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REMARKS ON A 'SERRIN CURVE' FOR SYSTEMS OF DIFFERENTIAL INEQUALITIES

DANIELE CASSANI *

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SUNTO. – Nel presente lavoro consideriamo sistemi ellittici semilineari di disequazioni differenziali in tutto $\mathbf{R}^N$, $N \geq 2$. In particolare, investighiamo la soglia per l'esistenza e non esistenza di soluzioni ultra deboli e senza condizioni di segno. Otteniamo un risultato di non esistenza ottimale utilizzando stime di capacità non lineare.

1. INTRODUCTION

Consider, as a model problem, the following system of differential inequalities

$$\begin{cases} -\Delta u \geq |v|^q \\ -\Delta v \geq |u|^p \end{cases} \tag{1}$$

in all of $\mathbf{R}^N$, $N \geq 2$, and where $p, q > 1$ (i.e. the super-linear case). We address the problem of determining the threshold, depending on p, q and the dimension N, between existence and non-existence of solution for (1) under *minimal* conditions. Before stating our main result, for the convenience of the reader we resume some well known facts

* Università degli Studi di Milano, Dipartimento di Matematica «F. Enriques»; e-mail: Daniele.Cassani@unimi.it

regarding the scalar version of system (1), that is when $u = v$ and $p = q$, besides briefly recalling the *critical* framework when in (1) inequalities are replaced by equations.

So far, semilinear equations of the form

$$-\Delta u = f(u), \quad \text{on} \quad \mathbf{R}^N$$

where $\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplace operator on $\mathbf{R}^N$ and $f\colon \mathbf{R} \to \mathbf{R}$ is a nonlinear function, have been intensively studied (see e.g. [11] for a survey). Indeed, such problems arise in a variety of situations, such as in astrophysics, studying the gravitational equilibrium of stars, in quantum field theory and mechanical statistics (see [20] and references therein). Concerning existence and nonexistence of solutions in the case of a pure power nonlinearity, that is for the equation

$$-\Delta u = |u|^p, \quad \text{on} \quad \mathbf{R}^N \tag{2}$$

with $p > 1$, it is well known that in dimension $N \geq 3$ a key role is played by the critical power [1]

$$\boxed{p_c := \frac{N+2}{N-2}}$$

($p_c = 2^* - 1$, where 2^* is the Sobolev critical exponent). Precisely, it was proved by Gidas and Spruck in [8] that for $1 < p < p_c$, equation (2) has no positive solutions, whence for $p > p_c$ one has existence results (see [17], [20] and also [3]).

As developed in [7], [16] and [15] passing from the differential equation to the differential inequality

$$-\Delta u \geq |u|^p \tag{3}$$

gives rise to the so-called Serrin exponent

$$\boxed{p_s := \frac{N}{N-2}}$$

[1] We mention that the critical power p_c appears naturally in Yang-Mills equations when $N = 4$ and in differential geometry in the context of Yamabe's problem.

which represents the new threshold between existence and nonexistence of solutions for (3), in the sense that (3) has no solutions if $1 < p \le p_s$, with $p_s < p_c$. The result is also sharp since for $p > p_s$ a class of solutions for (3) is explicitly known.

Coming to semilinear elliptic systems, the situation is quite different and in particular the notion of criticality extends. Consider the following system of coupled Poisson's equations:

$$\begin{cases} -\Delta u = |v|^{q-1}v \\ -\Delta v = |u|^{p-1}u \end{cases} \tag{4}$$

in all of $\mathbf{R}^N$ ($N \ge 3$). System (4) consists of the Euler-Lagrange equations related to the functional

$$J(u,v) = \int \nabla u \nabla v \, dx - \frac{1}{p+1}\int |u|^{p+1}\, dx - \frac{1}{q+1}\int |v|^{q+1}\, dx \tag{5}$$

namely, weak solutions of (4) correspond to critical points of the functional J, which posses a strongly indefinite *quadratic* part. Of course, when $u = v$ and $p = q$ we have that (4) and J reduce to

$$-\Delta u = |u|^{p-1}u, \quad j(u) = \frac{1}{2}\int |\nabla u|^2\, dx - \frac{1}{p+1}\int |u|^{p+1}\, dx \tag{6}$$

Denoting with $\mathcal{D}^{1,\alpha}(\mathbf{R}^N)$ the completion of $\mathcal{C}_c^\infty(\mathbf{R}^N)$ with respect to the L^α-norm $\|\nabla\cdot\|_\alpha$, it is then natural to look for solutions of the variational problem in the Sobolev space $\mathcal{D}^{1,2}(\mathbf{R}^N)$, where the functional j is of class $\mathcal{C}^1$ provided $p+1 \le 2^*$, the critical Sobolev exponent; in particular, the critical growth is given by $p+1 = 2^*$, that is $p = p_c$. For system (4) the choice of the function space to set up the problem is of particular interest because of its connection with *criticality*. Indeed, if we consider the functional in (5) defined on $\mathcal{D}^{1,2}(\mathbf{R}^N) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ again we find that maximal growth is given by $p+1 = q+1 = 2^*$. On the other hand, the functional (5) is well defined for $(u,v) \in \mathcal{D}^{1,\alpha}(\mathbf{R}^N) \times \mathcal{D}^{1,\beta}(\mathbf{R}^N)$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, by using Hölder's inequality in the term

$$\int \nabla u \nabla v \, dx$$

and J is of class $\mathcal{C}^1$ by Sobolev's inequalities, provided the nonlinearities satisfy $p+1 \le \alpha^*$, $q+1 \le \beta^*$. Here $\alpha^* := \frac{N\alpha}{N-\alpha}$, $\beta^* := \frac{N\beta}{N-\beta}$ are the critical Sobolev exponents in the embeddings

$$\mathcal{D}^{1,\alpha}(\mathbf{R}^N) \hookrightarrow L^{\alpha^*}(\mathbf{R}^N) \quad \text{and} \quad \mathcal{D}^{1,\beta}(\mathbf{R}^N) \hookrightarrow L^{\beta^*}(\mathbf{R}^N)$$

In this case, new critical growth phenomena occur; in fact, the maximal growth now is given by setting $p+1=\alpha^*$ and $q+1=\beta^*$. This extends the notion of *criticality* and gives rise to a *critical continuum* represented by the so-called *critical hyperbola* [2]:

$$\boxed{\frac{1}{p+1}+\frac{1}{q+1}=1-\frac{2}{N}} \tag{7}$$

in the (p,q) plane with asymptotes $p=\frac{2}{N-2}$ and $q=\frac{2}{N-2}$; see *Fig. 1*. Then, one of the nonlinearities may have *super-critical* growth (with respect to the situation in (6)) provided the other one has a suitable sub-critical growth.

However, the analogy with the scalar case is not yet complete, and several basic questions are still open. In [12], [10] it was shown that, for (p,q) belonging to the *critical hyperbola* (7), system (4) has a (ground state) solution. Moreover, Serrin and Zou in [19] proved the existence of solutions even in the super-critical case (above the critical hyperbola). The conjecture on the validity of a non-existence result, the analogous of the Gidas-Spruck [8] result in the scalar case, for (p,q) in the whole region between the Serrin curve and the critical hyperbola seems to be unsettled at the moment even though there are evidences in this direction. In particular, for (p,q) below the critical hyperbola, it turns out that there are no positive solutions with radial symmetry or satisfying some decay conditions at infinity (see [4] for a survey).

Nevertheless, one may expect that for the system of inequalities (1) the Serrin exponent has to be replaced by a *continuum*. Indeed, the following result was proved by E. Mitidieri in [13]:

[Corollary 2.1, p. 468] *Suppose that $p,q>1$, $N>3$ and*

$$\frac{1}{p+1}+\frac{1}{q+1}\geq 1-\frac{2}{N-2}\max\left(\frac{1}{p+1},\frac{1}{q+1}\right) \tag{8}$$

Then the problem (1) has no positive solutions of class $C^2(\mathbf{R}^N)$.

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[2] This was introduced in [13], see also [2], [9], [5] and [6] for a more general approach by using Sobolev-Orlicz spaces.

Remark 1.1. The equality in condition (8) yields a curve which, as we are going to show, plays the role of the Serrin exponent in the scalar case (notice that when $p = q$ we get the Serrin exponent p_s; see *Fig. 1*), and for this reason we take the liberty to call it *Serrin curve*.

2. Main results

The argument used in [14] relies on the method of spherical means. Exploiting the *nonlinear capacity* methods introduced by S.I. Pohožaev in [18], we next show that the regularity assumptions in the above result can be weakened, as well as the positivity condition removed. The argument we use was developed by Mitidieri-Pohožaev in [15] where many different classes of differential inequalities and in particular quasi-linear systems were considered. Our aim is to exploit this flexible technique to show how the argument becomes simpler in the case of the differential system of inequalities (1) and at the same time giving optimal results.

Remark 2.1. The regularity assumptions on solutions of inequalities are in general more meaningful with respect to equations. Indeed, in contrast to equations, there exists no regularity theory for solutions of differential inequalities, and in fact such a theory cannot exist!

We confine ourselves to the following heuristic but persuasive argument. It is well known from classical elliptic regularity theory that any solution of the equation:

$$-\Delta u = f(u) + g(x)$$

for a suitable function f and where g is a positive function, inherits the features of the right-hand side and belongs to a function space depending on g. On the other hand this solution satisfies the inequality

$$-\Delta u \geq f(u)$$

Therefore, if we have only the inequality and know nothing about its origin, we cannot recover the regularity properties of this solution. Thus the definition of a class of solutions for differential inequalities plays an important role.

Before stating our main result we make precise the notion of solution we deal with:

Definition 2.1. A couple $(u,v) \in L^p_{loc}(\mathbf{R}^N) \times L^q_{loc}(\mathbf{R}^N)$ is a weak solution of (1) provided the following hold

$$\int_{\mathbf{R}^N} |v|^q \varphi \, dx \leq \int_{\mathbf{R}^N} u(-\Delta\varphi)\, dx \tag{9}$$

$$\int_{\mathbf{R}^N} |u|^p \varphi \, dx \leq \int_{\mathbf{R}^N} v(-\Delta\varphi)\, dx \tag{10}$$

for all test functions $\varphi \in \mathcal{C}^2_0(\mathbf{R}^N; \mathbf{R}^+)$.

The main result we prove is the following:

Theorem 1. *Let be $N \geq 3$ and suppose that $p, q > 1$ satisfy (8), namely*

$$\frac{1}{p+1} + \frac{1}{q+1} \geq 1 - \frac{2}{N-2} \max\left(\frac{1}{p+1}, \frac{1}{q+1}\right)$$

Then problem (1) has no nontrivial weak solutions. Moreover, if $N=2$ then the system (1) has no nontrivial weak solutions for any $p, q > 1$.

3. Nonlinear capacity estimates

We first consider the case of dimension $N \geq 3$. Let $(u,v) \in$ $\in L^p_{loc}(\mathbf{R}^N) \times L^q_{loc}(\mathbf{R}^N)$ be a weak solution of (1). Then if $\varphi \in \mathcal{C}^2_0(\mathbf{R}^N; \mathbf{R}^+)$ we have by (9) and Hölder's inequality [3]

$$\int |v|^q \varphi \, dx \leq \left(\int |u|^p \varphi \, dx\right)^{\frac{1}{p}} \left(\int \frac{|\Delta\varphi|^{p'}}{\varphi^{p'-1}}\, dx\right)^{\frac{1}{p'}} \leq \text{ (by (10))} \tag{11}$$

$$\leq \left(\int v(-\Delta\varphi)\, dx\right)^{\frac{1}{p}} \left(\int \frac{|\Delta\varphi|^{p'}}{\varphi^{p'-1}}\, dx\right)^{\frac{1}{p'}} \leq \text{ (again by Hölder)}$$

$$\leq \left(\int |v|^q \varphi \, dx\right)^{\frac{1}{pq}} \left(\int \frac{|\Delta\varphi|^{q'}}{\varphi^{q'-1}}\, dx\right)^{\frac{1}{pq'}} \left(\int \frac{|\Delta\varphi|^{p'}}{\varphi^{p'-1}}\, dx\right)^{\frac{1}{p'}}$$

[3] We use the notation p' to denote the conjugate exponent of p i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, from now on the integration domain is $\mathbf{R}^N$ unless otherwise stated.

and then (12) yields the following estimate

$$\int |v|^q \varphi \, dx \leq \left(\int \frac{|\Delta\varphi|^{q'}}{\varphi^{q'-1}} \, dx \right)^{\frac{q}{q'(pq-1)}} \left(\int \frac{|\Delta\varphi|^{p'}}{\varphi^{p'-1}} \, dx \right)^{\frac{pq}{p'(pq-1)}} \qquad (12)$$
$$= \left\{ \left[\text{cap}\,(\varphi, q')\right]^{q-1} \left[\text{cap}\,(\varphi, p')\right]^{q(p-1)} \right\}^{\frac{1}{pq-1}}$$

where we have set

$$\text{cap}\,(\varphi, \alpha) := \int \frac{|\Delta\varphi|^\alpha}{\varphi^{\alpha-1}} \, dx$$

Remark 3.1. If $K \subset \Omega$ is a compact set and we define

$$\mathcal{C}_0^2(K, \Omega) := \left\{ \psi \in \mathcal{C}_0^2(\Omega) \mid 0 \leq \psi \leq 1, \; \psi \equiv 1 \text{ on } K \right\}$$

then, the nonlinear capacity induced by the operator $A(u) := -\Delta u - u^\alpha$ is given by

$$\text{cap}_A(K, \Omega) = \inf_{\psi \in \mathcal{C}_0^2(K,\Omega)} \text{cap}\,(\psi, \alpha)$$

Now we specialize the test function φ by setting

$$\varphi(x) := \varphi_0^\gamma(x), \quad \text{where} \quad \varphi_0(x) := \psi\left(\frac{|x|}{R}\right) \qquad (13)$$

for a scaling parameter $R > 0$ and where ψ is a smooth and positive standard cut-off function e.g. such that $\psi(s) = 1$ for $0 \leq s \leq 1$ and $\psi(s) = 0$ for $s > 2$. The real parameter $\gamma > 0$ has to be suitably chosen in order to have finite capacity.

Lemma 3.1. *If γ satisfies the condition*

$$\gamma > \max\left\{2q', 2p'\right\} \qquad (14)$$

then the following estimates hold

$$\text{cap}\,\left(\varphi, q'\right) \leq \frac{C_1}{R^{2q'-N}} \quad \textit{and} \quad \text{cap}\,\left(\varphi, p'\right) \leq \frac{C_2}{R^{2p'-N}} \qquad (15)$$

where C_i are positive constants which do not depend on R.

Proof. It is straightforward by the following calculations, where we denote by C_i positive constants depending only on γ, q', N and the choice of ψ but not on the parameter R:

$$\operatorname{cap}\left(\varphi_0^\gamma, q'\right) = \int \frac{|\Delta(\varphi_0^\gamma)|^{q'}}{\varphi_0^{\gamma(q'-1)}}\, dx \leq C \int \varphi_0^{\gamma-2q'} |\nabla\varphi_0|^{2q'} + \varphi_0^{\gamma-q'} |\Delta\varphi_0|^{q'}\, dx$$

$$\leq C \int \frac{\psi^{\gamma-2q'}|\psi'|^{2q'}}{R^{2q'}} + \frac{2^{q'-1}(N-1)^{q'}}{R^{q'}} \frac{\psi^{\gamma-q'}|\psi'|^{q'}}{|x|^{q'}} + \frac{2^{q'-1}}{R^{2q'}} |\psi''|^{q'} \psi^{\gamma-q'}\, dx$$

and performing the substitution $x = \xi R$, the integration domain becomes $1 \leq |\xi| \leq 2$ and one easily gets the first claim

$$\operatorname{cap}\left(\varphi_0^\gamma, q'\right) \leq \frac{C}{R^{2q'-N}}$$

The second estimate in (15) follows in a similar fashion. □

By (12) and Lemma 3.1 we obtain the estimate

$$\int |v|^q \varphi\, dx \leq \left(\frac{C_1^{q-1}}{R^{(2q'-N)(q-1)}} \frac{C_2^{q(p-1)}}{R^{(2p'-N)q(p-1)}} \right)^{\frac{1}{pq-1}} \tag{16}$$

$$= \left(\frac{C}{R^{2q-N(q-1)+2pq-N(p-1)q}} \right)^{\frac{1}{pq-1}}$$

Therefore letting $R \to +\infty$, the monotone convergence theorem yields $v \in L^q(\mathbf{R}^N)$ and $v \equiv 0$ (hence $u \equiv 0$) provided:

$$\boxed{2q - N(q-1) + 2pq - N(p-1)q > 0}$$

By using the same argument but starting with u in place of v, i.e. starting with (11) in (12), we get the analogous estimate for u, namely the following

$$\int |u|^p \varphi\, dx \leq \left(\frac{C}{R^{2p-N(p-1)+2pq-N(q-1)p}} \right)^{\frac{1}{pq-1}}$$

and again, if the following condition is satisfied

$$\boxed{2p - N(p-1) + 2pq - N(q-1)p > 0}$$

we obtain that u is identically zero as well as v by (9).

Remark 3.2. Observe that

$$\begin{cases} 2q - N(q-1) + 2pq - N(p-1)q = 0, & q \geq p \\ 2p - N(p-1) + 2pq - N(q-1)p = 0, & q \leq p \end{cases} \tag{17}$$

yields surprisingly the same curve as in (8)!

It remains to show how to handle the case when p and q lie on the curve (17). This is achieved by observing that, e.g. in the case $p \geq q$, estimate (16) yields the bound

$$\int_{\mathbf{R}^N} |v|^q \varphi \, dx \leq C$$

where C is independent of R and passing to the limit as $R \to +\infty$ we get

$$\int_{\mathbf{R}^N} |v|^q \, dx \leq C \tag{18}$$

Therefore resuming the calculations carried out in (12) we have

$$\int_{\mathbf{R}^N} |v|^q \varphi \, dx \leq C \left(\int_{R \leq |x| \leq \sqrt{2}R} |v|^q \, dx \right)^{\frac{1}{pq}}$$

Thus, by (21), letting $R \to +\infty$ we obtain $v \equiv 0$ also in this case.

Remark 3.3. Condition (9) is sharp, in the sense that for (p, q) satisfying

$$\frac{1}{p+1} + \frac{1}{q+1} < 1 - \frac{2}{N-2} \max\left(\frac{1}{p+1}, \frac{1}{q+1} \right)$$

one can easily check that the the following functions

$$u(x) = \frac{\varepsilon}{(1+|x|^2)^{\frac{q+1}{pq-1}}}, \quad v(x) = \frac{\varepsilon}{(1+|x|^2)^{\frac{p+1}{pq-1}}}$$

yield a solution to system (1), provided that $\varepsilon > 0$ is sufficiently small.

3.1. *The case of dimension* $N=2$

It is a remarkable fact that the computations carried out in section 3 are still meaningful when we come to dimension $N=2$. In particular estimates (16) are still valid. On the other hand we see from (8) that the asymptotes of the Serrin curve, which are the same of that of the critical hyperbola (8), go to infinity and the situation degenerates. Nevertheless, by the previous argument, we can state the following

Corollary 3.1. *Let the dimension* $N=2$ *and consider the following system in all of* $\mathbf{R}^2$

$$\begin{cases} -\Delta u \geq f(v) \\ -\Delta v \geq g(u) \end{cases} \tag{19}$$

where the nonlinearities f and g satisfy, for positive constants C_i,

$$f(t) \geq C_1|t|^q \quad \text{and} \quad g(t) \geq C_2|t|^p, \quad t \in \mathbf{R} \tag{20}$$

Then, problem (19) has no nontrivial weak solutions provided $p, q > 1$.

Remark 3.4. Observe that Definition 2.1 of weak solution now requires

$$f(v),\ g(u) \in L^1_{loc}(\mathbf{R}^2)$$

Remark 3.5. Notice that we do not require any lower bound on the solutions which of course would yield the classical Liouville's theorem because of the parabolic character of the manifold $\mathbf{R}^2$.

This concludes the proof of Theorem 1 stated in Section 2. As a concrete example, one may think of the following

Example 3.1. Let $\lambda, \mu > 0$. Then any weak solution of the following system

$$\begin{cases} -\Delta u \geq \lambda \left(e^{v^2} - 1\right) \\ \\ -\Delta v \geq \mu \left(e^{u^2} - 1\right) \end{cases} \quad \text{in} \quad \mathbf{R}^2 \tag{21}$$

vanishes identically.

Remark 3.6. Observe that the local integrability of the right hand side in (21) holds e.g. for all $(u,v) \in H^1(\mathbf{R}^2) \times H^1(\mathbf{R}^2)$ by means of the Trudinger-Moser inequality (see [1]).

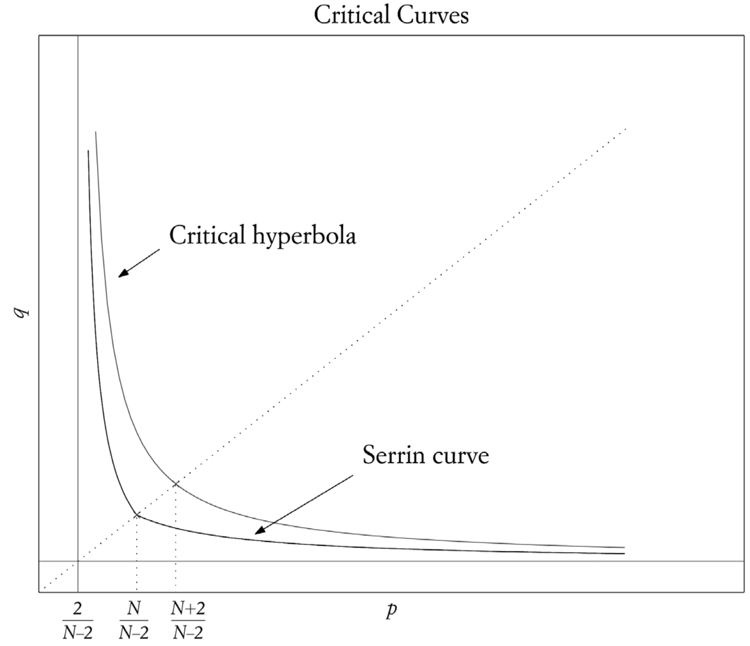

Fig. 1. – The critical hyperbola and the Serrin curve.

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