

Long-Lasting Exponential Spreading in Periodically Driven Quantum Systems

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Using a dynamical model relevant to cold-atom experiments, we show that long-lasting exponential spreading of wave packets in momentum space is possible. Numerical results are explained via a pseudoclassical map, both qualitatively and quantitatively. Possible applications of our findings are also briefly discussed.

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Apart from idealized exceptions like inverted oscillators [1], exponential growth of observable quantities is thought to be a rare occurrence in small quantum systems. Exponential instability, which is the landmark of classical chaos, does not give rise to long-lasting, physically observable exponential growth, not even in classical mechanics, because exponentially fast motion along unstable manifolds is concealed as such manifolds fold in homoclinic or heteroclinic tangles. Moreover, it does not survive in quantum mechanics beyond a short time scale $t_E \approx -\ln(\hbar)/\lambda$ [2], where λ is the maximal Lyapunov exponent. Using a simple dynamical model relevant to today's cold-atom experiments, we show in this work that exponential quantum spreading (EQS) of wave packets in momentum space does exist for a significantly long time scale. This is achieved by exploiting two intriguing mechanisms at once. First, our quantum system is set close to a quantum resonance condition, in a “pseudoclassical” (yet strictly quantum) regime where the quantum dynamics follows trajectories of a *fictitious* classical system, which emerges at small values of a “pseudo-Planck” constant \hbar^* . This “constant” \hbar^* measures the detuning from exact quantum resonance and can easily be made quite small by varying physical parameters while keeping the system in a deeply quantum regime. Second, appropriate driving fields are chosen such that a pseudoclassical stable manifold of an unstable fixed point can support easy-to-prepare quantum states. As shown below, these factors concur in eventually generating EQS in momentum space (or exponential energy increase) as a transient but long-lasting dynamical phenomenon, whose duration is given by $(A/\hbar^*) \ln(C/\hbar^*)$ (A and C are constants for small \hbar^*) and therefore can be enlarged to a very long time scale as parameter \hbar^* decreases. The exponential rate of EQS scales linearly with \hbar^* and is typically small for close-to-resonance situations, yet the EQS momentum scale can

be huge because it scales with $1/\hbar^*$. Such long-lasting EQS does not rest on the chaotic motion which is present in the *classical* limit but on quasi-integrable motion which is instead present in the *pseudoclassical* limit.

The dynamical model used here is a simple modification of the well-known kicked-rotor model [2], a paradigm of quantum chaos that has been extensively studied both theoretically and experimentally. For all previous variants of such a type of driven systems, quantum spreading is known to range from ballistic propagation to superdiffusion as well as standard linear diffusion followed by dynamical localization [2]. Our finding of long-lasting EQS is thus an intriguing result in quantum chaos. In a broader context, the EQS reported here is different from other forms of fast wave-packet spreading like, e.g., the “superballistic” diffusion found in a lattice model [3] (where the wave-packet spreading can be a cubic function of time) and the transient “hyperdiffusion” found in generalized Brownian motion [4] (where a square variance may increase at the fifth power of time). Furthermore, analogous to the novel exponential acceleration of classical particles [5], the possibility of long-lasting EQS might be useful for designing some robust acceleration methods for quantum systems.

Consider then the following Floquet propagator for a double-kicked-rotor model:

$$\hat{U} = e^{-i(T-T_0)(\hat{p}^2/2\hbar)} e^{-i(K/\hbar)\cos(\hat{q})} e^{-iT_0(\hat{p}^2/2\hbar)} e^{-i(K/\hbar)\cos(\hat{q})}, \quad (1)$$

with all quantities in dimensionless units. Throughout, \hat{q} and \hat{p} are the canonical coordinate and momentum operators, respectively. This system was first experimentally studied in Ref. [6] by subjecting cold atoms to two period- T δ -kicking sequences of an optical lattice potential, time-shifted by $T_0 < T$ with respect to each other. Under the so-called quantum main resonance condition

$T\hbar = 4\pi$, achievable in a number of cold-atom experiments [7–11], we obtain an “on-resonance double-kicked-rotor” model (RDKR) [12,13], described by the Floquet propagator:

$$\hat{U}_R = e^{-i(4\pi/\hbar - T_0)(\hat{p}^2/2\hbar)} e^{-i(K/\hbar)\cos(\hat{q})} e^{-iT_0(\hat{p}^2/2\hbar)} e^{-i(K/\hbar)\cos(\hat{q})}. \quad (2)$$

Because of spatial periodicity of the kicking potential $\cos(q)$, changes in momentum p must occur in integer multiples of \hbar . The momentum eigenvalues of the system are hence given by $m\hbar + \beta\hbar$, where $\beta \in [0, 1)$ is a quasi-momentum variable determined by the initial state and conserved by the dynamics. In cold-atom experimental realizations of the kicked-rotor model, the initial state is a mixture of states with a certain spread of β , where the spread can be made small by loading a Bose-Einstein condensate into the lattice such that the initial-state coherence spans over many lattice constants [9–11]. For $\beta = 0$, the Floquet spectrum of \hat{U}_R plotted against $T_0\hbar \in [0, 4\pi]$ yields the famous Hofstadter’s butterfly structure [13], with the spectra identical with that of the kicked-Harper model [14] for arbitrary irrational values of $T_0\hbar/\pi$ [15].

Representative numerical EQS results for the quantum map \hat{U}_R are depicted in Fig. 1, with the time dependence of the mean value $\langle n^2 \rangle \equiv \langle \hat{p}^2/\hbar^2 \rangle$ plotted in a log scale for a zero-momentum initial state (hence $\beta = 0$). As seen clearly from Fig. 1, apart from a short transient initial stage, $\log(\langle n^2 \rangle)$ displays a striking linear dependence on the number of iterations of \hat{U}_R (denoted by t). That is,

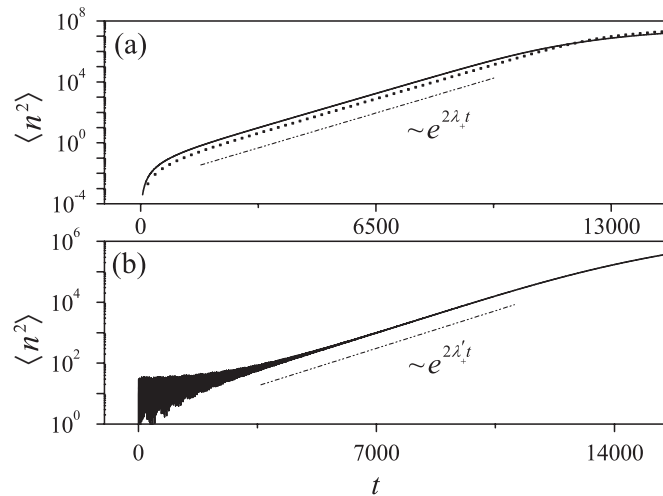


FIG. 1. Two examples of long-lasting exponential quantum spreading (solid lines) in RDKR [see Eq. (2)]. (a) $K = 5$, $T_0 = 1$, and $T_0\hbar = 2\pi + \hbar^*$ with $\hbar^* = 10^{-3}$. The dotted line represents the pseudoclassical result, and the dashed-dotted line represents $\sim e^{2\lambda_+ t}$ with λ_+ found analytically. (b) $K = 9$, $T_0 = 1$, and $T_0\hbar = 30\pi/13 + 1.7 \times 10^{-4}$. The dash-dotted line represents the best exponential fitting (shifted) that suggests an exponential rate of $\lambda'_+ = 4.1 \times 10^{-4}$. Here and in all other figures, plotted quantities are in dimensionless units.

Fig. 1 demonstrates *exponential* spreading in momentum space, with the exponential behavior sustaining over a wide range of time and momentum scales. For example, in Fig. 1(a), the exponential spreading occurs from $t \approx 10^3$ up to $t \approx 10^4$, and $\langle n^2 \rangle$ increases from 10^{-1} to 10^6 . For the case in Fig. 1(b), the exponential spreading also brings $\langle n^2 \rangle$ from 10^2 to 10^5 , over a time scale of the order of 10^4 . In all cases when such exponential behavior is observed in our extensive numerical studies, $\hbar T_0$ is close to $2\pi M/N$, with M and N being odd integers. For instance, $\hbar T_0$ is close to 2π for the case in Fig. 1(a), whereas for the case of Fig. 1(b) it is slightly away from $30\pi/13$.

To explain the EQS discovered in our numerical work, in our theoretical analysis we exploit the above-mentioned close-to-resonance condition. To that end, we consider cases when $\hbar T_0$ is close to a low-order resonance, i.e., $\hbar T_0 = 2\pi + \hbar^*$ with $|\hbar^*| \ll 1$. The reason for the notation \hbar^* will become manifest in Eq. (3). We note that under the precise condition $\hbar T_0 = 2\pi$ and $\beta = 0$ we have a so-called quantum antiresonance and that interesting aspects of long-time quantum diffusion in the vicinity of quantum antiresonance were previously analyzed for a different model, describing a kicked charge in a magnetic field [16]. The long-time asymptotic behavior of quantum motion is not, however, in the scope of the present work, as we are instead interested in the transient, though long-lasting EQS. We now proceed by adopting a pseudoclassical approximation which was successfully implemented in the study of other close-to-resonance phenomena observed in cold-atom optics [17]. That is, quantum dynamics under close-to-resonance conditions, and hence far away from its proper classical limit, may be nevertheless extremely close to the behavior of a fictitious classical system, unrelated to the classical limit. Cases with $\beta = 0$ will be studied first. For convenience, we also set $\hbar^* > 0$. With such specifications, the first factor in \hat{U}_R in Eq. (2) reduces to $e^{iT_0(\hat{p}^2/2\hbar)}$, a time reversal of the normal free-rotor propagator and a feat experimentally achieved in Ref. [10]. Next we introduce the rescaled kicking strength $\tilde{K} \equiv K\hbar^*/\hbar$ and the rescaled momentum $\tilde{p} \equiv \hat{p}\hbar^*/\hbar$. As $\beta = 0$, \tilde{p} is quantized in integer multiples of \hbar^* , so the relation $e^{\pm iT_0(\hat{p}^2/2\hbar)} = e^{\pm(i/\hbar^*)(\tilde{p}^2/2 + \pi\tilde{p})}$ holds true, and using it in Eq. (2) yields

$$\hat{U}_R^{(\beta=0)} = e^{(i/\hbar^*)(\tilde{p}^2/2 + \pi\tilde{p})} e^{-(i/\hbar^*)\tilde{K}\cos(\hat{q})} e^{-(i/\hbar^*)(\tilde{p}^2/2 + \pi\tilde{p})} \times e^{-(i/\hbar^*)\tilde{K}\cos(\hat{q})}. \quad (3)$$

In Eq. (3), \hbar^* manifestly plays a Planck-constant-like role, so that in the pseudoclassical limit when $\hbar^* \rightarrow 0$, the quantum dynamics described by (3) comes closer and closer to that of a classical double-kicked rotor whose kinetic energy term alternates between $\pm(\frac{\tilde{p}^2}{2} + \pi\tilde{p})$ but with a fixed kicking potential $\tilde{K}\cos(q)$. The associated pseudoclassical map can then be obtained directly:

$$\begin{aligned} q_{n+1} &= q_n + \tilde{K} \sin(\tilde{p}_{n+1} + q_{n+1}), \\ (\tilde{p}_{n+1} + q_{n+1}) &= (\tilde{p}_n + q_n) + \tilde{K} \sin(q_n). \end{aligned} \quad (4)$$

Remarkably, this map can be recognized as a kicked-Harper model [14], with the canonical pair identified as q and $\tilde{p} + q$. This map has unstable fixed points at $q = m\pi$, and $\tilde{p} + q = n\pi$, with $m + n$ being an even number, and Lyapunov exponents $\lambda_{\pm} = \ln[(\tilde{K}^2 + 2 \pm \sqrt{\tilde{K}^4 + 4\tilde{K}^2})/2]$. In the limit of $\tilde{K} \rightarrow 0$, stable and unstable manifolds of such fixed points connect in a periodic network, and phase space is divided into cells in the shape of parallelograms, with two sides parallel to the q (zero-momentum) axis and the other two inclined by $\arctan(2)$. For small nonzero \tilde{K} , splitting of such separatrices occurs, giving birth to an exponentially thin stochastic web. Figure 2(a) depicts a typical phase-space structure, which is indeed a portrait deformed from that of a kicked-Harper model [14]. The almost straight lines in Fig. 2(a) are the stochastic web.

The exponential spreading seen in Fig. 1(a) can now be understood with our pseudoclassical map. The initial state $p = 0$ corresponds to an ensemble placed almost perfectly astride the stable manifold of the unstable fixed point at $(q, \tilde{p}) = (\pi, 0)$. The dynamics first pulls this ensemble towards this fixed point and then repels the ensemble away from the fixed point at the exponential rate λ_+ along the unstable manifold. A key point is that, although this exponential increase involves a maximum change of 2π of the pseudoclassical momentum \tilde{p} , it translates into exponential behavior of the physical momentum p over a huge scale for $|\hbar^*| \ll 1$, because $p \sim \tilde{p}\hbar/\hbar^*$.

Figure 1(a) shows that results of pseudoclassical and quantum calculations nicely agree on the ensemble level. It is interesting to note that because the separatrix structure in the standard kicked-Harper model [14] is tilted away (45° for vanishing kicking strength) from the

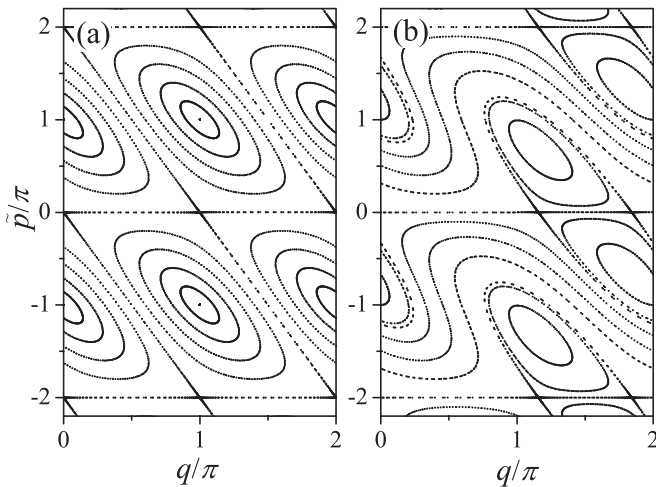


FIG. 2. Phase-space portrait of the pseudoclassical limit of RDKR for $K = 5$, $T_0 = 1$, and $T_0\hbar = 2\pi + \hbar^*$ with $\hbar^* = 3\sqrt{2} \times 10^{-4}\pi$. $\beta = 0$ in (a) and $\beta = 4 \times 10^{-5}$ in (b).

zero-momentum axis, our exponential spreading mechanism cannot be generated there, because a state sitting on the stable branch of a separatrix would be delocalized on a huge momentum scale right at the start. It should also be stressed that this mechanism does not rest on chaos; although chaotic diffusion inside the thin stochastic web is itself an intriguing phenomenon [18], EQS is instead an outcome of quasi-integrability.

The time scale of the found exponential spreading, denoted by t_{exp} , can also be estimated. Here we sketch a crude heuristic argument, leaving a more formal version for the Supplemental Materials [19]. For a pseudoclassical ensemble aligned with the q (zero-momentum) axis with a momentum width $\delta\tilde{p}$, exponential momentum growth $\tilde{p}(t) \propto \delta\tilde{p} \exp(\lambda_+ t)$ along the unstable manifold continues until $\tilde{p}(t) \sim \pi$, and therefore $t_{\text{exp}} \sim \lambda_+^{-1} \ln(\pi/\delta\tilde{p})$. It can be shown that $\delta\tilde{p} \propto \hbar^*$ for the zero-momentum quantum initial state, so, on account of $\lambda_+ \approx \tilde{K} = K\hbar^*/\hbar$ (for small \tilde{K}), we obtain $t_{\text{exp}} \sim (\hbar/K\hbar^*) \ln(\pi/\hbar^*)$. Figure 3 shows that this theoretical scaling with \hbar^* is in excellent agreement with the numerically found time scale of EQS. Additional numerical results (not shown) of the K dependence of t_{exp} also agree with our theory. It is now clear that as \hbar^* decreases and exact antiresonance is approached, t_{exp} increases and λ_+ decreases [20].

To motivate experimental interest, we now examine the effects of a nonzero quasimomentum β . Let us generalize our previous definition of \tilde{p} , i.e., $\tilde{p} \equiv |\hbar^*|(p - \beta\hbar)/\hbar$. Still focusing on $T_0\hbar = 2\pi + \hbar^*$ as an example, we find the following pseudoclassical map associated with U at $\beta \neq 0$:

$$\begin{aligned} \tilde{p}'_n &= \tilde{p}_n + \tilde{K} \sin(q_n), \\ q'_n &= q_n + \tilde{p}'_n + (2\beta + 1)\pi, \\ \tilde{p}_{n+1} &= \tilde{p}'_n + \tilde{K} \sin(q'_n), \\ q_{n+1} &= q'_n - \tilde{p}_{n+1} + (2\beta + 1)\pi. \end{aligned} \quad (5)$$

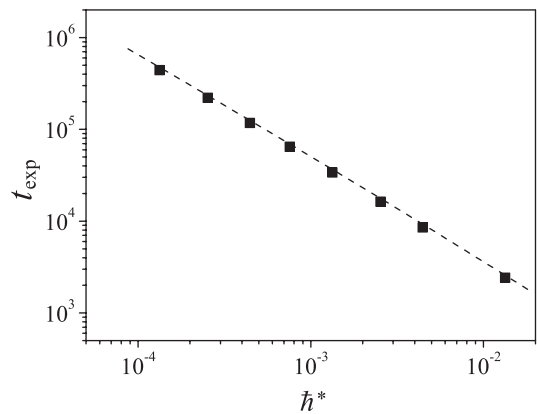


FIG. 3. Numerical t_{exp} (after which the spreading in momentum space deviates from the exponential law) for RDKR (black squares) with $K = 1$, $\beta = 0$, $T_0\hbar = 2\pi + \hbar^*$, and a varying \hbar^* . The dashed line represents the theoretical prediction.

Here \tilde{p}'_n and q'_n are two intermediate variables. Note also that this mapping reduces to Eq. (4) if $\beta = 0$. The phase-space structure of the mapping in Eq. (5) is still periodic in \tilde{p} with a period 2π , with one example shown in Fig. 2(b). Within the regime $-\pi \leq \tilde{p} < \pi$, there are two unstable fixed points located at $\tilde{p} = 2\beta\pi$, $q = 2\pi - \xi$, or $q = \pi + \xi$, with $\xi \equiv \arcsin(4\pi\beta/\tilde{K})$. As indicated by Fig. 2(b), for a sufficiently small β , the stable direction of the two unstable fixed points is still almost parallel to the q axis, and the unstable direction is still extended in \tilde{p} . The exponential spreading mechanism hence survives. The exponential divergence rate can be found from a linear stability analysis of the fixed points. A simple calculation gives that the rate is now dependent on β , i.e., $\lambda_+(\beta) = \ln[(\tilde{K}^{\prime 2} + 2 + \sqrt{\tilde{K}^{\prime 4} + 4\tilde{K}^{\prime 2}})/2]$, with $\tilde{K}^{\prime 2} = \tilde{K}^2 - 16\pi^2\beta^2$. As such, $\lambda_+(\beta)$ is maximal for $\beta = 0$. This indicates that, in the presence of a distribution of values of β centered at $\beta = 0$, the spreading dynamics will be dominated by the $\beta = 0$ component. On the other hand, when $|\beta| \geq \beta_{\text{cr}} \equiv \tilde{K}/4\pi = K\hbar^*/(4\pi\hbar)$, the fixed points disappear, and then the pseudoclassical mechanism for EQS is lost.

In Fig. 4(a), we numerically examine how a nonzero β affects the exponential spreading behavior. There it is seen that an exponential spreading law eventually turns into a periodic oscillation as β increases. Using this transition, we have also numerically extracted critical β values as a function of K and then compared them with our theoretical analysis of β_{cr} . Such a comparison is shown in Fig. 4(b), with excellent agreement. The important message from Fig. 4 is then the following: To observe exponential spreading with nonzero β values, it is advantageous to increase β_{cr} , which can be achieved by increasing the kicking strength K . Assuming that experimental uncertainty in β is around $\Delta\beta = 10^{-3}$ and that the near-resonance condition can be achieved at an accuracy of $\hbar^*/(T_0\hbar) \approx 10^{-3}$, then this requires $KT_0 \geq 4\pi$, which is in the appropriate regime for current cold-atom experiments [8–11].

As illustrated in Fig. 1(b), EQS is also observed near higher-order resonances, i.e., $T_0\hbar \approx 2\pi M/N$. There the pseudoclassical approach requires nontrivial modifications [21], which are not examined in the present work. Finally, we briefly mention some interesting applications of the EQS found here. First, if we introduce a phase shift between the two δ -kicking potentials as in Ref. [22] and then tune $T_0\hbar$ close to a resonance condition, we find that not only $\langle n^2 \rangle$ but even the current (i.e., the momentum expectation value) can increase exponentially over a long time scale. A novel type of ratchet accelerator can hence be formed. Second, because the dynamical model used here can be equally realized in spin chain models kicked by an external parabolic magnetic field [12,23], an exponential propagation of quantum excitation along a spin chain is in principle possible.

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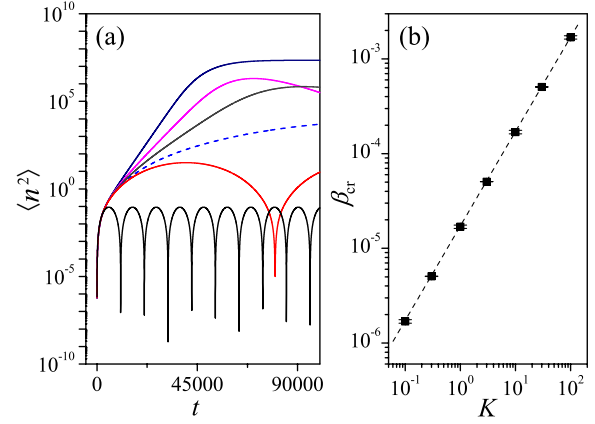


FIG. 4 (color online). (a) Effects of nonzero β on exponential quantum spreading. $K = 1$, $T_0 = 1$, and $T_0\hbar = 2\pi + \hbar^*$ with $\hbar^* = 3\sqrt{2} \times 10^{-4}\pi$. Solid lines (from top to bottom) are for $\beta/10^{-5} = 0, 1.2, 1.5, 1.8$, and 5 , respectively. The dashed line is for $\beta = \beta_{\text{cr}} \approx 1.688 \times 10^{-5}$. The initial state is a momentum eigenstate with $p = \beta\hbar$. (b) Comparison of the numerical β_{cr} (black squares) with theoretical β_{cr} for $\hbar^* = 3\sqrt{2} \times 10^{-4}\pi$ and a varying K .

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