Results in Mathematics

Quasi-Arithmetic Means Ad Libitum

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Abstract. Let $\alpha_1, \ldots, \alpha_m$ be two or more positive reals with sum 1, let $C \subseteq \mathbb{R}^k$ be an open convex set, and $f : C \to \mathbb{R}^k$ be a continuous injection with convex image. For each nonempty set $S \subseteq C$, let $\mathscr{M}(S)$ be the family of quasi-arithmetic means of all m-tuples of vectors in C with respect to f and the weights $\alpha_1, \ldots, \alpha_m$, that is, the family

$$\mathscr{M}(S) = \left\{ f^{-1}\left(\alpha_1 f(x_1) + \cdots + \alpha_m f(x_m)\right) : x_1, \ldots, x_m \in S \right\}.$$

We provide a simple necessary and sufficient condition on S for which the infinite iteration $\bigcup_n \mathscr{M}^n(S)$ is relatively dense in the convex hull of S.

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1. Introduction

Let C be a nonempty convex subset of a real vector space X and let $f : C \to X$ be an injective map with convex image. Let also $\alpha_1, \ldots, \alpha_m$ be two or more positive reals with sum 1. Then the *quasi-arithmetic mean* of $x_1, \ldots, x_m \in C$ with respect to the function f and the weights $\alpha_1, \ldots, \alpha_m$ is the "mean vector"

$$\mathsf{m}(x_1, \ldots, x_m) := f^{-1}\left(\alpha_1 f(x_1) + \cdots + \alpha_m f(x_m)\right),$$

so that m is a map $C^m \to C$.

Quasi-arithmetic means, which are commonly known also as quasi-linear means, provide a generalization of classical and weighted means (say, the arithmetic mean, the quadratic mean, the harmonic mean, and others). They were first considered by Kolmogorov [17], Nagumo [23], de Finetti [7], and Kitagawa [16], and have been proved to be useful in a large variety of contexts, see e.g. [1, Section 7.3 and Chapters 15, 17, and 20] and [18, Chapter 5]. Their lattice

structure has been recently studied in [24]. In addition, the investigation of functional equations involving quasi-arithmetic means dates back at least to Aumann [2] and it has been the subject of intense research for about eighty years, see, e.g., the textbook expositions [14, Chapter III] and [1, Chapters 15 and 17], the articles [6,10,11,15,19–21], and references therein; in turn, the latter works revealed connections with the study of fixed point theory, permutable mappings, and dynamical systems.

For each nonempty $S \subseteq C$, we write $\mathscr{M}(S)$ for the set of all quasi-arithmetic means of vectors in S with respect to f and $\alpha_1, \ldots, \alpha_m$, that is,

$$\mathscr{M}(S) := \{\mathsf{m}(x_1, \ldots, x_m) : x_1, \ldots, x_m \in S\}.$$

Considering that $\mathsf{m}(x, \ldots, x) = x$ for all $x \in C$, $\mathscr{M}(\cdot)$ is a monotone increasing operator (i.e., $S \subseteq \mathscr{M}(S)$ for all $S \subseteq C$), so it makes sense to define

$$\mathscr{M}^\omega(S) := \bigcup_{n \geq 0} \mathscr{M}^n(S),$$

where $\mathscr{M}^0(S) := S$ and $\mathscr{M}^{n+1}(S) := \mathscr{M}(\mathscr{M}^n(S))$ for all integers $n \geq 0$ and sets $S \subseteq C$. Of course, each $\mathscr{M}^n(S)$ is a subfamily of finite convex linear combinations of elements of S, hence the limit $\mathscr{M}^\omega(S)$ is contained in the convex hull of S, hereafter denoted by $\mathrm{conv}(S)$.

Remark 1.1. Note that $\mathscr{M}^\omega(S)$ is the smallest set $W \subseteq C$ containing S such that $\mathscr{M}(W) = W$. For, pick $x_1, \ldots, x_m \in \mathscr{M}^\omega(S)$. Then there exists an integer $n \geq 0$ such that $x_1, \ldots, x_m \in \mathscr{M}^n(S)$. Hence the quasi-arithmetic mean $\mathsf{m}(x_1, \ldots, x_m)$ belongs to $\mathscr{M}^{n+1}(S)$, with the conclusion that $\mathscr{M}(\mathscr{M}^\omega(S)) = \mathscr{M}^\omega(S)$.

As a related notion, in the case where f is the identity function id, following Mesikepp [22], the set $\mathscr{M}(S)$ can be regarded as the Minkowski M-sum

$$\oplus_M(S, \ldots, S) := \{a_1 x_1 + \cdots + a_m x_m : x_1, \ldots, x_m \in S, (a_1, \ldots, a_m) \in M\},$$

where M is the singleton $\{(\alpha_1, \ldots, \alpha_m)\}$; cf. also [3,9].

Here and after, we suppose that $X = \mathbb{R}^k$, for some integer $k \geq 1$, and that C is a nonempty open convex set. Let us also denote by $\mathrm{int}(S)$ and $\mathrm{cl}(S)$ the interior and the closure of a subset $S \subseteq \mathbb{R}^k$ in the relative topology of C, respectively. A subset $S \subseteq \mathbb{R}^k$ is said to be k-dimensional whenever it does not lie in an affine $(k-1)$-dimensional hyperplane, or, equivalently, $\mathrm{int}(\mathrm{conv}(S))$ is nonempty. Every subset of $\mathbb{R}^k$ is endowed with its relative topology. Hence, given subsets $A, B \subseteq \mathbb{R}^k$, we say that A is relatively dense in B if the closure of $A \cap B$ in the relative topology of B coincides with B. Lastly, let $|S|$ denote the cardinality of a set S.

In the case where f is the identity function and $C = \mathbb{R}^k$, the sets $\mathscr{M}^\omega(S)$ have been studied by Green and Gustin in [12]. Remarkably, they proved in [12, Theorem 2.2] the following density result.

Theorem 1.2. *Suppose that f is the identity function* id *on $\mathbb{R}^k$ and fix two or more positive weights $\alpha_1, \ldots, \alpha_m$ with sum 1. Then $\mathscr{M}^\omega(S)$ is relatively dense in* conv(S) *for every set $S \subseteq \mathbb{R}^k$.*

Theorem 1.2 is the starting point of this work. Our main question is to understand which continuous injections f may replace the identity map id in the above result.

2. Main Result

Our main result follows.

Theorem 2.1. *Let $C \subseteq \mathbb{R}^k$ be an open convex set and $f : C \to \mathbb{R}^k$ be a continuous injection with convex image. Fix also two or more positive weights $\alpha_1, \ldots, \alpha_m$ with sum 1, and a k-dimensional subset $S \subseteq C$. Then the following are equivalent*:

(i). $\mathscr{M}^\omega(S)$ is relatively dense in conv(S);
(ii). $f[\mathrm{int}(\mathrm{conv}(S)] \subseteq \mathrm{int}(\mathrm{conv}(f[S]))$.

If, in addition, S is compact, then they are also equivalent to:

(iii). $f[\mathrm{conv}(S)] \subseteq \mathrm{conv}(f[S])$.

Before we proceed to the proof of Theorem 2.1, some remarks are in order. First of all, if $C = \mathbb{R}^k$ and $f = $ id, then condition (ii) clearly holds, hence we recover Green and Gustin's Theorem 1.2 (in the case where S is not k-dimensional, it is enough to consider, modulo homeomorphisms, the problem replacing $\mathbb{R}^k$ with the affine hull of S; we omit details).

Second, the hypothesis that f has convex image is not automatically satisfied if $k \geq 2$: for instance, consider the continuous injection $f : C \to \mathbb{R}^2$ with $C := (0,1)^2$ defined by $f(x,y) := (x, x^2 + y)$ for every $(x,y) \in C$.

Lastly, by condition (iii), one may think that there exists some relationship between $f[\mathrm{conv}(S)]$ and $\mathrm{conv}(f[S])$, apart from the trivial inclusion

$$f[S] \subseteq f[\mathrm{conv}(S)] \cap \mathrm{conv}(f[S]).$$

But this is not the case: we may have equality, for instance, if $k = 2$, $C = (0,\infty)^2$, $S = \{(1,1),(2,2)\}$, and $f : C \to \mathbb{R}^2$ is the continuous injection defined by $f(x,y) := (x, x^2 + y^2)$ for all $(x,y) \in C$. Indeed, $f[\mathrm{conv}(S)] = \{(x, 2x^2) : x \in [1,2]\}$ and $\mathrm{conv}(f[S])$ is the line segment between $(1,2)$ and $(2,4)$.

Remark 2.2. The equivalence between (i) and (iii) does not hold without any conditions on S. For, suppose that $m = k = 2$, $\alpha_1 = \alpha_2 = 1/2$, and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an homeomorphism with the property that $f[D] = B$ and $f(1,0) = (1,0)$, where $D := [-1,1]^2$ and B is the closed unit ball B with center $(0,0)$ and radius 1 (for instance, it is sufficient to define $f(0,0) := (0.0)$ and $f(x,y) := (\max\{|x|,|y|\} \cdot \cos(\theta), \max\{|x|,|y|\} \cdot \sin(\theta))$ otherwise, where θ is the unique angle in $[0, 2\pi)$ such that $x = k \cos(\theta)$ and $y = k \sin(\theta)$ for some $k \in \mathbb{R}$).

Then, setting $S := D \setminus \{(1,0)\}$, it follows that $\mathscr{M}^\omega(S)$ is relatively dense in $\mathrm{conv}(S) = D$, hence condition (i) holds. On the other hand, $f[\mathrm{conv}(S)] = B$ is not contained in $\mathrm{conv}(f[S]) = B \setminus \{(1,0)\}$, hence condition (iii) fails.

As a consequence, we obtain an extension of Green and Gustin's Theorem 1.2 in the case of boxes S^k; here, an interval $I \subseteq \mathbb{R}$ may be unbounded.

Corollary 2.3. *Let $f_1, \ldots, f_k : I \to \mathbb{R}$ be continuous injections such that $I \subseteq \mathbb{R}$ is an open interval. Define the map $f : I^k \to \mathbb{R}^k$ by*

$$\forall x \in I^k, \quad f(x) := (f_1(x_1), \ldots, f_k(x_k))$$

Fix also two or more positive weights $\alpha_1, \ldots, \alpha_m$ with sum 1. Then $\mathscr{M}^\omega(S^k)$ is relatively dense in $\mathrm{conv}(S^k)$ for each $S \subseteq I$ with nonempty interior.

We will need a series of intermediate lemmas given in Sect. 3. We will deduce Theorem 2.1 and Corollary 2.3 in Sect. 4.

3. Intermediate Lemmas

In what follows, $C \subseteq \mathbb{R}^k$ stands for an open convex set, $f : C \to \mathbb{R}^k$ for a continuous injection with convex image, and S for a k-dimensional subset of C (as in the statement of Theorem 2.1).

Lemma 3.1. *The set $f[C]$ is open in $\mathbb{R}^k$ and the function f is an homeomorphism onto its image.*

Proof. It follows by the Brouwer's invariance of domain theorem, see [5].   □

Lemma 3.2. *Fix a subset $U \subseteq f[C]$. Then $f^{-1}[\mathrm{int}(U)] = \mathrm{int}(f^{-1}[U])$ and $f^{-1}[\mathrm{cl}(U)] = \mathrm{cl}(f^{-1}[U])$.*

Proof. On the one hand, by [8, Proposition 1.4.1], the continuity of f is equivalent to $f^{-1}[\mathrm{int}(V)] \subseteq \mathrm{int}(f^{-1}[V])$ for all $V \subseteq f[C]$. On the other hand, in view of Lemma 3.1, the restriction of the inverse f^{-1} on $\mathrm{int}(U)$ is continuous and injective. This implies that $f[\mathrm{int}(f^{-1}[U])]$ is an open set contained in $f[f^{-1}[U]] = U$. Hence $f[\mathrm{int}(f^{-1}[U])] \subseteq \mathrm{int}(U)$ and we conclude that $\mathrm{int}(f^{-1}[U]) \subseteq f^{-1}[\mathrm{int}(U)]$.

For the second part, set $V^c := f[C] \setminus V$ for each $V \subseteq f[C]$. Then

$$f^{-1}[\mathrm{cl}(U)] = f^{-1}[(\mathrm{int}(U^c))^c] = \left(f^{-1}[\mathrm{int}(U^c)]\right)^c$$
$$= \left(\mathrm{int}\left(\left(f^{-1}[U]\right)^c\right)\right)^c = \mathrm{cl}\left(f^{-1}[U]\right),$$

which concludes the proof.   □

Let us recall the classical result of Carathéodory:

Lemma 3.3 [25, p.73]. *Let $V \subseteq \mathbb{R}^k$ be an arbitrary set. Then*

$$\mathrm{conv}(V) = \bigcup_{U \subseteq V : |U| \leq k+1} \mathrm{conv}(U).$$

We will need also the following characterization of interior of convex hulls:

Lemma 3.4 [13]. *Let $V \subseteq \mathbb{R}^k$ be an arbitrary set. Then*

$$\operatorname{int}(\operatorname{conv}(V)) = \bigcup_{U \subseteq V \colon |U| \leq 2k} \operatorname{int}(\operatorname{conv}(U)).$$

Note that the constant $2k$ in the above lemma cannot be improved: indeed, if $V := \{e_1, -e_1, \ldots, e_k, -e_k\}$, where $\{e_1, \ldots, e_k\}$ stands for the canonical basis of $\mathbb{R}^k$, then the zero vector belongs to $\operatorname{int}(\operatorname{conv}(V))$, but it does not belong to $\operatorname{int}(\operatorname{conv}(U))$ for all $U \subseteq V$ with $|U| = 2k - 1$.

Lemma 3.5. *The following inclusion holds*:

$$f^{-1}\left[\operatorname{int}\left(\operatorname{conv}(f[S])\right)\right] \subseteq \bigcup_{T \subseteq S \colon |T| \leq 2k} \operatorname{cl}\left(f^{-1}\left[\operatorname{int}\left(\operatorname{conv}(f[T])\right)\right]\right). \qquad (1)$$

Proof. Note that the claim is obvious if $f[S]$ is not k-dimensional. Hence, let us assume hereafter that $\operatorname{int}(\operatorname{conv}(f[S])) \neq \emptyset$. By the injectivity of f and Lemma 3.4, the left hand side of (1) can be rewritten as

$$f^{-1}\left[\bigcup_{U \subseteq S \colon |U| \leq 2k} \operatorname{int}\left(\operatorname{conv}(f[U])\right)\right].$$

Now, fix $T \subseteq S$ with $|T| \leq 2k$ and note that $\operatorname{int}\left(\operatorname{conv}(f[T])\right)$ is an open set contained in $f[C]$, so by the continuity of f the set $f^{-1}[\operatorname{int}\left(\operatorname{conv}(f[T])\right)]$ is open and contained in C. Thus $\operatorname{int}\left(\operatorname{conv}(f[T])\right) \subseteq f[C]$, therefore by Lemma 3.2 the right hand side of (1) can be rewritten as

$$f^{-1}\left[\bigcup_{T \subseteq S \colon |T| \leq 2k} \operatorname{cl}\left(\operatorname{int}\left(\operatorname{conv}(f[T])\right)\right)\right].$$

At this point, the conclusion is immediate. $\qquad \square$

Lemma 3.6. *Let $V \subseteq \mathbb{R}^k$ a be k-dimensional convex set. If $H \subseteq \mathbb{R}^k$ is relatively dense in $\operatorname{int}(V)$, then H is relatively dense in V.*

Proof. It is sufficient to show that $\operatorname{int}(V)$ is relatively dense in V. For, let U be an open ball centered at some boundary point of V, so that $U \cap V \neq \emptyset$, and let us say $u \in U \cap V$. Since $\operatorname{int}(V) \neq \emptyset$, there exists a closed ball W with positive radius contained in $\operatorname{int}(V)$. If u belongs to $\operatorname{int}(V)$, the claim holds. Otherwise u has to be a boundary point of V. Now, since V is convex, the subset $D := \{\gamma u + (1-\gamma)w : \gamma \in [0,1], w \in W\}$ is contained in V. In addition, u is an interior point of U, hence there exists an open ball B centered in u with $B \subseteq U$. By monotonicity, we obtain $\operatorname{int}(D) \cap B \subseteq \operatorname{int}(D) \cap U \subseteq \operatorname{int}(V) \cap U$. The claim follows from the fact the left hand side is clearly nonempty. $\qquad \square$

Lastly, let us recall the following useful property:

Lemma 3.7 [4, Chapter II, Section 6, Proposition 16 and Corollaire 1]. *Let $U, V \subseteq \mathbb{R}^k$ be k-dimensional convex sets with U open and V closed. Then*

$$\operatorname{int}(\operatorname{cl}(U)) = U \quad and \quad \operatorname{cl}(\operatorname{int}(V)) = V.$$

4. Proofs Section

We proceed now to the proof of Theorem 2.1 and Corollary 2.3.

Proof of Theorem 2.1. (i) $\implies$ (ii). Let us suppose on the contrary that condition (i) holds, while (ii) does not. To simplify the notation, define

$$P := \operatorname{int}\left(\operatorname{conv}(S)\right), \ Q := f^{-1}\left[\operatorname{int}\left(\operatorname{conv}(f[S])\right)\right], \ \text{and} \ R := \operatorname{int}\left(\operatorname{conv}(f[S])\right).$$

Hence $P \setminus Q$ is nonempty. Note all P, Q, and R are open subsets of $\mathbb{R}^k$.

Claim 1. $P \setminus \operatorname{cl}(Q)$ has nonempty interior.

Proof. If (ii) does not hold then $f[P]$ is not contained in R. Let us suppose that $f[P]$ is contained in $\operatorname{cl}(R)$. Since $f[P]$ is open by Lemma 3.1, it follows that $f[P] \subseteq \operatorname{int}(\operatorname{cl}(R))$. Since the interior of the closure of an open convex set coincides with the set itself by Lemma 3.7, it follows that $f[P] \subseteq R$, contradicting our assumption. Therefore $f[P] \backslash \operatorname{cl}(R) \neq \emptyset$.

Let g be the restriction of the inverse map f^{-1} to R and note that $R = f[Q]$. By Lemma 3.1, g is a homeomorphism onto its image Q. It follows by Lemma 3.2 that $f[P]$ is not contained in $\operatorname{cl}(f[Q]) = \operatorname{cl}(g^{-1}[Q]) = g^{-1}[\operatorname{cl}(Q)] = f[\operatorname{cl}(Q)]$. By the injectivity of f this is equivalent to saying that the open set $P \setminus \operatorname{cl}(Q)$ is nonempty, which proves the claim. □

Let us suppose that $\mathscr{M}^\omega(S)$ is relatively dense in $\operatorname{conv}(S)$. Since $\mathscr{M}^\omega(S) \subseteq f^{-1}[\operatorname{conv}(f[S])]$ and $P \subseteq \operatorname{conv}(S)$, we obtain that $f^{-1}[\operatorname{conv}(f[S])]$ is relatively dense in P, that is, $V \cap f^{-1}[\operatorname{conv}(f[S])] \neq \emptyset$ for each nonempty open subset $V \subseteq P$. Taking into account Claim 1, we obtain

$$(P \setminus \operatorname{cl}(Q)) \cap f^{-1}[\operatorname{conv}(f[S])] \neq \emptyset. \tag{2}$$

Since S is k-dimensional and f is an homeomorphism by Lemma 3.1, $f[S]$ is k-dimensional, so that R is a nonempty open set. It follows by Lemma 3.2 and Lemma 3.7 that

$$f^{-1}\left[\operatorname{conv}(f[S])\right] \subseteq f^{-1}\left[\operatorname{cl}(\operatorname{conv}(f[S]))\right] = f^{-1}[\operatorname{cl}(R)] = \operatorname{cl}(Q).$$

However, this contradicts (2).

(ii) $\implies$ (i). Since S is k-dimensional, $|S| \geq k + 1 \geq 2$. Let us a fix a subset $T \subseteq S$, let us say $T = \{t_1, \ldots, t_h\}$, with $2 \leq h \leq 2k$. Let Δ be the $(h-1)$-dimensionalsimplex, that is, the set of vectors $\delta = (\delta_1, \ldots, \delta_h) \in [0,1]^h$

with $\sum_{i \leq h} \delta_i = 1$. Moreover, define $W := \bigcup_n W_n$ where, for each integer $n \geq 1$,

$$W_n := \left\{ \gamma \in \Delta : f^{-1}\left(\gamma_1 f(t_1) + \cdots + \gamma_h f(t_h)\right) \in \mathscr{M}^n(T) \right\}.$$

Claim 2. W is dense in Δ.

Proof. For each $V \subseteq \mathbb{R}^h$, let $\pi(V)$ be the projection of V on its first $h - 1$ coordinates, that is,

$$\pi(V) := \{ x \in \mathbb{R}^{h-1} : (x_1, \ldots, x_{h-1}, y) \in V \text{ for some } y \in \mathbb{R} \}.$$

Since the vectors δ in Δ satisfy the linear constraint $\sum_{i \leq h} \delta_i = 1$, it is sufficient to show that $\pi(W)$ is relatively dense in $\pi(\Delta) = \{ \delta \in [0,1]^{h-1} : \sum_{i < h} \delta_i \leq 1 \}$.

Define $\alpha := \max\{\alpha_1, \ldots, \alpha_m\}$. Consider the cube $J_0 := [0,1]^{h-1}$ and let us construct recursively a sequence $(\mathscr{J}_n)_{n \geq 1}$ of increasingly finer partitions of J_0 into $2^{n(h-1)}$ parallelepipeds as it follows. Set for convenience $\mathscr{J}_0 := \{J_0\}$ and suppose that $\mathscr{J}_{n-1}$ has been defined for some $n \geq 1$. At the nth step, let us divide each parallelepiped $J \in \mathscr{J}_{n-1}$ with $h-1$ hyperplanes with dimension $h - 2$, each one perpendicular to an edge and dividing the edge itself in two segments with lenghts proportional to α and $1 - \alpha$. Therefore we obtain a partition $\mathscr{J}_n$ of the original cube J_0 into $2^{n(h-1)}$ parallelepipeds of dimension $h - 1$, each one with volume $\leq \alpha^{n(h-1)}$ and diagonals of length $\leq \alpha^n \sqrt{h-1}$. Let V_n be the set of vertices of these parallelepipeds which belong also to $\pi(\Delta)$, and note that $V_n \subseteq \pi(W_n)$. Since $V := \bigcup_n V_n$ is contained in $\pi(W)$, we obtain

$$\limsup_{n \to \infty} \max_{x \in \pi(W_n)} \min_{y \in \pi(W_n), x \neq y} \|x - y\| \leq \limsup_{n \to \infty} \max_{J \in \mathscr{J}_n} \max_{x, y \in J, x \neq y} \|x - y\|$$
$$\leq \limsup_{n \to \infty} \alpha^n \sqrt{h-1} = 0.$$

Therefore V (and, hence, $\pi(W)$) is relatively dense in $\pi(\Delta)$. $\qquad\square$

At this point, let $g_T : \Delta \to \mathbb{R}^k$ be the function defined by

$$\forall \delta \in \Delta, \quad g_T(\delta) := \delta_1 f(t_1) + \cdots \delta_h f(t_h).$$

Since g_T is continuous with image $g_T[\Delta]$, it follows by Claim 2 that $g_T[W]$ is relatively dense in $g_T[\Delta]$ (this easy fact can found, e.g., in the proof of [8, Theorem 1.4.10]). In particular, $g_T[W]$ is relatively dense in $\mathrm{int}(\mathrm{conv}(f[T]))$. Taking into account that f is a homeomorphism, we obtain that $f^{-1}[g_T[W]]$ is relatively dense in $f^{-1}[\mathrm{int}(\mathrm{conv}(f[T]))]$. Since $\mathscr{M}^\omega(T)$ contains $f^{-1}[g_T[W]]$, it follows that

$$\mathscr{M}^\omega(T) \text{ is relatively dense in } \mathrm{cl}(f^{-1}[\mathrm{int}(\mathrm{conv}(f[T]))]). \tag{3}$$

Considering that (3) holds for all subsets $T \subseteq S$ with $2 \leq |T| \leq 2k$, it follows by monotonicity that

$$\mathscr{M}^\omega(S) \text{ is relatively dense in } \bigcup_{T \subseteq S : |T| \leq 2k} \mathrm{cl}(f^{-1}[\mathrm{int}(\mathrm{conv}(f[T]))]).$$

In view of Lemma 3.5, we get

$$\mathscr{M}^{\omega}(S) \text{ is relatively dense in } f^{-1}[\mathrm{int}(\mathrm{conv}(f[S]))].$$

Hence, by condition (ii) and the injectivity of f, we have

$$\mathscr{M}^{\omega}(S) \text{ is relatively dense in } \mathrm{int}(\mathrm{conv}(S)).$$

The conclusion follows by Lemma 3.6 and the hypothesis that the convex hull of S has nonempty interior.

   (iii) $\implies$ (ii). Condition (iii) implies that

$$f\left[\mathrm{int}(\mathrm{conv}(S))\right] \subseteq f\left[\mathrm{conv}(S)\right] \subseteq \mathrm{conv}(f[S]).$$

Moreover, f is an open map by Lemma 3.1, hence $f\left[\mathrm{int}(\mathrm{conv}(S))\right]$ is contained in $\mathrm{int}(\mathrm{conv}(f[S]))$, i.e., condition (ii) holds.

   Lastly, suppose that S is compact.

   (ii) $\implies$ (iii). Since S is k-dimensional, $f[S]$ is k-dimensional as well by condition (ii). Moreover, by Lemma 3.1, f is a homeomorphism onto its image. Since both continuity and convex hull operator preserve compacteness (see e.g. [8, Theorem 3.1.10] and [25, Theorem 3.20(d)], respectively), then $\mathrm{conv}(S)$, $\mathrm{conv}(f[S])$, and $f^{-1}[\mathrm{conv}(f[S])]$ are compact sets. In addition, by condition (ii) and the monotonicity of the closure operator, we have

$$\mathrm{cl}\left(\mathrm{int}\left(\mathrm{conv}(S)\right)\right) \subseteq \mathrm{cl}\left(f^{-1}\left[\mathrm{int}\left(\mathrm{conv}(f[S])\right)\right]\right).$$

Since a k-dimensional closed convex set coincides with the closure of its interior by Lemma 3.7, we obtain by Lemma 3.2 that

$$\begin{aligned}
\mathrm{conv}(S) = \mathrm{cl}\left(\mathrm{int}\left(\mathrm{conv}(S)\right)\right) &\subseteq \mathrm{cl}\left(f^{-1}\left[\mathrm{int}\left(\mathrm{conv}(f[S])\right)\right]\right)\\
&= \mathrm{cl}\left(\mathrm{int}\left(f^{-1}\left[\mathrm{conv}(f[S])\right]\right)\right) = f^{-1}\left[\mathrm{conv}(f[S])\right],
\end{aligned}$$

which is equivalent to condition (iii) by the injectivity of f. $\square$

Remark 4.1. As it is clear from the proof above, the implication (iii) $\implies$ (ii) holds independently of the compactness assumption on S.

Proof of Corollary 2.3. The image of an interval of a continuous real-valued function of a real variable is also an interval. Moreover, every interval is a convex set. Since $\mathrm{conv}(S)$ is an interval of $\mathbb{R}$ and the functions f_i are continuous injections, we obtain $f_i[\mathrm{conv}(S)] = \mathrm{conv}(f_i[S])$ for each $i \in \{1, \ldots, k\}$. However, each inverse function f_i^{-1} is a continuous injection, hence $f_i[\mathrm{int}(\mathrm{conv}(S))]$ is an open subsets of $f_i[\mathrm{conv}(S)]$, cf. Lemma 3.1. Therefore

$$f_i[\mathrm{int}(\mathrm{conv}(S))] \subseteq \mathrm{int}\left(f_i[\mathrm{conv}(S)]\right) = \mathrm{int}\left(\mathrm{conv}(f_i[S])\right)$$

for each $i \in \{1, \ldots, k\}$. By [8, Proposition 1.4.1], the map f_i^{-1} is continuous if and only if $f_i[\text{int}(V)] \subseteq \text{int}(f_i[V])$ for all $V \subseteq I$. Therefore, we obtain that

$$f[\text{int}(\text{conv}(S^k))] = f\left[\text{int}(\text{conv}(S))^k\right] = \prod_{i=1}^{k} f_i\left[\text{int}(\text{conv}(S))\right]$$

$$\subseteq \prod_{i=1}^{k} \text{int}\left(f_i\left[\text{conv}(S)\right]\right) = \text{int}\left(f[(\text{conv}(S))^k]\right) = \text{int}\left(f[\text{conv}(S^k)]\right).$$

The conclusion follows from Theorem 2.1. □

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