Enhancement of Transmission Rates in Quantum Memory Channels with Damping

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We consider the transfer of quantum information down a single-mode quantum transmission line. Such a quantum channel is modeled as a damped harmonic oscillator, the interaction between the information carriers -a train of N qubits- and the oscillator being of the Jaynes-Cummings kind. Memory effects appear if the state of the oscillator is not reset after each channel use. We show that the setup without resetting is convenient in order to increase the transmission rates, both for the transfer of quantum and classical private information. Our results can be applied to the micromaser.

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Quantum communication channels [1,2] use quantum systems as carriers for information. One can employ them to transfer classical information, by encoding classical bits by means of quantum states. Furthermore there are some peculiar issues strictly related to quantum computation: to transfer an (unknown) quantum state between different subunits of a quantum computer, to hold in memory a quantum state waiting to process it later, to distribute entanglement among different parties.

A key problem in quantum information is the determination of the classical and quantum *capacities* of noisy quantum channels, defined as the maximum number of bits or qubits that can be reliably transmitted per use of the channel. These quantities characterize the channel, giving an upper bound to the channel efficiency per use.

In any realistic implementation, errors occur due to the unavoidable coupling of the transmitted quantum systems with an uncontrollable environment. Noise can have significant low-frequency components, which traduce themselves in memory effects, leading to relevant correlations in the errors affecting successive transmissions. Important examples in this context are photons traveling across fibers with birefringence fluctuating with characteristic time scales longer than the separation between consecutive light pulses [3] or low-frequency impurity noise in solid-state implementations of quantum hardware [4]. Memory effects become unavoidably relevant when trying to increase the *transmission rate* [5], that is, to reduce the time interval that separates two consecutive channel uses.

Quantum channels with memory attracted increasing attention in the last years, see [5–15] and references therein. Coding theorems have been proved for classes of quantum memory channels [6,7]. Memory effects have been modeled by Markov chains [8–12] and the quantum capacity has been exactly computed for a Markov chain dephasing channel [10–12]. Various kinds of memory channels have been studied, for example: purely dephasing channels [11–13], lossy bosonic channels [14]; also spin chains have been studied as models for the channel itself [15]. Hamiltonian models of memory channels [11–13,15] aim at a description directly referring to physical systems and enlight another important example of a noisy quantum channel, namely, the memory of a quantum computer [16].

In this work the quantum channel is modeled as a damped harmonic oscillator, and we consider transfer of quantum information through it. A train of N qubits is sent down the channel (initially prepared in its ground state) and interacts with it during the transit time. If the state of the oscillator is not reset after each channel use, then the action of the channel on the kth qubit depends on the previous $k-1$ channel uses. The oscillator acts as a *local* "unconventional environment" [5,6,9,17], coupled to a memoryless reservoir damping both its phases and populations, which mimics any cooling process resetting the oscillator to its ground state. The model is visualized by a qubit-micromaser [18] system, the qubit train being a stream of two-level Rydberg atoms injected at low rate into the cavity. Unconventional environments capture essentials features of solid-state circuit-quantum electrodynamics (QED) [19] devices and in this context the model may describe the architecture of a quantum memory. The low injection rate is required in order to avoid collective effects such as superradiance. Atoms interact with the photon field inside the cavity and memory effects are relevant, if the lifetime of photons is longer than the time interval between two consecutive channel uses.

In what follows we will show that the quantum information transmission worsens with increasing transmission frequency due to the increase of memory effects. However, the decrease is found to be only moderate, so that the quantum transmission rate increases with increasing transmission frequency. Therefore, operating the memory channel at high transmission frequency, thus accepting *prima facie* deleterious memory effects, will be more beneficial than using low frequency. This result applies both to the transfer of quantum and classical private information.

The quantum capacity.—N channel uses correspond to a N-qubit input state ρ, which may be chosen with probability $\{p_i\}$ from a given ensemble $\{\sigma_i\}$ of the N-qubit Liouville space ($\rho = \sum_i p_i \sigma_i$). Because of the coupling to uncontrollable degrees of freedom, the transmission is in general not fully reliable. The output is therefore described by a linear, completely positive, trace preserving (CPT) map for N uses, $\mathcal{E}_N(\rho)$. For memoryless channels $\mathcal{E}_N = \mathcal{E}_1^{\otimes N}$, where $\mathcal{E}_1$ indicates the single use, and the quantum capacity Q can be computed as [16,20,21]

$$Q = \lim_{N\to\infty} \frac{Q_N}{N}, \qquad Q_N = \max_{\rho} I_c(\mathcal{E}_N, \rho), \tag{1}$$

$$I_c(\mathcal{E}_N, \rho) = S[\mathcal{E}_N(\rho)] - S_e(\mathcal{E}_N, \rho). \tag{2}$$

Here $S(\rho) = -\mathrm{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy, $S_e(\mathcal{E}_N, \rho)$ is the *entropy exchange* [16], defined as $S_e(\mathcal{E}_N, \rho) = S[(\mathcal{I} \otimes \mathcal{E}_N)(|\psi\rangle\langle\psi|)]$, where $|\psi\rangle\langle\psi|$ is any purification of ρ. That is, we consider the system S, described by the density matrix ρ, as a part of a larger quantum system RS; $\rho = \mathrm{Tr}_{\mathsf{R}}|\psi\rangle\langle\psi|$ and the reference system R evolves trivially, according to the identity superoperator $\mathcal{I}$. The quantity $I_c(\mathcal{E}_N, \rho)$ is called *coherent information* [16] and must be maximized over all input states ρ. In general I_c is not subadditive [16], i.e. $Q_N/N \geq Q_1$. When memory effects are taken into account the channel does not act on each carrier independently, $\mathcal{E}_N \neq \mathcal{E}_1^{\otimes N}$, and Eq. (1) in general only provides an upper bound on the channel capacity. However, for the so-called *forgetful channels* [6], for which memory effects decay exponentially with time, a quantum coding theorem exists showing that the upper bound can be saturated [6].

The model.—The overall Hamiltonian governing the dynamics of the system (N qubits), a local environment (harmonic oscillator) and a reservoir is defined as ($\hbar = 1$)

$$\begin{aligned} \mathcal{H}(t) &= \mathcal{H}_0 + V + \delta\mathcal{H}, \\ \mathcal{H}_0 &= \nu\left(a^\dagger a + \frac{1}{2}\right) + \frac{\omega}{2}\sum_{k=1}^{N} \sigma_z^{(k)}, \\ V &= \lambda \sum_{k=1}^{N} f_k(t)(a^\dagger \sigma_-^{(k)} + a\sigma_+^{(k)}). \end{aligned} \tag{3}$$

The qubit-oscillator interaction V is of the Jaynes-Cummings kind, and we take λ real and positive. Coupling is switchable: $f_k(t) = 1$ when qubit k is inside the channel (transit time τ_p), $f_k(t) = 0$ otherwise. The term $\delta\mathcal{H}$ describes both the reservoir's Hamiltonian and the local environment-reservoir interaction and causes damping of the oscillator (the cavity mode in the micromaser), that is, relaxation and dephasing with time scales τ_d and τ_ϕ, respectively. Two consecutive qubits entering the channel are separated by the time interval τ. One can argue that the resonant regime $\nu \sim \omega$ is the most significant when describing the coupling to modes inducing damping. We work in the interaction picture, where the effective Hamiltonian at resonance is given by $\tilde{\mathcal{H}} = e^{i\mathcal{H}_0 t}(V + \delta\mathcal{H})e^{-i\mathcal{H}_0 t}$ (we will omit the tilde from now on).

We assume $\tau_p \ll \tau, \tau_\phi, \tau_d$, so that nonunitary effects in the evolution of the system and the oscillator can be ignored during the crossing time τ_p. Between two successive pulses the oscillator evolves according to the standard master equation (obtained after tracing over the reservoir)

$$\dot{\rho}_{\mathsf{C}} = \Gamma(a\rho_{\mathsf{C}}a^\dagger - \tfrac{1}{2}a^\dagger a \rho_{\mathsf{C}} - \tfrac{1}{2}\rho_{\mathsf{C}} a^\dagger a). \tag{4}$$

The asymptotic decay (channel reset) to the ground state $|0\rangle$ takes place with rate Γ, so that $\tau_d = 1/\Gamma$. We introduce the memory parameter $\mu \equiv \tau_d/(\tau + \tau_d)$: fast decay $\tau_d \ll \tau$ yields the memoryless limit $\mu \ll 1$, whereas $\mu \lesssim 1$ when memory effects come into play.

The memoryless limit.—In this limit damping acts as a built-in reset for the oscillator to its ground state $\rho_{\mathsf{C}}(0) = |0\rangle\langle 0|$ after each channel use. We consider a generic single-qubit input state, $\rho_1(0) = (1-p)|g\rangle\langle g| + r|g\rangle \times \langle e| + r^\star|e\rangle\langle g| + p|e\rangle\langle e|$, with $\{|g\rangle, |e\rangle\}$ orthogonal basis for the qubit, p real and $|r| \leq \sqrt{p(1-p)}$. Given the initial, separable qubit-oscillator state $\rho_1(0) \otimes \rho_{\mathsf{C}}(0)$, we have

$$\mathcal{E}_1[\rho_1(0)] = \mathrm{Tr}_{\mathsf{C}}\{U(\tau_p)[\rho_1(0) \otimes \rho_{\mathsf{C}}(0)]U^\dagger(\tau_p)\}, \tag{5}$$

with $U(\tau_p)$ unitary time-evolution operator determined by the undamped Jaynes-Cummings Hamiltonian. It is easy to obtain [22], in the $\{|g\rangle, |e\rangle\}$ basis,

$$\mathcal{E}_1[\rho_1(0)] = \begin{bmatrix} 1 - p\cos^2(\lambda\tau_p) & r\cos(\lambda\tau_p) \\ r^\star\cos(\lambda\tau_p) & p\cos^2(\lambda\tau_p) \end{bmatrix}, \tag{6}$$

with λ frequency of the Rabi oscillations between levels $|e, 0\rangle$ and $|g, 1\rangle$. Equation (6) corresponds to an amplitude-damping channel: $\mathcal{E}_1[\rho_1(0)] = \sum_{k=0}^{1} E_k \rho_1(0) E_k^\dagger$, where the Kraus operators [1,2] $E_0 = |g\rangle\langle g| + \sqrt{\eta}|e\rangle\langle e|$, $E_1 = \sqrt{1-\eta}|g\rangle\langle e|$, with $\eta = \cos^2(\lambda\tau_p) \in [0, 1]$. This channel is degradable [23,24] and therefore to compute its quantum capacity it is sufficient to maximize the coherent information over a single channel use. Maximization is achieved by classical states ($r = 0$) and one obtains $Q = \max_{p\in[0,1]}\{H_2(\eta p) - H_2[(1-\eta)p]\}$ if $\eta > \frac{1}{2}$, where $H_2(x) = -x\log_2 x - (1-p)\log_2(1-x)$ is the binary Shannon entropy, while $Q = 0$ when $\eta \leq \frac{1}{2}$ [24].

Memory channels: Validity of Eq. (1).—Memory appears in our model when τ is finite. To show that the regularized coherent information still represents the true quantum capacity we follow the arguments made for forgetful channels in Ref. [6]. The key point is the use of a double-blocking strategy mapping, with a negligible error, the memory channel into a memoryless one. We consider blocks of $N + L$ uses of the channel and do the actual coding and decoding for the first N uses, ignoring the remaining L idle uses. We call $\bar{\mathcal{E}}_{N+L}$ the resulting CPT map. If we consider M uses of such blocks, the corresponding CPT map $\bar{\mathcal{E}}_{M(N+L)}$ can be approximated by the memoryless setting $(\bar{\mathcal{E}}_{(N+L)})^{\otimes M}$. One can use Eq. (1) to compute the quantum capacity if [6]

$$\|\bar{\mathcal{E}}_{M(N+L)}(\rho_{\mathrm{S}}) - (\bar{\mathcal{E}}_{(N+L)})^{\otimes M}(\rho_{\mathrm{S}})\|_1 \leq h(M-1)c^{-L}, \quad (7)$$

where ρ_{S} is a MN input state, $h > 0$, $c > 1$ are constant and $\|\rho\|_1 = \mathrm{Tr}\sqrt{\rho^\dagger \rho}$ is the trace norm [1] (note that c and h are independent of the input state ρ_{S}). One can prove [25] that, due to the exponentially fast channel (cavity) reset to the ground state, inequality (7) is fulfilled. Therefore, quantum capacity can be computed from the maximization (1) of coherent information.

Lower bound for the quantum capacity.—For the model Hamiltonian (3) computation of the coherent information for a large number N of channel uses is a difficult task, both for analytical and numerical investigations, even for separable input states, $\rho = \rho_1(0)^{\otimes N}$. Indeed, interaction with the oscillator entangles initially independent qubits. Nevertheless, a lower bound to the quantum capacity can be computed if $\tau_\phi \ll \tau$. This happens when additional mechanisms of pure dephasing (without relaxation) not explicitly included in Eq. (4) dominate the short time dynamics, the typical situation, e.g., in the solid state. The net effect, in the limit of strong dephasing of the cavity mode, is that separable input qubits do not become entangled after their interaction with the cavity. This enables us to address the problem for a very large number of channel uses, at least numerically. We have $I_c[\mathcal{E}_N, \rho = \rho_1(0)^{\otimes N}] = \sum_{k=1}^N I_c^{(k)}$, where $I_c^{(k)} \equiv I_c[\mathcal{E}_1^{(k)}, \rho_1(0)]$ and the CPT map $\mathcal{E}_1^{(k)}$ depends on k due to memory effects in the populations of the oscillator.

Steady state.—In the above strongly dephased regime the oscillator state $\rho_{\mathrm{C}}^{(k)}$ after k channel uses is diagonal and determined by the populations $\{w_n^{(k)}\}$. The buildup of the map that governs the populations dynamics requires the computation of the intermediate populations $\{\tilde{w}_n^{(k)}\}$, obtained after the Jaynes-Cummings interaction of the kth qubit with the oscillator:

$$\tilde{w}_n^{(k)} = w_{n-1}^{(k-1)} p S_n^2 + w_n^{(k-1)}[(1-p)C_n^2 + pC_{n+1}^2] + w_{n+1}^{(k-1)}(1-p)S_{n+1}^2, \qquad n \geq 0 \quad \text{and} \quad w_{-1}^{(k)} \equiv 0, \quad (8)$$

where we have used the shorthand notation $S_n = \sin(\Omega_n \tau_p)$ and $C_n = \cos(\Omega_n \tau_p)$, with $\Omega_n = \lambda\sqrt{n}$. Then the mapping from $\{\tilde{w}_n^{(k)}\}$ to $\{w_n^{(k)}\}$ is obtained after analytically solving the master equation (4) for the populations [26]. The overall mapping $\{w_n^{(k-1)}\} \to \{w_n^{(k)}\}$ is then numerically iterated. As shown in Fig. 1 (top) a steady-state distribution is eventually reached.

Following Eq. (2) we compute $I_c^{(k)} = S(\rho_1^{(k)}) - S(\rho_{1\mathrm{R}}^{(k)})$, with $\rho_1^{(k)}$ and $\rho_{1\mathrm{R}}^{(k)}$ output states for the kth qubit and for the kth qubit plus its reference system, respectively. Since $\rho_{\mathrm{C}}^{(k)}$ reaches a steady state, the same must happen for $I_c^{(k)}$. This expectation is confirmed by the numerical data shown in Fig. 1 (bottom). The optimization of the regularized coherent information (1) over separable input states is then simply obtained by maximizing the stationary value of the coherent information over $\rho_1(0)$. The obtained I_c-value provides a lower bound to the quantum capacity of the channel.

Transmission rates.—The (numerical) optimization is achieved when $r = 0$ (we have checked it for several values of η and Γ) and $p = p_{\mathrm{opt}}$ in $\rho_1(0)$. Note that p_{opt} may strongly depend on the time separation τ between consecutive channel uses [see Fig. 2 (top right)], namely, on the degree of memory of the channel. As shown in Fig. 2 (top left) the coherent information, optimized over separable input states, turns out to be a growing function of τ, that is, a decreasing function of the degree of memory of the channel. The memoryless setting $\tau \gg \tau_d$ might appear to be the optimal choice. However, long waiting times $\tau \gg \tau_d$ are required to reset the quantum channel (cool the harmonic oscillator/cavity) to its ground state after each channel use, thus reducing the transmission rate. It appears preferable to consider the quantum transmission rate $R_Q \equiv Q/\tau$, defined as the maximum number of qubits that can be reliably transmitted per unit of time [5]. Figure 2 (bottom) shows that in order to enhance R_Q it is convenient to choose $\tau \ll \tau_d$, namely, memory factors μ close to 1. In other words, by taking into account memory effects, one can make more efficient the use of the available transmitting resource.

These results are relevant also for the *secure* transmission of classical information, then for cryptographic purposes. The reference quantity is, for this case, the *private classical capacity* C_p, defined as the capacity for transmitting classical information protected against an eaves-

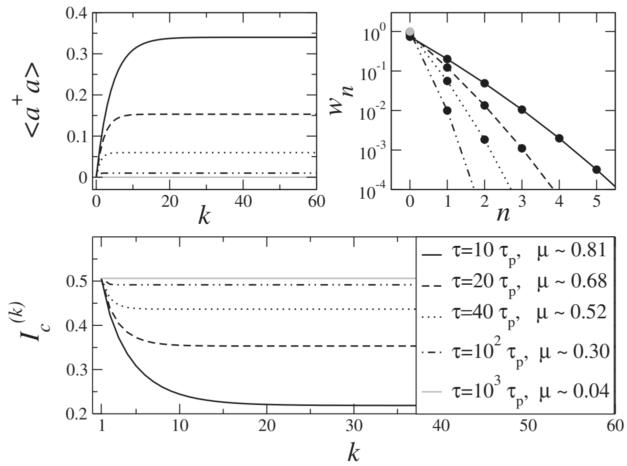

FIG. 1. Top: $\langle a^\dagger a\rangle$ vs the number k of channel uses (left) and steady-state populations w_n (right, numerically computed at $k = 200$). Bottom: coherent information $I_c^{(k)}$ vs k. Parameter values: $\eta = 0.8$ (i.e., $\lambda\tau_p \approx 0.46$) and $\lambda\tau_d = 20$.

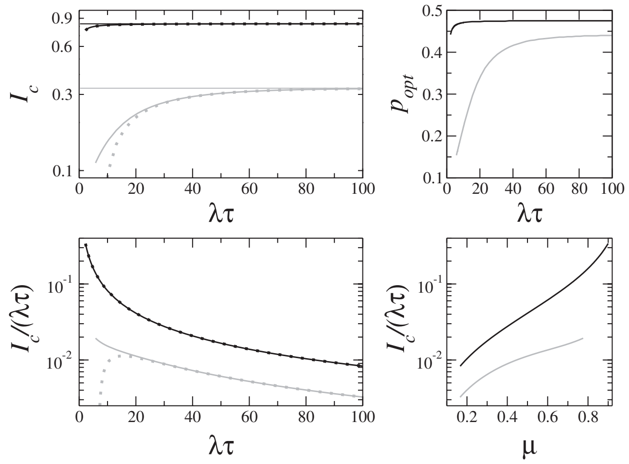

FIG. 2. Top left: steady-state coherent information I_c (optimized over separable input states) as a function of the dimensionless time separation $\lambda\tau$ between consecutive channel uses. Top right: optimal input state parameter $p_{\rm opt}$ vs $\lambda\tau$. Bottom left: same data as in the top left panel, but for the transmission rates $I_c/(\lambda\tau)$. Bottom right: transmission rates vs the memory parameter μ. Parameter values: $\eta = 0.95$ ($\lambda\tau_p \approx 0.22$) (black curves), $\eta = 0.7$ ($\lambda\tau_p \approx 0.58$) (gray curves), $\lambda\tau_d = 20$. Dotted curves correspond to $p = p_{\rm opt}(\tau \to \infty)$, and show, in the case with lower performances of the channel ($\eta = 0.7$), the importance of optimization.

dropper [21]. It was recently shown [27] that for degradable channels, as it is the case of our model in the memoryless limit, $C_p = Q$. Since the private classical capacity is always lower bounded by the coherent information [28], our results also show that the setup without resetting is convenient to increase the transmission rate $R_p \equiv C_p/\tau$ of private classical information.

Discussion.—Equation (3) models for instance dephasing in a micromaser emerging from fluctuations in the laser field. In the solid-state scenario it may describe communication by electrons or chiral quasiparticles [29] sent down a mesoscopic channel where they interact with optical phonons. As an effective model Eq. (3) has a broad range of applications since the unconventional environment [17] describes the most relevant part of the interaction with a bunch of phonon modes producing qubit radiative decay. In such solid-state systems the phonon dephasing time τ_ϕ is expected to be much shorter than the phonon decay time scale τ_d. In these cases we have shown that a setup without memory resetting is convenient in order to increase the rate of transmission of quantum information and private classical information. The noisy quantum channel Eq. (3) also describes the dynamics of a quantum memory [16], which may be implemented by coupling N superconducting qubits to a microstrip cavity, in a circuit-QED [19] architecture. In this case, the use of cavities with moderate quality factor [30] might be a good trade-off between reducing decoherence and avoiding cross-talks generating entanglement between the qubits crossing the channels. Our results show that in such situation it is convenient to use the channel without resetting to increase the rate of sequential processing of each qubit.

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