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BEST CONSTANTS FOR MOSER TYPE INEQUALITIES IN ZYGMUND SPACES

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Dedicated to Professor J. V. Gonçalves on the occasion of his 60^{th} birthday

Abstract

We first survey some recent results on optimal embeddings for the space of functions with $\Delta u \in L^1(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain. The target space in the embeddings turns out to be a *Zygmund space* and the best constants are explicitly known. Remarkably, the best constant in the case of zero boundary data is twice the best constant in the case of compactly supported functions. Then, following the same strategy, we establish a new version of the celebrated Trudinger–Moser inequality, as embedding into the Zygmund space $Z_0^{1/2}(\Omega)$, and we prove that, in contrast to the Moser case, here the best embedding constant is not attained.

1 Introduction

Brezis and Merle in the seminal paper [5] considered the following problem

$$\begin{cases} -\Delta u = f(x) \in L^1(\Omega) \\ u = 0, \quad \text{on } \partial\Omega \end{cases} \tag{1}$$

for a bounded domain $\Omega \subset \mathbb{R}^2$. In the *non-borderline* case in which $f \in L^p(\Omega)$, $p > 1$, elliptic regularity theory yields $u \in W^{2,p}(\Omega)$; in analogy, one may conjecture that in the *borderline case* $p = 1$, one has $u \in W^{2,1}(\Omega)$: this is false! Indeed, the Sobolev embedding for the space $W^{k,p}(\Omega)$ when $p = 1$ and $N = k = 2$ gives $W^{2,1}(\Omega) \hookrightarrow L^\infty(\Omega)$, whereas the maximal summability for solutions to (1) is of exponential type and namely

$$\int_\Omega e^{\lambda|u|}\,dx < \infty, \quad \forall\,\lambda > 0 \tag{2}$$

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as established in [5], where actually the following stronger uniform bound has been proved

$$\sup_{\|\Delta u\|_1=\|f\|_1=1} \int_\Omega e^{\lambda|u|}dx \begin{cases} \leq C(\lambda)\ , & \text{if } \lambda < 4\pi \\ = +\infty\ , & \text{if } \lambda \geq 4\pi \end{cases} \tag{3}$$

Inequality (3) is in the spirit of the Moser improvement [14] of the Pohožaev [16] and Trudinger [19] embedding in the so-called *Sobolev limiting case* $N = kp$, $p > 1$, which for $k = 1$ and $N = p = 2$ reads

$$\sup_{\substack{u\in W_0^{1,2}(\Omega)\\ \|\nabla u\|_2\leq 1}} \int_\Omega e^{\alpha u^2}\,dx \begin{cases} \leq C(\alpha)\ , & \text{if } \alpha \leq 4\pi \\ = +\infty\ , & \text{if } \alpha > 4\pi \end{cases} \tag{4}$$

Actually, from the point of view of Sobolev regularity, one has that solutions to (1) belong to $W_0^{1,q}(\Omega)$ for all $q < 2$ (sometimes called the *grand Sobolev space* $W_0^{1,2)}(\Omega)$), see [8, 11], but not to $W_0^{1,2}(\Omega)$, see [2, 15]; this explains the gap in the degree of integrability between (3) and (4).

Besides the above considerations, in the case of compactly supported functions, a celebrated result by D.R. Adams [1], which extends Moser's result to the higher order space $W_0^{k,\frac{N}{k}}(\Omega)$, $k > 1$, entails in particular the following

$$\sup_{\substack{u\in W_0^{2,\frac{N}{2}}(\Omega)\\ \|\Delta u\|_{\frac{N}{2}}\leq 1}} \int_\Omega e^{\beta|u|^{\frac{N}{N-2}}} \begin{cases} \leq C(\beta)\ , & \text{if } \beta \leq \beta_N \\ = +\infty\ , & \text{if } \beta > \beta_N \end{cases} \qquad \boxed{N\geq 3} \tag{5}$$

where β_N is explicit. Such results seem uninteresting when $N = 2$ since, as we have already recalled, $W_0^{2,1}(\Omega) \hookrightarrow L^\infty(\Omega)$.

However, the discussion carried out so far motivates the introduction of the following function spaces

$$\begin{aligned} W_\Delta^{2,p}(\Omega) &:= cl\Big\{u \in \mathcal{C}^\infty(\Omega)\cap\mathcal{C}^0(\overline{\Omega}),\ u|_{\partial\Omega} = 0\ :\ \|\Delta u\|_p < \infty\Big\} \\ W_{\Delta,0}^{2,p}(\Omega) &:= cl\Big\{u \in \mathcal{C}_c^\infty(\Omega)\ :\ \|\Delta u\|_p < \infty\Big\} \end{aligned}$$

which do coincide (with equivalence of norms, see [12]) respectively with $W^{2,p}\cap W_0^{1,p}(\Omega)$ and $W_0^{2,p}(\Omega)$ if and only if $p > 1$.

In [7] we addressed the problem of establishing optimal embeddings for the spaces $W_\Delta^{2,1}(\Omega)$ and $W_{\Delta,0}^{2,1}(\Omega)$ into the class of exponentially integrable functions (2): those results, which we summarize in Section 2, improve the Brezis–Merle result (3) and give a natural extension of the Adams inequality (5) to the case $N = 2$.

Notice that the integrability condition (2) is exactly the one which fulfills the definition of *Zygmund space* $Z_0(\Omega)$. More in general, for $\alpha > 0$ the Zygmund space $Z_0^\alpha(\Omega)$ consists of all measurable functions $u(x)$ on $\Omega \subset \mathbb{R}^N$, such that

$$\int_\Omega e^{\lambda|u|^{\frac{1}{\alpha}}}\,dx < \infty, \quad \forall\,\lambda > 0 \tag{6}$$

It is easy to verify (see [3]) that

$$Z_0^\alpha(\Omega) = \left\{u\ :\ \lim_{t\to 0}\frac{u^*(t)}{\left[1+\log\left(\frac{|\Omega|}{t}\right)\right]^\alpha} = 0\right\}$$

where $u^* : [0,\infty) \longrightarrow [0,\infty]$ denotes the Hardy–Littlewood positive decreasing rearrangement of $u : \Omega \subset \mathbb{R}^N \longrightarrow \mathbb{R}$, namely

$$u^*(t) := \sup\left\{s > 0\ :\ \left|\left\{x \in \mathbb{R}^N\ :\ |u(x)| > s\right\}\right| > t\right\}, \quad t \geq 0$$

The quantity

$$\|u\|_{Z^\alpha} := \sup_{t\in(0,|\Omega|)} \frac{u^*(t)}{\left[1+\log\left(\frac{|\Omega|}{t}\right)\right]^\alpha}$$

defines a quasinorm on $Z_0^\alpha(\Omega)$ which turns out to be equivalent to a real norm (by replacing $u^*(t)$ with the Hardy–Littlewood maximal function $\frac{1}{t}\int_0^t u^*(s)\,ds$). The space Z_0^α can be viewed as the limiting case of *Lorentz–Zygmund spaces* $L^{p,q}(\log L)^{-\alpha}$, as $p, q \to \infty$, see [3] and also [4], as well as the Orlicž space $L_{e^{|u|}}$, see [17], equipped with a different quasinorm.

Our main purpose here is to revisit the Pohožaev and Trudinger embedding in the limiting Sobolev case $H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain.

Indeed, the strategy used in [7] suggests to investigate the embedding

$$H_0^1(\Omega) \hookrightarrow L_{e^{u^2}}(\Omega) \tag{7}$$

from a different point of view than that of Moser, who focuses the attention more on the optimal uniform bound (4) than on the sharp version of the embedding inequality in (7). At our knowledge, the best constant for the embedding (7) has not been investigated before; recently, closely related results appeared in [9]. Moreover, it is well known that Moser's inequality is attained [6, 10]; in contrast, we show that the best constant in the embedding inequality for (7) is not attained. We borrow some ideas of [7] in order to prove the following main results:

Theorem 1 *Let Ω be a bounded domain in $\mathbb{R}^2$. Then, the following inequality holds*

$$\|u\|_{Z^{1/2}} \leq \frac{1}{\sqrt{4\pi}} \|\nabla u\|_2 \tag{8}$$

for any $u \in H_0^1(\Omega)$. Moreover, the constant appearing in (8) is sharp.

Theorem 2 *Let $\Omega = B_R$, the ball in $\mathbb{R}^2$ centered at the origin and radius R. Then, the best constant in (8), namely*

$$\frac{1}{\sqrt{4\pi}} = \sup_{\substack{u \in H_0^1(B_R) \\ \|\nabla u\|_2 = 1}} \|u\|_{Z^{1/2}}$$

is not achieved.

2 On borderline cases in second order Moser inequalities

Next we summarize the results obtained in [7] in the case $N = 2$, and for the sake of simplicity we set $Z_0^1(\Omega) = Z(\Omega)$. It is worth to point out that the notation $L_{exp}(\Omega)$ sometimes refers in literature to the Zygmund space of

functions which enjoy the integrability condition (2) just for some $\lambda > 0$; this is a strictly larger space than $Z(\Omega)$ in which $L^\infty(\Omega)$ is not dense, while the closure of $L^\infty(\Omega)$ in $L_{exp}(\Omega)$ is precisely $Z(\Omega)$; see [3, Theorem D].

Theorem 3 *Let N=2 and $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then, the following inequalities hold:*

$$\|u\|_Z \leq \frac{1}{4\pi} \|\Delta u\|_1, \qquad \forall\, u \in W_\Delta^{2,1}(\Omega) \tag{9}$$

$$\|u\|_Z \leq \frac{1}{8\pi} \|\Delta u\|_1, \qquad \forall\, u \in W_{\Delta,0}^{2,1}(\Omega), \quad u \geq 0 \tag{10}$$

Moreover, the constants in (9)-(10) are the best possible for any domain.

The next result deals with maximal summability properties for solutions to (1). From one side, it is an improvement of the Brezis–Merle result (3), on the other side, in the case of compactly supported functions, it yields a natural extension of Adams' result (5), to the case $p = 1$.

Corollary 1 *Let Ω be a bounded domain in $\mathbb{R}^2$. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $e^{\alpha t}\Phi(t)$, $\alpha > 0$, is increasing as $t \to +\infty$. Then there exists a constant $C = C(\alpha, \Phi) > 0$ such that*

$$\sup_{\|\Delta u\|_1 = 1} \int_\Omega e^{\alpha|u|}\Phi(|u|)\, dx \leq C|\Omega| \begin{cases} \text{if } \alpha \leq 4\pi, & \forall\, u \in W_\Delta^{2,1}(\Omega) \\ \\ \text{if } \alpha \leq 8\pi, & \forall\, u \in W_{\Delta,0}^{2,1}(\Omega), \quad u \geq 0 \end{cases} \tag{11}$$

if and only if $\Phi(t)$ is integrable near infinity.

Actually the validity of (10) and (11) in the case of compactly supported functions, extends, in the sharp form, also to the whole subspace of $W_{\Delta,0}^{2,1}(\Omega)$ consisting of radially symmetric functions. Let us sketch the proof of (9)–(10) in the case of radially symmetric functions:

- Assume first $v \in \mathcal{C}^2(B_R) \cap \mathcal{C}^0(\overline{B_R})$ and vanishing at the boundary ∂B_R. Let us define

$$w(t) := 4\pi\, v(Re^{-t/2}) \;,\; r = Re^{-t/2} \in (0, R]$$

Then $w \in C^2(0,\infty), w(0) = 0$ and we have

$$\begin{aligned} w'(t) &= -2\pi v_r(Re^{-t/2})Re^{-t/2}, \\ w'(\infty) &= 0, \\ w''(t) &= \pi v_{rr}(Re^{-t/2})R^2e^{-t} + \pi v_r(Re^{-t/2})Re^{-t/2} \\ &= \pi R^2 e^{-t}\left[v_{rr}(Re^{-t/2}) + \frac{v_r(Re^{-t/2})}{Re^{-t/2}}\right] \\ &= \pi R^2 e^{-t}\Delta v \mid_{|x|=Re^{-t/2}} \end{aligned}$$

so that

$$\begin{aligned} \|\Delta v\|_1 &\leq 2\pi \int_0^R \left|\left(v_{rr} + \frac{v_r}{r}\right)\right| r dr \\ &= \pi \int_0^\infty \left|R^2e^{-t}v_{rr}(Re^{-t/2}) + Re^{-t/2}v_r(Re^{-t/2})\right| dt \\ &= \int_0^\infty |w''(t)|\, dt \end{aligned}$$

Next notice that

$$w(t) = \int_0^t w'(s)ds = \int_0^t \left(-\int_s^\infty w''(z)dz\right)ds \leq t\|\Delta v\|_1$$

Therefore

$$v(r) = \frac{1}{4\pi} w\left(2\log\left(\tfrac{R}{r}\right)\right) \leq \frac{\|\Delta v\|_1}{2\pi}\log\left(\tfrac{R}{r}\right) \tag{12}$$

Since the decreasing rearrangement is monotone and positively homogeneous (see [4]), we obtain from (12)

$$v^*(t) \leq \frac{\|\Delta v\|_1}{2\pi}\left[\log\left(\tfrac{R}{r}\right)\right]^*(t) = \frac{\|\Delta v\|_1}{4\pi}\log\left(\tfrac{\pi R^2}{t}\right)$$

which implies (9).

- In the case of compactly supported functions $w \in W^{2,1}_{\Delta,0}(\Omega)$, notice that thanks to the homogeneous boundary conditions we have

$$w'(t) = -\int_t^{+\infty} w''(z)dz = \int_0^t w''(z)dz$$

since $w'(0) = w'(\infty) = 0$. Therefore,

$$\begin{aligned} 2|w'(t)| &= \left|-\int_t^{+\infty} w''(z)dz + \int_0^t w''(z)dz\right| \\ &\leq \int_t^{+\infty} |w''(z)|dz + \int_0^t |w''(z)|dz = \|\Delta v\|_1 \end{aligned}$$

and we proceed like in the previous case and obtain half the constant.

We mention that to cover the general case we use the Talenti Comparison Principle [18] and then we exhibit explicit sequences to test the sharpness. For the proof of Corollary 1 we refer to [7].

3 A new version of the Trudinger–Moser inequality

Inspired by the results of the previous section, we now go back to prove our main results concerning the sharp version of the Pohožaev and Trudinger embedding.

3.1 Proof of Theorem 1

Let $u \in H^1_0(\Omega)$, and let $u^\sharp(x)$ denote the spherically symmetric rearrangement of u. As in [14], set

$$w(t) = \sqrt{4\pi}\, u^\sharp(Re^{-t/2}) \tag{13}$$

where $\pi R^2 = |\Omega|$. Then w is monotone increasing on $(0,\infty)$ and, by the Polya–Szego inequality, we get

$$\int_0^\infty (w')^2 dt = \int_{\Omega^\sharp} |\nabla u^\sharp|^2 dx \leq \int_\Omega |\nabla u|^2 dx$$

By a density argument, we may suppose that $w(t)$ is C^1, so that

$$w(t) = \int_0^t w'(s)ds \leq \|\nabla u\|_2 \sqrt{t}$$

and hence

$$u^\sharp(x) \leq \frac{\|\nabla u\|_2}{\sqrt{2\pi}}\sqrt{\log\left(\frac{R}{|x|}\right)}$$

Then

$$u^*\left(\pi|x|^2\right) = u^\sharp(x) \le \frac{\|\nabla u\|_2}{\sqrt{4\pi}}\sqrt{\log\left(\frac{\pi R^2}{t}\right)} = \frac{\|\nabla u\|_2}{\sqrt{4\pi}}\sqrt{\log\left(\frac{|\Omega|}{t}\right)}$$

so that

$$\|u\|_{Z^{1/2}} = \sup_{t\in(0,|\Omega|)} \frac{u^*(t)}{\sqrt{1+\log\left(\frac{|\Omega|}{t}\right)}} \le \frac{\|\nabla u\|_2}{\sqrt{4\pi}}$$

The inequality (8) is sharp, and this can be achieved by using the Moser truncated functions [14]:

$$w_n(t) = \begin{cases} \frac{t}{\sqrt{n}}, & t \le n \\ \sqrt{n}, & t \ge n \end{cases}$$

by which one has $\|\nabla u_n\|_2 = 1$ and

$$\begin{aligned} \sqrt{4\pi}\|u_n\|_{Z^{1/2}} &= \sqrt{4\pi} \sup_{t\in(0,|\Omega|)} \frac{u_n^*(t)}{\sqrt{1+\log\left(\frac{|\Omega|}{t}\right)}} \\ &= \sup_{t\in(0,\infty)} \frac{w_n(t)}{\sqrt{1+t}} = \frac{\sqrt{n}}{\sqrt{n+1}} \to 1, \quad \text{as } n\to\infty \qquad (14) \end{aligned}$$

3.2 Proof of Theorem 2

Let $\{u_n\}$ be a normalized maximizing sequence namely,

$$\|\nabla u_n\|_2 = 1 \qquad \text{and} \qquad \|u_n\|_{Z^{1/2}} \to \frac{1}{\sqrt{4\pi}}, \qquad \text{as } n\to\infty$$

We may assume that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $|\nabla u_n|^2 \to \mu$ weakly in the sense of measures. By the concentration-compactness result of P.L. Lions [13], the sequence $\{u_n\}$ enjoys one of the following alternatives:

- there exists $q > 4\pi$ such that the family $v_n = e^{u_n^2}$ is uniformly bounded in $L^q(\Omega)$, and thus $\int_\Omega e^{4\pi u_n^2} \to \int_\Omega e^{4\pi u^2}$ as $n\to+\infty$. In particular, this is the case when u is different from zero.

or

- $\mu = \delta_{x_0}$, for some $x_0 \in \overline{\Omega}$, and $u \equiv 0$.

We will show that we are in the latter situation i.e. the non compact case. Thanks to the Polya–Szego inequality, we may restrict our attention to radial and nondecreasing normalized maximizing sequences: indeed,

$$\|u_n\|_{Z^{1/2}} = \|u_n^\sharp\|_{Z^{1/2}} \text{ and } \|\nabla u_n\|_2 \ge \|\nabla u_n^\sharp\|_2$$

so that, if $\{u_n\}$ is a maximizing sequence, then $\|\nabla u_n\|_2 = \|\nabla u_n^\sharp\|_2=1$ and $\{u_n^\sharp\}$ is a normalized maximizing sequence. Note that $\{u_n^\sharp\}$ is concentrating at the origin if and only if $\{u_n\}$ is concentrating at some point $x_0\in\Omega$.
Let $\{w_n\}$ be the sequence associated to $\{u_n\}$ via the change of variable (13). Then

$$w_n(0) = 0, \quad w_n'(t) \ge 0 \quad \text{and} \quad \int_0^{+\infty} (w_n'(t))^2\, dt = 1$$

By (14) we have

$$\sqrt{4\pi}\,\|u_n\|_{Z^{1/2}} = \sup_{t\in(0,\infty)} \frac{w_n(t)}{\sqrt{1+t}}$$

so that

$$\sup_{t\in(0,\infty)} \frac{w_n(t)}{\sqrt{1+t}} \to 1^-, \qquad \text{as } n\to\infty$$

In particular, for any $\varepsilon > 0$ there exist $n_\varepsilon \in \mathbb{N}$ and $t_\varepsilon \in (0,+\infty)$ such that

$$w_{n_\varepsilon}(t_\varepsilon) > (1-\varepsilon)\sqrt{1+t_\varepsilon}$$

On the other hand, for any $A\in(0,t)$ and for any $t>0$

$$w(t) - w(A) = \int_A^t w'(s)ds \le \sqrt{\int_A^t (w')^2\, ds}\,\sqrt{t-A}$$

so that, for any $\varepsilon>0$ and for any $A\in(0,t_\varepsilon)$

$$(1-\varepsilon)\sqrt{1+t_\varepsilon} < \sqrt{\int_A^{t_\varepsilon} (w_{n_\varepsilon}')^2}\sqrt{t_\varepsilon - A} + w_{n_\varepsilon}(A) \le \sqrt{\int_A^{t_\varepsilon} (w_{n_\varepsilon}')^2}\sqrt{t_\varepsilon - A} + \sqrt{A} \quad (15)$$

Choosing $A = 0$ in (15), we have

$$t_\varepsilon \longrightarrow \infty \text{ as } \varepsilon \to 0$$

Furthermore, for any $A > 0$ we also have

$$\int_0^A (w'_{n_\varepsilon})^2\, ds \longrightarrow 0$$

indeed, if not $\int_A^{t_\varepsilon} (w'_{n_\varepsilon})^2 < 1 - \delta$ for some $\delta > 0$, contradicting (15).
Therefore, up to extracting a subsequence, $\{w_n\}$ is concentrating at $+\infty$ and in turn, $\{u_n\}$ is a normalized maximizing sequence concentrating at 0: this completes the proof of Theorem 2.

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