Simple representation of quantum process tomography

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We show that the Fano representation leads to a particularly simple and appealing form of the quantum process tomography matrix χ_F, in that the matrix χ_F is real, the number of matrix elements is exactly equal to the number of free parameters required for the complete characterization of a quantum operation, and these matrix elements are directly related to evolution of the expectation values of the system's polarization measurements. These facts are illustrated in the examples of one- and two-qubit quantum noise channels.

I. INTRODUCTION

The characterization of physical, generally noisy processes in open quantum systems is a key issue in quantum information science [1,2]. Quantum process tomography (QPT) provides, in principle, full information on the dynamics of a quantum system and can be used to improve the design and control of quantum hardware. Several QPT methods have been developed including the standard QPT [2–5], ancilla-assisted QPT [6–8], and direct characterization of quantum dynamics [9]. In recent years QPT has been experimentally demonstrated with up to three-qubit systems in a variety of different implementations including quantum optics [8,10–16], nuclear magnetic resonance quantum processors [17–19], atoms in optical lattices [20], trapped ions [21,22], and solid-state qubits [23,24].

Any quantum state ρ can be expressed in the Fano form [25–27] (also known as Bloch representation). Since the density operator ρ is Hermitian, the parameters of the expansion over the Fano basis are real. Furthermore, due to the linearity of quantum mechanics, any quantum operation $\rho \rightarrow \rho' = \mathcal{E}(\rho)$ is represented, in the Fano basis, by an affine map.

In this paper, we point out that in standard QPT it is convenient to compute the QPT matrix in the Fano basis. Such process matrix, χ_F, has the following advantages: (i) the matrix elements of χ_F are real and (ii) the number of matrix elements in χ_F is exactly equal to the number of free parameters needed in order to determine a generic quantum operation. Furthermore, the χ_F-matrix elements are directly related to the modification, induced by the quantum operation $\mathcal{E}$, of the expectation values of the system's polarization measurements. We will illustrate our results in the examples of one- and two-qubit quantum noise. In particular, we will determine in the χ_F matrix the specific patterns of various quantum noise processes. Finally, we will discuss the number of free parameters physically relevant in determining a quantum operation for a two-qubit system exposed to weak local noise.

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II. FANO REPRESENTATION OF THE STANDARD QPT

To simplify writing, we discuss the Fano representation of the standard QPT only for qubits, even though the obtained results can be readily extended to qudit systems. Any n-qubit state ρ can be written in the Fano form as follows [25–27]:

$$\rho = \frac{1}{N} \sum_{\alpha_1,\ldots,\alpha_n = x,y,z,I} c_{\alpha_1\ldots\alpha_n} \sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_n}, \quad (1)$$

where $N = 2^n$, σ_x, σ_y, and σ_z are the Pauli matrices, $\sigma_I \equiv 1$, and

$$c_{\alpha_1\ldots\alpha_n} = \mathrm{Tr}(\sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_n} \rho). \quad (2)$$

Note that the normalization condition $\mathrm{Tr}(\rho) = 1$ implies $c_{I\ldots I} = 1$. Moreover, the generalized Bloch vector $\mathbf{b} = \{b_\alpha\}_{\alpha=1,\ldots,N^2-1}$ is real due to the hermiticity of ρ. Here $b_\alpha \equiv c_{\alpha_1\ldots\alpha_n}$, with $\alpha \equiv \sum_{k=1}^n i_k 4^{n-k}$, where we have defined $i_k = 1$, 2, 3, and 4 in correspondence to $\alpha_k = x$, y, z, and I. Note that from 1 to n qubits run from the most significant to the least significant. For instance, for two qubits ($n=2$), the $N^2 - 1 = 15$ components of vector $\mathbf{b}$ are ordered as follows:

$$\mathbf{b}^T = (b_1, b_2, \ldots, b_{15})$$
$$= (c_{xx}, c_{xy}, c_{xz}, c_{xI}, c_{yx}, c_{yy}, c_{yz}, c_{yI}, c_{zx}, c_{zy}, c_{zz}, c_{zI}, c_{Ix}, c_{Iy}, c_{Iz}). \quad (3)$$

Due to the linearity of quantum mechanics any quantum operation $\rho \rightarrow \rho' = \mathcal{E}(\rho)$ is represented in the Fano basis $\{\sigma_{\alpha_1} \otimes \ldots \otimes \sigma_{\alpha_n}\}$ by an affine map:

$$\begin{bmatrix} \mathbf{b}' \\ 1 \end{bmatrix} = \mathcal{M} \begin{bmatrix} \mathbf{b} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M} & \mathbf{a} \\ \hline \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ 1 \end{bmatrix}, \quad (4)$$

where $\mathbf{M}$ is a $(N^2-1) \times (N^2-1)$ matrix, $\mathbf{a}$ is a column vector of dimension N^2-1, and $\mathbf{0}$ is the null vector of the same dimension.

All information about the quantum operation $\mathcal{E}$ is contained in the $N^4 - N^2$ free elements of matrix $\mathcal{M}$, namely, in the matrix

$$\chi_F = [\mathbf{M} \mid \mathbf{a}]. \tag{5}$$

To obtain the QPT matrix χ_F from experimental data, one needs to prepare N^2 linearly independent initial states $\{\rho_i\}$, let them evolve according to the quantum operation $\mathcal{E}$, and then measure the resulting states $\{\rho_i' = \mathcal{E}(\rho_i)\}$. If we call $\mathcal{R}$ the $N^2 \times N^2$ matrix whose columns are given by the Fano representation of states ρ_i and $\mathcal{R}'$ the corresponding matrix constructed from states ρ_i', we have

$$\mathcal{R}' = \mathcal{M}\mathcal{R}, \tag{6}$$

and therefore

$$\mathcal{M} = \mathcal{R}'\mathcal{R}^{-1}. \tag{7}$$

As it is well known [2], the standard QPT can be performed with initial states being product states and local measurements of the final states. As initial states $\{\rho_i\}$ we choose the 4^n tensor-product states of the four single-qubit states

$$|0\rangle, \quad |1\rangle, \quad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle). \tag{8}$$

To estimate $\mathcal{R}'$, one needs to prepare many copies of each initial state ρ_i, let them evolve according to the quantum operation $\mathcal{E}$, and then measure observables $\sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_n}$. Of course, such measurements can be performed on the computational basis $\{|0\rangle, |1\rangle\}^{\otimes n}$ provided each measurement is preceded by suitable single-qubit rotations.

III. SINGLE-QUBIT SYSTEMS

The matrix $\mathcal{R}$ corresponding to basis (8) reads

$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{9}$$

Therefore,

$$\mathcal{R}^{-1} = \begin{bmatrix} -\dfrac{1}{2} & -\dfrac{1}{2} & \dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & -\dfrac{1}{2} & -\dfrac{1}{2} & \dfrac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{10}$$

The coefficients (c_x, c_y, c_z) in the Fano form (1) are the Bloch-vector coordinates of the density matrix ρ in the Bloch-ball representation of single-qubit states. We need $N^4 - N^2 = 12$ parameters to characterize a generic quantum operation acting on a single qubit. Each parameter describes a particular noise channel (such as bit flip, phase flip, amplitude damping, etc.) and can be most conveniently visualized as associated with rotations, deformations, and displacements of the Bloch ball [1,2,28]. Here we point out that these noise channels lead to specific patters in the state process matrix χ_F.

For instance, for the phase-flip channel,

$$\rho' = \mathcal{E}(\rho) = p\sigma_z\rho\sigma_z + (1-p)\rho, \quad \left(0 \leq p \leq \frac{1}{2}\right), \tag{11}$$

we have

$$\mathcal{R}' = \begin{bmatrix} 0 & 0 & 1-2p & 0 \\ 0 & 0 & 0 & 1-2p \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{12}$$

We can then compute $\mathcal{M} = \mathcal{R}'R^{-1}$ and the first three lines of $\mathcal{M}$ correspond to the state matrix

$$\chi_F^{(\text{pf})} = \begin{bmatrix} 1-2p & 0 & 0 & 0 \\ 0 & 1-2p & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{13}$$

Therefore, the Bloch ball is mapped into an ellipsoid with z as symmetry axis:

$$c_x \rightarrow c_x' = (1-2p)c_x,$$

$$c_y \rightarrow c_y' = (1-2p)c_y,$$

$$c_z \rightarrow c_z' = c_z. \tag{14}$$

As a further example, we consider the amplitude damping channel,

$$\rho' = \sum_{k=0}^{1} E_k\rho E_k^\dagger, \tag{15}$$

with the Kraus operators

$$E_0 = |0\rangle\langle0| + \sqrt{1-p}|1\rangle\langle1|, \quad E_1 = \sqrt{p}|0\rangle\langle1|, \quad (0 \leq p \leq 1). \tag{16}$$

In this case we obtain

$$\chi_F^{(\text{ad})} = \begin{bmatrix} \sqrt{1-p} & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & 0 & 0 \\ 0 & 0 & 1-p & p \end{bmatrix}. \tag{17}$$

The Bloch ball is deformed into an ellipsoid, with its center displaced along the z axis:

$$c_x \rightarrow c_x' = \sqrt{1-p}c_x,$$

$$c_y \rightarrow c_y' = \sqrt{1-p}c_y,$$

$$c_z \rightarrow c_z' = (1-p)c_z + p. \tag{18}$$

IV. TWO-QUBIT SYSTEMS

Matrices $\mathcal{R}$ and $\mathcal{R}^{-1}$ corresponding to the 16 tensor-product states of single-qubit states [Eq. (8)] read as follows:

$$
\mathcal{R} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix},
\tag{19}
$$

$$
\mathcal{R}^{-1} = \frac{1}{4}
\begin{bmatrix}
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 \\
-2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\
-2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 \\
-2 & -2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & -2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & -2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\tag{20}
$$

Note that the 16 columns of $\mathcal{R}$ correspond, from left to right, to the states $|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, \ldots, \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, that is, states are ordered with the first qubit being the most significant one.

The coordinates $\{c_{\alpha_1 \alpha_2}\}$ in the Fano form (1) are the expectation values of the polarization measurements $\{\sigma_{\alpha_1} \otimes \sigma_{\alpha_2}\}$. The coefficients in the state matrix χ_F represent-ing a quantum operation $\mathcal{E}$ can therefore be interpreted in terms of modification of these expectation values.

For instance, let us assume that the two qubits are inde-pendently exposed to pure dephasing, that is, to quantum noise described by the phase-flip channel (11), with the same noise strength p for both qubits. The process matrix for such uncorrelated dephasing channel is given by

$\chi_F^{(ud)}$

$$
= \begin{bmatrix}
g^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \tag{21}
$$

where $g \equiv 1 - 2p$. Correspondingly, the mapping for the expectation values of the polarization measurements reads

$$
c'_{\alpha_1 \alpha_2} = g^{m_1 + m_2} c_{\alpha_1 \alpha_2}, \tag{22}
$$

where $m_i = 1$ for $\alpha_i = x, y$ and $m_i = 0$ for $\alpha_i = z, I$.

As an example of nonlocal quantum noise, we consider a model of fully correlated pure dephasing. We model the interaction of the two qubits with the environment as a phase kick rotating both qubits through the same angle θ about the z axis of the Bloch ball. This rotation is described in the $\{|0\rangle, |1\rangle\}$ basis by the unitary matrix

$$
R_z(\theta) = \begin{bmatrix} e^{-i(\theta/2)} & 0 \\ 0 & e^{i(\theta/2)} \end{bmatrix} \otimes \begin{bmatrix} e^{-i(\theta/2)} & 0 \\ 0 & e^{i(\theta/2)} \end{bmatrix}. \tag{23}
$$

We assume that the rotation angle is drawn from the random distribution

$$
p(\theta) = \frac{1}{\sqrt{4\pi\lambda}} e^{-\theta^2/4\lambda}. \tag{24}
$$

Therefore, the final state ρ', obtained after averaging over θ, is given by

$$
\rho' = \int_{-\infty}^{+\infty} d\theta\, p(\theta) R_z(\theta) \rho R_z^\dagger(\theta). \tag{25}
$$

For this correlated dephasing channel we obtain the process matrix

$\chi_F^{(cd)}$

$$
= \begin{bmatrix}
h & 0 & 0 & 0 & 0 & k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & h & 0 & 0 & -k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -k & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \tag{26}
$$

where $g \equiv e^{-\lambda}$, $h \equiv \frac{1}{2}(1 + g^4)$, and $k \equiv \frac{1}{2}(1 - g^4)$. It is clear that process matrix (26) for correlated dephasing has a pattern that allows to clearly distinguish it from process matrix (21) for the uncorrelated dephasing.

It is also obvious that, if there exists partial previous knowledge of the dominant noise sources, it is not necessary to construct the whole state process matrix χ_F in order to characterize the quantum operation. For instance, if we know *a priori* that dephasing is the main source of noise and we wish to estimate its degree of correlation, it is sufficient to prepare, for instance, the initial state $\rho = \frac{1}{2}(|0\rangle + |1\rangle)^{\otimes 2}$ and measure the x and y polarizations of both qubits for the final state ρ'. The initial state is fully polarized along x and therefore

$$
c_{xx} = 1,
$$

$$
c_{yy} = 0. \tag{27}
$$

For the final state, in the case of fully correlated dephasing

$$
(c'_{xx})^{(cd)} = h = \frac{1}{2}[1 + g^4],
$$

$$
(c'_{yy})^{(cd)} = k = \frac{1}{2}[1 - g^4], \tag{28}
$$

while the expectation values of the xx- and yy-polarization measurements are remarkably different for uncorrelated dephasing:

$$
(c'_{xx})^{(ud)} = g^2,
$$

$$
(c'_{yy})^{(ud)} = 0. \tag{29}
$$

V. DISCUSSION

While in general two-qubit quantum operations depend on $N^4 - N^2 = 240$ real parameters, an important question is how many parameters are physically significant. The answer of course depends on the specific noise processes. However, a clear answer can be given assuming that external noise is weak and local, that is to say, it acts independently on the two qubits. In this case, local noise is described by 24 parameters, 12 for each qubit. Undesired coupling effects (cross-talk) between qubits can be characterized with only three additional parameters, θ_x, θ_y, and θ_z. Indeed, any two-qubit unitary transformation U can be decomposed as [29–31]

$$U = (A_1 \otimes B_1) e^{i(\theta_x \sigma_x \otimes \sigma_x + \theta_y \sigma_y \otimes \sigma_y + \theta_z \sigma_z \otimes \sigma_z)} (A_2 \otimes B_2), \quad (30)$$

with A_1, A_2, B_1, and B_2 appropriate single-qubit unitaries. In the limit of weak noise, the state matrix χ_F is simply given by the sum of the contributions of each noise channel. Therefore, the local unitaries A_1, A_2, B_1, and B_2 only change the 24 local noise parameters and overall we need $24 + 3 = 27 \ll 240$ parameters to describe the quantum noise. In the symmetric case in which the local noise parameters are the same for both qubits the number of free parameters further reduces to $12 + 3 = 15$. The above argument can be easily extended to many-qubit systems. Due to the two-body nature of interactions, we need to determine $\mathcal{N} = 12n + 3\frac{n(n-1)}{2}$ parameters to characterize noise. Note that $\mathcal{N} = O(\log N) \ll N^4 - N^2$. Of course, cases with strong or nonlocal noise would require a larger number of free parameters.

It might be useful to briefly compare our approach with other representations of standard QPT [2–5], which are based on the operator-sum representation of quantum operations:

$$\mathcal{E}(\rho) = \sum_{m,n=1}^{N^2} \chi_{mn} E_m \rho E_n^\dagger, \quad (31)$$

where $\{E_m\}$ forms a basis for the set of operators acting on the N-dimensional Hilbert space of the system. In standard QPT usually $E_{m=Ni+j} = |i\rangle\langle j|$ is chosen, with $\{|i\rangle\}$ orthonormal basis of the system. Such choice does not exploit the fact that ρ is Hermitian and therefore leads to complex matrix elements χ_{mn}. In short, χ is a $N^2 \times N^2$ complex matrix, while in our approach based on the Fano representation we deal with $N^4 - N^2$ independent real parameters. Finally, we note that our approach requires inversion of the $N^2 \times N^2$ matrix $\mathcal{R}$, while the standard QPT algorithm described in Ref. [2] involves the calculation of a generalized inverse for a $N^4 \times N^4$ matrix.

To summarize, we have shown that the Fano representation of the standard QPT is convenient, since the process matrix χ_F is real and the number of matrix elements is exactly equal to the number of free parameters required for the complete characterization of a generic quantum operation. Moreover, the matrix elements of χ_F are directly related to the evolution, induced by the quantum operation, of the system's polarization measurements. We have also shown that quantum noise channels have specific patterns in the Fano representation of χ_F. Finally, we have shown that in the case, of interest for quantum information processing, of weak and local noise the number of relevant noise parameters is $\mathcal{N} = O(\log N) \ll N^4 - N^2$, that is, much smaller than the number of parameters needed to determine a generic quantum operation. In this case, the χ_F matrix is very sparse and therefore the number of polarization measurements needed to reconstruct it is much smaller than for a generic quantum operation, thus considerably reducing the QPT complexity.

[1] G. Benenti, G. Casati, and G. Strini, *Principles of Quantum Computation and Information* (World Scientific, Singapore, 2004), Vol. I; *Principles of Quantum Computation and Information* (World Scientific, Singapore, 2007), Vol. II.

[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[3] I. L. Chuang and M. A. Nielsen, J. Mod. Opt. **44**, 2455 (1997).

[4] J. F. Poyatos, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **78**, 390 (1997).

[5] A. G. Kofman and A. N. Korotkov, e-print arXiv:0903.0671.

[6] G. M. D'Ariano and P. Lo Presti, Phys. Rev. Lett. **86**, 4195 (2001).

[7] D. W. Leung, J. Math. Phys. **44**, 528 (2003).

[8] J. B. Altepeter, D. Branning, E. Jeffrey, T. C. Wei, P. G. Kwiat, R. T. Thew, J. L. O'Brien, M. A. Nielsen, and A. G. White, Phys. Rev. Lett. **90**, 193601 (2003).

[9] M. Mohseni and D. A. Lidar, Phys. Rev. Lett. **97**, 170501 (2006); M. Mohseni, A. T. Rezakhani, and D. A. Lidar, Phys. Rev. A **77**, 032322 (2008).

[10] M. W. Mitchell, C. W. Ellenor, S. Schneider, and A. M. Stein-

berg, Phys. Rev. Lett. **91**, 120402 (2003).

[11] F. De Martini, A. Mazzei, M. Ricci, and G. M. D'Ariano, Phys. Rev. A **67**, 062307 (2003).

[12] J. L. O'Brien, G. J. Pryde, A. Gilchrist, D. F. V. James, N. K. Langford, T. C. Ralph, and A. G. White, Phys. Rev. Lett. **93**, 080502 (2004).

[13] Y. Nambu and K. Nakamura, Phys. Rev. Lett. **94**, 010404 (2005).

[14] N. K. Langford, T. J. Weinhold, R. Prevedel, A. Gilchrist, J. L. O'Brien, G. J. Pryde, and A. G. White, Phys. Rev. Lett. **95**, 210504 (2005).

[15] N. Kiesel, C. Schmid, U. Weber, R. Ursin, and H. Weinfurter, Phys. Rev. Lett. **95**, 210505 (2005).

[16] Z.-W. Wang, Y.-S. Zhang, Y.-F. Huang, X.-F. Ren, and G.-C. Guo, Phys. Rev. A **75**, 044304 (2007).

[17] A. M. Childs, I. L. Chuang, and D. W. Leung, Phys. Rev. A **64**, 012314 (2001).

[18] N. Boulant, T. F. Havel, M. A. Pravia, and D. G. Cory, Phys. Rev. A **67**, 042322 (2003).

[19] Y. A. Weinstein *et al.*, J. Chem. Phys. **121**, 6117 (2004).

[20] S. H. Myrskog, J. K. Fox, M. W. Mitchell, and A. M. Stein-

berg, Phys. Rev. A **72**, 013615 (2005).

[21] M. Riebe, K. Kim, P. Schindler, T. Monz, P. O. Schmidt, T. K. Korber, W. Hansel, H. Haffner, C. F. Roos, and R. Blatt, Phys. Rev. Lett. **97**, 220407 (2006).

[22] T. Monz, K. Kim, W. Hansel, M. Riebe, A. S. Villar, P. Schindler, M. Chwalla, M. Hennrich, and R. Blatt, Phys. Rev. Lett. **102**, 040501 (2009).

[23] M. Howard *et al.*, New J. Phys. **8**, 33 (2006).

[24] N. Katz *et al.*, Phys. Rev. Lett. **101**, 200401 (2008).

[25] U. Fano, Rev. Mod. Phys. **29**, 74 (1957); **55**, 855 (1983).

[26] F. T. Hioe and J. H. Eberly, Phys. Rev. Lett. **47**, 838 (1981).

[27] J. Schlienz and G. Mahler, Phys. Rev. A **52**, 4396 (1995).

[28] G. Benenti, S. Felloni, and G. Strini, Eur. Phys. J. D **38**, 389 (2006).

[29] N. Khaneja, R. Brockett, and S. J. Glaser, Phys. Rev. A **63**, 032308 (2001).

[30] B. Kraus and J. I. Cirac, Phys. Rev. A **63**, 062309 (2001).

[31] M. A. Nielsen, C. M. Dawson, J. L. Dodd, A. Gilchrist, D. Mortimer, T. J. Osborne, M. J. Bremner, A. W. Harrow, and A. Hines, Phys. Rev. A **67**, 052301 (2003).