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SUBDIFFERENTIABILITY WITH RESPECT TO min-TYPE FUNCTIONS

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Abstract

A characterization of the $\mathcal{L}_k^0$-subdifferentiable functions (k positive integer) is obtained. Recall that a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called $\mathcal{L}_k$-subdifferentiable (Rubinov [4]) if it admits a $\mathcal{L}_k$-subgradient at any point $x^0 \in \operatorname{dom} f$, that is a functional $\ell : \mathbb{R}^n \to \mathbb{R}$, $\ell(x) = \min_{1 \le i \le k} \langle l^i, x \rangle$, such that $f(x) \ge \ell(x) - \ell(x^0) + f(x^0)$, $\forall x \in \mathbb{R}^n$. We call f $\mathcal{L}_k^0$-subdifferentiable if f admits at any $x^0 \in \operatorname{dom} f$ an $\mathcal{L}_k$-subgradient of special form (as explained in the paper).

Key words: abstract convexity, abstract subdifferentiability, min-type functions

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1. Introduction. The abstract convex analysis, grown after the monographs of Pallaschke, Rolewicz [2], Singer [7] and Rubinov [4] to a mathematical discipline with its own problems, aims to generalize the results of convex analysis to abstract convex functions on the base of global aspects of the subdifferential (generalizations on the base of local aspects lead to nonsmooth analysis). Its importance is due mainly to applications to global optimization. One of the problems of abstract convex analysis is to characterize the class of abstract subdifferentiable functions. In this paper we characterize the class of $\mathcal{L}_k^0$-subdifferentiable functions, k positive integer. The $\mathcal{L}_k^0$-subdifferentiability is defined in the next section. The paper continues the research from [1] where this problem is solved for positively homogeneous (PH) functions in the special case $k = n$, and from [4] where the case $k \ge n + 1$ is studied.

2. Preliminaries. Denote by $\mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$. Let X be a given set and L be a set of functions $\ell : X \to \mathbb{R}$. The functions from L are called abstract linear functions. For a function $f : X \to \mathbb{R}_{+\infty}$ the function $\ell \in L$, such that $f(x) \geq \ell(x) - \ell(x^0) + f(x^0)$, $\forall x \in X$, is called an L-subgradient of f at x^0. The set $\partial_L f(x^0)$ of all subgradients of f at x^0 is called the L-subdifferential of f at x^0. If $\partial_L f(x^0) \neq \varnothing$, then f is called L-subdifferentiable at x^0. The function f is called L-subdifferentiable, if it is L-subdifferentiable at any point $x^0 \in \operatorname{dom} f := \{x \in X \mid f(x) \neq +\infty\}$.

The set $H = H_L = \{h : X \to \mathbb{R} \mid h(x) = \ell(x) - c,\ \ell \in L,\ c \in \mathbb{R}\}$ forms the set of abstract affine functions. A function $f : X \to \mathbb{R}_{+\infty}$ is said H-convex at x^0, if $f(x^0) = \sup\{h(x^0) \mid h \in H,\ h \leq f\}$, and H-convex if it is H-convex at any $x^0 \in X$. Here $h \leq f$ means $h(x) \leq f(x)$ for all $x \in X$.

In the sequel we consider the case $X = \mathbb{R}^n$. Then $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$ denotes the scalar product of the vectors $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, and $\|x\| = \langle x, x\rangle^{1/2}$ stands for the Euclidean norm of x.

In this paper we are interested in abstract subdifferentiability with respect to min-type functions. For a positive integer k we define the class of abstract linear functions $\mathcal{L}_k$ (min-type functions) as the set of the functionals $\ell : \mathbb{R}^n \to \mathbb{R}$ such that $\ell(x) = \min_{1 \leq i \leq k} \langle l^i, x\rangle$ for some $l^1, \dots, l^k \in \mathbb{R}^n$. Abstract convexity, and in particular abstract subdifferentiability, with respect to min-type functions is studied in [4,5] and [6]. The original aim of the present paper was to characterize the class of $\mathcal{L}_k$-subdifferentiable functions. This aim underwent some change as it is explained below. When $k \geq n+1$ the problem finds a satisfactory solution (compare with Rubinov [4], Theorem 5.19). So, the interesting case is $k \leq n$. Confining to PH functions in [1], we consider the case $k = n$ as a crucial one. Observe also that $\mathcal{L}_1$ is the class of the linear functions, so the case $k = 1$ leads to the usual convexity and subdifferentiability.

For a given function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ and $x^0 \in \operatorname{dom} f$ the problem to find a generic $\mathcal{L}_k$-subgradient ℓ meets with constructive difficulties. So, we will confine to look for $\mathcal{L}_k$-subgradiends $\ell(x) = \min_{1 \leq i \leq k} \langle l^i, x\rangle$ from $\partial_{\mathcal{L}_k} f(x^0)$ having the property

$$\langle l^1, x^0\rangle = \langle l^2, x^0\rangle = \cdots = \langle l^k, x^0\rangle \quad (= \ell(x^0))\,. \tag{1}$$

The set of all these subgradients is denoted by $\mathcal{L}_k^0(x^0)$. By definition $\mathcal{L}_k(x^0) \subset \mathcal{L}_k$. We say that f is $\mathcal{L}_k^0$-subdifferentiable at x^0 if it is $\mathcal{L}_k^0(x^0)$-subdifferentiable at x^0, and we use to write often $\partial_{\mathcal{L}_k^0} f(x^0)$ for this subdifferential instead of $\partial_{\mathcal{L}_k^0(x^0)} f(x^0)$.

In the sequel we use the notion of calmness originating from [3]. The function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is called (globally) calm at the point $x^0 \in \operatorname{dom} f$ if

$$\operatorname{Calm} f(x^0) := \inf \left\{ \frac{f(x) - f(x^0)}{\|x - x^0\|} \mid x \in \mathbb{R}^n,\ x \neq x^0 \right\} > -\infty\,.$$

The value $\operatorname{Calm} f(x^0)$ is called the (global) calmness of f at x^0.

Proposition 1. *If f is $\mathcal{L}_k^0$-subdifferentiable at $x^0 \in \operatorname{dom} f$, then* $\operatorname{Calm} f(x^0) > -\infty$.

Proof. If $\ell \in \partial_{\mathcal{L}_k^0} f(x^0)$ and $\ell(x) = \min_{1\le i\le k}\langle l^i,\, x\rangle$, then for $x \neq x^0$ holds

$$\frac{f(x)-f(x^0)}{\|x-x^0\|} \ge \frac{\ell(x)-\ell(x^0)}{\|x-x^0\|} = \frac{\min_{1\le i\le k}\langle l^i,\, x-x^0\rangle}{\|x-x^0\|} \ge -\max_{1\le i\le k}\|l^i\|\,,$$

whence $\operatorname{Calm} f(x^0) \ge -\max_{1\le i\le k}\|l^i\| > -\infty$. □

3. $\mathcal{L}'_{\|}$-subdifferentiability. Given a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$, a point $x^0 \in \mathbb{R}^n$, an m-dimensional subspace $L \subset \mathbb{R}^n$, and a vector $\zeta \in \mathbb{R}^n$, we introduce the function

$$\tilde{f}_{x^0,L,\zeta}(x) = \begin{cases} f(x^0) + \langle \zeta,\, z\rangle\,, & x = x^0 + z,\ z \in L\,,\\ f(x)\,, & \text{otherwise}\,.\end{cases}$$

With the function f we relate the following condition:

$$\mathbb{C}(f, x^0, L, \zeta): \qquad \inf_{z\in L} \operatorname{Calm} \tilde{f}_{x^0,L,\zeta}(x^0+z) > -\infty\,.$$

Given a function $g : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ and a subspace L, we denote by $g|_L$ the restriction of g on L. The dual space of L is denoted by L^* (that is L^* stands for the set of the linear functionals on L). The notation $\partial g|_L(x^0)$ denotes the subdifferential of $g|_L$ at $x^0 \in L$, that is

$$\partial g|_L(x^0) = \{\ell^* \in L^* \mid g|_L(x) \ge \ell^*(x) - \ell^*(x^0) + g|_L(x^0) \text{ for all } x \in L\}\,.$$

The elements $\ell^* \in \partial g|_L(x^0)$ are called subgradients of $g|_L$ at x^0. From the representation $\ell^*(x) = \langle \zeta, x\rangle$ with appropriate $\zeta \in \mathbb{R}^n$, we can identify the functional ℓ^* with the vectors ζ, considering equivalent any two vectors ζ^1, ζ^2, which difference $\zeta^1 - \zeta^2$ is orthogonal to L. On the basis of this identification the subdifferential $\partial g|_L(x^0)$ is considered as a set of vectors $\zeta \in \mathbb{R}^n$. In this sense in the sequel we use to write $\zeta \in \partial g|_L(x^0)$.

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ and $x^0 \in \operatorname{dom} f$. Let k be integer with $1 \le k \le n$. Then f is $\mathcal{L}_k^0$-subdifferentiable at x^0 if and only if there exists an $(n+1-k)$-dimensional subspace $L \subset \mathbb{R}^n$ with $x^0 \in L$, and there exists $\zeta \in \partial f|_L$, such that condition $\mathbb{C}(f, x^0, L, \zeta)$ is satisfied.*

Proof. Necessity. Let f be $\mathcal{L}_k^0$-subdifferentiable at $x^0 \in \operatorname{dom} f$, and let $\ell \in \partial_{\mathcal{L}_k^0} f(x^0)$. Then $\ell(x) = \min_{1\le i\le k}\langle l^i, x\rangle$ and (1) holds. Actually (1) is a homogeneous linear system with $k-1$ equations, hence of rank at most $k-1$. Taking into account that x^0 also solves (1), we can find an $(n+1-k)$-dimensional subspace $L \subset \mathbb{R}^n$ such that $x^0 \in L$ and $\ell^* := \ell|_L \in L^*$. Restricting the inequality $f(x) - f(x^0) \ge \ell(x) - \ell(x^0)$, $x \in \mathbb{R}^n$, to L, we get $\ell^* \in \partial f|_L(x^0)$. Using this inequality and the representation $\ell^*(x) = \langle \zeta, x\rangle$, $x \in L$, we get

$$\mathrm{Calm}\, \tilde{f}_{x^0,L,\zeta}(x^0+z) = \inf_{\substack{x\in\mathbb{R}^n\\ x\neq x^0+z}} \frac{\tilde{f}_{x^0,L,\zeta}(x) - \tilde{f}_{x^0,L,\zeta}(x^0+z)}{\|x-x^0-z\|}$$

$$\geq \min\left(\inf_{\substack{x\in\mathbb{R}^n\\ x\neq x^0+z}} \frac{f(x)-f(x^0)-\langle\zeta,\, z\rangle}{\|x-x^0-z\|}\,,\ \inf_{\substack{x=x^0+w,\, w\in L\\ w\neq z}} \frac{\tilde{f}_{x^0,L,\zeta}(x)-f(x^0)-\langle\zeta,\, z\rangle}{\|x-x^0-z\|}\right)$$

$$\geq \min\left(\inf_{\substack{x\in\mathbb{R}^n\\ x\neq x^0+z}} \frac{\ell(x)-\ell(x^0)-\langle\zeta,\, z\rangle}{\|x-x^0-z\|}\,,\ \inf_{\substack{w\in L\\ w\neq z}} \frac{\langle\zeta, w-z\rangle}{\|w-z\|}\right)$$

$$\geq \min\left(-\max_{1\leq i\leq k}\|l^i\|\,,\ -\|\zeta\|\right).$$

The right-hand side of this inequality is finite and does not depend on z, whence condition $\mathbb{C}(f, x^0, L, \zeta)$ is satisfied.

Sufficiency. Due to condition $\mathbb{C}(f, x^0, L, \zeta)$ there exists a constant $C > 0$ such that

$$\inf_{z\in L} \mathrm{Calm}\, \tilde{f}_{x^0,L,\zeta}(x^0+z) \geq -C > -\infty\,. \tag{2}$$

Consider the subspace $M = \{x \in \mathbb{R}^n \mid \langle l,\, x\rangle = 0 \text{ for all } l \in L\}$ of $\mathbb{R}^n$ orthogonal to the subspace L. Since M is a $(k-1)$-dimensional subspace, we can find k vectors $m^1, \dots, m^k$ such that their convex hull S, which is a simplex, contains the ball $B = \{x \in M : \|x\| \leq 1\}$. Let $q(x) = \max_{1\leq i\leq k}\langle m^i, x\rangle$ be the support function of S. Since $S \supset B$ and the support function of B is equal to $\|x\|$, it follows that

$$q(x) := \max_{1\leq i\leq k}\langle m^i, x\rangle \geq \|x\|, \quad x \in M\,. \tag{3}$$

Fix $x \in \mathbb{R}^n$ and let $\bar{x}$ be the orthogonal projection of x on L. Then $\bar{x} = \sum_{i=1}^{n+1-k} \langle u^i, x\rangle\, u^i$, where $\{u^1, \dots, u^{n+1-k}\}$ is an orthonormal basis of L. Since $\bar{x} = x^0 + (\bar{x} - x^0) \in x^0 + L$, we have

$$\tilde{f}_{x^0,L,\zeta}(\bar{x}) = f(x^0) + \langle\zeta,\, \bar{x} - x^0\rangle\,. \tag{4}$$

Since $\bar{x} \in L$, from (2) we have

$$\tilde{f}_{x^0,L,\zeta}(x) - \tilde{f}_{x^0,L,\zeta}(\bar{x}) \geq -C\, \|x-\bar{x}\|\,.$$

Due to (3) and $x - \bar{x} \in M$ we get

$$\|x-\bar{x}\| \leq \max_{1\leq i\leq k}\langle m^i, x-\bar{x}\rangle\,,$$

so that

$$\tilde f_{x^0,L,\zeta}(x) - \tilde f_{x^0,L,\zeta}(\bar x) \ge -C\,\|x-\bar x\| \ge -C \max_{1\le i\le k}\langle m^i, x-\bar x\rangle\,.$$

Since $m^i \in M$, $i=1,\dots,k$, and $\bar x$ belongs to the subspace L being orthogonal to M, it follows that $\langle m^i,\bar x\rangle = 0$ for $i=1,\dots,k$. Using these equalities and (4) we obtain

$$\begin{aligned} f(x) \ge \tilde f_{x^0,L,\zeta}(x) &= \Big(\tilde f_{x^0,L,\zeta}(x) - \tilde f_{x^0,L,\zeta}(\bar x)\Big) + \tilde f_{x^0,L\zeta}(\bar x)\\ &\ge -C \max_{1\le i\le k}\langle m^i, x\rangle + f(x^0) + \langle \zeta,\, \bar x - x^0\rangle\\ &\ge -C \max_{1\le i\le k}\langle m^i, x\rangle + f(x^0) + \sum_{i=1}^{n+1-k}\langle u^i,\, x\rangle\,\langle\zeta,\, u^i\rangle - \langle\zeta,\, x^0\rangle\,, \end{aligned}$$

or equivalently

$$f(x) - f(x^0) \ge \min_{1\le i\le k}\langle -C\,m^i + \sum_{i=1}^{n+1-k}\langle\zeta,\, u^i\rangle\, u^i\,,\, x\,\rangle - \langle\zeta,\, x^0\rangle\,. \tag{5}$$

Here we have used the inequality $f(x) \ge \tilde f_{x^0,L,\zeta}(x)$ which needs to be explained only when $x = x^0 + z$, $z\in L$. Then this inequality reduces to

$$f(x^0+z) - f(x^0) \ge \langle\zeta,\, z\rangle\,,\quad z\in L\,,$$

which is true by definition since $\zeta \in \partial f|_L(x^0)$ is a subgradient of the convex function $f|_L$.

Put now

$$l^i = -C\,m^i + \sum_{i=1}^{n+1-k}\langle\zeta,\, u^i\rangle\, u^i\,,\quad i=1,\dots,k\,,$$

and observe that these vectors do not depend on x (from here on x could be considered an arbitrary vector). Define the functional $\ell:\mathbb{R}^n\to\mathbb{R}$ by $\ell(x) = \min_{1\le i\le k}\langle l^i, x\rangle$. We have obviously

$$\begin{aligned} \langle l^i, x^0\rangle &= \langle -C\,m^i\,,\, x^0\,\rangle + \sum_{i=1}^{n+1-k}\langle\zeta,\, u^i\rangle\,\langle u^i,\, x^0\rangle\\ &= \langle \sum_{i=1}^{n+1-k}\langle\zeta,\, u^i\rangle\, u^i\,,\, x^0\,\rangle = \langle\bar\zeta,\, x^0\rangle\,,\quad i=1,\dots,k\,, \end{aligned}$$

where $\bar\zeta = \sum_{i=1}^{n+1-k}\langle\zeta,\, u^i\rangle\, u^i$ is the orthogonal projection of ζ on L. These equalities show that $\ell\in\mathcal{L}_k^0(x^0)$ and

$$\ell(x^0) = \langle\bar\zeta,\, x^0\rangle = \langle\zeta,\, x^0\rangle - \langle\zeta-\bar\zeta,\, x^0\rangle = \langle\zeta,\, x^0\rangle\,.$$

Now inequality (5) can be written as

$$f(x) - f(x^0) \ge \ell(x) - \ell(x^0)\,,$$

which shows that $\ell \in \partial_{\mathcal{L}_k^0(x^0)}$, that is f is $\mathcal{L}_k^0$-subdifferentiable at x^0. $\square$

The next two theorems characterize the $\mathcal{L}_k^0$-subdifferentiability in the case $k \geq n+1$. For shortness the proofs are omitted.

Theorem 2. *The function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is $\mathcal{L}_{n+1}^0$-subdifferentiable at* 0 (*provided* $0 \in \operatorname{dom} f$) *if and only if* $\operatorname{Calm} f(0) > -\infty$. *It is $\mathcal{L}_k^0$-subdifferentiable at* 0 *with $k > n+1$ if and only if it is $\mathcal{L}_{n+1}^0$-subdifferentiable at* 0.

Theorem 3. *The function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is $\mathcal{L}_k^0$-subdifferentiable with $k \geq n+1$ at $x^0 \in \operatorname{dom} f$, $x^0 \neq 0$, if and only if f is $\mathcal{L}_n^0$-subdifferentiable at x^0.*

The following example shows that in Theorem 1 condition $\mathbb{C}(f, x^0, L, \zeta)$ cannot be substituted by $\operatorname{Calm} \tilde{f}_{x^0,L,\zeta}(x^0 + z) > -\infty\,, \quad \forall z \in L$.

Example 1. *The function $f : \mathbb{R}^2 \to \mathbb{R}_{+\infty}$ given by*

$$f(x_1, x_2) = \begin{cases} -\sqrt{|x_1 x_2|}\,, & x_1 \geq x_2^2 \text{ and } |x_2| \geq x_1^2\,, \\ \sqrt{|x_1 x_2|}\,, & x_1 \leq -x_2^2 \text{ and } |x_2| \geq x_1^2\,, \\ 0\,, & x_1\, x_2 = 0\,, \\ +\infty\,, & \text{otherwise}\,. \end{cases}$$

is calm at any $x \in \operatorname{dom} f$, but not $\mathcal{L}_2^0$-subdifferentiable at any nonzero point x^0 of the coordinate axes (*at such a point, choosing L and ζ as in Theorem* 1, *we have* $L = \{tx^0 \mid t \in \mathbb{R}\}$, $\langle \zeta, z \rangle = 0$ *when* $z = tx^0 \in L$, $\tilde{f}_{x^0,L,\zeta}(x) = f(x)$ *whence* $\operatorname{Calm} \tilde{f}_{x^0,L,\zeta}(x^0 + z) = \operatorname{Calm} f(x^0 + z) > -\infty$ *for* $z \in L$, *but* $\inf_{z \in L} \operatorname{Calm} \tilde{f}_{x^0,L,\zeta}(x^0 + z) = -\infty$).

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