SPECTRAL AND STOCHASTIC PROPERTIES OF THE f-LAPLACIAN, SOLUTIONS OF PDE'S AT INFINITY AND GEOMETRIC APPLICATIONS

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ABSTRACT. The aim of this paper is to suggest a new viewpoint to study qualitative properties of solutions of semilinear elliptic PDE's defined outside a compact set. The relevant tools come from spectral theory and from a combination of stochastic properties of the relevant differential operators. Possible links between spectral and stochastic properties are analyzed in detail.

CONTENTS

INTRODUCTION

In this paper we suggest a new perspective to study qualitative properties of solutions of semilinear elliptic PDE's, especially when these are defined only outside a compact set. In order to enlarge the range of applicability of the techniques, we decide to place our treatment in the setting of weighted Riemannian manifolds and corresponding drifted Laplacians.

The germ of the present investigation is contained in the very recent paper [20] which is devoted to a systematic treatment of the Feller property of a Riemannian manifold. In fact, using a suitable comparison theory, we shall show how (weighted) manifolds which are both stochastically complete and Feller represent a natural framework where solutions of PDE's at infinity can be studied. The fact that transience and stochastic completeness of the underlying manifolds have PDE's counterpart is well understood. However, due to the nature of these stochastic properties, so far only global solutions have been considered. The introduction of the Feller property in combination with the stochastic completeness, will enable us to get important information even in the case of solutions at infinity.

Manifolds which are both stochastically complete and Feller belong to a pretty wide class containing complete Ricci solitons, complete manifolds with controlled Ricci tensor and Cartan-Hadamard manifolds with at most quadratic exponential volume growth. The usefulness of the technique is visible in the geometric applications which include submanifold theory and the Yamabe problem. It is well known that the spectral theory of diffusion operators is affected by stochastic properties of the corresponding diffusion process. For instance, using capacitary arguments, it is readily seen that recurrence forces the bottom of the spectrum of the diffusion operator to be zero. Recent works [4, 13] have emphasized an intriguing link between the stochastic completeness of certain specific manifolds and the essential spectrum of the operator. We shall analyze in more details possible relations between stochastic and spectral properties of weighted manifolds. On the route we will prove a generalized and abstract version of the discreteness of the spectrum of bounded minimal submanifolds recently obtained in [4]. The nature of the essential spectrum also suggests that it could fit in very well in the main topic of the paper. Indeed, by the decomposition principle, the bottom of the essential spectrum is sensitive only on the geometry at infinity of the underlying manifold and Barta's classical characterization leads naturally to solutions at infinity of PDE's.

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1. Notation

Throughout this note, we shall always use the symbol M_f to denote the m-dimensional weighted manifold

$$M_f = (M, \langle, \rangle, d\mathrm{vol}_f),$$

where $(M, \langle, \rangle)$ is a Riemannian manifold, $f : M \to \mathbb{R}$ is a selected smooth function on M, $d\mathrm{vol}$ denotes the Riemannian measure of $(M, \langle, \rangle)$ and, finally, $d\mathrm{vol}_f = e^{-f} d\mathrm{vol}$ is the weighted measure. The f-Laplacian associated to the

weighted manifold M_f is the operator

$$\Delta_f u = \operatorname{div}_f (\nabla u) := e^f \operatorname{div} \left(e^{-f}\nabla u\right),$$

which is symmetric on $L^2 (M, d\mathrm{vol}_f)$.

The Bakry-Emery Ricci curvature of the weighted manifold M_f is the 2-tensor

$$Ric_f = Ric + \operatorname{Hess}(f).$$

In case

$$Ric_f = \lambda \langle, \rangle,$$

for some constant $\lambda \in \mathbb{R}$, then the weighted manifold M_f is called a Ricci soliton. The Ricci soliton M_f is said to be shrinking, steady or expanding according to the fact that $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

2. STOCHASTIC COMPLETENESS AND MAXIMUM PRINCIPLE

We say that the weak maximum principle at infinity holds on a weighted manifold M_f if for every $u \in C^2 (M)$, with $\sup_M u = u^* < +\infty$, there exists a sequence $\{x_k\}$ along which

$$\text{(i) } u (x_k) > u^* - \frac{1}{k}, \quad \text{(ii) } \Delta_f u (x_k) < \frac{1}{k}.$$

It is known, [17], [18], that this principle is equivalent to the stochastic completeness of the diffusion process associated to Δ_f. This means that the heat kernel of the f-Laplacian $p_f (x, y, t)$ satisfies the conservation property

$$\int_M p_f (x, y, t) \, d\mathrm{vol}_f (y) = 1,$$

for every $x \in M$ and $t > 0$. The f-stochastic completeness, in turn, is implied by the volume condition

$$\frac{R}{\log \mathrm{vol}_f (B_R)} \notin L^1 (+\infty).$$

Accordingly, since for a complete weighted manifold M_f satisfying $Ric_f \geq \lambda$ we have, for constants A, B, C, the Qian-Wei-Wylie volume estimate [23], [27],

$$(1) \qquad \mathrm{vol}_f (B_R) \leq A + B \int_{R_0}^R e^{-\lambda t^2 + Ct} dt, \quad R >> 1.$$

It follows then that every complete gradient Ricci soliton is f-stochastically complete, therefore, enjoys the weak maximum principle for the f-Laplacian.

In fact, using estimates for the potential function due to Z.-H. Zhang, [28], that will be described in Section 4.1 below, and a general result contained in [19], one proves that on every gradient Ricci soliton, the full Omori-Yau

maximum principle holds, both for the f-Laplacian and for the ordinary Laplacian. Namely,

Theorem 1. *Let $(M, \langle , \rangle)$ be any gradient Ricci soliton. Then for every function $u \in C^2(M)$ bounded above, there exists a sequence $\{x_k\}$ such that $u(x_n) \to u^*$, $|\nabla u(x_k)| < 1/k$ and $\Delta_f u(x_k) < 1/k$ (resp. $\Delta u(x_k) < 1/k$).*

We point out that the case of shrinking solitons was recently obtained by M. Fernández-López and E. García-Río, [10].

3. The Feller property

It is known, see [17], that the weak maximum principle at infinity for the f-Laplacian is a powerful tool to deduce qualitative information on the solutions of differential inequalities of the form

$$(2) \qquad\qquad \Delta_f u \geq \Lambda(u).$$

Accordingly, every bounded above solution u of (2), on the whole manifold M, satisfies

$$\Lambda(u^*) \leq 0.$$

This fact has many applications in geometric analysis. Our aim is now to investigate qualitative properties of solutions of (2) which are defined only in a neighborhood of infinity. This requires the introduction of new tools that can be developed under the validity of a further stochastic property of the underlying manifold, namely, the Feller property.

Note that if M is stochastically complete for Δ_f, then a bounded solution $u > 0$ of the differential inequality

$$\Delta_f u \geq \lambda u$$

outside a smooth domain $\Omega \subset\subset M$ satisfies

$$u(x) \leq c \cdot h(x), \text{ on } M \backslash \Omega,$$

where $c > 0$ is a suitable constant and $h > 0$ is the minimal solution of the problem

$$\begin{cases} \Delta_f h &= \lambda \cdot h & \text{in } M \backslash \overline{\Omega} \\ h &= 1 & \text{on } \partial\Omega, \end{cases}$$

(which is constructed by means of an exhaustion procedure).

Indeed, let $c = \sup_{\partial\Omega} u$. Then, for every $\epsilon > 0$,

$$\Delta_f(u - c \cdot h - \epsilon) \geq \lambda(u - c \cdot h) \geq \lambda(u - c \cdot h - \epsilon) \text{ on } M \setminus \overline{\Omega}$$

and $u - c \cdot h - \epsilon \leq -\epsilon$ on $\partial\Omega$. Therefore the function $v_\epsilon = \max\{0, u - c \cdot h - \epsilon\}$ is bounded, non-negative and satisfies $\Delta_f v_\epsilon \geq \lambda v_\epsilon$. Since M is stochastically complete with respect to Δ_f, $v_\epsilon \equiv 0$, that is, $u \leq ch + \epsilon$, and the conclusion follows letting $\epsilon \to 0$.

In particular, if $h(x) \to 0$ as $x \to \infty$, we can deduce that the same holds for the original function u.

According to a characterization by R. Azencott [1], it happens that the required decay property of h is equivalent to the Feller property on M with respect to Δ_f, that is, that the heat semigroup generated by $-\Delta_f$ maps the space $C_o(M)$ of continuous functions vanishing at infinity into itself, or equivalently, that for every relatively compact open set Ω in M, the heat kernel p_f of $-\Delta_f$ satisfies

$$\int_\Omega p_f(x, y, t) d\mathrm{vol}_f(y) \to 0 \text{ as } x \to +\infty.$$

We thus obtain the following result.

Theorem 2. *Let M_f be f-stochastically complete. If M_f is Feller, then every bounded solution $v > 0$ of*

$$\Delta_f v \geq \lambda v, \text{ on } M \backslash \overline{\Omega}$$

satisfies

$$v(x) \to 0, \text{ as } x \to \infty.$$

On the basis of these observations, we prove the following theorem.

Theorem 3. *Let M_f be a stochastically complete and Feller manifold for Δ_f. Consider the differential inequality*

$$(3) \qquad \Delta_f u \geq \Lambda(u), \text{ on } M \backslash \Omega,$$

where $\Omega \subset\subset M$ and $\Lambda : [0, +\infty) \to [0, +\infty)$ is either continuous or it is non-decreasing function which satisfies the following conditions:

$$(a)\ \Lambda(0) = 0;\ (b)\ \Lambda(t) > 0,\ \forall t > 0;\ (c)\ \liminf_{t \to 0+} \frac{\Lambda(t)}{t^\xi} > 0,$$

for some $0 \leq \xi \leq 1$. Then, every bounded solution $u > 0$ of (3) satisfies

$$\lim_{x \to \infty} u(x) = 0.$$

Proof. Let us consider the case where f is not continuous, the other case is easier. By assumption, there exists $0 < \varepsilon < 1/2$ and $c > 0$ such that

$$\Lambda(t) \geq c\, t^\xi, \text{ on } (0, 2\varepsilon).$$

Since

$$t^\xi \geq t, \text{ on } (0, 1],$$

and Λ is non-decreasing, then

$$\Lambda(u(x)) \geq \Lambda_\varepsilon(u(x)) = \begin{cases} c\, u, & \text{if } u(x) < \varepsilon \\ c\, \varepsilon, & \text{if } u(x) \geq \varepsilon. \end{cases}$$

On the other hand, since $u > 0$ is bounded, if we set $u^* = \sup_{M \setminus \Omega} u$, then

$$c\varepsilon \geq \frac{c\,\varepsilon}{u^*} u^* \geq \frac{c\,\varepsilon}{u^*} u.$$

It follows that

$$\Delta_f u \geq \Lambda_\varepsilon (u) \geq \lambda u,$$

where

$$\lambda = c \min \left\{ 1, \frac{\varepsilon}{u^*} \right\} > 0.$$

Using the Feller property we now conclude that $u(x) \to 0$, as $x \to \infty$.   □

4. Estimates for the f-Laplacian of the distance function and Comparison results

Because of comparison arguments and radialization techniques, many of the properties of solutions of differential in(equalities) involving the f-Laplacian, and in particular, the stochastic properties of Δ_f may be deduced imposing suitable bounds on $\Delta_f r$ where $r(x)$ denotes the distance function from a reference point $o \in M$. We are going to collect some results along these lines concerning stochastic completeness, the Feller property, the full Omori-Yau maximum principle and the compact support principle. Items (i) and (ii) in the next theorem are weighted versions of Corollary 15.2 (c) and (d) in [12].

Theorem 4. *Let M_f be a weighted Riemannian manifold, and let $r(x)$ be the distance function from a fixed point $o \in M$. Let also g be a C^2 odd function on $\mathbb{R}$ satisfying $g(0) = 0$, $g'(0) = 1$ and $g(t) > 0$ for all $t > 0$.*
(i) Assume that there exist $R_0 \geq 0$ and constant $n > 1$ such that for every x within the cut locus of o with $r(x) > R_0$ we have

$$(4) \qquad \Delta_f r(x) \leq (n-1) \frac{g'}{g}(r(x)) \;\; with \;\; \frac{\int_0^r g(t)^{n-1} dt}{g(r)^{n-1}} \notin L^1(+\infty),$$

then M_f is stochastically complete.
(ii) Assume that o is a pole and that there exist $R_0 \geq 0$ and a constant $n > 1$ such that, for $r(x) > R_0$, we have

$$(5) \qquad \Delta_f r(x) \geq (n-1) \frac{g'}{g}(r(x)) \;\; with \;\; \frac{\int_0^r g(t)^{n-1} dt}{g((r)^{n-1}} \in L^1(+\infty),$$

then M_f is not stochastically complete.
(iii) Assume that o is a pole, that there exist $R_0 \geq 0$ and a constant $n > 1$ such that for $r(x) > R_0$ we have

$$(6) \qquad \qquad \Delta_f r(x) \geq (n-1) \frac{g'}{g}(r(x))$$

and that either

(7)
$$\frac{1}{g^{n-1}(r)} \in L^1(+\infty)$$

or

(8) *(a)* $\dfrac{1}{g^{n-1}(r)} \notin L^1(+\infty)$ *and*   *(b)* $\dfrac{\int_r^{+\infty} g^{n-1}(t)\, dt}{g^{n-1}(r)} \notin L^1(+\infty)$,

then M_f is Feller.

(iv) Assume there exist $R_0 \geq 0$ and a constant $n > 1$ such that if x is within the cut locus of o with $r(x) > R_0$ we have

(9)
$$\Delta_f r(x) \leq (n-1)\frac{g'}{g}(r(x))$$

and that g does not satisfy the conditions in (iii). Then M_f is not Feller.

Proof. We outline the proof, which follows the lines of arguments valid for the ordinary Laplacian. To prove (i) and (ii), let $\alpha(r)$ be the function defined by

$$\alpha(r) = \int_0^r \frac{\int_0^t g(s)^{n-1} ds}{g((t)^{n-1}}\, dt$$

and note that

$$\alpha'(r) > 0 \qquad \alpha''(r) + (n-1)\frac{g'}{g}\alpha'(r) = 1.$$

Let also $v(x) = \alpha(r(x))$, so that v is C^2 within the cut locus of o and there

$$\Delta_f v(x) = \alpha''(r(x)) + \Delta_f r(x)\alpha'(r(x)).$$

Now, in case (ii) v is C^2 and bounded above on M, and since $\alpha' > 0$ it does not attain a maximum. Thus, if R is large enough that (5) holds in B_R^c and $\gamma < u^*$ is sufficiently close to u^* that $\Omega_\gamma = \{x : u(x) > \gamma\} \subset B_R^c$, then, in Ω_γ

$$\Delta_f v \geq \alpha''(r(x)) + (n-1)\frac{g'}{g}\alpha'(r(x)) = 1,$$

so that v violates the weak maximum principle at infinity, and M_f is not stochastically complete.

To prove (i), let u be C^2 and bounded above on M. We claim that for every $\gamma < u^*$, $\inf_{\Omega_\gamma} \Delta_f u \leq 0$ which clearly implies that the weak maximum principle at infinity holds for Δ_f and M_f is stochastically complete.

Note that in the present situation v tends to infinity as $r(x) \to \infty$ and satisfies

$$\Delta_f v \leq \alpha'' + (n-1)\frac{g'}{g}\alpha' \leq 1$$

for x within the cut locus of o and such that $r(x) > R_0$.

Assume by contradiction that there exists $\gamma < u^*$ such that

$$\inf_{\Omega_c} \Delta_f u \geq 2c > 0.$$

Clearly u does not attain its supremum, and, by taking γ close enough to u^*, we may arrange that (4) holds on Ω_γ. Let $x_o \in \Omega_\gamma$ and choose $0 < \delta < c$ small enough that the function $\tilde{u} = u - \gamma - \delta v$ is positive at x_o. Since $\tilde{u} < 0$ on Ω_γ, and tends to $-\infty$ as $r(x) \to +\infty$, it attains a positive maximum at $\bar{x} \in \Omega_\gamma$, and using the Calabi trick we may assume that $r(x)$ be smooth at $\bar{x}$. Then, at $\bar{x}$,

$$\Delta_f \tilde{u}(\bar{x}) = \Delta_f u(\bar{x}) - \delta \Delta_f v(\bar{x}) \geq 2c - \delta \geq c > 0,$$

which yields the required contradiction.

We now come to the Feller property. In case (iii), the conditions satisfied by g imply that the model manifold M_g^n defined as $\mathbb{R}^n$ endowed with the metric

$$\langle , \rangle = dr^2 + g(r)^2 \, d\theta^2,$$

is Feller (with respect to the ordinary Laplacian), and, according to [20] Theorem 4.4 and Lemma 5.1, the radial minimal solution β of the exterior boundary value problem

(10)
$$\begin{cases} \beta'' + (n-1)\dfrac{g'}{g}\beta' = \lambda\beta \\ \beta(R_0) = 1 \end{cases}$$

tends to 0 as r tends to infinity and satisfies $h'(r) < 0$. As above, define $v(x) = \beta(r(x)$, so that $v(x) \to 0$ as $r(x) \to \infty$ and

$$\Delta_f v = \beta'' + \Delta_f r \beta' \leq \beta'' + (n-1)\frac{g'}{g}\beta' = \lambda v.$$

For every $R > R_0$, let h_R be the solution of the exterior boundary value problem

$$\begin{cases} \Delta_f h_R = \lambda h_R \\ h_R(R_0) = 1, \ h_R(R) = 1, \end{cases}$$

and note that by the comparison principle $h_R < v$ in $B_R \setminus B_{R_0}$. As $R \to +\infty$, h_R tends to the minimal solution h of the problem

(11)
$$\begin{cases} \Delta_f h = \lambda h \\ h(R_0) = 1. \end{cases}$$

Since clearly $0 < h < v$ in $M \setminus B_{R_0}$, h tends to 0 as $r(x)$ tends to ∞ and M_f is Feller.

Finally, assume that (9) holds and that g does not satisfy the conditions in (iii) hold. Note that in particular, $g^{-n+1} \notin L^1(+\infty)$ while $g^{n-1} \in L^1(+\infty)$, and according to i) M_f is stochastically complete. By [20] Theorem 4.4, the

model manifold M_g^n is not Feller, and therefore the minimal radial solution β of the exterior problem (10) does not tend to zero as $r \to +\infty$. Since $\beta' \leq 1$ by [20] Corollary 5.1, it follows that the function $v(x) = \beta(r(x))$ is a bounded solution of

$$\Delta_f v \geq \lambda v$$

which satisfies $v = 1$ on R_0 and which does not tend to zero at infinity. It follows from the discussion preceding Theorem 2 that the minimal solution of the exterior problem (11) satisfies $v \leq c \cdot h$ for some constant $c > 0$, so that h does not tend to zero at infinity and M_f is not Feller. $\qquad\square$

Remark 5. In the case that n is an integer, the differential inequalities satisfied by $\Delta_f r$ can be interpreted as comparison with the Laplacian of the distance function of model M_g^n. It is also interesting to observe that in the case of the Feller property the inequalities assumed for $\Delta_f r$ go in the opposite direction than those assumed in the case of stochastic completeness.

In the case of the ordinary Laplacian, upper and lower estimates for Δr may be obtained via the Laplacian comparison theorem imposing lower bounds on the Ricci curvature, or upper bounds on the sectional curvature, respectively. In the case of the f-Laplacian, there does not seem to be an analogue of the sectional curvature whose control allows to obtain lower estimates for $\Delta_f r$. As for upper estimates, the most effective way to obtain upper bounds for $\Delta_f r$ is to impose lower bounds on the modified Bakry-Emery Ricci tensor

$$Ric_\alpha = Ric + Hess\, f - \frac{1}{\alpha} df \otimes df$$

with $\alpha > 0$, but in view of applications to Ricci solitons it is important to try and obtain estimates for $\Delta_f r$ assuming lower bound on the Bakry-Emery Ricci tensor Ric_f corresponding to $\alpha = +\infty$, together with some control on the weight f and or its gradient.

Indeed, it was shown by Qian, [23] Theorem 2.1, in the more general case of operators of the form $\Delta + X$ with a drift which is not necessarily a gradient, that upper estimates for $\Delta_f r$ follow from imposing lower bounds on Ric_f and a control on the drift term X. More precisely we have:

Theorem 6. *Let M_f be a weighted manifold, let o be a reference point in M and let $r(x) = d(x, o)$ be the Riemannian distance function from o.*
(i) Assume that

$$\mathrm{Ric}_f \geq -k^2$$

for some constant $k \geq 0$, then there exists a constant C depending only on M, on ∇f and on o such that

$$\Delta_f r(x) \leq C + \frac{m-1}{r(x)} + k^2 \cdot r(x) \quad on \quad M \setminus \mathrm{cut}(o).$$

(ii) Assume that

$$\begin{cases} \mathrm{Ric}_f(x) \geq -k_1^2(r(x)) \\ |\nabla f|(x) \leq k_2(r(x)), \end{cases}$$

where $k_i(r)$ are continuous non-decreasing functions satisfying $k_i(r) \to +\infty$ as $r \to +\infty$. Then

$$\Delta_f r(x) \leq m \frac{g'(r(x))}{g(r(x))} \quad \text{on} \quad M \setminus \mathrm{cut}(o),$$

where $g : [0, +\infty) \to [0, +\infty)$ is the solution of the initial value problem

$$(12) \qquad \begin{cases} g''(r) - \dfrac{k_1(r)^2 + k_2(r)^2}{m} g(r) = 0 \\ g(0) = 0, \ g'(0) = 1. \end{cases}$$

(iii) Assume that there are nonnegative constants k and C such that

$$\begin{cases} \mathrm{Ric}_f \geq -k \\ |\nabla f| \leq C(d(x,o) + 1) \quad \forall x \in M. \end{cases}$$

Then for every $p \in M$, and every $x \in M \setminus (\{p\} \cup \mathrm{cut}(p))$, if $\rho(x) = d(x,p)$, then

$$\Delta_f \rho(x) \leq \frac{m-1}{\rho} + \frac{1}{3}(k + 2C)\rho + C(1 + d(o,p)).$$

$$\square$$

Applying Theorem 4 we obtain a versions for the Δ_f of results by Qian, [23], Theorems 1.5 and 1.6, on the stochastic completeness of Laplacians with drift. We point out that the proof we present is entirely deterministic.

Theorem 7. *Let M_f be a complete weighted Riemannian manifold, and assume that either*

$$(13) \qquad \mathrm{Ric}_f \geq -k^2 \quad \text{for some constant} \ \ k \geq 0,$$

or

$$(14) \qquad \begin{cases} \mathrm{Ric}_f(x) \geq -k_1^2(r(x)) \\ |\nabla f|(x) \leq k_2(r(x)), \end{cases}$$

where $k_i(r)$ are continuous non-decreasing functions satisfying $k_i(r) \to +\infty$ as $r \to +\infty$ and

$$(15) \qquad \frac{1}{\sqrt{k_1^2(t) + k_2^2(t)}} \notin L^1(+\infty).$$

Then M_f is stochastically complete for Δ_f.

Proof. If (13) holds, then $\Delta_f r$ satisfies the estimate (i) in Theorem 6, and it is easily seen that this implies (4) with $n = m$ and

$$g(r) = r \exp\Big[\frac{1}{m-1}\big(Cr + \frac{k}{2}r^2\big)\Big],$$

and it is clear that $g(r)$ satisfies the non-integrability in (4), and M_f is stochastically complete by Theorem 4 (i).

Assuming that (14) and (15) hold, then $\Delta_f r(x)$ satisfies the estimate in Theorem 6 (ii) with g solution of (12). This implies that $\psi = g'/g$ satisfies the Riccati equation

$$\psi' + \psi^2 = k^2, \quad \text{with } k(r)^2 = \frac{k_1(r)^2 + k_2(r)^2}{m},$$

and $\psi(r) = \dfrac{1}{r} + O(1)$ as $r \to 0$. Thus $\psi' < 0$ whenever $\psi > k$ and since $k(r)$ is increasing and tends to $+\infty$ as $r \to +\infty$, standard arguments show that there exists r_o such that $\psi(r_o) = k(r_o)$ and $\psi(r) \le k(r)$ for all $r \ge r_o$.

This in turn implies that there exists R sufficiently large and a constant $C > 0$ such that

$$\frac{\int_0^r g(t)^m dt}{g(r)^m} \ge \frac{C}{k(r)} \quad \text{if } r \ge R.$$

Indeed, $g(r) \to +\infty$ as $r \to +\infty$ and, using de l'Hospital's rule, and the fact that k is non-decreasing we have

$$\liminf_{r \to \infty} \frac{k(r) \int_0^r g(t)^m dt}{g(r)^m} \ge \liminf_{r \to \infty} \frac{k(r)g(r)}{mg'(r)} \ge \frac{1}{m}.$$

By (15), $1/k(r) \notin L^1(+\infty)$ and, again by Theorem 4 (i), M_f is stochastically complete. $\qquad\square$

We stress that, in order to deduce the validity of the Feller property using Theorem 4 (iii), one needs a control from below on $\Delta_f r$, while lower bounds on Ric_f typically produce upper estimates on $\Delta_f r$.

Using a probabilistic technique which extends to the Laplacian with a drift term a previous result of P. Hsu, [14] for Δ, and which essentially consists in genuine estimates on hitting time probabilities, Qian [23, Theorem 1.7] proves the following theorem. Note that, again, Qian's result actually applies to the more general operators of the form $L = \Delta + X$.

Theorem 8. *Let M_f be a complete weighted manifold and assume that for some $o \in M$ the conclusion of Theorem 6 (iii) holds, namely, there exist constants k and $C >$ such that, for every $p \in M$, if $\rho(x) = d(x, p)$ denotes the distance function from p, we have*

$$\Delta_f \rho(x) \le \frac{m-1}{\rho} + \frac{1}{3}(k + 2C)\rho + C(1 + d(o, p)).$$

Then M_f is Feller with respect to Δ_f. In particular, if Ric_f and ∇f satisfy the estimates in Theorem 6 (iii), then M_f is Feller with respect to Δ_f.

We next describe a result which extends a condition on the validity of the full Omori-Yau maximum principle for the operator Δ_f proved in [19]. The argument we are going to use is an adaptation of a recent elegant proof of the Omori-Yau maximum principle due to A. Borbély, [5] and [6]. We are grateful to A. Borbély for sending us a copy of [6].

Theorem 9. *Let M_f be a complete weighted manifold and assume that there exists a non-negative C^2 function γ satisfying the following conditions*

(16) $\gamma(x) \to +\infty$ *as* $x \to \infty$

(17) $\exists A > 0$ *such that* $|\nabla\gamma| \leq A$ *off a compact set*

(18) $\exists B > 0$ *such that* $\Delta_f \gamma \leq BG(\gamma)$ *off a compact set*

where G is a smooth function on $[0, +\infty)$ satisfying

(19) *(i)* $G(0) > 0$ *(ii)* $G'(t) \geq 0$ *on* $[0, +\infty)$ *(iii)* $G(t)^{-1} \notin L^1(+\infty)$.

*Then the Omori-Yau maximum principle for Δ_f holds. The same conclusion holds if $\gamma(r) = r(x)$ is the distance function from a reference point o and we assume (**??**) and that inequality (18) holds in the complement of the cut locus of o (while of course (16) and (17) are automatically satisfied).*

Proof. We outline the proof which follows closely Borbely's argument.

Let u be a C^2 function such that $u^* = \sup_M u < +\infty$. We want to show that there exists a sequence $\{x_n\}$ such that $u(x_n) \to u^*$, $\lim_n |\nabla u(x_n)| = 0$ and $\limsup_n \Delta_f u(x_n) \leq 0$. We may assume that u does not attain its supremum for otherwise the conclusion is obvious. Define

$$F(t) = \exp\{\int_0^t G(s)^{-1} ds\},$$

so that F is non-decreasing and $\lim_{t \to +\infty} F(t) = +\infty$. For every t, let $\Omega_t = \{x : \gamma(x) > t\}$. Since γ is proper, there exists t_o such that inequalities (17) and (18) hold in the set Ω_{t_o}. Let $\epsilon < \min\{1, u^* - \sup_{\Omega_{t_o}^c} u\}$, and define $h_\lambda(x) = \lambda F(\gamma(x)) + u^* - \epsilon$. Since $F \geq 1$, if $\lambda > \epsilon$ then $h_\lambda > u^* > u$ on M. Let $\lambda_o = \inf\{\lambda : h_\lambda(x) > u(x) \, \forall x \in M\}$, and note that since $u < u^*$ on M then $\lambda_o > 0$, and, by continuity, $h_{\lambda_o}(x) \geq u(x)$ for every x in M.

We claim that there exists x_ϵ such that $h_{\lambda_o}(x_\epsilon) = u(x_\epsilon)$. Note that since $h_{\lambda_o} > u^* - \epsilon > \sup_{\Omega_{t_o}^c} u$, x_ϵ lies necessarily in Ω_{t_o}.

To prove the claim, we will show that if $h_\lambda > u$ on M for some $\lambda > 0$ then there exists $\lambda' < \lambda$ such that $h_{\lambda'} > u$ on M. Indeed, since $F(\gamma) \to +\infty$ as $\gamma \to +\infty$, and γ is proper, there exists $t_1 > t_o$ such that $h_\lambda > u^* + 1$ in Ω_{t_1}. Since $\Omega_{t_1}^c$ is compact, and $h_\lambda(x) > u(x)$ on M, we may choose $\lambda' < \lambda$ sufficiently close to λ so that $h_{\lambda'} > u$ on $\Omega_{t_1}^c$, and $h_{\lambda'} > u^*$ on $\partial\Omega_{t_1}$. Since F is increasing, $h_{\lambda'} > u^*$ in Ω_{t_1} and therefore $h_{\lambda'} > u$ on M, as required.

Next we claim that h_{λ_o} is smooth at x_ϵ. This is clear if γ is C^2 on M, while if $\gamma(x) = r(x)$ is the Riemannian distance function, the proof in Borbely's paper, which only uses the fact that $u - h_{\lambda_o}$ attains a maximum at x_ϵ, and properties of the function $r(x)$ applies without changes.

Thus, since $u(x_\epsilon) = h_{\lambda_o}(x_\epsilon) = \lambda_o F(\gamma(x_\epsilon)) + u^* - \epsilon < u^* = \sup u$ we have

$$u(x_\epsilon) > u^* - \epsilon \quad \text{and} \quad \lambda_o F(\gamma(x_\epsilon)) < \epsilon.$$

Also, since $u - h_{\lambda_o}$ attains a maximum at x_ϵ,

$(i)\ \nabla u(x_\epsilon) = \lambda_o F'(\gamma(x_\epsilon)) \nabla \gamma(x_\epsilon)$

$(ii)\ \Delta_f u(x_\epsilon) \leq \Delta_f h_{\lambda_o}(x_\epsilon) = \lambda_o [F''(\gamma(x_\epsilon))|\nabla \gamma(x_\epsilon)|^2 + F'(\gamma(x_\epsilon))\Delta_f \gamma(x_\epsilon)].$

Easy computations show that $F' = F/G$ and $F'' \leq F/G^2$ so that (i) above and (17) yield

$$|\nabla u(x_\epsilon)| \leq A\lambda_o \frac{F(\gamma(x_\epsilon))}{G(\gamma(x_\epsilon))} < \frac{A}{G(0)}\epsilon,$$

while using (ii) and (18) we get

$$\Delta_f u(x_\epsilon) \leq \lambda_o F(\gamma(x_\epsilon))[\frac{1}{G^2(\gamma(x_\epsilon))} + \frac{\Delta_f u(x_\epsilon)}{G(\gamma(x_\epsilon))}] \leq (1+B)\lambda_o F(x_\epsilon) < (1+B)\epsilon.$$

$\square$

We conclude this section with a brief discussion of the compact support principle for the operator Δ_f (see, [21] [22], where more general quasilinear elliptic operators in divergence form are considered).

A function u is said to be a semiclassical solution of the differential inequality

(20) $$\Delta_f u \geq \lambda(u)$$

in a domain Ω if $u \in C^1(\Omega)$ and u satisfies the inequality in weak sense, that is

$$\int_\Omega [\langle \nabla f, \nabla \phi \rangle + \lambda(u)\phi] \leq 0$$

for every non-negative $\phi \in C_c^1(\Omega)$.

The compact support principle is said to hold for the differential inequality (20) if, whenever Ω is an exterior domain, namely $\Omega \supset M \setminus B_R(o)$ for some $R > 0$, and $u \geq 0$ is a semiclassical solution of (20) in Ω with the property that $u(x) \to 0$ as $r(x) \to +\infty$ that u vanishes identically outside a compact set.

We are grateful to M. Rigoli for pointing that the validity of the compact support principle depends essentially on the properties of the operator and of the function λ, and that the effect of the geometry is once again encoded by the behavior of $\Delta_f r(x)$.

Indeed, the proof of Theorem 1.1 in [22], may be adapted with minor changes to obtain the following

Theorem 10. *Let M_f be a complete weighted manifold, and let λ be a continuous function on $[0,+\infty)$ which vanishes in 0 and is non-decreasing and strictly positive on some interval $(0,\delta)$, $\delta > 0$. Set $\Lambda(t) = \int_0^t \lambda(s)ds$. If*

$$\tag{21} \frac{1}{\sqrt{\Lambda(s)}} \in L^1(0+),$$

and there exists $C > 0$ such that the differential inequality

$$\tag{22} \Delta_f r(x) \geq -C^2 > -\infty$$

holds weakly on M, then the compact support principle holds for the differential inequality (20).

As an immediate consequence, as noted in [22], if (21) holds, $\Delta_f = \Delta$, and M is a Cartan-Hadamard manifold, then the compact support principle holds for (20).

4.1. **Feller property on Ricci solitons.** In this section we are going to prove that Ricci solitons are Feller with respect to the weighted Laplacian Δ_f and with respect to the ordinary Laplacian Δ.

The fact that for every gradient Ricci soliton the weighted Laplacian Δ_f is Feller is then a consequence of the results of the previous section and of the following estimates for the gradient of the potential function which have been obtained by Zhang, [28] (see also H.-D. Cao and D. Zhou, [9]). These in turn depend on lower estimates for the scalar curvature of Ricci solitons obtained by Zhang, and by a different method in [18]. We are going to briefly describe the latter approach.

We recall from [18] the following "a-priori" estimate for weak solutions of semi-linear elliptic inequalities under volume assumptions (see also [16], Theorem B).

Theorem 11. *Let $\left(M, \langle,\rangle, e^{-f}dvol\right)$ be a complete, weighted manifold. Let $a(x)$, $b(x) \in C^0(M)$, set $a_-(x) = \max\{-a(x), 0\}$ and assume that*

$$\sup_M a_-(x) < +\infty$$

and

$$b(x) \geq \frac{1}{Q(r(x))} \ \text{on} \ M,$$

for some positive, non-decreasing function $Q(t)$ such that $Q(t) = o(t^2)$, as $t \to +\infty$. Assume furthermore that, for some $H > 0$,

$$\frac{a_-(x)}{b(x)} \leq H, \ \text{on} \ M.$$

Let $u \in Lip_{loc}(M)$ be a non-negative solution of

$$\tag{23} \Delta_f u \geq a(x)u + b(x)u^\sigma,$$

weakly on $\left(M, e^{-f} d\mathrm{vol}\right)$, *with* $\sigma > 1$. *If*

(24) $$\liminf_{r \to +\infty} \frac{Q\left(r\right) \log \mathrm{vol}_f\left(B_r\right)}{r^2} < +\infty,$$

then

$$u\left(x\right) \le H^{\frac{1}{\sigma-1}}, \ \text{on } M.$$

Using the volume estimate (1) we deduce the validity of the next

Corollary 12. *Let* $(M, \langle,\rangle, \nabla f)$ *be a complete Ricci soliton. Then the scalar curvature S of M satisfies the lower estimate*

(25) $$S(x) \ge \begin{cases} 0 & \text{if } \lambda \ge 0 \\ m\lambda & \text{if } \lambda < 0 \end{cases}$$

Proof. Indeed,

$$\Delta_f S = \lambda S - |Ric|^2.$$

and since, by the Cauchy-Schwarz inequality $|\mathrm{Ric}|^2 \ge \frac{1}{m} S^2$, we have that $S_-(x) = \min\{0, -S(x)\}$ is a weak solution of the differential inequality

$$\Delta_f S_- \ge \lambda S_- + \frac{1}{m} S_-^2.$$

On the other hand, by (1), the condition (24) above is satisfied, and an application of Theorem 11 with $a(x) = \lambda$ and $b(x) = 1/m$ shows that

$$S_- \le m\lambda_-$$

and the conclusion follows. $\qquad\qquad\square$

Using these estimates into the basic equation

(26) $$S + |\nabla f|^2 - 2\lambda f = C$$

and integrating along minimizing geodesics one obtains the estimates for the potential function and its gradient described in [28] and [9]:

Lemma 13. *Let* $(M, \langle,\rangle, \nabla f)$ *be a complete Ricci soliton. Then there exist positive constants a and b depending only on the soliton such that*

(27) $$|\nabla f| \le b + |\lambda| d(x, o) \ \text{ and } \ |f(x)| \le a + bd(x, o) + \frac{|\lambda|}{2} d(x, o).$$

Using these results we deduce the following

Proposition 14. *Let* $(M, \langle,\rangle, \nabla f)$ *be a complete Ricci soliton. Then Δ_f and Δ are both stochastically complete and Feller.*

Proof. Since $Ric_f = Ric + Hess\, f = \lambda g$ by definition, and

$$|\nabla f| \leq C_1 + |\lambda| d(x, o)$$

by the lemma above, the conditions of Theorem 8 are satisfied and the statement concerning the f-Laplacian Δ_f follows.

In fact, as mentioned above, the above conditions imply that, if $p \in M$ and $\rho(x) = d(x, p)$ then $\Delta_f \rho$ satisfies the estimate

$$\Delta_f \rho \leq \frac{m-1}{\rho} + C_1 \rho + C_2(1 + d(o, p)).$$

On the other hand

$$|\langle \nabla f, \nabla \rho \rangle| \leq |\nabla f| \leq b + |\lambda| d(o, x)) \leq b + |\lambda|(d(+, p) + \rho(x))$$

so that

$$\Delta \rho = \Delta_f \rho + \langle \nabla f, \nabla \rho \rangle \leq \frac{m-1}{\rho} + [C_1 + |\lambda|]\rho + (C_2 + b + |\lambda|)(1 + d(o, p))$$

and, again by Theorem 8 the Laplacian Δ is Feller.   □

The estimates for the potential function described above allow us to prove Theorem 1.

Proof of Theorem 1. Both statements follow applying Theorem 9 with the choice $\gamma(x) = r(x)^2$ and using the estimates for ∇f described above together with Qian's estimates for $\Delta_f r$ and Δr as in the proof of Proposition 14.   □

4.2. **Applications of the Feller property to geometry and PDE's.**
As alluded to at the beginning of Section 3, and formalized in Theorem 3, using the Feller property on a stochastically complete manifold enables one to extend the investigation of qualitative properties of solution of PDE's to the case where these are defined only in a neighborhood at infinity. In this section, we will exemplify the use of this viewpoint in a number of different geometric and analytic settings. We stress that the needed stochastic completeness assumptions are enjoyed by a very rich family of examples. For instance, according to Proposition 14, a natural and important framework is represented by Ricci solitons. In the non-weighted setting, we have the class of complete manifolds such that $Ric \geq -G(r)$, where $G(r) > 0$ is a smooth increasing function satisfying $1/\sqrt{G} \notin L^1(+\infty)$; [14]. Another admissible category for the application of Theorem 3 is given by Cartan-Hadamard manifolds with at most quadratic exponential volume growth. Actually, the (radial) sectional curvature assumption can be considerably relaxed as explained in Theorem 5.9 of [20].

4.2.1. *Isometric immersions.* Recall that if a Riemannian manifold $(M, \langle, \rangle)$ is stochastically complete, then the mean curvature $\mathbf{H}$ of a bounded isometric immersion $f : M \to \mathbb{B}_R(0) \subset \mathbb{R}^n$ must satisfy

$$\sup_M |\mathbf{H}| \, R \geq 1.$$

In particular, a stochastically complete minimal submanifold in Euclidean space is necessarily unbounded. The next result show that this can be extended to the case where we have an isometric immersion of an end E of M with respect to a given compact subset K of M. We observe that the concepts of stochastic completeness and Feller property can be localized on one end E simply requiring that its double $\mathcal{D}(E)$ be stochastically complete or Feller, respectively. It is then easy to verify maximum principle at infinity holds for every function $f \colon E \to \mathbb{R}$ which is bounded above and does not attain its supremum on ∂E. It is easily verified that M is stochastically complete if and only if so are all ends.

Similarly, the end E is Feller if and only if one (and therefore all) relatively compact domain with smooth boundary $\Omega \supset K$, the minimal positive solution of the boundary value problem

$$\begin{cases} \Delta h & = & \lambda h & in \ M \setminus \overline{\Omega} \\ h & = & 1 & on \ \partial\Omega \end{cases}$$

tends to zero as $x \to \infty$ within the given end. Again, it is easy to see that an end E satisfies this condition if and only if its double is Feller, (see [20] Section 7 for details).

We are now ready to state our result. We are grateful to R. Haslhofer and T. Ilmanen for helpful comments related to the formulation of the theorem.

Theorem 15. *Let $(M, \langle, \rangle)$ be a Riemannian manifold and let E be an end of M with respect to a compact set K. Assume that E is stochastically complete and Feller, and that there exists a bounded isometric immersion $f \colon E \to \mathbb{R}$. Then then mean curvature of f satisfies*

$$\sup_E |\mathbf{H}| \, R \geq 1.$$

Proof. Assume by contradiction that

(28)
$$\sup_E |\mathbf{H}| \, R < 1.$$

Straightforward computations show that $u = |f|^2 \geq 0$ satisfies

$$\Delta u \geq c, \ on \ E,$$

where we have set

$$c = 2m \left(1 - \sup_E |\mathbf{H}| \, R \right) > 0.$$

If follows that

$$\Delta u \geq \lambda u$$

on E with

$$\lambda = \frac{c}{R}.$$

Now, let Ω be a bi-collared relatively compact neighborhood of ∂E in $\mathcal{D}(E)$. We use the same letters f and u to denote the obvious extensions of f and u to $\mathcal{D}$. We clearly have $f(\mathcal{D}(E) \setminus \Omega) \subset \mathbb{B}_R(0)$ and

$$\Delta u \;\; \geq \;\; \lambda u \;\; on \;\; \mathcal{D}(E) \setminus \Omega$$

An application of Theorem 3 shows that $u(x) \to 0$, this is, $f(x) \to 0$ as $x \to +\infty$ in M. On the other hand, since the strict inequality holds in (28), for $R' > R$ sufficiently close to R we have $\sup_E |\mathbf{H}| \, R' < 1$, and clearly $f(E) \subset \mathbb{B}(0', R')$ provided $|0' - 0| < R' - R$. Thus we can repeat the argument with $u'(x) = |f(x) - 0'|^2$ for which again we have

$$\Delta u' \geq c$$

with the same value c, and then $u'(x) \to 0$, i.e., $f(x) \to 0' \neq 0$, as $x \to \infty$. This yields the required contradiction and the theorem is proved. $\square$

4.2.2. *Conformal deformations.* Given a Riemannian manifold $(M, \langle, \rangle)$ of dimension $m \geq 3$ consider the conformally related metric $\overline{\langle, \rangle} = v^{\frac{4}{m-2}} \langle, \rangle$ where $v > 0$ is a smooth function. Thus, the conformality factor v obeys the Yamabe equation

$$c_m^{-1} \Delta v - Sv = -\overline{S} v^{\frac{m+2}{m-2}},$$

where S and $\overline{S}$ denote the scalar curvatures of $\langle, \rangle$ and $\overline{\langle, \rangle}$, respectively. Assume that M is stochastically complete and that

$$\sup_M S(x) \leq S^*, \quad \inf_M \overline{S}(x) \geq \overline{S}_*,$$

for some constants $S^* \geq 0$ and $\overline{S}_* > 0$. An application of the weak minimum principle at infinity to the Yamabe equation shows that

$$\left(\frac{S^*}{\overline{S}_*} \right)^{\frac{m-2}{4}} \geq v_* = \inf_M v.$$

In particular, if $S(x) \leq 0$ on M, then $v_* = 0$. Actually, since the infimum of v cannot be attained,

$$\inf_{M \setminus \Omega} v = 0,$$

for every $\Omega \subset\subset M$. Clearly, to reach these conclusions the scalar curvature bound must hold on M.

Theorem 16. *Let $(M, \langle,\rangle)$ be a stochastically complete, Feller manifold of dimension $m \geq 6$ and scalar curvature satisfying*

$$\sup_{M \backslash \Omega} S(x) \leq 0,$$

for some compact domain $\Omega \subset M$. Let $\overline{\langle,\rangle} = v^{\frac{4}{m-2}} \langle,\rangle$ be a conformally related metric such that

$$\inf_M v = v_* > 0.$$

If the scalar curvature of $(M, \overline{\langle,\rangle})$ satisfies

$$\inf_{M \backslash \Omega} \overline{S}(x) = \overline{S}_* > 0,$$

then

$$v(x) \to +\infty, \text{ as } x \to \infty.$$

With respect to the assumption that S is nonnegative at infinity, one may wonder if it could be made nonnegative everywhere on M with a conformal change of metric. Note however this in general would require a control on the positive part S^+ of S in the set Ω, which moreover may depend on the metric itself in a rather implicit way (see, e.g., Prop 1.2 in [24]).

Proof. Just note that the positive, bounded function $u(x) = v(x)^{-1}$ satisfies

$$c_m^{-1} \Delta u \geq -Su + \overline{S} u^{\frac{m-6}{m-2}} \geq \overline{S} u^{\frac{m-6}{m-2}}.$$

Since

$$0 \leq \frac{m-6}{m-2} < 1,$$

Theorem 3 yields

$$u(x) \to 0, \text{ as } x \to \infty.$$

$\square$

As an immediate consequence, we obtain the following non-existence result. Note that this applies, for instance, to an expanding, gradient Ricci soliton M. Indeed, in this case, the scalar curvature assumption is compatible with the restriction $\inf_M S \leq 0$ imposed by the soliton structure.

Corollary 17. *On a stochastically complete and Feller manifold $(M, \langle,\rangle)$ of dimension $m \geq 6$ and*

$$\sup_{M \backslash \Omega} S(x) \leq 0$$

one cannot perform a conformal change $\overline{\langle,\rangle} = v^{\frac{4}{m-2}} \langle,\rangle$ in such a way that

$$0 < v_* \leq v(x) \leq v^* < +\infty$$

and

$$\inf_{M \backslash \Omega} \overline{S}(x) = \overline{S}_* > 0.$$

4.2.3. *Compact support property of bounded solutions of PDE's.* Recall that a certain PDE satisfies the compact support principle if a solution, in the exterior of a compact set, which is non-negative and decays at infinity, must have compact support. We are going to analyze some situations where the decay assumption can be relaxed. This has applications to the Yamabe problem.

Theorem 18. *Let* $(M, \langle, \rangle)$ *be a complete and stochastically complete, Cartan-Hadamard manifold. Let* $u > 0$ *be a bounded solution of*

(29) $$\Delta u \geq \lambda(u), \ on \ M \backslash \Omega$$

for some domain $\Omega \subset\subset M$ *and for some non-decreasing function* $\lambda : [0, +\infty) \rightarrow [0, +\infty)$ *satisfying the following conditions:*

(30) $$(a) \ \lambda(0) = 0; \ (b) \ \lambda(t) > 0 \ \forall t > 0; \ (c) \ \liminf_{t \to 0+} \frac{\lambda(t)}{t^\xi} > 0,$$

for some $0 \leq \xi < 1$. *Then* u *has compact support.*

Proof. Recall that a Cartan-Hadamard manifold is Feller (see, [1], [20]). By Theorem 3, we know that $u(x) \rightarrow 0$, as $x \rightarrow \infty$. The conclusion now follows from the compact support principle which, is valid under the stated assumptions on M and λ (see [22], Theorem 1.1, and Theorem 10 above). $\square$

The above theorem can be applied to obtain nonexistence results. For instance, combining Theorem 18 and Corollary 17 we get

Corollary 19. *Let* $(M, \langle, \rangle)$ *be a stochastically complete Cartan-Hadamard manifold of dimension* $m \geq 6$. *Then the metric of* M *cannot be conformally deformed to a new metric* $\overline{\langle, \rangle} = v^2 \langle, \rangle$ *with* $v_* > 0$ *and scalar curvature* $\overline{S}$ *satisfying* $\liminf_{x \to \infty} \overline{S} > 0$.

Of course for the conclusion of Theorem 18 to hold it suffices that M be stochastically complete, Feller and that the compact support principle holds for solutions of (29). Theorem 18 can be therefore generalized as follows.

Theorem 20. *Let* $(M, \langle, \rangle)$ *be a complete Riemannian manifold with a pole* o *and set* $r(x) = d(x, o)$. *Assume that*

$$Ric \geq -K(r(x)),$$

where $K > 0$ *is an increasing function satisfying*

$$\frac{1}{\sqrt{K}} \notin L^1(+\infty).$$

Assume also that

$$Sec_{rad} \leq G(r(x))$$

where G is a smooth even function such that the unique solution g of the Cauchy problem

$$\begin{cases} g'' + Gg = 0 \\ g(0) = 0, \ g'(0) = 1. \end{cases}$$

satisfies

$$\inf \frac{g'}{g}(t) > -\infty.$$

If $u \geq 0$ is a bounded solution of

$$\Delta u \geq \lambda(u), \ on \ M \backslash \Omega$$

where λ satisfies (30), then u has compact support.

In a similar view, using the comparison results established in Theorem 4 we obtain the following

Theorem 21. *Let M_f be a geodesically complete, stochastically complete, weighted manifold with a pole o, and let by $r(x)$ be the Riemannian distance function from M. Suppose that there exists an integer n and an even function $g : \mathbb{R} \rightarrow [0, +\infty)$ such that $g(0) = 0$ $g'(0) = 1$ and $g(r) > 0$ for $r > 0$ such that*

$$\Delta_f r(x) \geq (n-1)\frac{g'}{g}(r(x)) \quad for \ r(x) >> 1.$$

Suppose moreover

$$\inf \frac{g'}{g}(t) > -\infty$$

and that either

(31) $$\frac{1}{g^{n-1}(r)} \in L^1(+\infty)$$

or

(32) $\quad$ *(i)* $\dfrac{1}{g^{n-1}(r)} \notin L^1(+\infty) \quad and \quad$ *(ii)* $\dfrac{\int_r^{+\infty} g^{n-1}(t)\, dt}{g^{n-1}(r)} \notin L^1(+\infty).$

If $u > 0$ is a bounded solution of

$$\Delta_f u \geq \lambda(u), \ on \ M \backslash \Omega$$

where λ satisfies (30), then u has compact support.

Proof. Indeed, since g satisfies the conditions (31) or (32) it follows from Theorem 4 (iii) that M_f is Feller. On the other hand, by Theorem 10 the compact support principle holds for solutions of (29). Therefore the conclusion follows as in Theorem 18. $\qquad \square$

5. Spectral theory of weighted Laplacians

5.1. **Basic theory.** In this section we collect some results on the spectral properties of the f-Laplacian. Generally the proofs may be obtained by adapting those valid for the ordinary Laplacian, and therefore they will be mostly omitted.

The first basic observation is that the f-Laplacian, is associated to the f-Diriclet form

$$(33) \qquad Q_f(u) = \int_M |\nabla u|^2 d\mathrm{vol}_f,$$

originally defined on $C_c^\infty(M)$. The form Q_f is closable and its closure induces a non-negative self-adjoint operator on $L^2(\mathrm{vol}_f)$, still denoted with $-\Delta_f$. The same proof valid for the usual Laplacian can be adapted to show that $-\Delta_f$ is essentially self-adjoint on $C_c^\infty(M)$. It is also useful to note that under the unitary transformation $T(u) = e^{-f/2}u$ of $L^2(d\mathrm{vol})$ onto $L^2(d\mathrm{vol}_f)$, the operator Δ_f is unitarily equivalent to the Schrödinger operator

$$\Delta + \left(\frac{1}{2}\Delta f - \frac{1}{4}|\nabla u|^2\right),$$

More generally, if Ω is any open set in M, we will denote with $-\Delta_f^\Omega$ the Friedrichs extension of the operator $-\Delta_f$ originally on $C_c^\infty(\Omega)$. Its domain is given by

$$Dom(-\Delta_f^\Omega) = \{u \in H_0^1(\Omega, d\mathrm{vol}_f) : (\Delta_f)_{dist}u \in L^2(\Omega, d\mathrm{vol}_f)\}.$$

The operator $-\Delta_f^\Omega$ is a positive operator, its spectrum is a subset of $[0, +\infty)$ and its bottom admits the usual variational characterization

$$\lambda_1(-\Delta_f^\Omega) = \inf \frac{\int_\Omega |\nabla u|^2 d\mathrm{vol}_f}{\int_\Omega |u|^2 d\mathrm{vol}_f},$$

where the infimum is taken over $u \in C_c^\infty(\Omega)$, or equivalently, in $H_0^1(\Omega)$. Also, $\sigma(-\Delta_f^\Omega)$ can be decomposed into the disjoint union $\sigma_d(-\Delta_f^\Omega) \cup \sigma_{ess}(-\Delta_f^\Omega)$, where σ_d is the set of isolated eigenvalues of finite multiplicity, called the discrete spectrum, and its complement σ_{ess}, called the essential spectrum, is the set of eigenvalues of infinite multiplicity and of accumulations points of the spectrum.

Adapting the arguments valid for the ordinary Laplacian (or using the above mentioned unitary equivalence with a Schrödinger operator, see, [26]) one shows that the following decomposition principle holds.

Theorem 22. *For every relatively compact domain* Ω

$$\sigma_{ess}(-\Delta_f^M) = \sigma_{ess}(-\Delta_f^{M\setminus\overline{\Omega}}).$$

In particular,

$$\inf \sigma_{ess}(-\Delta_f^M) = \sup_{\Omega \subset\subset M} \lambda_1(-\Delta_f^{M\backslash\overline{\Omega}}).$$

Similarly, one may generalize a result of R. Brooks, [7], [8], to obtain the following upper bound for the infimum of the essential spectrum in terms of the weighted volume growth of the manifold (see [26]).

Theorem 23. *Let M_f be a complete weighted manifold.*

(a) *If $\mathrm{vol}_f(M) = +\infty$, then*

$$\lim_{R\to+\infty} \frac{\log \mathrm{vol}_f(B_R)}{R} \geq \inf \sigma_{ess}\left(-\Delta_f^M\right) \geq \lambda_1\left(-\Delta_f^M\right) \geq 0.$$

(b) *If $\mathrm{vol}_f(M) < +\infty$, then*

$$\lim_{R\to+\infty} \frac{-\log\left(\mathrm{vol}_f(M) - \mathrm{vol}_f(B_R)\right)}{R} \geq \inf \sigma_{ess}\left(-\Delta_f^M\right) \geq \lambda_1\left(-\Delta_f^M\right) \geq 0.$$

The following Barta-type lower estimate for $\lambda_1(-\Delta_f^\Omega)$ is a weighted version of a result in [3]. Its proof is obtained following exactly the arguments in [3] using a weighted version of the divergence theorem.

Theorem 24. *Let M_f be a weighted manifold and let $\Omega \subset M$ be a domain. Then, for every vector field X on Ω*

$$\lambda_1\left(-\Delta_f^\Omega\right) \geq \inf_\Omega \left\{\mathrm{div}_f(X) - |X|^2\right\}.$$

The equality holds if Ω is a compact domain with smooth boundary.

A classical consequence is represented by the next

Corollary 25. *Let M_f be a weighted manifold and let $\Omega \subset M$ be a domain. Then, for every domain Ω and for every $0 < u \in C^2(\Omega)$,*

$$\lambda_1\left(-\Delta_f^\Omega\right) \geq \inf_\Omega \left(-\frac{\Delta_f u}{u}\right).$$

In particular, recalling Theorem 22, we deduce

Corollary 26. *Let M_f be a weighted manifold. Then, for every domain $\Omega \subset\subset M$ and for every $0 < u \in C^2(M\backslash\Omega)$, it holds*

$$\inf \sigma_{ess}\left(-\Delta_f^M\right) \geq \inf_{M\backslash\Omega} \left(-\frac{\Delta_f u}{u}\right).$$

The following version of the classical Cheng eigenvalue comparison was pointed out in [25].

Theorem 27. *Assume that the complete weighted manifold M_f satisfies*

$$Ric_f \geq -\alpha,$$

for some $\alpha \geq 0$. Then

$$\lambda_1 \left(-\Delta_f^{B_R}\right) \leq \lambda_1 \left(-\Delta^{\mathbb{B}_R^{m+1}}\right),$$

where $\mathbb{B}_R^{m+1}$ is the ball of radius $R > 0$ in the $(m+1)$-dimensional spaceform $\mathbb{M}^{m+1} \left(\alpha m^{-1}\right)$ of constant curvature αm^{-1}.

5.2. **Essential spectrum and stochastic properties.** The purpose of this section is to understand possible connections between the structure of the spectrum of the Laplacian and the stochastic properties of noncompact Riemannian manifold, namely, stochastic (in)completeness and Feller propery. The starting point of the investigation is represented by the (proof of the) following recent result, [4], which answers in the affirmative a question raised by S.T. Yau.

Theorem 28. *Let M be a geodesically complete manifold which admits a proper minimal immersion $f : M \to \mathbb{B}_R (0)$ into an open ball $\mathbb{B}_R (0) \subset \mathbb{R}^N$. Then $\sigma_{ess} \left(-\Delta^M\right) = \emptyset$.*

Proof. According to the decomposition principle,

$$\inf \sigma_{ess} \left(-\Delta^M\right) = \sup_{\substack{\Omega_j \text{ cpt} \\ \Omega_j \nearrow M}} \lambda_1 \left(-\Delta^{M \backslash \Omega_j}\right),$$

where, by Barta theorem,

$$\lambda_1 \left(-\Delta^{M \backslash \Omega_j}\right) \geq \sup \inf_{M \backslash \Omega_j} \left(-\frac{\Delta v}{v}\right),$$

the supremum being taken with respect to all smooth (say C^2) functions $v > 0$ on $M \backslash \Omega_j$. In particular, choosing

$$\Omega_j = \left\{x \in M : |f(x)|^2 \leq R^2 - \frac{1}{j}\right\} \subset\subset M$$

and

$$v(x) = R^2 - |f(x)|^2 > 0$$

gives

$$\inf \sigma_{ess} \left(\Delta^M\right) \geq \lambda_1 \left(\Delta^{M \backslash \Omega_j}\right) \geq \frac{2m}{1/j} \to +\infty, \text{ as } j \to +\infty.$$

$\square$

Remark 29. In the assumptions of Theorem 28, M is stochastically incomplete. Indeed

$$u(x) = |f(x)|^2$$

is bounded and satisfies

$$\Delta u = 2m,$$

thus proving that u violates the weak maximum principle at infinity (in the terminology of [2], u is a woymp violating function). By the maximum

principle characterization of stochastic completeness, it follows that M is stochastically incomplete, as claimed. Moreover, $0 \leq u(x) < \sup_M u = R^2 < +\infty$ and $\Omega_\delta = \{x \in M : -\infty < u(x) \leq R^2 - \delta\}$ defines a compact exhaustion of M, that is, $u : M \to [0, R^2)$ is a proper function. We are going to prove that these ingredients suffice to conclude the discreteness of the spectrum, thus establishing an abstract and generalized version of the main Theorem in 28.

Definition 30. *Say that a function $u : M \to (-\infty, u^*)$, $u^* < +\infty$, is an exhaustion function if, for every $\delta > 0$, the set*

$$\Omega_\delta = \{x \in M : -\infty < u(x) \leq u^* - \delta\}$$

is compact and $\Omega_\delta \nearrow M$ as $\delta \to 0$. Note that, in case M is noncompact, necessarily, $u^* = \sup_M u$.

Definition 31. *Let M_f be a weighted manifold. A woymp violating function for the f-Lpalcian, is a C^2 function $u : M \to \mathbb{R}$ satisfying $\sup_M u = u^* < +\infty$ such that, for any sequence $\{x_k\}$ along which $u(x_k) \to u^*$, it holds $\limsup_{k \to +\infty} \Delta_f u(x_k) > 0$.*

Theorem 32. *Let M_f be a (f-stochastically incomplete) noncompact weighted manifold. If M_f supports a woymp violating exhaustion function then*

$$\sigma_{ess}\left(-\Delta_f^M\right) = \emptyset.$$

Proof. Let $u : M \to (-\infty, u^*)$, $u^* = \sup_M u < +\infty$, be a woymp violating exhaustion function. Arguing exactly as above, we consider $\Omega_j = \{x \in M : u(x) \leq u^* - 1/j\} \nearrow M$ and $v(x) = u^* - u(x) > 0$. Note that, since u is woymp violating,

$$\inf_{M \setminus \Omega_j} \Delta_f u = c_j > 0,$$

where, by the obvious monotonicity property of the infimum, the sequence c_j is increasing. Therefore,

$$\inf \sigma_{ess}\left(-\Delta_f^M\right) \geq \lambda_1\left(-\Delta_f^{M \setminus \Omega_j}\right) \geq \inf_{M \setminus \Omega_j}\left(-\frac{\Delta_f v}{v}\right) \geq jc_j \to +\infty, \text{ as } j \to +\infty.$$

$\square$

At this point, a natural question is to what extent the existence of a woymp violating exhaustion function characterize the spectrum of the Laplacian of the underlying manifold. Beside proper bounded submanifolds with controlled mean curvature, are there natural examples of manifolds supporting woymp violating exhaustion functions? Which geometric conditions ensure that such functions exist? Some examples will help us to focus some important aspects.

Example 33. Let $M_g^m \approx \mathbb{R}^m$ be a complete, noncompact model manifold endowed with the metric

$$\langle , \rangle = dr^2 + g(r)^2 d\theta^2,$$

where the smooth function $g : [0, +\infty) \to [0, +\infty)$ satisfies

$$\begin{cases} g(r) > 0, & r > 0 \\ g^{(2k)}(0) = 0, & k \in \mathbb{N} \\ g'(0) = 1. \end{cases}$$

According to a well known characterization M_g^m is stochastically incomplete if and only if

(34) $$\int_0^{+\infty} \frac{\int_0^t g^{m-1}}{g^{m-1}(t)} = u^* < +\infty.$$

Indeed, the function,

$$u(x) = \int_0^{r(x)} \frac{\int_0^t g^{m-1}}{g^{m-1}(t)} dt : M_g^m \to [0, u^*),$$

satisfies

$$\Delta u = 1,$$

and it is bounded if and only (34) holds, in which case u is in fact a woymp violating exhaustion function, and $\sigma_{ess}\left(-\Delta^{M_g^m}\right) = \emptyset$.

Thus a model is stochastically incomplete if and only if it admits a woymp violating exhaustion function. In particular, for a stochastically incomplete manifold which is a model, the condition that $\sigma_{ess}\left(-\Delta^{M_g^m}\right) = \emptyset$ is equivalent with the existence of a woymp violating exhaustion.

Example 34. In general it is not true that the stochastic incompleteness of a manifold is equivalent to the discreteness of the spectrum of its Laplacian. Indeed, condition $\sigma_{ess}\left(-\Delta^M\right) = \emptyset$ is invariant under bilipschiz diffeomorphisms whereas, according to a result by T. Lyons, [15], the stochastic (in)completeness is not. More concretely, consider the Riemannian product $N = M_g^m \times \mathbb{R}$ where M_g^m is stochastically incomplete model. Then, N is stochastically incomplete but the essential spectrum of the Laplacian $\Delta_N = \Delta_M + d^2/dt^2$ is nonempty. Note that a natural voymp violating function on N is given by $v(x, t) = u(x)$ where $u(x)$ is defined in the previous example. Clearly, $v : N \to [0, u^*)$ is not an exhaustion function. This shows that the assumptions of Theorem 32 are necessary.

One may also investigate whether the condition that the bottom of the spectrum of exterior balls grows at a specified rate, namely

(35) $$\lambda_1(-\Delta^{M \setminus \overline{B}_R}) \geq f(R),$$

where $f(R)$ is a monotone nondecreasing and diverges as $R \to +\infty$, forces stochastic incompleteness. However, without additional global assumptions

even this implication fails. To see this, we begin observing that, by standard arguments, if $(M, \langle\,,\rangle)$ has a pole o and $|\Delta r| \geq c > 0$ on $M \setminus \overline{\Omega}$, then $\lambda_1(-\Delta^{M\setminus\overline{\Omega}}) \geq c^2/4$. Indeed, by continuity Δr has constant sign in every connected component of $M \setminus \overline{\Omega}$. For every $v \in C_c^\infty(M \setminus \overline{\Omega})$ with support in one of such connected components we have

$$\int v^2 \leq \frac{1}{c} \int |\Delta r| v^2 = \pm\frac{2}{c} \int v\langle \nabla v, \nabla r\rangle \leq \frac{2}{c} \left(\int v^2\right)^{1/2} \left(\int |\nabla v|^2\right)^{1/2},$$

and

$$(36) \qquad \frac{c^2}{4} \int v^2 \leq \int |\nabla v|^2.$$

Note that, in particular, it follows from the decomposition principle, that if $|\Delta r| \to +\infty$ as $x \to \infty$, then $\sigma_{ess}(-\Delta^M) = \emptyset$.

Now, given a function f as above, assume that g satisfies

$$g(r) = \exp\left(-2\int_1^r \sqrt{f(t)}\, dt\right)$$

for sufficiently large r. Then the model M_g^m has the property that

$$\Delta r \leq -2\sqrt{f(R)} \quad \text{in the complement of } B_R,$$

and (35) follows. On the other hand, since $g(r)$ is decreasing, $\mathrm{vol}(B_R)$ grows at most linearly, and M_g^m is necessarily stochastically complete.

We note in passing that this shows that the assumption that M be a Cartan-Hadamard manifold plays a fundamental role in Problem 9 of Grigor'yan's survey [12].

Example 35. In general a stochastically incomplete manifold with discrete spectrum can support woymp violating functions both exhaustion and not. Indeed, take $N = M \times M$ where M supports a positive woymp violating exhaustion function $u : M \to [0, u^*)$ (for instance, M is one of the bounded minimal surfaces of $\mathbb{R}^3$ by Martin-Morales). Then $\sigma_{ess}\left(-\Delta^M\right) = \emptyset$. Moreover, $v(x, y) = u(x) + u(y)$ is woymp violating and exhaustion whereas $w(x, y) = u(x)$ is woymp violating but not exhaustion.

As a consequence, for a stochastically incomplete manifold M, condition $\sigma_{ess}\left(-\Delta^M\right) = \emptyset$ could imply at most that there exists one woymp violating exhaustion function.

In a similar view, since both σ_{ess} and the Feller property are only sensitive to the properties of the manifold off a compact set, it is also natural to investigate to what extent they are related. However, we are going to see that, without further assumptions, there is no link between the Feller property and the discreteness of the spectrum.

Example 36. For the sake of simplicity, we restrict ourselves to the case of the ordinary Laplacian Δ, even though much of the ensuing discussion could be generalized to the f-Laplacian.

As noted in the proof of Theorem 21, the model manifold M_g^m is Feller is and only if either

$$\frac{1}{g^{m-1}(r)} \in L^1(+\infty)$$

or

$$\text{(i)} \ \frac{1}{g^{m-1}(r)} \notin L^1(+\infty) \qquad \text{and} \qquad \text{(ii)} \ \frac{\int_r^{+\infty} g^{m-1}(t)\,dt}{g^{m-1}(r)} \notin L^1(+\infty).$$

Recalling that $\Delta r = (m-1)\frac{g'}{g}$, we deduce from the discussion in Example 34 that, if $g(r) = e^{-r^\alpha}$ for $r >> 1$, then M_g^m has discrete spectrum for every $\alpha > 1$, and it is Feller for $\alpha \leq 2$ and non-Feller for $\alpha > 2$. Note that all these manifolds have finite volume, and so are automatically stochastically complete, showing that even in the case of models there is no equivalence between discreteness of the spectrum, and stochastic incompleteness.

It follows that if M is the connected sum of $\mathbb{R}^m$ with a non-Feller model M_g^m, then M is non-Feller. Since the essential spectrum of $\mathbb{R}^m$ is the entire interval $[0, +\infty)$ an easy argument based on characteristic sequences supported in the end of M isometric to an exterior domain of $\mathbb{R}^m$ shows that $\sigma_{ess}(-\Delta^M) = [0, +\infty)$, and M is therefore a non-Feller manifold with non-empty essential spectrum. Of course, $\mathbb{R}^m$ itself is a trivial example of a Feller manifold with non-empty essential spectrum.

Example 37. As seen in Example 33, a stochastically incomplete model has discrete spectrum. Since such model has necessarily infinite volume, by the characterization of the previous example it is Feller. Small modifications to the arguments described above show that even for non-Feller, stochastically incomplete manifolds there is in general no connection with the discreteness of the spectrum.

Again note that one may extend the definition of stochastic completeness (incompleteness) to an end of a manifold, by requiring that the double of the end be stochastically complete (incomplete). Then it follows easily from the weak maximum principle that a manifold is stochastically complete incomplete if and only so is at least one of its ends.

The connected sum $M = M_{g_1}^m \# M_{g_2}^m$ of a stochastically incomplete model $M_{g_1}^m$ with a non-Feller model $M_{g_2}^m$ with discrete spectrum as described in Example 36, provides an example of a stochastically incomplete, non-Feller manifold with $\sigma_{ess}(-\Delta^M) = \emptyset$.

On the other hand, the connected sum $M = M_{g_1}^m \# M_{g_2}^m \# \mathbb{R}^m$ is stochastically incomplete, non-Feller, and $\sigma_{ess}(-\Delta^M) = [0, +\infty)$.

Note that all these examples have more than one end, and the case of manifolds with only one end remains open.

5.3. **Spectrum and semilinear PDE's.** In this section we use very easy spectral considerations to deduce information on nonnegative solutions of the differential inequality

$$(37) \qquad \Delta_f u \leq au - bu^\sigma,$$

in the exterior of a compact set for some constants $a \geq 0, b > 0$ and $\sigma > 1$. By the strong minimum principle, $u > 0$ unless it is identically zero in each connected component of every point where it vanishes. If u satisfied the inequality on all of M_f, and the weighted manifold was stochastically complete with respect to the f-Laplacian, then a direct application of the weak maximum principle would imply that either $\inf_M u = 0$ or

$$\inf_M u \geq \left(\frac{a}{b}\right)^{\frac{1}{\sigma-1}}.$$

For instance, when applied to the scalar curvature $S(x)$ of a complete, shrinking Ricci soliton, this procedure gives the estimate

$$0 \leq \inf_M S \leq m\lambda,$$

where λ denotes the soliton constant and $m = \dim M$. Indeed, it is well known that S satisfies

$$(38) \qquad \frac{1}{2}\Delta_f S = \lambda S - |Ric|^2 \leq \lambda S - \frac{S^2}{m}.$$

In the spirit of the previous sections, we are going to extend these considerations outside a compact set. This time, however, we use spectral assumptions instead of stochastic properties.

Proposition 38. *Let $u > 0$ be a solution of (37) in a neighborhood at infinity, for some constants $a \geq 0$, $b > 0$ and $\sigma > 1$. Then, for every domain $\Omega \subset\subset M$,*

$$\inf_{M\setminus\Omega} u \geq \left(\frac{a}{b}\right)^{\frac{1}{\sigma-1}} - \inf \sigma_{ess}\left(-\Delta_f^M\right).$$

Remark 39. In particular, if $\inf \sigma_{ess}\left(-\Delta_f^M\right) = 0$, we recover the above conclusion outside every large compact set, thus proving that $\inf_M u$ cannot be achieved. According to Brooks estimates, this happens under suitable volume growth assumptions. Conversely, if $\inf_{M\setminus\Omega} u > (a/b)^{\frac{1}{\sigma-1}}$, we deduce a gap in the essential spectrum and, therefore, a volume growth estimate.

Proof. It is a trivial application of Corollary 26. Indeed

$$\inf \sigma_{ess}\left(-\Delta_f^M\right) \geq \inf_{M\setminus\Omega}\left(-\frac{\Delta_f u}{u}\right) \geq -a + b \inf_{M\setminus B_R} u^{\sigma-1}.$$

$$\square$$

Coming back to the scalar curvature $S(x)$ of a gradient shrinker M_f with soliton constant $\lambda > 0$, recall that $S \geq 0$, the equality holding at some point if and only if $M_f = \mathbb{R}_f^m$ and $f(x) = A|x|^2 + \langle B, x \rangle + c$. Assume $S > 0$, for otherwise there is nothing to prove. Then, for every $R > 0$, we obtain the estimates

$$\inf_{M \backslash B_R} \frac{|Ric|^2}{S} - \lambda \leq \inf \sigma_{ess} \left(-\Delta_f^M \right)$$

and

$$\inf_{M \backslash B_R} S - m\lambda \leq m \inf \sigma_{ess} \left(-\Delta_f^M \right).$$

In conclusion of this section, we remark that Proposition 38 follows essentially from a suitable application of Barta's theorem. Further use of this result, but in a slightly different direction, yields a different kind of information on the solutions at infinity of (37).

Proposition 40. *Let M_f be a complete, m-dimensional weighted manifold satisfying*

$$Ric_f \geq -\mu,$$

for some constant $\mu > 0$. Then, there exists a constant $c = c(m, \mu) > 0$ such that the following holds. If u is a solution of (37) outside a compact set K, for some constants $a \geq 0$, $b > 0$ and $\sigma > 1$, then, for every $B_R(x) \in M \setminus K$,

$$\left(\frac{a}{b} + \frac{c}{b} \frac{1 + R^2}{R^2} \right)^{\frac{1}{\sigma - 1}} \geq \inf_{B_R(x)} u.$$

In particular, for any fixed $R > 0$,

$$\limsup_{x \to \infty} \left\{ \inf_{B_R(x)} u \right\} < +\infty.$$

Proof. Obviously, the only interesting case is

$$\inf_{B_R(x)} u > 0.$$

Then,

$$\lambda_1 \left(-\Delta_f^{B_R(x)} \right) \geq \inf_{B_R(x)} \left(-\frac{\Delta_f u}{u} \right) \geq -a + b \inf_{B_R(x)} u^{\sigma-1}$$

On the other hand, by Theorem 27,

$$\lambda_1 \left(-\Delta_f^{B_R(x)} \right) \leq \lambda_1 \left(-\Delta^{\mathbb{B}_R^{m+1}} \right) \leq c_1(\mu, m) \left(1 + \frac{1}{R^2} \right)$$

with $\mathbb{B}_R^{m+1} \subset \mathbb{H}^{m+1}(\mu/m)$. Combining these two inequalities completes the proof. $\square$

The above result is relevant in the study of scalar curvature estimates for expanding solitons. Recall that the scalar curvature of a complete gradient expander M_f satisfies

$$m\lambda \leq \inf_M S \leq 0,$$

where λ is the soliton constant and $m = \dim M$. Since, by definition, $Ric_f = \lambda < 0$ and recalling that $S(x)$ satisfies equation (38), we conclude that

$$c \geq \frac{R^2}{1 + R^2} \inf_{B_R(x)} S,$$

for some constant $c = c(m, \lambda) > 0$ and for every $R > 0$. In particular

$$\limsup_{x \to \infty} \left\{ \inf_{B_R(x)} S \right\} < +\infty,$$

for every $R > 0$. This compares with Zhang estimate

$$\frac{|S(x)|}{r(x)^2} \leq d(m, \lambda)$$

which, in turn, follows applying the estimates of the potential function to the basic equation (26).

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