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J. Math. Anal. Appl. 309 (2005) 353–368

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Isolated minimizers and proper efficiency for $C^{0,1}$ constrained vector optimization problems

Ivan Ginchev [a], Angelo Guerraggio [b], Matteo Rocca [b,*]

[a] *Department of Mathematics, Technical University of Varna, Studentska Str. 9010, Varna, Bulgaria*
[b] *Department of Economics, University of Insubria, via Ravasi, 2, 21100, Varese, Italy*

Received 13 May 2004

Available online 2 March 2005

Submitted by B.S. Mordukhovich

Abstract

We consider the vector optimization problem $\min_C f(x)$, $g(x) \in -K$, where $f:\mathbb{R}^n \to \mathbb{R}^m$ and $g:\mathbb{R}^n \to \mathbb{R}^p$ are $C^{0,1}$ (i.e. locally Lipschitz) functions and $C \subseteq \mathbb{R}^m$ and $K \subseteq \mathbb{R}^p$ are closed convex cones. We give several notions of solution (efficiency concepts), among them the notion of properly efficient point (p-minimizer) of order k and the notion of isolated minimizer of order k. We show that each isolated minimizer of order $k \geqslant 1$ is a p-minimizer of order k. The possible reversal of this statement in the case $k = 1$ is studied through first order necessary and sufficient conditions in terms of Dini derivatives. Observing that the optimality conditions for the constrained problem coincide with those for a suitable unconstrained problem, we introduce sense I solutions (those of the initial constrained problem) and sense II solutions (those of the unconstrained problem). Further, we obtain relations between sense I and sense II isolated minimizers and p-minimizers.

Keywords: Vector optimization; Locally Lipschitz data; Properly efficient points; Isolated minimizers; Optimality conditions

* Corresponding author.
E-mail addresses: ginchev@ms3.tu-varna.acad.bg (I. Ginchev), aguerraggio@eco.uninsubria.it (A. Guerraggio), mrocca@eco.uninsubria.it (M. Rocca).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.01.041

1. Introduction

In this paper we consider the vector optimization problem

$$\min_C f(x), \quad g(x) \in -K, \tag{1}$$

where $f:\mathbb{R}^n \to \mathbb{R}^m$, $g:\mathbb{R}^n \to \mathbb{R}^p$. Here n, m and p are positive integers and $C \subseteq \mathbb{R}^m$ and $K \subseteq \mathbb{R}^p$ are closed convex cones. Often in the literature the consideration of vector optimization problems is restricted to the case where C is a closed convex pointed cone with nonempty interior. For the investigations of this paper both conditions C pointed and with nonempty interior are too restrictive (they need not be satisfied in problem (14) below) and we prefer to get rid of them at the beginning.

Usually the solutions of problem (1) are called points of efficiency. We prefer, as in scalar optimization, to call them minimizers.

The solutions of a vector problem are often studied through a scalarization, i.e. reducing the vector problem to an equivalent scalar problem. A well-known approach is linear scalarization, but several other ad hoc scalarization techniques have been used. It has been shown [1–3] that more restrictive definitions of minimality for the considered scalarized problem correspond to more restrictive notions of efficiency. In this paper we consider a particular kind of scalarization which makes use of the so called "oriented distance" from a point to a set. In terms of oriented distance the notion of isolated minimizer (i-minimizer) of a given order is introduced in [4], extending to the vector case a notion known in scalar optimization [5–7]. Under the name of strict minimizer the same concept appears in [8–10]. In this paper we prefer the original name of isolated minimizer given by Auslender [5]. We observe in Section 2, that the isolated minimizers for the vector problem are isolated minimizers of an appropriate scalar problem. In this work we are interested in the links between isolated minimizers of the scalarized problem and properly efficient points (p-minimizers) of the constrained problem (1). We will assume that f and g are of class $C^{0,1}$, i.e. locally Lipschitz functions and for such functions we apply some first-order necessary and sufficient optimality conditions to clarify the relations between these concepts.

Observing that the optimality conditions of the constrained problem coincide with those of a suitable unconstrained problem, we introduce sense I solutions (those of the initial constrained problem) and sense II solutions (those of the unconstrained problem). We establish some relations between sense I and sense II p-minimizers and i-minimizers, which give also a motivation for the "duplication" of the notions of solution (one prefers probably to deal with the simpler unconstrained problem instead of the constrained one).

The outline of the paper is the following. Section 2 is devoted to notions of optimality for problem (1) and their scalarization. Section 3 generalizes the notion of a p-minimizer to a p-minimizer of order k and starts the investigations of the links between isolated minimizers and proper efficiency by showing (Theorems 3.1 and 3.2) that each i-minimizer is a p-minimizer. The possible reversal of this statement in the case $k = 1$ is the main subject of investigation in the paper. In Section 4 with reference to $C^{0,1}$ functions, we recall some first order necessary and sufficient optimality conditions in terms of Dini derivatives, obtained in [3]. Section 5 discusses a reversal of Theorem 3.2, shows that the given optimality conditions are important to solve this problem and they lead to two approaches toward efficiency concepts, defining sense I and sense II concepts. The relation between sense I and sense II i-minimizers and p-minimizers is investigated.

2. Vector optimality concepts and scalar characterizations

We denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the Euclidean norm and the scalar product in the considered finite-dimensional spaces. The open unit ball is denoted by B. From the context it should be clear to which spaces these notations are applied. The results of the paper can be immediately extended to finite-dimensional real Banach spaces.

There are different concepts of solution for problem (1). In the following definitions we assume that the considered point x^0 is feasible, i.e. $g(x^0) \in -K$ (equivalently $x^0 \in g^{-1}(-K)$). The definitions below are given in a local sense. We omit this specification in the text.

Definition 2.1. (i) The feasible point x^0 is said to be weakly efficient (efficient), if there is a neighbourhood U of x^0, such that if $x \in U \cap g^{-1}(-K)$ then $f(x) - f(x^0) \notin -\operatorname{int} C$ (respectively $f(x) - f(x^0) \notin -(C \setminus \{0\})$).

(ii) The feasible point x^0 is said to be properly efficient if there exist a closed (but not necessarily convex) cone $\tilde{C} \subseteq \mathbb{R}^n$, with $C \setminus \{0\} \subseteq \operatorname{int} \tilde{C}$ and a neighbourhood U of x^0, such that if $x \in U \cap g^{-1}(-K)$, then $f(x) - f(x^0) \notin -\operatorname{int} \tilde{C}$.

In this paper the weakly efficient, the efficient and the properly efficient points of problem (1) are called respectively w-minimizers, e-minimizers and p-minimizers. The following chain of implications is known:

$$p\text{-minimizer} \quad \Rightarrow \quad e\text{-minimizer} \quad \Rightarrow \quad w\text{-minimizer}.$$

Remark 2.1. If we assume that C is a pointed cone, in virtue of Definition 2.1 a p-minimizer can be defined in the following way: the feasible point x^0 is said to be properly efficient for the constrained problem (1) if there exists a closed convex cone $\tilde{C}$, such that $C \setminus \{0\} \subseteq \operatorname{int} \tilde{C}$ and x^0 is weakly efficient for the problem $\min_{\tilde{C}} f(x)$, $g(x) \in -K$. This is the commonly accepted definition of a properly efficient point (see, e.g., Henig [11]). The latter however does not work with non-pointed cones, hence it is too restrictive with regard to our hypotheses.

Definition 2.2. The feasible point x^0 is said a strong e-minimizer if there exists a neighborhood U of x^0, such that $f(x) - f(x^0) \notin -C$, for $x \in U \setminus \{x^0\} \cap g^{-1}(-K)$.

Obviously, every strong e-minimizer is e-minimizer.

The unconstrained problem

$$\min_C f(x), \quad x \in \mathbb{R}^n, \tag{2}$$

is a particular case of problem (1) and the defined notions of optimality concern also this problem.

For the cone $M \subseteq \mathbb{R}^k$ its positive polar cone M' is defined by $M' = \{\zeta \in \mathbb{R}^k \mid \langle \zeta, \phi \rangle \geqslant 0$ for all $\phi \in M\}$. The cone M' is closed and convex and it is well known that $M'' := (M')' = \operatorname{cl} \operatorname{co} M$; see, e.g., [12, Chapter III, §15]. In particular, for a closed convex cone M we have $M = M'' = \{\phi \in \mathbb{R}^k \mid \langle \zeta, \phi \rangle \geqslant 0$ for all $\zeta \in M'\}$.

If $\phi \in -\operatorname{cl} \operatorname{co} M$, then $\langle \zeta, \phi \rangle \leqslant 0$ for all $\zeta \in M'$. We set $M'(\phi) = \{\zeta \in M' \mid \langle \zeta, \phi \rangle = 0\}$. Then $M'(\phi)$ is a closed convex cone and $M'(\phi) \subseteq M'$. Consequently its positive polar cone $M(\phi) = (M'(\phi))'$ is a closed convex cone, $M \subseteq M(\phi)$ and its positive polar cone satisfies $(M(\phi))' = M'(\phi)$. In this paper we apply this notation for $M = K$ and $\phi = -g(x^0)$. Then we write for short $K'(x^0)$ instead of $K'(-g(x^0))$ (and call this cone the index set of problem (1) at x^0) and $K(x^0)$ instead of $K(-g(x^0))$.

Next results characterize the solutions of problem (1) in terms of a suitable scalarization.

Proposition 2.1 [3]. *Define*

$$\varphi(x) = \max\{\langle \xi, f(x) - f(x^0) \rangle \mid \xi \in C',\ \|\xi\| = 1\}. \tag{3}$$

The feasible point $x^0 \in \mathbb{R}^n$ is a w-minimizer for problem (1)*, if and only if x^0 is a minimizer for the scalar problem*

$$\min \varphi(x), \quad g(x) \in -K. \tag{4}$$

Proposition 2.2 [3]. *The feasible point x^0 is a strong e-minimizer of problem* (1) *if and only if x^0 is a strong minimizer of problem* (4)*, i.e. if and only if there exists a neighborhood U of x^0, such that $\varphi(x) - \varphi(x^0) > 0$ for all $x \in (U \setminus \{x^0\}) \cap g^{-1}(-K)$.*

Recall that the feasible point x^0 is said to be an isolated minimizer of order $k > 0$ of problem (4) when there exists a constant $A > 0$ such that $\varphi(x) \geqslant \varphi(x^0) + A\|x - x^0\|^k$ for all $x \in U \cap g^{-1}(-K)$. The concept of an isolated minimizer for scalar problems has been popularized by Auslender [5]. It is natural to introduce a similar concept of optimality for the vector problem (1).

Definition 2.3. We say that the feasible point x^0 is an isolated minimizer (i-minimizer) of order k for the vector problem (1) if it is an isolated minimizer of order k for the scalar problem (4).

To interpret geometrically the property that x^0 is a minimizer of problem (1) of certain type, we introduce the so called oriented distance. Given a set $A \subseteq \mathbb{R}^k$, then the distance from $y \in \mathbb{R}^k$ to A is given by $d(y, A) = \inf\{\|a - y\| \mid a \in A\}$. This definition works also for $A = \emptyset$ putting $d(y, \emptyset) = \inf \emptyset = +\infty$. The oriented distance from y to A is defined by $D(y, A) = d(y, A) - d(y, \mathbb{R}^k \setminus A)$. This definition gives $D(y, A) = +\infty$ when $A = \emptyset$ and $D(y, A) = -\infty$ when $A = \mathbb{R}^k$.

Function D is introduced in Hiriart-Urruty [13,14] and is used later in Ciligot-Travain [15], Amahroq and Taa [16], Miglierina [17], Miglierina and Molho [18]. Zaffaroni [1] gives different notions of efficiency and uses the function D for their scalarization and comparison. Ginchev and Hoffmann [19] use the oriented distance to study approximation of set-valued functions by single-valued ones and in the case of a convex cone C show the representation $D(y,-C)=\sup_{\|\xi\|=1,\xi\in C'}\langle\xi,y\rangle$. Turn attention that this formula works also in the case of the improper cones $C=\{0\}$ (then $D(y,-C)=\sup_{\|\xi\|=1}\langle\xi,y\rangle=\|y\|$) and $C=\mathbb{R}^m$ (then $D(y,-C)=\sup_{\xi\in\emptyset}\langle\xi,y\rangle=-\infty$).

In particular function φ in (4) is expressed by $\varphi(x)=D(f(x)-f(x^0),-C)$. Propositions 2.1 and 2.2 are easily reformulated in terms of the oriented distance, namely:

$$x^0\ w\text{-minimizer} \quad\Leftrightarrow\quad D\big(f(x)-f(x^0),-C\big)\geqslant 0 \quad \text{for } x\in U\cap g^{-1}(-K),$$

$$x^0 \text{ strong } e\text{-minimizer} \quad\Leftrightarrow$$
$$D\big(f(x)-f(x^0),-C\big)>0 \quad \text{for } x\in\big(U\setminus\{x^0\}\big)\cap g^{-1}(-K).$$

The definition of i-minimizer gives

$$x^0\ i\text{-minimizer of order } k \quad\Leftrightarrow$$
$$D\big(f(x)-f(x^0),-C\big)\geqslant A\|x-x^0\|^k \quad \text{for } x\in U\cap g^{-1}(-K).$$

We see that an i-minimizers is a strong e-minimizer. In the next section we explore the links between i-minimizers and p-minimizers. The next proposition has an immediate proof and we omit it.

Proposition 2.3. *The point x^0 is an i-minimizer of order k for problem* (1) *if and only if there exists a constant $A>0$ and a neighborhood U of x^0, such that*

$$\big(f(x)+C\big)\cap B\big(f(x^0),A\|x-x^0\|^k\big)=\emptyset,\quad \forall x\in U\setminus\{x^0\} \tag{5}$$

(*here $B(f(x^0),\delta)$ denotes the open ball with center in $f(x^0)$ and radius δ*).

Remark 2.2. Points satisfying (5) are called strict efficient points of order k in [8–10].

Remark 2.3. In the important case $C=\mathbb{R}^n_+$ it can be shown (see [2,3]) that statements like those of Propositions 2.1 and 2.2 remain true if function φ is substituted by

$$\varphi_0(x)=\max\nolimits_{1\leqslant i\leqslant n}\big(f_i(x)-f_i(x^0)\big). \tag{6}$$

In fact, there exist constants $\alpha,\beta>0$ such that $\alpha\varphi(x)\leqslant\varphi_0(x)\leqslant\beta\varphi(x)$.

3. Isolated minimizers and proper efficiency

Applying the oriented distance function we can generalize the concept of proper efficiency. For given $k\geqslant 1$ and $a>0$ we define the set

$$C^k(a)=\big\{y\in\mathbb{R}^m \mid D(y,C)\leqslant a\|y\|^k\big\}.$$

It is easily seen that when $k=1$ the set $C^1(a)$ is a closed cone (not necessarily convex; see, e.g., [20]).

Definition 3.1. We say that the feasible point x^0 is a properly efficient point (p-minimizer) of order $k\geqslant 1$ for problem (1) if there exist a neighbourhood U of x^0 and a constant $a>0$ such that if $x\in U\cap g^{-1}(-K)$ then $f(x)-f(x^0)\notin -\operatorname{int} C^k(a)$.

The previous definition cannot be extended mechanically to the case $0 < k < 1$. In this case, for arbitrary $a > 0$ and all sufficiently small $\|y\|$ we would have $D(y, C) \leqslant \|y\| \leqslant a\|y\|^k$. Therefore, assuming f is continuous, for x sufficiently close to x^0, the inclusion $f(x) - f(x^0) \notin -\operatorname{int} C^k(a)$ could not hold.

Proposition 3.1. *The point x^0 is a p-minimizer for problem* (1) *if and only if it is a p-minimizer of order* 1.

Proof. If x^0 is a p-minimizer of order 1 then x^0 satisfies Definition 2.1 with respect to the cone $\tilde{C} = C^1(a)$, hence x^0 is a p-minimizer.

Conversely, let x^0 be a p-minimizer and $\tilde{C}$ be the cone from Definition 2.1. Since the set $F = \{y \in C \mid \|y\| = 1\}$ is compact and disjoint from the closed set $\mathbb{R}^n \setminus \tilde{C}$, therefore $a := \operatorname{dist}(F, \mathbb{R}^n \setminus \operatorname{int} \tilde{C}) > 0$. Now obviously $C^1(a) \subseteq \tilde{C}$. Since $x \in U \cap g^{-1}(-K)$ implies $f(x) - f(x^0) \notin -\operatorname{int} \tilde{C}$ and $-\operatorname{int} \tilde{C} \supset -\operatorname{int} \tilde{C}^1(a)$, we get $f(x) - f(x^0) \notin -\operatorname{int} C^1(a)$. Therefore x^0 is a p-minimizer of order 1. □

Definition 3.1 can be equivalently rephrased according to the following results.

Proposition 3.2. *The feasible point x^0 is a p-minimizer of order k for problem* (1) *if and only if there exist a neighbourhood U of x^0 and a constant $a > 0$ such that for all $\varepsilon > 0$ and all $x \in U \cap g^{-1}(-K)$ satisfying $\|f(x) - f(x^0)\| \geqslant \varepsilon$ it holds $D(f(x) - f(x^0), -C) \geqslant a\varepsilon^k$.*

Proof. Let x^0 be a p-minimizer of order k. Then there exist a neighbourhood U of x^0 and a constant $a > 0$ such that for all $x \in U \cap g^{-1}(-K)$ it holds $f(x) - f(x^0) \notin -\operatorname{int} C^k(a)$. Taking into account the definition of $C^k(a)$, we obtain $D(f(x) - f(x^0), -C) \geqslant a\|f(x) - f(x^0)\|^k$. Then $\|f(x) - f(x^0)\| \geqslant \varepsilon$ gives $D(f(x) - f(x^0), -C) \geqslant a\varepsilon^k$.

Conversely, let x^0 satisfy the given condition. In particular, if we fix $x \in U \cap g^{-1}(-K)$ in advance, the inequality $\|f(x) - f(x^0)\| \geqslant \varepsilon$ is satisfied for $\varepsilon = \|f(x) - f(x^0)\|$. Hence, we get $D(f(x) - f(x^0), -C) \geqslant a\|f(x) - f(x^0)\|^k$ which can be rephrased as $f(x) - f(x^0) \notin -\operatorname{int} C^k(a)$. □

Proposition 3.3. *The feasible point x^0 is a p-minimizer of order $k \geqslant 1$ for problem* (1) *if and only if there exist a neighbourhood U of x^0 and a constant $a > 0$ such that for all $\varepsilon > 0$ it holds*

$$\left(f\left(U \cap g^{-1}(-K)\right) - f(x^0)\right) \cap (a\varepsilon^k B - C) \subseteq \varepsilon B. \tag{7}$$

Proof. Let x^0 be a p-minimizer of order k and let the neighbourhood U of x^0 and the constant $a > 0$ be those from Proposition 3.2. We show that (7) holds for all $\varepsilon > 0$. Assume, on the contrary, that there exists $x \in U \cap g^{-1}(-K)$ such that $f(x) - f(x^0) \in a\varepsilon^k B - C$, or equivalently $D(f(x) - f(x^0), -C) < a\varepsilon^k$, but $f(x) - f(x^0) \notin \varepsilon B$, or equivalently $\|f(x) - f(x^0)\| \geqslant \varepsilon$. This inequality, according to Proposition 3.2 implies $D(f(x) - f(x^0), -C) \geqslant a\varepsilon^k$, a contradiction.

Assume now that for x^0 there exist a neighbourhood U and a constant $a > 0$ for which (7) holds. We show that also the condition in Proposition 3.2 is satisfied. Assume, on the contrary, that there exists $\varepsilon > 0$ and $x \in U \cap g^{-1}(-K)$ satisfying $\|f(x) - f(x^0)\| \geqslant \varepsilon$, but $D(f(x) - f(x^0), -C) < a\varepsilon^k$. This means that $f(x) - f(x^0)$ belongs to the left-hand side of (7) but not to the right-hand side, a contradiction. □

As far as we know, the definition of proper efficiency of order $k \geqslant 1$ is a new one. Let us however mention that from Proposition 3.3, it follows that p-minimizers of order k are strictly efficient points in the sense of Bednarczuk [21].

When f is a $C^{0,1}$ function, the following relation holds between i-minimizers of order k and p-minimizers of order k.

Theorem 3.1. *Let f be of class $C^{0,1}$. If a point x^0 is an i-minimizer of order $k \geqslant 1$ for problem* (1) *then x^0 is a p-minimizer of order k.*

Proof. Assume, on the contrary, that x^0 is an i-minimizer of order k but not p-minimizer of order k. Let f be Lipschitz with constant L in $x^0 + r\,\mathrm{cl}\,B$. Take sequences $\delta_\nu \to 0+$ and $\varepsilon_\nu \to 0+$ and consider the sets $C^k(\varepsilon_\nu)$. Since x^0 is not a p-minimizer of order k it follows that there exists a sequence of feasible points $x^\nu \in (x^0 + \delta_\nu B) \cap g^{-1}(-K)$ such that $f(x^\nu) - f(x^0) \in -\operatorname{int} C^k(\varepsilon_\nu)$, and in particular $f(x^\nu) - f(x^0) \neq 0$. From the definition of $C^k(\varepsilon_\nu)$ we get

$$D\big(f(x^\nu) - f(x^0), -C\big) < \varepsilon_\nu \big\|f(x^\nu) - f(x^0)\big\|^k \leqslant \varepsilon_\nu L^k \|x^\nu - x^0\|^k$$

which contradicts to x^0 i-minimizer of order k. □

We formulate separately the particular case obtained by Theorem 3.1 for $k = 1$.

Theorem 3.2. *Let f be of class $C^{0,1}$. If x^0 is an i-minimizer of first order for problem* (1) *then x^0 is a p-minimizer.*

Next Examples 3.1 and 3.2 show respectively that the Lipschitz assumption in Theorems 3.1 and 3.2 cannot be dropped and the result of Theorems 3.1 and 3.2 in general cannot be reverted. As for the used notations, let us say that we prefer to denote the fixed value of the variable x by x^0 when x is vector-valued (then x_i^0 stands for the ith coordinate of x^0) and x_0 when x is real-valued.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}^2$, $g : \mathbb{R} \to \mathbb{R}$, be defined as $f(x) = (\sqrt{|x|}), -\sqrt[4]{|x|})$ and $g(x) = x$. Let $C = \mathbb{R}^2_+$ and $K = \mathbb{R}_+$. The point $x_0 = 0$ is an i-minimizer of first order, but not a p-minimizer for problem (1).

From $f(x) = f(-x)$ we see that the condition $g(x) \equiv x \leqslant 0$ does not introduce changes on the efficiency properties of $x_0 = 0$ for the constrained problem (1) in comparison with the unconstrained problem (2). It is obvious from the definition that x_0 is not a p-minimizer. Since $D(f(x) - f(x_0), -\mathbb{R}^2_+) \geqslant \sqrt{|x|} \geqslant |x|$ for $|x| < 1$, the point x_0 is an i-minimizer of first order. Thus, the conclusion of Theorem 3.2 does not hold, but obviously f is not $C^{0,1}$.

Example 3.2. (i) Let $f:\mathbb{R}\to\mathbb{R}^2$, $g:\mathbb{R}\to\mathbb{R}$ be defined as $f(x)=(x^2,-x^2)$ and $g(x)=x$. Let $C=\mathbb{R}^2_+$ and $K=\mathbb{R}_+$. Hence, f and g are of class $C^{0,1}$, $x_0=0$ is a p-minimizer, but x_0 is not an i-minimizer of first order.

(ii) Consider the function $f:\mathbb{R}\to\mathbb{R}^2$, $f(x)=(f_1(x),f_2(x))$, with $f_1(x)=-x^2\sin\frac{1}{x}-x^2$ and $f_2(x)=[f_1(x)]^2$, if $x\neq 0$, and $f_1(0)=f_2(0)=0$. The point $x_0=0$ is a p-minimizer of any order $k>2$, but there exists no positive number k, such that x_0 is an i-minimizer of order k.

For Example 3.2(i), as an application of Proposition 3.2 we observe that

$$D\big(f(x)-f(x_0),-C\big)=D\big((x^2,-x^2),-\mathbb{R}^2_+\big)\geqslant x^2=\frac{1}{\sqrt{2}}\big\|(x^2,-x^2)\big\|$$
$$=\frac{1}{\sqrt{2}}\big\|f(x)-f(x_0)\big\|.$$

Therefore $f(x)-f(x_0)\notin -\operatorname{int}C^1(1/\sqrt{2})$, whence x_0 is a p-minimizer. On the other hand, x_0 is not an i-minimizer of first order for problem (1), since x_0 is not an isolated minimizer of first order for the scalar problem $\varphi(x)\to\min$, $x\leqslant 0$, which easily seen from $x^2\leqslant\varphi(x)\leqslant\sqrt{2}\,x^2$.

For Example 3.2(ii), we observe that, for every $k>2$, we have

$$D\big(f(x)-f(x_0),-C\big)\geqslant f_2(x)=\big[f_1(x)\big]^2\geqslant\big|f_1(x)\big|^k\big(\big[f_1(x)\big]^2+1\big)^{k/2}$$
$$=\big\|f(x)-f(x_0)\big\|^k,$$

for x in a suitable neighbourhood of x_0. Hence, $f(x)-f(x_0)\notin -\operatorname{int}C^k(1)$, and x_0 is a p-minimizer of order $k>2$. On the other hand, it is easily seen that x_0 is not an i-minimizer of any order $k>0$.

In the sequel we consider only i-minimizers of first order and for this reason sometimes we call them simply i-minimizers. Similarly, we consider only p-minimizers of first order, which as we know are just p-minimizers.

4. Dini derivatives and first-order optimality conditions

Problem (1) has been investigated in [3] under the hypothesis that f and g are of class $C^{0,1}$. The authors obtained optimality conditions in terms of the first-order Dini directional derivative.

Given a $C^{0,1}$ function $\Phi:\mathbb{R}^n\to\mathbb{R}^k$ we define the Dini directional derivative (we use to say just Dini derivative) $\Phi'_u(x^0)$ of Φ at x^0 in direction $u\in\mathbb{R}^n$, as the set of the cluster points of $(1/t)(\Phi(x^0+tu)-\Phi(x^0))$ as $t\to 0+$, that is as the Kuratowski limit

$$\Phi'_u(x^0)=\operatorname*{Lim\,sup}_{t\to 0+}\frac{1}{t}\big(\Phi(x^0+tu)-\Phi(x^0)\big).$$

It can be shown (see, e.g., [3]) that if Φ is of class $C^{0,1}$, then $\Phi'_u(x^0)$ is a nonempty compact subset of $\mathbb{R}^k$, whatever $u\in\mathbb{R}^n$.

In connection with problem (1) we deal with the Dini directional derivative of the function $\Phi:\mathbb{R}^n \to \mathbb{R}^{m+p}$, $\Phi(x) = (f(x), g(x))$ and then we use to write $\Phi'_u(x^0) = (f,g)'_u(x^0)$. If at least one of the derivatives $f'_u(x^0)$ and $g'_u(x^0)$ is a singleton, then $(f,g)'_u(x^0) = (f'_u(x^0), g'_u(x^0))$. Let us turn attention that always $(f,g)'_u(x^0) \subseteq f'_u(x^0) \times g'_u(x^0)$, but in general these two sets do not coincide.

Theorem 4.1 gives first-order optimality conditions in terms of the Dini derivative and is useful in clarifying the links between i-minimizers and p-minimizers. It uses the conditions denoted below by $\mathcal{N}'_{0,1}$ and $\mathcal{S}'_{0,1}$, in which all cluster points of the differential quotient of (f,g) play a role. This justifies the usage of the set-valued Dini derivative $(f,g)'_u(x^0)$. The set-valuedness appears in fact when in problem (1) we consider arbitrary functions f and g of class $C^{0,1}$. When the considerations are restricted to directionally differentiable functions f and g, the Dini derivative is a singleton and is expressed through the directional derivative, i.e. $(f,g)'_u(x^0) = (f'(x^0,u), g'(x^0,u))$. Let us mention that the use of set-valued derivatives (of first and second order) in vector optimization is known in the literature (see, e.g., [22–25]). The importance of set-valued derivatives for vector functions is stressed also in Rockafellar and Wets [26, p. 327], where the authors define the notion of graphical derivative. In opposite to the introduced Dini derivative, the graphical derivative involves in its definition also a variation in the direction (compare with formula 8.20, p. 327 in [26]). In a simplified setting Demyanov and Rubinov [27] apply the name of Dini derivative when a variation in the direction does not appear and of Hadamard derivative otherwise. Following this convention we use the name of Dini derivative for the notion defined in this section, while the graphical derivative in [26] is in fact an Hadamard type derivative.

In the formulation of Theorem 4.1 we use the following constraint qualification, which is a generalization for $C^{0,1}$ constraints of the well-known Kuhn–Tucker constraint qualification (compare with Mangasarian [28, p. 102]):

$$\begin{aligned}\mathcal{Q}_{0,1}(x^0):\quad & \text{if } g(x^0) \in -K \text{ and } \frac{1}{t_k}\big(g(x^0+t_k u^0) - g(x^0)\big) \to z^0 \in -K(x^0)\\ & \text{then } \exists u^k \to u^0\colon \exists k_0 \in \mathbb{N}\colon \forall k > k_0\colon\ g(x^0 + t_k u^k) \in -K.\end{aligned}$$

Theorem 4.1 [3]. *Let f, g be $C^{0,1}$ functions.*

Necessary conditions. *Let x^0 be a w-minimizer for problem* (1). *Then for each $u \in \mathbb{R}^n$ the following condition is satisfied*:

$$\begin{aligned}\mathcal{N}'_{0,1}:\quad & \forall (y^0, z^0) \in (f,g)'_u(x^0)\colon\ \exists(\xi^0,\eta^0) \in C' \times K'\colon\\ & (\xi^0,\eta^0) \neq (0,0), \qquad \big\langle \eta^0, g(x^0)\big\rangle = 0 \quad \textit{and} \quad \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle \geqslant 0.\end{aligned}$$

Sufficient conditions. *Let $x^0 \in \mathbb{R}^n$ and suppose that for each $u \in \mathbb{R}^n \setminus \{0\}$ the following condition is satisfied*:

$$\begin{aligned}\mathcal{S}'_{0,1}:\quad & \forall (y^0, z^0) \in (f,g)'_u(x^0)\colon\ \exists(\xi^0,\eta^0) \in C' \times K'\colon\\ & (\xi^0,\eta^0) \neq (0,0), \qquad \big\langle \eta^0, g(x^0)\big\rangle = 0 \quad \textit{and} \quad \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle > 0.\end{aligned}$$

Then x^0 is an i-minimizer of first order for problem (1).

Conversely, if x^0 is an i-minimizer of first order for problem (1) *and the constraint qualification $\mathcal{Q}_{0,1}(x^0)$ holds, then condition $\mathcal{S}'_{0,1}$ is satisfied.*

Theorem 4.1 is valid and simplifies in an obvious way when instead of (1) we consider the unconstrained problem (2). Let us underline that in this case the reversal of the sufficient conditions does not require the use of constraint qualifications.

Theorem 4.2. Necessary conditions. *Let f be a $C^{0,1}$ function. Let x^0 be a w-minimizer of problem* (2)*. Then for each $u \in \mathbb{R}^n$ the following condition is satisfied*:

$$\forall y^0 \in f'_u(x^0):\ \exists \xi^0 \in C':\quad \xi^0 \neq 0 \quad \text{and} \quad \langle \xi^0, y^0 \rangle \geqslant 0.$$

Sufficient conditions. *Let $x^0 \in \mathbb{R}^n$ and suppose that for each $u \in \mathbb{R}^n \setminus \{0\}$ the following condition is satisfied*:

$$\forall y^0 \in f'_u(x^0):\ \exists \xi^0 \in C':\quad \xi^0 \neq 0 \quad \text{and} \quad \langle \xi^0, y^0 \rangle > 0. \tag{8}$$

Then x^0 is an i-minimizer of first order for problem (2)*. Conversely, if x^0 is an i-minimizer of first order for problem* (2) *then condition* (8) *is satisfied.*

As an application of Theorem 4.1 we get the next Proposition 4.1.

Proposition 4.1. *Let f and g be $C^{0,1}$ functions. If for some pair $(\xi^0, \eta^0) \in (C' \times K'(x^0)) \setminus \{(0,0)\}$, the feasible point x^0 is an isolated minimizer of first order for the scalar function*

$$\gamma(x) = \langle \xi^0, f(x) \rangle + \langle \eta^0, g(x) \rangle, \tag{9}$$

then x^0 is a p-minimizer of (1)*.*

Proof. Let $u \in \mathbb{R}^n \setminus \{0\}$ and let $(y^0, z^0) \in (f, g)'_u(x^0)$. Hence, for some sequence $t_k \to 0+$, we have

$$y^0 = \lim_{k \to +\infty} \frac{f(x^0 + t_k u) - f(x^0)}{t_k}, \qquad z^0 = \lim_{k \to +\infty} \frac{g(x^0 + t_k u) - g(x^0)}{t_k}.$$

Since x^0 is an isolated minimizer of first order for the scalar function (9), there exists a number $A > 0$, such that $\gamma(x^0 + t_k u) - \gamma(x^0) \geqslant A t_k$, whence

$$\left\langle \xi^0, \frac{1}{t_k}\big(f(x^0 + t_k u) - f(x^0)\big) \right\rangle + \left\langle \eta^0, \frac{1}{t_k}\big(g(x^0 + t_k u) - g(x^0)\big) \right\rangle \geqslant A > 0.$$

Passing to the limit we get $\langle \xi^0, y^0 \rangle + \langle \eta^0, z^0 \rangle \geqslant A > 0$. Now the Sufficient condition in Theorem 4.1 gives that x^0 is an i-minimizer of first order for problem (1), and according to Theorem 3.2 it is also a p-minimizer. □

5. Two approaches toward proper efficiency

It is natural to put the question, under what condition Theorem 3.2 admits a reversal, that is under what condition x^0 p-minimizer implies x^0 i-minimizer. Example 3.2(i) shows that in general such a reversal does not hold. To answer the posed question we consider first the unconstrained problem (2). Then a crucial role plays the property $0 \notin f'_u(x^0)$.

Theorem 5.1. *Let f be a locally Lipschitz function and let x^0 be a p-minimizer for the unconstrained problem* (2)*, which has the property $0 \notin f'_u(x^0)$ for all $u \in \mathbb{R}^n \setminus \{0\}$. Then x^0 is an i-minimizer of first order for* (2)*.*

Proof. We prove separately the particular case when C is a pointed cone, in order to demonstrate an application of Theorem 4.2.

The case C pointed. According to Remark 2.1, we may assume that the cone $\tilde{C}$ in Definition 2.1 is closed and convex, such that $\operatorname{int} \tilde{C} \supset C \setminus \{0\}$ and x^0 is w-minimizer for the problem $\min_{\tilde{C}} f(x)$, $x \in \mathbb{R}^n$. According to the Necessary conditions of Theorem 4.2, this means that for each $u \in \mathbb{R}^n \setminus \{0\}$ and $y^0 \in f'_u(x^0)$, there exists $\tilde{\xi}^0 \in \tilde{C}' \setminus \{0\}$, such that $\langle \tilde{\xi}^0, y^0 \rangle \geqslant 0$. This inequality, together with $y^0 \neq 0$ (implied by property $0 \notin f'_u(x^0)$), shows that $y^0 \notin -\operatorname{int} \tilde{C} \cup \{0\}$. Since $C \subseteq \operatorname{int} \tilde{C} \cup \{0\}$, we see that $y^0 \notin -C$. This implies, that there exists $\xi^0 \in C'$, such that $\langle \xi^0, y^0 \rangle > 0$. According to the reversal of the Sufficient conditions of Theorem 4.2, the point x^0 is an i-minimizer of first order.

The general case. The general case assumes that the cone C is only closed and convex.

Assume on the contrary that x^0 is a p-minimizer for the unconstrained problem (2), but it is not an i-minimizer of first order. Choose a monotone decreasing sequence $\varepsilon_k \to 0+$. Hence, there exist sequences $t_k \to 0+$ and u^k, $\|u^k\| = 1$, such that

$$D\big(f(x^0 + t_k u^k) - f(x^0), -C\big) = \max_{\xi \in \Gamma_{C'}} \big\langle \xi, f(x^0 + t_k u^k) - f(x^0) \big\rangle < \varepsilon_k t_k, \tag{10}$$

where $\Gamma_{C'} = \{\xi \in C' \mid \|\xi\| = 1\}$. We may assume that $0 < t_k < r$ and f is Lipschitz with constant L in $x^0 + r \operatorname{cl} B$. Passing to a subsequence, we may assume also that $u^k \to u^0$, $\|u^0\| = 1$, and that $y^0 = \lim_k y^{0,k}$, where $y^{0,k} = (1/t_k)(f(x^0 + t_k u^0) - f(x^0))$. From the definition of the Dini derivative we have $y^0 \in f'_u(x^0)$ and from the assumptions $y^0 \neq 0$. We show that $y^k \to y^0$, where $y^k = (1/t_k)(f(x^0 + t_k u^k) - f(x^0))$. This follows from the estimation

$$\begin{aligned} \|y^k - y^0\| &\leqslant \frac{1}{t_k} \big\| f(x^0 + t_k u^k) - f(x^0 + t_k u^0) \big\| + \|y^{0,k} - y^0\| \\ &\leqslant L\|u^k - u^0\| + \|y^{0,k} - y^0\|. \end{aligned}$$

Let now $\xi \in \Gamma_{C'}$. Then

$$\begin{aligned} \langle \xi, y^k \rangle &= \frac{1}{t_k} \big\langle \xi, f(x^0 + t_k u^k) - f(x^0) \big\rangle \leqslant \frac{1}{t_k} \max_{\xi \in \Gamma_{C'}} \big\langle \xi, f(x^0 + t_k u^k) - f(x^0) \big\rangle \\ &= \frac{1}{t_k} D\big(f(x^0 + t_k u^k) - f(x^0), -C\big) < \frac{1}{t_k} \varepsilon_k t_k = \varepsilon_k. \end{aligned}$$

Passing to a limit with $k \to \infty$ we get $\langle \xi, y^0 \rangle \leqslant 0$ for arbitrary $\xi \in \Gamma_{C'}$, whence $D(y^0, -C) = \max_{\xi \in \Gamma_{C'}} \langle \xi, y^0 \rangle \leqslant 0$.

On the other hand, x^0 is a p-minimizer, which according to Definition 3.1 and Proposition 3.1 means that there exists a constant $a > 0$, such that for sufficiently large k it holds

$$f(x^0+t_ku^k)-f(x^0)\notin -\operatorname{int}C^1(a) \quad\Leftrightarrow$$
$$\frac{1}{t_k}D\big(f(x^0+t_ku^k)-f(x^0),-C\big)\geqslant a\left\|\frac{1}{t_k}\big(f(x^0+t_ku^k)-f(x^0)\big)\right\|.$$

Applying the positive homogeneity of the oriented distance and taking the limit as $k\to\infty$ we get the contradiction

$$0\geqslant D(y^0,-C)\geqslant a\|y^0\|>0, \tag{11}$$

which shows that x^0 is an i-minimizer. □

Now we generalize Theorem 5.1 for the constrained problem.

Theorem 5.2. *Let f and g be $C^{0,1}$ functions and let x^0 be a p-minimizer for the constrained problem* (1), *which has the property*

$$(y^0,z^0)\in(f,g)'_u(x^0)\quad \textit{and}\quad z^0\in -K(x^0)\quad \textit{implies}\quad y^0\neq 0. \tag{12}$$

Then x^0 is an i-minimizer of first order for (1).

Proof. Assume on the contrary that x^0 is a p-minimizer for the constrained problem (1) but it is not an i-minimizer. Choose a monotone decreasing sequence $\varepsilon_k\to 0+$. By assumption, there exist sequences $t_k\to 0+$ and u^k, $\|u^k\|=1$, such that $g(x^0+t_ku^k)\in -K$ and (10) holds. We may assume that $0<t_k<r$ and f and g are locally Lipschitz with constant L in $x^0+r\operatorname{cl}B$. Passing to a subsequence we may assume also that $u^k\to u^0$, $\|u^0\|=1$, and that $y^0=\lim_k y^{0,k}$ and $z^0=\lim_k z^{0,k}$. Here $y^{0,k}=(1/t_k)(f(x^0+t_ku^0)-f(x^0))$ and similarly $z^{0,k}=(1/t_k)(g(x^0+t_ku^0)-g(x^0))$. Obviously $(y^0,z^0)\in(f,g)'_{u^0}(x^0)$ and similarly to the general case proof of Theorem 5.1 we have $y^0=\lim_k y^k$ and $z^0=\lim_k z^k$, where $y^k=(1/t_k)(f(x^0+t_ku^k)-f(x^0))$ and $z^k=(1/t_k)(g(x^0+t_ku^k)-g(x^0))$. Further $z^0\in -K(x^0)$, which is true since $\eta\in K'(x^0)$ implies $\langle\eta,z^k\rangle=(1/t_k)\langle\eta,g(x^0+t_ku^k)\rangle\leqslant 0$. Therefore condition (12) gives $y^0\neq 0$. Repeating now the general case proof of Theorem 5.1, we get the contradictory chain of inequalities (11), which proves the thesis. □

As we see from Theorems 5.1 and 5.2, the condition $0\notin f'_u(x^0)$ plays an important role for the implication x^0 *p-minimizer implies that x^0 is an i-minimizer of first order*. However, as next Example 5.1 shows, in the constrained case this condition is not necessary for this implication (while it is in the unconstrained case as Theorem 4.2 shows).

Example 5.1. Consider the constrained problem (1) with $f:\mathbb{R}\to\mathbb{R}$, $f(x)=-x^2$, $C=\mathbb{R}_+$, $g:\mathbb{R}\to\mathbb{R}$, $g(x)=|x|$, $K=\mathbb{R}_+$. The point $x_0=0$ is the only feasible point and according to the definitions in Section 2 it is both a p-minimizer and an i-minimizer of first order. The Dini derivative at x_0 in direction u is $(f,g)'_u(x_0)=(0,|u|)$.

The sufficient condition $\mathcal{S}'_{0,1}$ in Theorem 4.1 involves in fact the condition

$$(0,0)\notin(f,g)'_u(x^0)\quad \text{for all } u\in\mathbb{R}^n\setminus\{0\} \tag{13}$$

(which is weaker than (12)). Indeed, if $(y^0, z^0) = (0, 0) \in (f, g)'_u(x^0)$, then the strong inequality $\langle \xi_0, y^0\rangle + \langle \eta^0, z^0\rangle > 0$ for $(\xi^0, \eta^0) \in C' \times K'(x^0)$ cannot be satisfied. Therefore, it seems natural, for the investigated implication, to apply condition (13), instead of condition (12). The next example shows however, that the conclusion of Theorem 5.2 does not hold, if we replace condition (12) with condition (13).

Example 5.2. Consider problem (1), with $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x^2, -x^2)$, $C = \mathbb{R}^2_+$, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = -|x|$, $K = \mathbb{R}_+$ and let $x_0 = 0$. For $u \in \mathbb{R} \setminus \{0\}$ we have $f'_u(x_0) = (0, 0)$ and $(f, g)'_u(x^0) = (0, 0; -|u|) \neq 0$. Therefore condition (13) holds, but (12) does not. Further $g(x_0) = 0$, whence $K(x_0) = \mathbb{R}_+$. The constraint qualification $\mathcal{Q}_{0,1}(x^0)$ is satisfied, since $g(x_0 + tu) = -t|u| \in -\mathbb{R}_+$ for every $u \in \mathbb{R}$ and $t > 0$. The point x_0 is a p-minimizer, but not an i-minimizer of first order. Therefore, the conclusion of Theorem 5.2 does not hold.

In virtue of Example 5.2, to obtain a result similar to Theorem 5.2 under condition (13) we need a new approach toward the concepts of i-minimizer and p-minimizer. For this purpose, we relate to the constrained problem (1) and the feasible point x^0, the unconstrained problem

$$\min_{C \times K(x^0)} \big(f(x), g(x)\big). \tag{14}$$

Definition 5.1. We say that x^0 is a p-minimizer of order k in sense II (or just p-minimizer in sense II, when $k = 1$) for the constrained problem (1) if x^0 is a p-minimizer of order k for the unconstrained problem (14).

Similarly, we say that x^0 is an isolated minimizer of order k in sense II for the constrained problem (1) if x^0 is an isolated minimizer of order k for the unconstrained problem (14).

We will preserve the names for the concepts used so far, but sometimes we will refer to them as sense I concepts, saying, e.g., p-minimizer in sense I, instead of just p-minimizer.

As an immediate application of Theorem 5.1 we get the following result.

Theorem 5.3. *Let f and g be $C^{0,1}$ functions and let x^0 be a p-minimizer in sense* II *for the constrained problem* (1)*, which has property* (13)*. Then x^0 is an i-minimizer of first order in sense* II *for* (1).

Next, under the hypotheses of Theorem 5.3, we show that x^0 is an i-minimizer in sense I. We state also relations between sense I and sense II, i-minimizers and p-minimizers.

Theorem 5.4. *Let f and g be $C^{0,1}$ functions and let x^0 be a p-minimizer in sense* II *for the constrained problem* (1)*, which has property* (13)*. Then x^0 is an i-minimizer of first order in sense* I *for* (1) *and hence x^0 is also a p-minimizer in sense* I.

Proof. According to Theorem 5.3, x^0 is an i-minimizer of first order for the unconstrained problem (14). The reversal of the Sufficient conditions part of Theorem 4.2 gives a condition, which coincides with the sufficient condition $\mathcal{S}'_{0,1}$ of Theorem 4.1, whence x^0 is an i-minimizer in sense I for the constrained problem (1). Theorem 3.2 gives now that x^0 is also a p-minimizer in sense I for (1). □

Thus, within the set of points satisfying (13), the set of the p-minimizers in sense II is a subset of the p-minimizers in sense I. The reversal does not hold. In fact, the following reasoning shows, that in Example 5.2 the point x_0 is a p-minimizer in sense I, but it is not a p-minimizer in sense II. Now, for the corresponding problem (14) we have

$$(f, g) : \mathbb{R} \to \mathbb{R}^3, \qquad \big(f(x), g(x)\big) = \big(x^2, -x^2, -|x|\big)$$

and $C \times K(x_0) = \mathbb{R}^2_+ \times \mathbb{R}_+ = \mathbb{R}^3_+$. Each point $x \in \mathbb{R}$ is feasible and we have $x^2 \leqslant \varphi(x) \leqslant \sqrt{2}\,x^2$, whence x_0 is an i-minimizer of order 2 in sense II, but it is not an i-minimizer of first order in sense II. Therefore, according to Theorem 5.3, in spite that x_0 is a p-minimizer in sense I, it is not a p-minimizer in sense II (the assumption that x_0 is a p-minimizer in sense II would imply that x_0 is an i-minimizer of first order in sense II).

Let us now make some comparison between Theorems 5.2 and 5.4. In spite that condition (13) is more general than condition (12), Theorem 5.4 does not imply Theorem 5.2. Indeed, the assumption in Theorem 5.4 is that x^0 is a p-minimizer in sense II, which does not imply the more general assumption in Theorem 5.2 that x^0 is a p-minimizer in sense I.

Next we compare the i-minimizers in senses I and II.

Theorem 5.5. *Let f and g be $C^{0,1}$ functions. If x^0 is an i-minimizer of first order in sense* II *for the constrained problem* (1), *then x^0 is an i-minimizer of first order in sense* I *for* (1). *If the constraint qualification $\mathcal{Q}_{0,1}(x^0)$ holds, then also the converse is true.*

Proof. Let x^0 be an i-minimizer of first order in sense II. The reversal of the Sufficient conditions part of Theorem 4.2 gives the sufficient condition $\mathcal{S}'_{0,1}$ of Theorem 4.1, whence x^0 is an i-minimizer in sense I.

Conversely, let x^0 be an i-minimizer of first order in sense I. Under the constraint qualification $\mathcal{Q}_{0,1}(x^0)$, we can apply the reversal of the Sufficient conditions part of Theorem 4.1, getting condition $\mathcal{S}'_{0,1}$, which is identical with the sufficient conditions of Theorem 4.2 applied to problem (14), whence x^0 is an i-minimizer in sense II. □

We conclude the paper with the following remark. The comparison of the p-minimizers and the i-minimizers has been a motivation to "duplicate" the notions of optimality introducing sense II concepts. As we have shown, sense II concepts are related to the usual concepts of optimality and they are simpler to some extent, since they are defined through an unconstrained problem. The complexity of the vector optimization problems has caused the appearance of many notions of optimality. Each of them stresses a particular quality of the minimizer. The p-minimizers enjoy stability properties as it is shown in Benson and Morin [29], Podinovskiy and Nogin [30] and Miglierina and Molho [18]. For scalar problems Auslender [5] shows that the isolated minimizers also obey some stability properties.

The study of the stability properties of the p-minimizers and i-minimizers of order k on one hand, and of sense I and sense II concepts on the other hand, is in our opinion an interesting subject for research. One can observe here a qualitative advantage of sense II concepts, which gives for them an additional "right for existence." Namely, sense I p-minimizers and i-minimizers obey stability with respect to the objective data, while sense II concepts obey stability with respect to both objective and constraint data. We intend to demonstrate this in a separate issue.

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