on the dual cycles:

$$\begin{aligned} \mathrm{Var}_{\tau_1}\nabla_1 &= -\Delta_1, & \mathrm{Var}_{\tau_1}\nabla_2 &= 0, \\ \mathrm{Var}_{\tau_2}\nabla_1 &= 0, & \mathrm{Var}_{\tau_2}\nabla_2 &= -\Delta_2. \end{aligned} \tag{A.38}$$

The monodromy group of the Morse function $f(x,y)$ is isomorphic to the group of non-singular 2×2 integer matrices with determinant 1. The group is generated by the action of the Picard-Lefschetz operators on the vanishing cycles, given by

$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{A.39}$$

These methods, explicitly shown in one or two complex dimensions, can in principle be extended to higher dimensions to determine vanishing cycles, their duals, and the action of the monodromy group on them.

A.2 Picard-Lefschetz theorem

The Picard-Lefschetz theorem establishes a relation between the variation of (co)-vanishing cycles due to the action of the monodromy operator with their intersection product in $H_{n-1} \times H^\vee_{n-1} \mapsto \mathbb{Z}$.

Let us start considering the simple loop τ_i associated to the path u_i connecting the non-critical reference point $t_0 \in \mathbb{C}_t$ with the critical value $t_i \in \mathbb{C}_t$. Let us assume that the critical value is $t_i = 0$, so that in some local coordinates around the critical point $P_i \in \mathbb{C}^n$, we can write the function f in the form

$$f(z_1,\ldots,z_n) = \sum_{j=1}^{n} z_j^2. \tag{A.40}$$

If we intersect $f^{-1}(t_0)$ with the ball $\sum_j |z_j|^2 \le 4\epsilon^2$, the non-critical value t_0 is sufficiently close to the critical value 0, say $|t_0| = \epsilon^2$. We can suppose that all other critical values of f are outside the disk of radius $4\epsilon^2$ in $\mathbb{C}_t$, so that our simple loop encircles just one singularity.

Let us define the ball $\overline{B}_{2\epsilon}$ of radius 2ϵ in the space $\mathbb{C}^n$,

$$\overline{B}_{2\epsilon} = \{(z_1,\ldots,z_n)\,|r \le 2\epsilon\} \tag{A.41}$$

and let us call $\tilde{F}_t$ the intersection of the level set F_t with this ball.

Lemma 1: *For $|t| < 4\epsilon^2$, the level set F_t is transverse to the $(2n-1)$-dimensional sphere $\partial\overline{B}_{2\epsilon}$.*

From this lemma it follows that for $0 < |t| < 4\epsilon^2$ the sets $\tilde{F}_t = F_t \cap \overline{B}_{2\epsilon}$ are diffeomorphic manifolds with boundary, while $\tilde{F}_0$ is a cone with vertex in zero.

Lemma 2: *For $0 < |t| < 4\epsilon^2$, the manifold $\tilde{F}_t$ is diffeomorphic to the disk sub-bundle of the tangent bundle of the standard $(n-1)$ dimensional sphere S^{n-1}.*

From this second lemma follows the following result:

Lemma 3: *The self-intersection number of vanishing cycle Δ in the complex manifold $\tilde{F}_{\epsilon^2}$ is equal to*

$$(\Delta \circ \Delta) = (-1)^{(n-1)(n-2)/2}\left(1 + (-1)^{n-1}\right) = \begin{cases} 0 & \text{for} \quad n = 0 \bmod 2, \\ +2 & \text{for} \quad n = 1 \bmod 4, \\ -2 & \text{for} \quad n = 3 \bmod 4. \end{cases} \tag{A.42}$$

Poincaré duality for a compact manifold X of dimension n states that $H^k(X) \sim H_{n-k}(X)$. If X is noncompact, while for cohomology it is not a problem, for homology one has to introduce Borel-Moore homology for which one has $H^k(X) \sim H^{BM}_{n-k}(X)$, see [109]. Hence, in our case, we get $H_k\left(\tilde{F}_{\epsilon^2}, \partial\tilde{F}_{\epsilon^2}\right) \sim H^{BM}_{n-k}\left(\tilde{F}_{\epsilon^2}\right)$, and $H_k\left(\tilde{F}_{\epsilon^2}, \partial\tilde{F}_{\epsilon^2}\right) \sim H^k\left(\tilde{F}_{\epsilon^2}\right) \sim H^k(S^{n-1})$. Therefore, the relative homology group $H_k(\tilde{F}_{\epsilon^2}, \partial\tilde{F}_{\epsilon^2})$ is zero for $k \neq n-1$, while $H_{n-1}(\tilde{F}_{\epsilon^2}, \partial\tilde{F}_{\epsilon^2})$ is isomorphic to the $\mathbb{Z}$. Moreover the latter is generated by the relative cycle ∇ dual to Δ such that $\Delta \circ \nabla = 1$.

In general, a relative cycle $\delta \in H_k\left(F_{\epsilon^2}, \partial F_{\epsilon^2}\right)$ can be represented in the form

$$\delta = \delta_1 + \delta_2 \tag{A.43}$$

where $\delta_1 \in H_k\left(\tilde{F}_{\epsilon^2}, \partial\tilde{F}_{\epsilon^2}\right)$ and δ_2 is a chain in $F_{\epsilon^2} \setminus B_{2\epsilon}$. The transformation $h_\tau = \Gamma_1$ is the identity in $F_{\epsilon^2} \setminus B_{2\epsilon}$, hence, it acts non-trivially only on the cycle δ_1. Therfore, $\mathrm{Var}_\tau(\delta) = \mathrm{Var}_\tau(\delta_1)$.
Since $H_k\left(\tilde{F}_{\epsilon^2}, \partial\tilde{F}_{\epsilon^2}\right) = \langle\nabla\rangle \simeq \mathbb{Z}$, then $\delta_1 = m \cdot \nabla$, with $m \in \mathbb{Z}$ and $m = \delta \circ \Delta$, and, in order to compute the action of the variation operator on $H_k\left(\tilde{F}_{\epsilon^2}, \partial\tilde{F}_{\epsilon^2}\right)$, it is sufficient to calculate its action on ∇.

Theorem [Picard-Lefschetz]:
Under the above hypotheses

$$\mathrm{Var}_\tau\left(\nabla\right) = (-1)^{n(n+1)/2}\Delta. \tag{A.44}$$

It follows from this:

Corollary:
For $a \in H\left(F_{t_0}, \partial F_{t_0}\right)$:

$$\mathrm{Var}_\tau(a) = (-1)^{n(n+1)/2}\left(a \circ \Delta\right)\Delta, \tag{A.45}$$

$$h^{(r)}_\tau(a) = a + (-1)^{n(n+1)/2}\left(a \circ \Delta\right) i_*\Delta; \tag{A.46}$$

for $a \in H_{n-1}(F_{t_0})$

$$h_\tau(a) = a + (-1)^{n(n+1)/2} (a \circ \Delta) \Delta, \tag{A.47}$$

where i_* *is the homomorphism* (A.17)*.*

A.3 Example: Pearcey integral (positive determinant case)

In this section we compute the monodromy matrices and intersection numbers, needed in section 3.4. Let $\Delta_1 = \{z_3\} - \{z_4\}$ be the vanishing cycle associated to the singular point σ_1, meaning the two points $\{z_3\}$ and $\{z_4\}$ coalesce when approaching to σ_1. The self intersection number

$$\Delta_1 \circ \Delta_1 = 2, \tag{A.48}$$

can be easily computed by setting $n = 1$ in equation (A.42).

For the sake of clarity, being careful not to make confusion among vanishing and co-vanishing cycles, is useful here to re-wright (A.47) with adapted notation:

$$h_j(\Delta_i) = \Delta_i + (-1)^{n(n+1)/2} (\Delta_i \circ \Delta_j) \Delta_j, \tag{A.49}$$

We can now apply (A.49), for evaluating the monodromy action on Δ_1 when going around the singular point σ_1:

$$h_1(\Delta_1) = \Delta_1 - (\Delta_1 \circ \Delta_1)\Delta_1 = -\Delta_1. \tag{A.50}$$

The transformed vanishing cycle

$$\Delta_1^{(1)} \equiv h_1(\Delta_1) = \{z_4\} - \{z_3\}. \tag{A.51}$$

reveals the flipping $\{z_3\} \leftrightarrow \{z_4\}$. We can now use this information to determine the action on the co-vanishing cycle $\Delta_2 = \{z_1\} - \{z_4\}$:

$$\Delta_2^{(1)} = \Delta_2(\{z_3\} \leftrightarrow \{z_4\}) = \{z_1\} - \{z_3\}. \tag{A.52}$$

In order to express $\Delta_2^{(1)}$ in terms of Δ_1 and Δ_2, we use the trick to insert $\{z_4\} - \{z_4\} = 0$ in the previous equation, getting:

$$\Delta_2^{(1)} = \{z_1\} + (\{z_4\} - \{z_4\}) - \{z_3\} = (\{z_1\} - (\{z_4\}) + (\{z_4\}) - \{z_3\}) = \Delta_2 - \Delta_1. \tag{A.53}$$

Proceeding analogously for Δ_3, we get

$$\Delta_3^{(1)} = \Delta_3 - \Delta_1. \tag{A.54}$$

Packaging relations (A.50), (A.53) and (A.54) in the representation of the monodromy, around σ_1, acting on the basis of vanishing cycles, i.e.

$$\Delta_i^{(1)} = (M_1)_{ij}\Delta_j, \tag{A.55}$$

we get the first of the matrices reported in (3.54):

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} . \tag{A.56}$$

In order to obtain M_2 and M_3 we simply repeat the very same procedure considering the singular points σ_2 and σ_3 cases, respectively.

Finally, we can use monodromy matrices to extract the intersection numbers among vanishing cycle. Inverting (A.49), we get

$$\Delta_i \circ \Delta_j = [\Delta_i - h_j(\Delta_i)]\Delta_j^{-1}. \tag{A.57}$$

By contruction, we have $h_j(\Delta_i) = (M_j)_{ik}\Delta_k$, so when can rewrite Picard-Lefschetz formula with an explicit dependence on the monodromy matrices as

$$\Delta_i \circ \Delta_j = [\Delta_i - (M_j)_{ik}\Delta_k]\Delta_j^{-1}. \tag{A.58}$$

Let us apply (A.58) to compute, for instance, the intersections

$$\begin{aligned} \Delta_1 \circ \Delta_2 &= [\Delta_1 - (M_2)_{1k}\Delta_k]\Delta_2^{-1} \\ &= [\Delta_1 - \Delta_1 + \Delta_2]\Delta_2^{-1} = 1, \end{aligned} \tag{A.59}$$

and

$$\begin{aligned} \Delta_2 \circ \Delta_1 &= [\Delta_2 - (M_1)_{2k}\Delta_k]\Delta_1^{-1} \\ &= [\Delta_2 + \Delta_1 - \Delta_2]\Delta_1^{-1} = 1, \end{aligned} \tag{A.60}$$

revealing the intersection product is symmetric, as expected for even $n - 1$.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

References

[1] M. Kontsevich and Y. Soibelman, *Holomorphic Floer theory I: exponential integrals in finite and infinite dimensions*, arXiv:2402.07343 [INSPIRE].

[2] A.V. Kotikov, *Differential equation method: The calculation of N point Feynman diagrams*, *Phys. Lett. B* **267** (1991) 123 [INSPIRE].

[3] A.V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett. B* **254** (1991) 158 [INSPIRE].

[4] F.A. Lunev, *Differential equations for definition and evaluation of Feynman integrals*, *Phys. Rev. D* **50** (1994) 6589 [hep-th/9407174] [INSPIRE].

[5] E. Remiddi, *Differential equations for Feynman graph amplitudes*, *Nuovo Cim. A* **110** (1997) 1435 [hep-th/9711188] [INSPIRE].

[6] M. Argeri and P. Mastrolia, *Feynman Diagrams and Differential Equations*, *Int. J. Mod. Phys. A* **22** (2007) 4375 [arXiv:0707.4037] [INSPIRE].

[7] T.W. Grimm and A. Hoefnagels, *Reductions of GKZ systems and applications to cosmological correlators*, *JHEP* **04** (2025) 196 [arXiv:2409.13815] [INSPIRE].

[8] T.W. Grimm, A. Hoefnagels and M. van Vliet, *A Reduction Algorithm for Cosmological Correlators: Cuts, Contractions, and Complexity*, arXiv:2503.05866 [INSPIRE].

[9] K. Aomoto and M. Kita, *Theory of Hypergeometric Functions*, Springer (2011) [DOI:10.1007/978-4-431-53938-4] [INSPIRE].

[10] M. Yoshida, *Hypergeometric Functions, My Love: Modular Interpretations of Configuration Spaces*, Vieweg+Teubner Verlag (1997) [DOI:10.1007/978-3-322-90166-8].

[11] M. Kita and M. Yoshida, *Intersection Theory for Twisted Cycles*, *Math. Nachr.* **166** (1994) 287.

[12] M. Kita and M. Yoshida, *Intersection Theory for Twisted Cycles II — Degenerate Arrangements*, *Math. Nachr.* **168** (1994) 171.

[13] K. Matsumoto, *Quadratic Identities for Hypergeometric Series of Type* (k, l), *Kyushu J. Math.* **48** (1994) 335.

[14] K. Cho and K. Matsumoto, *Intersection theory for twisted cohomologies and twisted Riemann's period relations I*, *Nagoya Math. J.* **139** (2016) 67 [INSPIRE].

[15] K. Aomoto, M. Kita, P. Orlik and H. Terao, *Twisted de Rham Cohomology Groups of Logarithmic Forms*, *Adv. Math.* **128** (1997) 119.

[16] K. Matsumoto, *Intersection numbers for logarithmic k-forms*, *Osaka J. Math.* **35** (1998) 873 [INSPIRE].

[17] A. Adolphson and S. Sperber, *On twisted de Rham cohomology*, *Nagoya Math. J.* **146** (1997) 55.

[18] K. Mimachi and M. Yoshida, *Intersection numbers of twisted cycles and the correlation functions of the conformal field theory. 2*, *Commun. Math. Phys.* **234** (2003) 339 [math/0208097] [INSPIRE].

[19] K. Ohara, *Intersection numbers of twisted cohomology groups associated with Selberg-type integrals*, http://www.math.kobe-u.ac.jp/HOME/ohara/Math/980523.ps.

[20] K. Mimachi and M. Yoshida, *Intersection numbers of twisted cycles associated with the Selberg integral and an application to the conformal field theory*, *Commun. Math. Phys.* **250** (2004) 23 [INSPIRE].

[21] K. Ohara, Y. Sugiki and N. Takayama, *Quadratic Relations for Generalized Hypergeometric Functions PFP-1*, *Funkcial. Ekvac.* **46** (2003) 213.

[22] Y. Goto, *Twisted cycles and twisted period relations for Lauricella's hypergeometric function fc*, *Int. J. Math.* **24** (2013) 1350094 [arXiv:1308.5535] [INSPIRE].

[23] Y. Goto, *Intersection Numbers and Twisted Period Relations for the Generalized Hypergeometric Function* ${}_{m+1}F_m$, *Kyushu J. Math.* **69** (2015) 203.

[24] Y. Goto and K. Matsumoto, *The monodromy representation and twisted period relations for Appell's hypergeometric function* F_4, *Nagoya Math. J.* **217** (2015) 61 [INSPIRE].

JHEP01(2026)050

[25] S.-J. Matsubara-Heo and N. Takayama, *An Algorithm of Computing Cohomology Intersection Number of Hypergeometric Integrals*, *Nagoya Math. J.* **246** (2022) 256 [arXiv:1904.01253] [INSPIRE].

[26] Y. Goto and S.-J. Matsubara-Heo, *Homology and cohomology intersection numbers of GKZ systems*, *Indag. Math.* **33** (2022) 546.

[27] S.-J. Matsubara-Heo, *Computing cohomology intersection numbers of GKZ hypergeometric systems*, *PoS* **MA2019** (2022) 013 [arXiv:2008.03176] [INSPIRE].

[28] S.-J. Matsubara-Heo, *Localization formulas of cohomology intersection numbers*, *J. Math. Soc. Jap.* **75** (2023) 909 [arXiv:2104.12584] [INSPIRE].

[29] P. Mastrolia and S. Mizera, *Feynman Integrals and Intersection Theory*, *JHEP* **02** (2019) 139 [arXiv:1810.03818] [INSPIRE].

[30] S.L. Cacciatori, M. Conti and S. Trevisan, *Co-Homology of Differential Forms and Feynman Diagrams*, *Universe* **7** (2021) 328 [arXiv:2107.14721] [INSPIRE].

[31] E. Cattani, *Mixed Lefschetz Theorems and Hodge-Riemann Bilinear Relations*, *Int. Math. Res. Not.* **2008** (2010) rnn025.

[32] S. Bloch and P. Vanhove, *The elliptic dilogarithm for the sunset graph*, *J. Number Theor.* **148** (2015) 328 [arXiv:1309.5865] [INSPIRE].

[33] P. Lairez, E. Pichon-Pharabod and P. Vanhove, *Effective homology and periods of complex projective hypersurfaces*, *Math. Comput.* **93** (2024) 2985 [arXiv:2306.05263] [INSPIRE].

[34] K. Bönisch et al., *Analytic structure of all loop banana integrals*, *JHEP* **05** (2021) 066 [arXiv:2008.10574] [INSPIRE].

[35] K. Bönisch et al., *Feynman integrals in dimensional regularization and extensions of Calabi-Yau motives*, *JHEP* **09** (2022) 156 [arXiv:2108.05310] [INSPIRE].

[36] A. Klemm, C. Nega and R. Safari, *The l-loop Banana Amplitude from GKZ Systems and relative Calabi-Yau Periods*, *JHEP* **04** (2020) 088 [arXiv:1912.06201] [INSPIRE].

[37] C. Duhr, A. Klemm, C. Nega and L. Tancredi, *The ice cone family and iterated integrals for Calabi-Yau varieties*, *JHEP* **02** (2023) 228 [arXiv:2212.09550] [INSPIRE].

[38] E. Pichon-Pharabod, *Periods of fibre products of elliptic surfaces and the Gamma conjecture*, arXiv:2505.07685.

[39] C. Bogner and S. Weinzierl, *Periods and Feynman integrals*, *J. Math. Phys.* **50** (2009) 042302 [arXiv:0711.4863] [INSPIRE].

[40] C.F. Doran, A. Harder, P. Vanhove and E. Pichon-Pharabod, *Motivic Geometry of two-Loop Feynman Integrals*, *Quart. J. Math. Oxford Ser.* **75** (2024) 901 [arXiv:2302.14840] [INSPIRE].

[41] K.G. Chetyrkin and F.V. Tkachov, *Integration by parts: The algorithm to calculate β-functions in 4 loops*, *Nucl. Phys. B* **192** (1981) 159 [INSPIRE].

[42] S. Laporta, *High-precision calculation of multiloop Feynman integrals by difference equations*, *Int. J. Mod. Phys. A* **15** (2000) 5087 [hep-ph/0102033] [INSPIRE].

[43] H. Frellesvig et al., *Decomposition of Feynman Integrals on the Maximal Cut by Intersection Numbers*, *JHEP* **05** (2019) 153 [arXiv:1901.11510] [INSPIRE].

[44] S. Mizera and A. Pokraka, *From Infinity to Four Dimensions: Higher Residue Pairings and Feynman Integrals*, *JHEP* **02** (2020) 159 [arXiv:1910.11852] [INSPIRE].

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[45] H. Frellesvig et al., *Vector Space of Feynman Integrals and Multivariate Intersection Numbers*, *Phys. Rev. Lett.* **123** (2019) 201602 [arXiv:1907.02000] [INSPIRE].

[46] H. Frellesvig et al., *Decomposition of Feynman Integrals by Multivariate Intersection Numbers*, *JHEP* **03** (2021) 027 [arXiv:2008.04823] [INSPIRE].

[47] S. Mizera, *Inverse of the String Theory KLT Kernel*, *JHEP* **06** (2017) 084 [arXiv:1610.04230] [INSPIRE].

[48] S. Mizera, *Scattering Amplitudes from Intersection Theory*, *Phys. Rev. Lett.* **120** (2018) 141602 [arXiv:1711.00469] [INSPIRE].

[49] A. Kaderli, *A note on the Drinfeld associator for genus-zero superstring amplitudes in twisted de Rham theory*, *J. Phys. A* **53** (2020) 415401 [arXiv:1912.09406] [INSPIRE].

[50] N. Kalyanapuram and R.G. Jha, *Positive Geometries for all Scalar Theories from Twisted Intersection Theory*, *Phys. Rev. Res.* **2** (2020) 033119 [arXiv:2006.15359] [INSPIRE].

[51] C. Ma et al., *Mixed QCD-EW corrections for Higgs leptonic decay via HW^+W^- vertex*, *JHEP* **09** (2021) 114 [arXiv:2105.06316] [INSPIRE].

[52] S. Weinzierl, *Correlation functions on the lattice and twisted cocycles*, *Phys. Lett. B* **805** (2020) 135449 [arXiv:2003.05839] [INSPIRE].

[53] F. Gasparotto, A. Rapakoulias and S. Weinzierl, *Nonperturbative computation of lattice correlation functions by differential equations*, *Phys. Rev. D* **107** (2023) 014502 [arXiv:2210.16052] [INSPIRE].

[54] J. Chen et al., *Baikov representations, intersection theory, and canonical Feynman integrals*, *JHEP* **07** (2022) 066 [arXiv:2202.08127] [INSPIRE].

[55] M. Giroux and A. Pokraka, *Loop-by-loop differential equations for dual (elliptic) Feynman integrals*, *JHEP* **03** (2023) 155 [arXiv:2210.09898] [INSPIRE].

[56] T. Ahmed et al., *Two-loop vertices with vacuum polarization insertion*, *JHEP* **01** (2024) 010 [arXiv:2308.05028] [INSPIRE].

[57] C. Duhr et al., *On the electron self-energy to three loops in QED*, *JHEP* **11** (2024) 020 [arXiv:2408.05154] [INSPIRE].

[58] G. Crisanti and S. Smith, *Feynman integral reductions by intersection theory with orthogonal bases and closed formulae*, *JHEP* **09** (2024) 018 [arXiv:2405.18178] [INSPIRE].

[59] G. Brunello et al., *Intersection numbers, polynomial division and relative cohomology*, *JHEP* **09** (2024) 015 [arXiv:2401.01897] [INSPIRE].

[60] G. Fontana and T. Peraro, *Reduction to master integrals via intersection numbers and polynomial expansions*, *JHEP* **08** (2023) 175 [arXiv:2304.14336] [INSPIRE].

[61] S. Caron-Huot and A. Pokraka, *Duals of Feynman integrals. Part I. Differential equations*, *JHEP* **12** (2021) 045 [arXiv:2104.06898] [INSPIRE].

[62] S. Caron-Huot and A. Pokraka, *Duals of Feynman Integrals. Part II. Generalized unitarity*, *JHEP* **04** (2022) 078 [arXiv:2112.00055] [INSPIRE].

[63] V. Chestnov et al., *Macaulay matrix for Feynman integrals: linear relations and intersection numbers*, *JHEP* **09** (2022) 187 [arXiv:2204.12983] [INSPIRE].

[64] V. Chestnov et al., *Intersection numbers from higher-order partial differential equations*, *JHEP* **06** (2023) 131 [arXiv:2209.01997] [INSPIRE].

[65] V. Chestnov, G. Fontana and T. Peraro, *Reduction to master integrals and transverse integration identities*, *JHEP* **03** (2025) 113 [arXiv:2409.04783] [INSPIRE].

[66] G. Brunello, V. Chestnov and P. Mastrolia, *Intersection numbers from companion tensor algebra*, *JHEP* **07** (2025) 045 [arXiv:2408.16668] [INSPIRE].

[67] S.L. Cacciatori and P. Mastrolia, *Intersection Numbers in Quantum Mechanics and Field Theory*, arXiv:2211.03729 [INSPIRE].

[68] M. Kontsevich and D. Zagier, *Periods*, in *Mathematics Unlimited — 2001 and Beyond*, Springer Berlin Heidelberg (2001), p. 771–808 [DOI:10.1007/978-3-642-56478-9_39].

[69] G. Brunello et al., *Fourier calculus from intersection theory*, *Phys. Rev. D* **109** (2024) 094047 [arXiv:2311.14432] [INSPIRE].

[70] S.L. Cacciatori, H. Epstein and U. Moschella, *Loops in de Sitter space*, *JHEP* **07** (2024) 182 [arXiv:2403.13145] [INSPIRE].

[71] S.L. Cacciatori, H. Epstein and U. Moschella, *Loops in anti de Sitter space*, *JHEP* **08** (2024) 109 [arXiv:2403.13142] [INSPIRE].

[72] E. Witten, *Analytic Continuation Of Chern-Simons Theory*, *AMS/IP Stud. Adv. Math.* **50** (2011) 347 [arXiv:1001.2933] [INSPIRE].

[73] P.D. Alvarez, S.L. Cacciatori, F. Canfora and B.L. Cerchiai, *Analytic SU(N) Skyrmions at finite Baryon density*, *Phys. Rev. D* **101** (2020) 125011 [arXiv:2005.11301] [INSPIRE].

[74] S.L. Cacciatori et al., *Analytic multi-Baryonic solutions in the SU(N)-Skyrme model at finite density*, *JHEP* **12** (2021) 150 [arXiv:2105.10789] [INSPIRE].

[75] S.L. Cacciatori et al., *Cooking pasta with Lie groups*, *Nucl. Phys. B* **976** (2022) 115693 [arXiv:2201.12598] [INSPIRE].

[76] S.L. Cacciatori, F. Canfora and F. Muscolino, *Pearcey integrals, Stokes lines and exact baryonic layers in the low energy limit of QCD*, *Nucl. Phys. B* **1000** (2024) 116477 [arXiv:2402.07551] [INSPIRE].

[77] F. Belgiorno, S.L. Cacciatori and A. Viganò, *Analog Hawking effect: A master equation*, *Phys. Rev. D* **102** (2020) 105003 [arXiv:2003.04236] [INSPIRE].

[78] F. Belgiorno, S.L. Cacciatori, A. Farahat and A. Viganò, *Analog Hawking effect: BEC and surface waves*, *Phys. Rev. D* **102** (2020) 105004 [arXiv:2003.04242] [INSPIRE].

[79] F. Belgiorno, S.L. Cacciatori and S. Trevisan, *Analogous Hawking Radiation in Dispersive Media*, *Universe* **10** (2024) 412 [INSPIRE].

[80] S. Trevisan, F. Belgiorno and S.L. Cacciatori, *Exact solutions for analog Hawking effect in dielectric media*, *Phys. Rev. D* **110** (2024) 085009 [arXiv:2406.04181] [INSPIRE].

[81] R. Angius, S.L. Cacciatori, A. Massidda and S. Noja, *Exponential integrals for Feynman diagrams*, in preparation.

[82] L. Nicolaescu, *An invitation to morse theory*, in *An Invitation to Morse Theory*, Springer New York (2007), p. 151–191 [DOI:10.1007/978-0-387-49510-1_4].

[83] J. Milnor, *Singular Points of Complex Hypersurfaces*, Annals of Mathematics Studies, Princeton University Press (1969).

[84] B.I. Dundas, *A Short Course in Differential Topology*, Cambridge University Press (2018) [DOI:10.1017/9781108349130].

JHEP01(2026)050

[85] W. Ebeling, *The Milnor Lattices of the Elliptic Hypersurface Singularities*, *Proc. Lond. Math. Soc.* **s3–53** (1986) 85.

[86] W. Ebeling, *The Monodromy Groups of Isolated Singularities of Complete Intersections*, Springer Berlin Heidelberg (1987) [DOI:10.1007/bfb0078929].

[87] W. Ebeling, *Monodromy*, in *Singularities and Computer Algebra*, Cambridge University Press (2006), p. 129–156 [DOI:10.1017/cbo9780511526374.008].

[88] A.M. Gabrielov, *Polar curves and intersection matrices of singularities*, *Invent. Math.* **54** (1979) 15.

[89] V.S. Kulikov, *Mixed Hodge Structures and Singularities*, Cambridge University Press (1998) [DOI:10.1017/cbo9780511758928].

[90] E. Brieskorn, *Die monodromie der isolierten singularitäten von hyperflächen*, *Manuscripta Mathematica* **2** (1970) 103.

[91] B. Malgrange, *Chap. IV: Regular connexions after Deligne*, in Borel et al, *Algebraic D-Modules*, Academic Press (1994) 151.

[92] S. Lefschetz, *L'analysis situs et la géométrie algébrique*, *Gauthier-Villars* (1924).

[93] S.M. Husein-Zade, *The monodromy groups of isoleted singularities of hypersurfaces*, *Russ. Math. Surv.* **32** (1977) 23.

[94] M. Kontsevich and Y. Soibelman, *Analyticity and resurgence in wall-crossing formulas*, *Lett. Math. Phys.* **112** (2022) 32 [arXiv:2005.10651] [INSPIRE].

[95] S. Cecotti and C. Vafa, *On classification of N=2 supersymmetric theories*, *Commun. Math. Phys.* **158** (1993) 569 [hep-th/9211097] [INSPIRE].

[96] F. Pham, *Introduction á l'étude topologique des singularités de landau*, *Mémorial des sciences mathématiques* (1967).

[97] V.I. Arnold, S.M. Gusein-Zade and A.N. Varchenko, *Singularities of Differentiable Maps, Volume 2: Monodromy and Asymptotics of Integrals*, Birkhäuser Boston (1988) [DOI:10.1007/978-0-8176-8343-6].

[98] M. Goresky, *Lecture notes on sheaves and perverse sheaves*, arXiv:2105.12045.

[99] F.W. Olver, *Asymptotics and Special Functions*, CRC Press (1997).

[100] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435 [INSPIRE].

[101] P.A. Baikov, *Explicit solutions of the multiloop integral recurrence relations and its application*, *Nucl. Instrum. Meth. A* **389** (1997) 347 [hep-ph/9611449] [INSPIRE].

[102] A.G. Grozin, *Integration by parts: An Introduction*, *Int. J. Mod. Phys. A* **26** (2011) 2807 [arXiv:1104.3993] [INSPIRE].

[103] C. Duhr and S. Maggio, *Feynman integrals, elliptic integrals and two-parameter K3 surfaces*, *JHEP* **06** (2025) 250 [arXiv:2502.15326] [INSPIRE].

[104] C. Duhr et al., *Three-loop banana integrals with four unequal masses*, *JHEP* **12** (2025) 034 [arXiv:2507.23061] [INSPIRE].

[105] S. Pögel, T. Teschke, X. Wang and S. Weinzierl, *The unequal-mass three-loop banana integral*, arXiv:2507.23594 [INSPIRE].

[106] C. Bogner, A. Schweitzer and S. Weinzierl, *Analytic continuation and numerical evaluation of the kite integral and the equal mass sunrise integral*, *Nucl. Phys. B* **922** (2017) 528 [arXiv:1705.08952] [INSPIRE].

[107] J. Broedel, C. Duhr, F. Dulat and L. Tancredi, *Elliptic polylogarithms and iterated integrals on elliptic curves II: an application to the sunrise integral*, *Phys. Rev. D* **97** (2018) 116009 [arXiv:1712.07095] [INSPIRE].

[108] W. Schmid, *Variation of hodge structure: The singularities of the period mapping*, *Invent. Math.* **22** (1973) 211 [INSPIRE].

[109] G.E. Bredon, *Sheaf Theory*, McGraw-Hill, New York (1967).