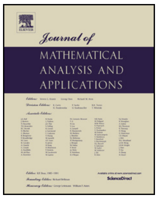
J. Math. Anal. Appl. 552 (2025) 129748

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa



Regular Articles

On maldistributed sequences and meager ideals

Paolo Leonetti

Department of Economics, Università degli Studi dell'Insubria, via Monte Generoso 71, 21100 Varese, Italy

ARTICLE INFO

Article history:
Received 30 December 2024
Available online 29 May 2025
Submitted by H. Frankowska

Keywords:
Maldistributed sequence
Lower semicontinuous submeasure
Analytic P-ideals
Meager sets
Meager ideals

ABSTRACT

We show that an ideal $\mathcal{I}$ on ω is meager if and only if the set of sequences (x_n) taking values in a Polish space X for which all elements of X are $\mathcal{I}$-cluster points of (x_n) is comeager. The latter condition is also known as ν-maldistribution, where $\nu : \mathcal{P}(\omega) \to \mathbb{R}$ is the $\{0,1\}$-valued submeasure defined by $\nu(A) = 1$ if and only if $A \notin \mathcal{I}$. It turns out that the meagerness of $\mathcal{I}$ is also equivalent to a technical condition given by Mišík and Tóth (2025) [19]. Lastly, we show that the analogue of the first part holds replacing ν with $\|\cdot\|_\varphi$, where φ is a lower semicontinuous submeasure.

© 2025 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

1. Introduction

Let ω stand for the set of nonnegative integers. We say that a map $\nu : \mathcal{P}(\omega) \to \mathbb{R}$ is a *diffuse capacity* if it is a monotone map (that is, $\nu(A) \le \nu(B)$ for all $A \subseteq B \subseteq \omega$) such that $\nu(F) = 0$ and $\nu(\omega \setminus F) = 1$ for all finite $F \subseteq \omega$. If, in addition, ν is subadditive (that is, $\nu(A \cup B) \le \nu(A) + \nu(B)$ for all $A, B \subseteq \omega$) then we call it *diffuse submeasure*. Given a topological space X, a sequence $\boldsymbol{x} = (x_n : n \in \omega) \in X^\omega$ is called *ν-maldistributed* if

$$\nu(\{n \in \omega : x_n \in U\}) = 1$$

for all nonempty open sets $U \subseteq X$, cf. [19, Definition 3.1] for the case of separable metric spaces. Moreover, define the set

$$\Sigma_\nu(X) := \{\boldsymbol{x} \in X^\omega : \boldsymbol{x} \text{ is } \nu\text{-maldistributed}\}$$

E-mail address: leonetti.paolo@gmail.com.

https://doi.org/10.1016/j.jmaa.2025.129748

0022-247X/© 2025 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

and endow X^ω with the product topology. It is worth noting, as a particular case, that if ν is the diffuse submeasure defined by $\nu(S) := 0$ if $S \subseteq \omega$ is finite and $\nu(S) := 1$ otherwise then a continuous map $T : X \to X$ is commonly known as hypercyclic if and only if there exists $x_0 \in X$ such that its orbit $(T^n x_0 : n \in \omega)$ is ν-maldistributed, cf. [3,15] and references therein.

Very recently, Mišík and Tóth proved a sufficient technical condition to guarantee that, from a topological viewpoint, most sequences with values in X are ν-maldistributed, namely, the complement of $\Sigma_\nu(X)$ is meager (hence, contained in a countable union of closed sets with empty interior). With the above premises, their main result [19, Theorem 3.1] can be formulated as follows:

Theorem 1.1. *Let X be a separable metric space and suppose that $\nu : \mathcal{P}(\omega) \to \mathbb{R}$ is a diffuse capacity which satisfies the condition*:

$$\begin{aligned} &\forall \alpha \in (0,1),\ \exists g_\alpha \in \omega^\omega,\ \forall A \subseteq \omega : \\ &\qquad \nu(\omega \setminus A) \le 1-\alpha \implies \exists n_{\alpha,A} \in \omega, \forall n \ge n_{\alpha,A} : A \cap [n, n+g_\alpha(n)] \neq \emptyset. \end{aligned} \tag{1}$$

Then $\Sigma_\nu(X)$ is comeager.

Results on the same spirit of Theorem 1.1 can be found also in [1,2,16,20]. In the same work, Mišík and Tóth asked whether the converse of Theorem 1.1 holds, namely, whether there exists a separable metric space X and a diffuse capacity ν such that $\Sigma_\nu(X)$ is comeager, while condition (1) does *not* hold, see [19, Open Problem 5.1]. Our aim is to answer it in the negative for a certain family of diffuse submeasures.

To this aim, recall that an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is a family of subsets stable under finite unions and subsets. Moreover, it is assumed that the family of finite sets Fin is contained in $\mathcal{I}$, and that $\omega \notin \mathcal{I}$. The dual filter of an ideal $\mathcal{I}$ is denoted by $\mathcal{I}^\star := \{S \subseteq \omega : \omega \setminus A \in \mathcal{I}\}$. Identifying $\mathcal{P}(\omega)$ with the Cantor space 2^ω, we can speak about the topological complexity of ideals (in particular, it makes sense to speak about meager ideals). Lastly, given a sequence $\boldsymbol{x}$ taking values in a topological space X, we say that $\eta \in X$ is an $\mathcal{I}$-cluster point of the sequence $\boldsymbol{x}$ if $\{n \in \omega : x_n \in U\} \in \mathcal{I}^+$ for all open neighborhoods U of η, where $\mathcal{I}^+ := \mathcal{P}(\omega) \setminus \mathcal{I}$ stands for the family of $\mathcal{I}$-positive sets. We refer to [17] for basic facts and characterizations of the set of $\mathcal{I}$-cluster points, which is denoted by $\Gamma_{\boldsymbol{x}}(\mathcal{I})$.

Our main result shows that condition (1) for the submeasure $\nu := \mathbf{1}_{\mathcal{I}^+}$ is equivalent to meagerness of the ideal $\mathcal{I}$, and also to the comeagerness of $\Sigma_\nu(X)$; this goes in the spirit of the characterizations of meagerness of $\mathcal{I}$ given in [2].

Theorem 1.2. *Let X be a Polish space with $|X| \ge 2$, let $\mathcal{I}$ be an ideal on ω, and define the diffuse submeasure $\nu := \mathbf{1}_{\mathcal{I}^+}$. Then the following are equivalent*:

(i) *ν satisfies condition* (1);
(ii) *$\mathcal{I}$ is meager*;
(iii) *$\Sigma_\nu(X)$ is comeager.*

It is worth noting, as it follows by [8, Proposition 2.11 and Theorem 6.2], that, if $\mathcal{I}$ is an ideal on ω and X is an infinite separable metric space, then the existence of a $\mathbf{1}_{\mathcal{I}^+}$-maldistributed sequence is equivalent to the fact that for every $n \in \omega$ there exists a sequence $\boldsymbol{x} \in X^\omega$ with at least n $\mathcal{I}$-cluster points, which happens if and only if $\mathcal{I}$ is not a Fubini sum of finitely many maximal ideals.

Lastly, we provide a really large class of diffuse submeasures which satisfy condition (1). To this aim, a monotone subadditive map $\varphi : \mathcal{P}(\omega) \to [0,\infty]$ is said to be a *lower semicontinuous submeasure* (in short, lscsm) if it satisfies $\varphi(F) < \infty$ for all finite $F \subseteq \omega$ and, in addition,

$$\forall A \subseteq \omega, \quad \varphi(A) = \sup\{\varphi(A \cap [0,n]) : n \in \omega\}.$$

Notice that the above property is precisely the lower semicontinuity of the submeasure φ, regarding its domain $\mathcal{P}(\omega)$ as the Cantor space 2^ω, that is, if $A_n \to A$ then $\liminf_n \varphi(A_n) \ge \varphi(A)$. Examples of lscsms include $\varphi(A) = |A|$ or $\varphi(A) = \sum_{n \in A} 1/(n+1)$ or $\varphi(A) = \sup_{n \ge 1} |A \cap [0,n]|/n$, cf. also [7, Chapter 1].

Given a lscsm $\varphi : \mathcal{P}(\omega) \to [0,\infty]$, define the family

$$\mathrm{Exh}(\varphi) := \{S \subseteq \omega : \|S\|_\varphi = 0\}, \quad \text{where } \|S\|_\varphi := \lim_{n\to\infty} \varphi(S \setminus [0,n]).$$

Informally, $\|S\|_\varphi$ stands for the φ-mass at infinity of the set S. A classical result of Solecki [21, Theorem 3.1] states that an ideal $\mathcal{I}$ on ω is an analytic P-ideal if and only if there exists a lscsm φ such that

$$\mathcal{I} = \mathrm{Exh}(\varphi) \quad \text{and} \quad \varphi(\omega) < \infty.$$

Here, $\mathcal{I}$ is said to be a P-ideal if for every sequence (A_n) with values in $\mathcal{I}$ there exists $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all $n \in \omega$.

We remark that the family of analytic P-ideals is large and includes, among others, all Erdős–Ulam ideals introduced by Just and Krawczyk in [11], ideals generated by nonnegative regular matrices [9,10], the Fubini products $\emptyset \times \mathrm{Fin}$, which can be defined as $\{A \subseteq \omega : \forall n \in \omega, A \cap I_n \in \mathrm{Fin}\}$, where (I_n) is a given partition of ω into infinite sets, certain ideals used by Louveau and Veličković [18], and, more generally, density-like ideals and generalized density ideals [5,13]. Additional pathological examples can be found in [22]. It has been suggested in [4,5] that the theory of analytic P-ideals may have some relevant yet unexploited potential for the study of the geometry of Banach spaces.

Proposition 1.3. *Let X be a separable metric space and $\varphi : \mathcal{P}(\omega) \to [0,\infty]$ be a lscsm such that $\|\omega\|_\varphi = 1$. Then $\|\cdot\|_\varphi$ is a diffuse submeasure which satisfies condition (1). Hence, $\Sigma_{\|\cdot\|_\varphi}(X)$ is comeager.*

The above simple result provides a generalization of [19, Proposition 4.1], which corresponds to the case of lscsm φ generating a density ideal as in [7, Section 1.13], which in turn extends the main results in [6].

To conclude, one might be tempted to conjecture that, at least in the case of the submeasures $\mathbf{1}_{\mathcal{I}^+}$, where $\mathcal{I}$ is an ideal on ω, the set $\Sigma_\nu(X)$ is either meager or comeager. For instance, if X is a compact metric space with $|X| \ge 2$ and $\mathcal{I}$ is a maximal ideal on ω (that is, the complement of a free ultrafilter), then every sequence with values in X would be $\mathcal{I}$-convergent, so that $\Sigma_\nu(X) = \emptyset$. However, the following example shows that this is not the case.

Example 1.4. Endow $X := \{0,1\}$ with the discrete topology, let $\mathcal{I}_0, \mathcal{I}_1$ be two maximal ideals on 2ω and $2\omega+1$, respectively, and define

$$\mathcal{I} := \{S \subseteq \omega : S \cap 2\omega \in \mathcal{I}_0 \text{ and } S \cap (2\omega+1) \in \mathcal{I}_1\}.$$

Then $\Sigma_{\mathbf{1}_{\mathcal{I}^+}}(X)$ is neither meager nor comeager. In fact, for each $i,j \in X$ define

$$\mathcal{S}_{i,j} := \{\boldsymbol{x} \in X^\omega : \mathcal{I}_0\text{-}\lim \boldsymbol{x} \upharpoonright 2\omega = i \text{ and } \mathcal{I}_1\text{-}\lim \boldsymbol{x} \upharpoonright (2\omega+1) = j\}.$$

Regarding X as the Abelian group $\mathbb{Z}/2\mathbb{Z}$, it is easy to see that the above sets $\mathcal{S}_{i,j}$ are homeomorphic. Since X^ω is Polish and $\Sigma_{\mathbf{1}_{\mathcal{I}^+}}(X) = \mathcal{S}_{0,1} \cup \mathcal{S}_{1,0}$, we conclude that $\Sigma_{\mathbf{1}_{\mathcal{I}^+}}(X)$ is neither meager nor comeager.

The proofs of our results are given in Section 2.

2. Proofs

Before the proofs of our main characterization, we start with the following intermediate result, cf. [2, Theorem 3.1]. This applies, in particular, to complete metric spaces X with $|X| \ge 2$.

Proposition 2.1. *Let X be a Hausdorff space with $|X| \ge 2$ and assume that X^ω is Baire. Suppose also that there exists $\eta \in X$ such that*

$$\mathcal{S}_\eta := \{\boldsymbol{x} \in X^\omega : \eta \in \Gamma_{\boldsymbol{x}}(\mathcal{I})\} \tag{2}$$

is comeager. Then $\mathcal{I}$ is meager.

Proof. Fix $\eta \in X$ such that $\mathcal{S}_\eta$ is comeager, and let $U, V \subseteq X$ be two disjoint nonempty open sets such that $\eta \in U$. Since X^ω is Baire, there exists a decreasing sequence (G_n) of dense open subsets of X^ω such that $\bigcap_n G_n$ is dense and contained in $\mathcal{S}_\eta$.

Now, consider the following game defined by Laflamme in [14]: Players I and II choose alternately subsets $C_0, F_0, C_1, F_1, \ldots$ of ω, where the sets $C_0 \supseteq C_1 \supseteq \ldots$, which are chosen by Player I, are cofinite and the sets $F_k \subseteq C_k$, which are chosen by Player II, are finite. Player II is declared to be the winner if and only if $\bigcup_k F_k \in \mathcal{I}^+$. We may suppose without loss of generality that $F_k \cap C_{k+1} = \emptyset$ and $C_k = [c_k, \infty)$ for all $k \in \omega$ (hence, the sequence (c_k) corresponds to arbitrary (large enough) choices made by Player I). By [14, Theorem 2.12], Player II has a winning strategy if and only if $\mathcal{I}$ is meager. The remaining part of the proof consists in showing that Player II has a winning strategy.

We will define recursively, together with the description of the strategy of Player II, also a decreasing sequence of basic open sets

$$A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \supseteq \cdots$$

in X^ω (recall that a basic open set in X^ω is a cylinder of the type $D = \{\boldsymbol{x} \in X^\omega : x_0 \in W_0, x_1 \in W_1, \ldots, x_n \in W_n\}$ for some open sets $W_0, \ldots, W_n \subseteq X$, and we set $m(D) := n$). Suppose that the sets $C_0, F_0, \ldots, C_{k-1}, F_{k-1}, C_k \subseteq \omega$ have been already chosen and that the open sets $A_0, B_0, \ldots, A_{k-1}, B_{k-1} \subseteq X^\omega$ have already been defined, for some $k \in \omega$, where we assume by convention that $B_{-1} := X^\omega$ and $m(X^\omega) := -1$. Then we define the sets A_k, B_k, and F_k as follows:

(i) $A_k := \{\boldsymbol{x} \in B_{k-1} : x_n \in V \text{ for all } n \text{ with } m(B_{k-1}) < n < c_k\}$;

(ii) B_k is a nonempty basic open set contained in $G_k \cap A_k$ (note that this is possible since G_k is open dense and A_k is nonempty open); in addition, for each $n \in [c_k, m(B_k)]$, let $W_n \subseteq X$ be the smallest nonempty open set such that if $\boldsymbol{x} \in B_k$ then $x_n \in W_n$ (equivalently, W_n is the unique open subset of X such that the projection of the cylinder B_k at the n-th coordinate is precisely $x_n \in W_n$). Replacing each W_n with the smaller open set $W_n \cap U$ if $W_n \cap U \neq \emptyset$, it is possible to assume without loss of generality that either $W_n \subseteq U$ or $W_n \cap U = \emptyset$ for all $n \in [c_k, m(B_k)]$.

(iii) $F_k := \{n \in [c_k, m(B_k)] : W_n \subseteq U\}$ (note that this is a finite set, possibly empty).

We obtain by construction that there exists a sequence $\boldsymbol{x} = (x_n : n \in \omega) \in X^\omega$ such that $\boldsymbol{x} \in \bigcap_k B_k \subseteq \bigcap_k G_k \subseteq \mathcal{S}_\eta$. This implies that η is an $\mathcal{I}$-cluster point of $\boldsymbol{x}$, hence $\{n \in \omega : x_n \in U\} \in \mathcal{I}^+$. At the same time, by the definitions above

$$\{n \in \omega : x_n \in U\} = \bigcup\nolimits_k \{n \in [c_k, m(B_k)] : W_n \subseteq U\} = \bigcup\nolimits_k F_k.$$

This proves that Player II has a winning strategy. Therefore $\mathcal{I}$ is meager. □

Note that Theorem 1.1 proves, in particular, the implication (i) $\Longrightarrow$ (iii) of Theorem 1.2. However, we provide below a self-contained proof.

Proof of Theorem 1.2. First of all, it is routine to check that, if $\mathcal{I}$ is an ideal on ω, then $\nu := \mathbf{1}_{\mathcal{I}^+}$ is a diffuse submeasure, and that it satisfies condition (1) if and only if:

$$\exists g \in \omega^\omega, \forall A \in \mathcal{I}^\star, \exists n_A \in \omega, \forall n \geq n_A : \quad A \cap [n, n+g(n)] \neq \emptyset. \tag{3}$$

Moreover, by Talagrand's characterization [23, Theorem 2.1], the meagerness of $\mathcal{I}$ is equivalent to the existence of a sequence $(I_n : n \in \omega)$ of intervals of ω such that $\max I_n < \min I_{n+1}$ for all $n \in \omega$ and that $S \in \mathcal{I}^+$ whenever $I_k \subseteq S$ for infinitely many $k \in \omega$.

(i) $\Longrightarrow$ (ii). Suppose that condition (1) holds for $\mathbf{1}_{\mathcal{I}^+}$ or, equivalently, condition (3) is satisfied. Observe that the latter is equivalent to the existence of $g \in \omega^\omega$ such that if $S := \omega \setminus A$ contains infinitely many $[n, n+g(n)]$, then $A \notin \mathcal{I}^\star$, i.e., $S \in \mathcal{I}^+$. At this point, define $I_0 := [0, g(0)]$ and $I_{n+1} := [a_n, a_n + g(a_n)]$ where $a_n := 1 + \max I_n$ for all $n \in \omega$. It follows that $S \in \mathcal{I}^+$ whenever S contains infinitely many intervals I_n. Hence $\mathcal{I}$ is meager by Talagrand's characterization.

(ii) $\Longrightarrow$ (i). Pick a sequence of intervals (I_n) as in Talangrand's characterization, and define $g(n) := \max I_k$ where k is the smallest nonnegative integer with $\min I_k \geq n$. Now, pick $A \subseteq \omega$ and suppose that there exists infinitely many $n \in \omega$ such that $A \cap [n, n+g(n)] = \emptyset$. Then $\omega \setminus A$ contains infinitely many I_n, so that it belongs to $\mathcal{I}^+$. Therefore $A \notin \mathcal{I}^\star$, and condition (3) holds.

(ii) $\Longrightarrow$ (iii). Pick a sequence of intervals (I_n) as in Talangrand's characterization. Let $A := \{a_n : n \in \omega\}$ be a countable dense subset of X and note that a sequence $\boldsymbol{x} \in X^\omega$ is ν-maldistributed if and only if $\Gamma_{\boldsymbol{x}}(\mathcal{I}) = X$. Taking into account that the set of $\mathcal{I}$-cluster points $\Gamma_{\boldsymbol{x}}(\mathcal{I})$ is closed, see e.g. [17, Lemma 3.1(iv)], then $\boldsymbol{x}$ is ν-maldistributed if and only if $A \subseteq \Gamma_{\boldsymbol{x}}(\mathcal{I})$. Since A is countable and the family of meager subsets of X is a σ-ideal, it is enough to show that

$$\forall \eta \in X, \quad \mathcal{S}_\eta := \{\boldsymbol{x} \in X^\omega : \eta \in \Gamma_{\boldsymbol{x}}(\mathcal{I})\}$$

is comeager. To this aim, fix $\eta \in X$. Consider the Banach–Mazur game defined as follows: Players I and II choose alternatively nonempty open subsets of X^ω as a nonincreasing chain

$$U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \cdots,$$

where Player I chooses the sets $U_0, U_1, \ldots$; Player II has a winning strategy if $\bigcap_n V_n \subseteq \mathcal{S}_\eta$. It follows by [12, Theorem 8.33] that Player II has a winning strategy if and only if $\mathcal{S}_\eta$ is comeager. In fact, let d denote the metric on X and suppose that the nonempty open set U_n has been chosen by Player I. Then U_n contains a nonempty basic open set B_n of X^ω with support in a subset of the coordinates $\{0, 1, \ldots, \kappa_n\}$. Pick $j_n \in \omega$ such that $\min I_{j_n} > \kappa_n$. Then it is enough that Player II chooses

$$V_n := \{\boldsymbol{x} \in B_n : \forall i \in I_{j_n}, \quad d(x_i, \eta) < 2^{-n}\}.$$

This is indeed a winning strategy for Player II: if $\boldsymbol{x} \in \bigcap_n V_n$ then for every $\varepsilon > 0$ we have that $\{n \in \omega : d(x_n, \eta) < \varepsilon\}$ contains infinitely many intervals I_k, hence it is an $\mathcal{I}$-positive set. Therefore $\boldsymbol{x} \in \mathcal{S}_\eta$. It follows that $\Sigma_\nu(X)$ is comeager.

(iii) $\Longrightarrow$ (ii). Suppose that $\Sigma_\nu(X)$ is comeager and fix $\eta \in X$. Since X is a complete metric space, then X^ω is Baire and the set $\mathcal{S}_\eta$ defined in (2) is comeager as it is a superset of $\Sigma_\nu(X)$. Therefore $\mathcal{I}$ is meager by Proposition 2.1. □

Remark 2.2. As it follows from the proof above, the implication (ii) $\Longrightarrow$ (iii) holds for all separable metric spaces X.

We conclude with the proof of Proposition 1.3.

Proof of Proposition 1.3. Fix $\alpha \in (0,1)$ and for each $n \in \omega$ define $g_\alpha(n) := \min\{k \in \omega : \varphi([n, n+k]) \geq 1-\alpha/4\}$. Now, pick $A \subseteq \omega$ such that $\|\omega \setminus A\|_\varphi \leq 1-\alpha$ and fix $n_A \in \omega$ such that $\varphi((\omega \setminus A) \cap [n, \infty)) \leq 1-\alpha/2$ for all $n \geq n_A$. Since φ is a submeasure, it follows that, for all integers $n \geq n_A$, we get

$$\begin{aligned}\varphi(A \cap [n, n+g_\alpha(n)]) &\geq \varphi([n, n+g_\alpha(n)]) - \varphi((\omega \setminus A) \cap [n, n+g_\alpha(n)]) \\ &\geq \varphi([n, n+g_\alpha(n)]) - \varphi((\omega \setminus A) \cap [n, \infty)) \geq {}^{\alpha}\!/_{4},\end{aligned}$$

hence $A \cap [n, n+g_\alpha(n)] \neq \emptyset$. The second part follows by Theorem 1.1. $\square$

Acknowledgments

The author is thankful to two anonymous reviewers for a really careful reading of the manuscript which allowed to fix a gap in a previous version of the manuscript. The author is also grateful to Marek Balcerzak (Lodz University of Technology, PL) for the suggestion of [19, Open Problem 5.1].

References

[1] A. Aveni, P. Leonetti, Most numbers are not normal, Math. Proc. Camb. Philos. Soc. 175 (1) (2023) 1–11.
[2] M. Balcerzak, S. Głąb, P. Leonetti, Another characterization of meager ideals, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. RACSAM 117 (2) (2023) 90.
[3] J. Bès, Q. Menet, A. Peris, Y. Puig, Recurrence properties of hypercyclic operators, Math. Ann. 366 (1–2) (2016) 545–572.
[4] P. Borodulin-Nadzieja, B. Farkas, Analytic P-ideals and Banach spaces, J. Funct. Anal. 279 (8) (2020) 108702.
[5] P. Borodulin-Nadzieja, B. Farkas, G. Plebanek, Representations of ideals in Polish groups and in Banach spaces, J. Symb. Log. 80 (4) (2015) 1268–1289.
[6] J. Bukor, J.T. Tóth, On topological properties of the set of maldistributed sequences, Acta Univ. Sapientiae Math. 12 (2) (2020) 272–279.
[7] I. Farah, Analytic quotients: theory of liftings for quotients over analytic ideals on the integers, Mem. Am. Math. Soc. 148 (702) (2000), xvi+177.
[8] R. Filipów, A. Kwela, P. Leonetti, Borel complexity of sets of ideal limit points, preprint, last updated: Apr 18, 20025, arXiv:2411.10866.
[9] R. Filipów, J. Tryba, Ideal convergence versus matrix summability, Stud. Math. 245 (2) (2019) 101–127.
[10] R. Filipów, J. Tryba, Representation of ideal convergence as a union and intersection of matrix summability methods, J. Math. Anal. Appl. 484 (2) (2020) 123760.
[11] W. Just, A. Krawczyk, On certain Boolean algebras $\mathscr{P}(\omega)/I$, Trans. Am. Math. Soc. 285 (1) (1984) 411–429.
[12] A.S. Kechris, Classical Descriptive Set Theory, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.
[13] A. Kwela, P. Leonetti, Density-like and generalized density ideals, J. Symb. Log. 87 (1) (2022) 228–251.
[14] C. Laflamme, Filter games and combinatorial properties of strategies, in: Set Theory, Boise, ID, 1992–1994, in: Contemp. Math., vol. 192, Amer. Math. Soc., Providence, RI, 1996, pp. 51–67.
[15] P. Leonetti, Strong universality, recurrence, and analytic P-ideals in dynamical systems, preprint, last updated: May 08, 2025, arXiv:2401.01131.
[16] P. Leonetti, Tauberian theorems for ordinary convergence, J. Math. Anal. Appl. 519 (2) (2023) 126798.
[17] P. Leonetti, F. Maccheroni, Characterizations of ideal cluster points, Analysis (Berlin) 39 (1) (2019) 19–26.
[18] A. Louveau, B. Veličković, A note on Borel equivalence relations, Proc. Am. Math. Soc. 120 (1) (1994) 255–259.
[19] L. Mišík, J.T. Tóth, Maldistributed sequences in metric spaces, J. Math. Anal. Appl. 541 (2) (2025) 128667.
[20] L. Olsen, Extremely non-normal numbers, Math. Proc. Camb. Philos. Soc. 137 (1) (2004) 43–53.
[21] S. Solecki, Analytic ideals and their applications, Ann. Pure Appl. Log. 99 (1–3) (1999) 51–72.
[22] M. Talagrand, A simple example of pathological submeasure, Math. Ann. 252 (2) (1979/1980) 97–102.
[23] M. Talagrand, Compacts de fonctions mesurables et filtres non mesurables, Stud. Math. 67 (1) (1980) 13–43.